GLOBAL BOUNDS FOR THE LYAPUNOV EXPONENT AND THE INTEGRATED DENSITY OF STATES OF RANDOM SCHRÖDINGER OPERATORS IN ONE DIMENSION

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ABSTRACT. In this article we prove an upper bound for the Lyapunov exponent $\gamma(E)$ and a two-sided bound for the integrated density of states $N(E)$ at an arbitrary energy $E > 0$ of random Schrödinger operators in one dimension. These Schrödinger operators are given by potentials of identical shape centered at every lattice site but with non-overlapping supports and with randomly varying coupling constants. Both types of bounds only involve scattering data for the single-site potential. They show in particular that both $\gamma(E)$ and $N(E) - \sqrt{E}/\pi$ decay at infinity at least like $1/\sqrt{E}$. As an example we consider the random Kronig-Penney model.

1. INTRODUCTION

In this article we will consider random Schrödinger operators $H(\omega)$ in $L^2(\mathbb{R})$ of the form

$$H(\omega) = H_0 + V_\omega, \quad H_0 = -\frac{d^2}{dx^2}, \quad V_\omega = \sum_{j\in\mathbb{Z}} \alpha_j(\omega) f(\cdot - j), \tag{1}$$

where $\{\alpha_j(\omega)\}_{j\in\mathbb{Z}}$ is a sequence of i.i.d. (independent, identically distributed) variables on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having a common distribution measure κ (i.e. $\mathbb{P}\{\alpha_j \in \Delta\} = \kappa(\Delta)$ for any Borel set $\Delta \subset \mathbb{R}$). In what follows we always suppose that κ is supported on a compact interval and the single-site potential f is integrable with support in the interval [-1/2,1/2]. Moreover, the random variables are assumed to form a stationary, metrically transitive random field, i.e. there are measure preserving ergodic transformations $\{T_j\}_{j\in\mathbb{Z}}$ such that $\alpha_j(T_k\omega) = \alpha_{j-k}(\omega)$ for all $\omega \in \Omega$. The spectral properties of the operator (1) were studied in detail in [9, 7, 11, 16, 22]. The results are most complete for the case when f is the point interaction (see [1]).

The integrated density of states $N(E)$ and the Lyapunov exponent $\gamma(E)$ are important quantities associated with operators of the form (1) (see e.g. [4]). In particular, according to Ishii-Pastur-Kotani theorem [15] the set $\{E : \gamma(E) = 0\}$ is the essential support of the absolute continuous part of the spectral measure for $H(\omega)$.

The main idea of our approach is to approximate the operator (1) by means of the sequence

$$H^{(n)}(\omega) = H_0 + \sum_{j=-n}^{n} \alpha_j(\omega) f(\cdot - j)$$

with unchanged H_0, which converges to $H(\omega)$ in the strong resolvent sense. This differs from the usual approach where one puts the whole system in a box, which then tends to infinity (see e.g. [4]). In [12] (see also [13]) we used this approximation to invoke scattering theory for the study the spectral properties of the limiting operator (1). Some other applications of scattering

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theory to the study of spectral properties of such type Schrödinger operators in one dimension can be found in [11] and [22].

One of the important ingredients of our approach developed in [12] is the Lifshitz-Krein spectral shift function. The spectral shift function naturally replaces the eigenvalue counting function usually used to construct the density of states for the operator (1). The celebrated Birman-Krein theorem (see e.g. [3]) relates the spectral shift function to scattering theory. In fact, up to a factor $-\pi^{-1}$ it may be identified with the scattering phase for the pair ($H^{(n)}(\omega)$, H_0), i.e. $\xi^{(n)}(E;\omega) = -\pi^{-1}\delta^{(n)}(E;\omega)$ when $E > 0$,

$$\delta^{(n)}(E;\omega) = \frac{1}{2i}\log\det S^{(n)}(E;\omega) = \frac{1}{2i}\log\det\begin{pmatrix} T_\omega^{(n)}(E) & R_\omega^{(n)}(E) \\ L_\omega^{(n)}(E) & T_\omega^{(n)}(E)\end{pmatrix}.$$

Here $|T^{(n)}(E)|^2$ and $|R^{(n)}(E)|^2 = |L^{(n)}(E)|^2$ have the meaning of transmission and reflection coefficients, respectively, such that $|T^{(n)}(E)|^2 + |R^{(n)}(E)|^2 = 1$. For $E < 0$ the quantity $\xi^{(n)}(E;\omega)$ equals minus the counting function for $H^{(n)}(\omega)$.

