Dynamically defined recurrence dimension

Petr Kůrka, Vincent Penné, Sandro Vaienti

May 18, 2000

Abstract

In this paper, we introduce and characterize the *dynamically defined