In particular in [12] we proved the almost sure existence of the limit

$$\xi(E) = \lim_{n\to\infty}\frac{\xi^{(n)}(E;\omega)}{2n+1}, \tag{2}$$

which we called the spectral shift density. Also we proved the equality $\xi(E) = N_0(E) - N(E)$, where $N(E)$ and $N_0(E) = \pi^{-1}[\max(0,E)]^{1/2}$ are the integrated density of states of the Hamiltonians $H(\omega)$ and H_0 respectively. This result also extends to higher dimension in the continuous [14] and discrete [5] cases. Also we showed that almost surely the Lyapunov exponent $\gamma(E)$ at energy $E > 0$ is given as

$$\gamma(E) = -\lim_{n\to\infty}\frac{\log|T^{(n)}(E;\omega)|}{2n+1}, \tag{3}$$

where $T^{(n)}(E,\omega)$ is the transmission amplitude for the pair of Hamiltonians ($H^{(n)}(\omega)$, H_0) at energy E. We recall that $\gamma(E)$ is defined as the upper Lyapunov exponent for the fundamental matrix at energy E of the Schrödinger operator $H(\omega)$. The connection between the Lyapunov exponent and the transmission coefficient $|T_\omega^{(n)}(E)|$ was recognized long ago [17, 18]. A complete proof has appeared in [12].

We note that the theory of the spectral shift function was also recently used to show that the integrated density of states is independent of the choice of boundary conditions [19] on the sides of a large box, in which the system is put.

The conditions on the random variables α_j and the single-site potential f stated above are slightly weaker than those in [12]. However the results of [12] which will be used below remain valid also in this more general case.

The aim of the present paper is to prove *global* bounds for the Lyapunov exponent and the integrated density of states, i.e. bounds which hold for all $E > 0$ and describe the correct asymptotic behavior in the limit $E \to \infty$. These results are formulated as Theorems 1 and 2 below. To the best of our knowledge the first article to look for the asymptotic behavior of $\gamma(E)$ and $N(E)$ in the limit $E \to \infty$ is [2]. The best known estimate for the integrated density of states is due to Kirsch and Martinelli [10, Corollary 3.1]. This bound however does not reproduce the correct asymptotic behavior of $N(E)$ in the large energy limit. Another estimate, which is due to Pastur and Figotin (see [20, Sec. V.11.B]), is valid for an $\mathbb{R}$-metrically transitive random field. Since our potential $V_\omega(x)$ is a $\mathbb{Z}$-metrically transitive field this estimate does not apply directly to the present situation. Our two-sided estimate leads to the bound (23) below which is very close to that of Pastur and Figotin.

In what follows C will denote a finite positive generic constant varying with the context, but which depends only on f and κ.

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2. THE LYAPUNOV EXPONENT

We recall that the scattering matrix $S(E)$ for a pair of Hamiltonians $(H,\, H_0)$ on $L^2(\mathbb{R})$ at fixed energy $E \geq 0$ is a 2×2 unitary matrix

$$S(E) = \begin{pmatrix} T(E) & R(E) \\ L(E) & T(E) \end{pmatrix}, \tag{4}$$

where $L(E)$ and $R(E)$ denote the left and right reflection amplitudes respectively. The transmission amplitude $T(E)$ can vanish only for $E = 0$ (see [8, 6]). To any S-matrix (4) we associate the unimodular matrix

$$\Lambda(E) = \begin{pmatrix} \frac{1}{T(E)} & -\frac{R(E)}{T(E)} \\ \frac{L(E)}{T(E)} & \frac{1}{\overline{T(E)}} \end{pmatrix}.$$

Let $T_\alpha(E)$, $R_\alpha(E)$, $L_\alpha(E)$ be the elements of the S-matrix at energy E for the pair of operators $(H_0 + \alpha f,\, H_0)$ and $\Lambda_\alpha(E)$ the corresponding Λ-matrix. Also let $\widetilde{\Lambda}_\alpha(E) = U_E^{1/2} \Lambda_\alpha(E) U_E^{1/2}$ with

$$U_E = \begin{pmatrix} e^{i\sqrt{E}} & 0 \\ 0 & e^{-i\sqrt{E}} \end{pmatrix}.$$

Explicitly we have

$$\widetilde{\Lambda}_\alpha(E) = \begin{pmatrix} \frac{e^{i\sqrt{E}}}{T_\alpha(E)} & -\frac{R_\alpha(E)}{T_\alpha(E)} \\ \frac{L_\alpha(E)}{T_\alpha(E)} & \frac{e^{-i\sqrt{E}}}{\overline{T(E)}} \end{pmatrix}.$$

Consider the matrix

$$A(E) = \mathbb{E}\left\{ \widetilde{\Lambda}_{\alpha(\omega)}(E)^\dagger \widetilde{\Lambda}_{\alpha(\omega)}(E) \right\} = \int \widetilde{\Lambda}_\alpha(E)^\dagger \widetilde{\Lambda}_\alpha(E) d\kappa(\alpha) \geq 0,$$

where for brevity we write $\alpha(\omega)$ instead of $\alpha_j(\omega)$ with some $j \in \mathbb{Z}$. Let $\beta_+(E)$ be the largest eigenvalue of $A(E)$ and $\beta_-(E)$ the smallest. It will turn out below that $\beta_+(E) \geq 1$. Set $\widetilde{\gamma}(E) = (\log \beta_+(E))/2 \geq 0$.

The first main result of the present article is

Theorem 1. *Given the Hamiltonian* (1) *and the distribution κ for the coupling constant α, for all $E > 0$ the resulting Lyapunov exponent satisfies the upper bound*

$$\gamma(E) \leq \widetilde{\gamma}(E). \tag{5}$$

In particular $\gamma(E)$ decays at least like $1/\sqrt{E}$ at infinity.

Proof. Let $\Lambda^{(n)}(E;\omega)$ denote the Λ-matrix for the pair $(H^{(n)}(\omega),\, H_0)$, which by the factorization property can be represented in the form

$$\Lambda^{(n)}(E;\omega) = U_E^{-n-1/2} \prod_{j=-n}^{n} \widetilde{\Lambda}_{\alpha_j(\omega)}(E) \cdot U_E^{-n-1/2}. \tag{6}$$

In fact, this factorization property is a consequence of the multiplicativity property of the fundamental matrix (see [12] for a proof and for references to earlier work). A short calculation gives

$$|T^{(n)}(E;\omega)|^{-2} = \frac{1}{4} \mathrm{tr} \left(\Lambda^{(n)}(E;\omega)^\dagger \Lambda^{(n)}(E;\omega) \right) + \frac{1}{2}. \tag{7}$$

With $\mathbb{E}$ denoting the expectation with respect to the measure $\mathbb{P}$, by Jensen's inequality and (7) we therefore have the estimate

$$\begin{aligned} e^{-2\mathbb{E}\{\log |T^{(n)}(E;\omega)|\}} &\leq \mathbb{E}\Big\{|T^{(n)}(E;\omega)|^{-2}\Big\} \\ &= \frac{1}{4}\mathbb{E}\Big\{\mathrm{tr}\,\Big(\Lambda^{(n)}(E;\omega)^\dagger\Lambda^{(n)}(E;\omega)\Big)\Big\} + \frac{1}{2}. \end{aligned} \tag{8}$$

From the factorization property (6) it follows that

$$\mathrm{tr}\,\Big(\Lambda^{(n)}(E;\omega)^\dagger\Lambda^{(n)}(E;\omega)\Big) = \mathrm{tr}\,\left(\prod_{j=n}^{-n}\widetilde{\Lambda}_{\alpha_j(\omega)}(E)^\dagger \prod_{j=-n}^{n}\widetilde{\Lambda}_{\alpha_j(\omega)}(E)\right). \tag{9}$$

We will now make use of the fact that the $\alpha_k(\omega)$ are i.i.d. random variables. For this purpose define the 2×2 matrices $A_j(E)\geq 0$ recursively by $A_0 = I$ and

$$A_j(E) = \int \widetilde{\Lambda}_\alpha(E)^\dagger A_{j-1}(E)\widetilde{\Lambda}_\alpha(E)d\kappa(\alpha), \tag{10}$$

such that in particular $A(E) = A_1(E)$. Now it is easy to see that

$$\begin{aligned} &\mathbb{E}\Big\{\mathrm{tr}\,\Big(\Lambda^{(n)}(E;\omega)^\dagger\Lambda^{(n)}(E;\omega)\Big)\Big\} \\ &= \mathrm{tr}\,\Big(\mathbb{E}\Big(\Lambda^{(n)}(E;\omega)^\dagger\Lambda^{(n)}(E;\omega)\Big)\Big) = A_{2n+1}(E). \end{aligned} \tag{11}$$

We now use the fact that the operator inequality $0\leq A\leq A'$ implies $0\leq \mathrm{tr}\,A\leq \mathrm{tr}\,A'$ and $B^\dagger AB\leq B^\dagger A'B$ for all B. In particular we have $A(E)\leq \beta_+(E)\,I$ from which we obtain the recursive estimates $A_j(E)\leq\beta_+(E)A_{j-1}(E)\leq\cdots\leq\beta_+(E)^j\,I$ and hence

$$\mathbb{E}\Big\{\mathrm{tr}\,\Big(\Lambda^{(n)}(E;\omega)^\dagger\Lambda^{(n)}(E;\omega)\Big)\Big\} \leq 2\beta_+(E)^{2n+1}. \tag{12}$$

We remark that with the same arguments one proves the lower bound

$$2\beta_-(E)^{2n+1}\leq \mathbb{E}\Big(\mathrm{tr}\,\Big(\Lambda^{(n)}(E;\omega)^\dagger\Lambda^{(n)}(E;\omega)\Big)\Big).$$

The relation (3), the estimate (12) combined with (8) and Fatou's lemma imply now

$$\begin{aligned} \gamma(E) &\leq \frac{1}{2}\lim_{n\to\infty}\frac{\log\mathbb{E}\Big\{|T^{(n)}(E;\omega)|^{-2}\Big\}}{2n+1} \\ &\leq \frac{1}{2}\lim_{n\to\infty}\frac{\log\left(\beta_+(E)^{2n+1}/2+1/2\right)}{2n+1} = \frac{1}{2}\log\beta_+(E), \end{aligned}$$

which proves the claim (5).

To establish the last claim of the theorem we recall the following well known estimates (see e.g. [8, 6])

$$|T_\alpha(E)-1| + |R_\alpha(E)| \leq C\frac{1}{\sqrt{E}} \tag{13}$$

valid for all large $E>0$ uniformly for all α in the (compact) support of κ for fixed f. Using the estimate (13) in (14) gives the estimate $\beta_+(E)\leq 1 + C/\sqrt{E}$ for all large E. Since $\widetilde{\gamma}(E) = (\log\beta_+(E))/2$, this concludes the proof of the theorem. □

Since $\gamma(E)\geq 0$, we obviously have the inequality $\beta_+(E)\geq 1$ for almost all E. We will give now a direct independent proof of this fact and simultaneously obtain an expression for $\beta_+(E)$. The matrix $A(E)$ may be written in the form

$$A(E) = \begin{pmatrix} a(E) & b(E) \\ \overline{b(E)} & a(E)\end{pmatrix}$$

with

$$a(E) = \int \left(\frac{2}{|T_\alpha(E)|^2} - 1 \right) d\kappa(\alpha) \tag{14}$$

$$b(E) = -e^{i\sqrt{E}} \int \frac{R_\alpha(E)}{T_\alpha(E)^2} d\kappa(\alpha). \tag{15}$$

This gives the two eigenvalues of $A(E)$ in the form

$$\beta_\pm(E) = a(E) \pm |b(E)| \tag{16}$$

Obviously $a(E) \geq 1$ and hence $\beta_+(E) \geq 1$. In fact, $a(E) = 1$ is possible if and only if $R_\alpha(E) = 0$ for almost all α in the support of κ. Then also $b(E) = 0$ and $\beta_+(E) = 1$. Actually (if $\operatorname{supp}\kappa$ has at least one non-isolated point) we do not believe there are nontrivial f and E for which this holds but in any case for such E's the Lyapunov exponent vanishes as is easily verified (see also [12]), so this is a trivial confirmation of estimate (5) in this case. In the remaining case we trivially have $\beta_+(E) > 1$.

As an example we consider the random Kronig - Penney model which is formally obtained from $H(\omega)$ by replacing f with the Dirac δ-function at the origin. Then we have (correcting for a misprint on page 232 of [12])

$$T_\alpha(E) = \left(1 + i\frac{\alpha}{2\sqrt{E}}\right)^{-1} \tag{17}$$

$$R_\alpha(E) = -i\frac{\alpha}{2\sqrt{E}} \left(1 + i\frac{\alpha}{2\sqrt{E}}\right)^{-1} \tag{18}$$

and our method still applies. This gives

$$a(E) = 1 + \frac{\langle \alpha^2 \rangle}{4E} \tag{19}$$

$$b(E) = i\frac{\langle \alpha \rangle}{2\sqrt{E}} - \frac{\langle \alpha^2 \rangle}{4E}. \tag{20}$$

Here for brevity by $\langle\ \rangle$ we denote the mean with respect to the probability measure $\mathbb{P}$ such that

$$\langle \alpha \rangle = \mathbb{E}\{\alpha(\omega)\} = \int \alpha d\kappa(\alpha), \qquad \langle \alpha^2 \rangle = \mathbb{E}\{\alpha(\omega)^2\} = \int \alpha^2 d\kappa(\alpha).$$

In particular (19) gives

$$\beta_+(E) = 1 + \frac{\langle \alpha^2 \rangle}{4E} + \frac{1}{2\sqrt{E}} \left(\frac{\langle \alpha^2 \rangle}{4E} + \langle \alpha \rangle^2 \right)^{1/2}. \tag{21}$$

So also in this case $\gamma(E)$ decays at least like $1/\sqrt{E}$ as $E \to \infty$ and at least like $1/E$ if the mean $\langle \alpha \rangle$ of α vanishes, i.e. if on average the coupling constant is zero.

3. The integrated density of states

We denote by $\xi_\alpha(E)$ the spectral shift function for the pair $(H_0 + \alpha f, H_0)$. The second main result of this article is given by

Theorem 2. *For all $E > 0$ the spectral shift density $\xi(E)$ for the operator* (1) *satisfies the following two-side bound*

$$\mathbb{E}\{\xi_{\alpha(\omega)}(E)\} - r(E) \leq \xi(E) \leq \mathbb{E}\{\xi_{\alpha(\omega)}(E)\} + r(E), \tag{22}$$

where

$$r(E) = \min\left\{ \frac{1}{2}, \frac{1}{\pi}\mathbb{E}\left\{ \frac{|R_{\alpha(\omega)}(E)}{1 - |R_{\alpha(\omega)}(E)|} \right\} \right\}.$$

In particular $\mathbb{E}\{\xi_{\alpha(\omega)}(E)\}$ *and* $r(E)$ *decays at least like* $1/\sqrt{E}$ *at infinity.*

Remarks. 1. One can easily prove the following estimate

$$\mathbb{E}\{\xi_{\alpha(\omega)}(E)\} - 1 \leq \xi(E) \leq \mathbb{E}\{\xi_{\alpha(\omega)}(E)\} + 1,$$

which is valid for all $E \in \mathbb{R}$.

2. By the monotonicity of the spectral shift function with respect to perturbation $\xi(E) \geq 0$ if $\operatorname{supp} \kappa \subset \mathbb{R}_+$ and $\xi(E) \leq 0$ if $\operatorname{supp} \kappa \subset \mathbb{R}_-$ for almost all $E > 0$.

3. For large $E > 0$ by (13)

$$r(E) = \min\left\{\frac{1}{2}, \frac{1}{\pi}\mathbb{E}\left\{\frac{|R_{\alpha(\omega)}(E)|}{1 - |R_{\alpha(\omega)}(E)|}\right\}\right\} = \frac{1}{\pi}\mathbb{E}\left\{\frac{|R_{\alpha(\omega)}(E)|}{1 - |R_{\alpha(\omega)}(E)|}\right\} \leq \frac{C}{\sqrt{E}}.$$

4. In [12] we proved the relation $\xi(E) = N_0(E) - N(E) = \sqrt{E}/\pi - N(E)$, where $N_0(E)$ is the integrated density of states for the free operator H_0. Theorem 2 then gives the following two-sided bound for the integrated density of states

$$\frac{\sqrt{E}}{\pi} - \mathbb{E}\{\xi_{\alpha(\omega)}(E)\} - r(E) \leq N(E) \leq \frac{\sqrt{E}}{\pi} - \mathbb{E}\{\xi_{\alpha(\omega)}(E)\} + r(E), \quad E > 0. \tag{23}$$

There are some other upper bounds on the integrated density of states. A well-known result is a one-sided bound due to Kirsch and Martinelli [10, Corollary 3.1],

$$N(E) \leq \frac{C}{\sqrt{\eta}}\,\mathbb{E}\left\{\int_{-1/2}^{1/2} (E + \eta - V_\omega(x))_+ dx\right\}$$

for any $\eta > 0$ and all $E \in \mathbb{R}$. This bound however does not reproduce the correct asymptotic behavior of $N(E)$ in the large energy limit.

5. The bounds (5) and (22) are of interest in the context of the Thouless formula (see e.g. [20])

$$\gamma(E) - \gamma_0(E) = -\int_{\mathbb{R}} \log|E - E'|\, d\xi(E'), \qquad E \in \mathbb{R}, \tag{24}$$

where $\gamma_0(E) = [\max(0, -E)]^{1/2}$ is the Lyapunov exponent for H_0. The Thouless formula in the form (24) can be viewed as a subtracted dispersion relation (see e.g. [12]).

Proof. In [12] we proved (see Theorem 3.3 there and its proof) that for any two potentials V_1 and V_2 with (compact) disjoint supports one has

$$\xi(E; H_0 + V_1 + V_2, H_0) = \xi(E; H_0 + V_1, H_0) + \xi(E; H_0 + V_2, H_0) + \xi_{12}(E)$$

with

$$\xi_{12}(E) = -\frac{1}{2\pi i}\log\frac{1 - R_1(E)\,L_2(E)}{1 - \overline{R_1(E)}\,\overline{L_2(E)}},$$

where $R_k(E)$ and $L_k(E)$ are the right and left reflection coefficients for the Schrödinger equation with the potential V_k, $k = 1, 2$. Actually Theorem 3.3 in [12] states that $|\xi_{12}(E)| \leq 1/2$ for all $E \geq 0$. Now we improve on this estimate. As in [12] we set

$$L_k(E) = a_k(E)e^{i\delta_k^{(L)}}, \quad R_k(E) = a_k(E)e^{i\delta_k^{(R)}}, \quad k = 1, 2$$

with $0 \leq a_k(E) \leq 1$. Moreover $a_k(E) = 1$ only when $T_k(E) = 0$, which we recall can happen only if $E = 0$. Therefore

$$\begin{aligned}
&\log \frac{1 - R_1(E)\, L_2(E)}{1 - \overline{R_1(E)}\ \overline{L_2(E)}} \\
&= \log \frac{1 - a_1(E)a_2(E)e^{i(\delta_1^{(R)} + \delta_2^{(L)})}}{1 - a_1(E)a_2(E)e^{-i(\delta_1^{(R)} + \delta_2^{(L)})}} \\
&= -2i \arctan \frac{a_1(E)a_2(E)\sin(\delta_1^{(R)} + \delta_2^{(L)})}{1 - a_1(E)a_2(E)\cos(\delta_1^{(R)} + \delta_2^{(L)})}.
\end{aligned}$$

By means of the inequality $|\arctan x| \leq |x|$ we immediately obtain

$$|\xi_{12}(E)| \leq \min\left\{\frac{1}{2}, \frac{1}{\pi}\frac{a_1(E)a_2(E)}{1 - a_1(E)a_2(E)}\right\}. \tag{25}$$

Since $0 \leq a_k(E) < 1$ we can replace $a_1(E)a_2(E)(1 - a_1(E)a_2(E))^{-1}$ either by $a_1(E)(1 - a_1(E))^{-1}$ or by $a_2(E)(1 - a_2(E))^{-1}$.

Now let us consider the operator $H^{(n)}(\omega)$ for finite n. Applying the inequality (25) we obtain

$$\begin{aligned}
&\left|\xi^{(n)}(E;\omega) - \xi_{\alpha_n(\omega)}(E) - \xi_{\alpha_{-n}(\omega)}(E) - \xi^{(n-1)}(E;\omega)\right| \\
&\leq \quad \min\left\{\frac{1}{2}, \frac{1}{\pi}\frac{|R_{\alpha_n(\omega)}(E)|}{1 - |R_{\alpha_n(\omega)}(E)|}\right\} + \min\left\{\frac{1}{2}, \frac{1}{\pi}\frac{|R_{\alpha_{-n}(\omega)}(E)|}{1 - |R_{\alpha_{-n}(\omega)}(E)|}\right\}
\end{aligned}$$

Repeating this procedure recursively we obtain

$$\left|\xi^{(n)}(E;\omega) - \sum_{j=-n}^{n} \xi_{\alpha_j(\omega)}(E)\right| \leq \sum_{j=-n}^{n} \min\left\{\frac{1}{2}, \frac{1}{\pi}\frac{|R_{\alpha_j(\omega)}(E)|}{1 - |R_{\alpha_j(\omega)}(E)|}\right\}.$$

From the existence of the spectral shift density (2) by the Birkhoff ergodic theorem it follows that

$$\left|\xi(E) - \mathbb{E}\left\{\xi_{\alpha(\omega)}(E)\right\}\right| \leq \mathbb{E}\left\{\min\left\{\frac{1}{2}, \frac{1}{\pi}\frac{|R_{\alpha(\omega)}(E)|}{1 - |R_{\alpha(\omega)}(E)|}\right\}\right\}.$$

From the obvious inequality

$$\mathbb{E}\left\{\min\left\{\frac{1}{2}, \frac{1}{\pi}\frac{|R_{\alpha(\omega)}(E)|}{1 - |R_{\alpha(\omega)}(E)|}\right\}\right\} \leq \min\left\{\frac{1}{2}, \frac{1}{\pi}\mathbb{E}\left\{\frac{|R_{\alpha(\omega)}(E)|}{1 - |R_{\alpha(\omega)}(E)|}\right\}\right\}$$

the bound (22) follows.

For large E we have the following asymptotics [6] uniformly in α on compact sets:

$$\begin{aligned}
R_\alpha(E) &= \frac{\alpha}{2i\sqrt{E}}\int_{\mathbb{R}} e^{2i\sqrt{E}t} f(t)dt + O(E^{-1}), \\
L_\alpha(E) &= \frac{\alpha}{2i\sqrt{E}}\int_{\mathbb{R}} e^{-2i\sqrt{E}t} f(t)dt + O(E^{-1})
\end{aligned}$$

such that $R_\alpha(E) = O(1/\sqrt{E})$ and $L_\alpha(E) = O(1/\sqrt{E})$. If the single-site potential f has p derivatives in $L^1(\mathbb{R})$ then $L_\alpha(E) = O(E^{-(p+1)/2})$ and $R_\alpha(E) = O(E^{-(p+1)/2})$ as $E \to \infty$ [6]. The estimate $\mathbb{E}\{\xi_{\alpha(\omega)}(E)\} = O(1/\sqrt{E})$ is Proposition 3 below. □

As an example we consider again the random Kronig-Penney model. The single-site spectral shift function is given in this case by

$$\xi_\alpha(E) = \frac{1}{\pi}\arctan\left(\frac{\alpha}{2\sqrt{E}}\right), \quad E > 0.$$

Therefore

$$\mathbb{E}\Big\{\xi_{\alpha(\omega)}(E)\Big\} = \frac{1}{\pi}\int_{\mathbb{R}} \arctan\left(\frac{\alpha}{2\sqrt{E}}\right) d\kappa(\alpha)$$

and thus

$$\left|\mathbb{E}\Big\{\xi_{\alpha(\omega)}(E)\Big\}\right| \leq \frac{\langle|\alpha|\rangle}{2\pi\sqrt{E}}.$$

Using the explicit expression for the reflection amplitude one can easily show that

$$\frac{\langle|\alpha|\rangle}{2\sqrt{E}} + \frac{\langle\alpha^2\rangle}{4E} \leq \mathbb{E}\left\{\frac{|R_{\alpha(\omega)}(E)|}{1-|R_{\alpha(\omega)}(E)|}\right\} \leq \frac{\langle|\alpha|\rangle}{2\sqrt{E}} + \frac{\langle\alpha^2\rangle}{2E}.$$

We complete this section with an estimate on $\mathbb{E}\{\xi_{\alpha(\omega)}(E)\}$ in the general case. We will prove

Proposition 3. *There is a constant $c > 0$ independent of E, f, and κ such that for all $E > 0$*

$$\left|\mathbb{E}\Big\{\xi_{\alpha(\omega)}(E)\Big\}\right| \leq \frac{C}{2\sqrt{E}}\mathbb{E}\Big\{|\alpha(\omega)|^{1/2}\Big\}^2\int_{-1/2}^{1/2}|f(x)|dx.$$

Let $l^{1/2}(L^1)$ denote the Birman-Solomyak class of measurable functions V for which

$$\|V\|_{l^{1/2}(L^1)} = \left[\sum_{j=-\infty}^{\infty}\left(\int_{j-1/2}^{j+1/2}|V(x)|dx\right)^{1/2}\right]^2 < \infty.$$

The claim of the proposition immediately follows from the following

Lemma 4. *Let $V \in l^{1/2}(L^1)$. There is a constant c_1 independent of V and E such that*

$$|\xi(E; H_0+V, H_0)| \leq \frac{c_1}{2\sqrt{E}}\|V\|_{l^{1/2}(L^1)}$$

for all $E > 0$.

Proof. As proved in [23] there is a constant $c_2 > 0$ independent of E and V such that

$$|\xi(E; H_0+V, H_0)| \leq C_1\|V^{1/2}R_0(E+i0)|V|^{1/2}\|_{\mathcal{J}_1},$$

where $V^{1/2} = \mathrm{sign}V\ |V|^{1/2}$, $R_0(z) = (H_0 - z)^{-1}$, and $\|\cdot\|_{\mathcal{J}_1}$ denotes the trace class norm (see e.g. [21]). From the proof of Proposition 5.6 in [21] it follows that

$$\|V^{1/2}R_0(E+i0)|V|^{1/2}\|_{\mathcal{J}_1} \leq \frac{c_3}{\sqrt{E}}\|V\|_{l^{1/2}(L^1)}$$

for all $E > 0$. □

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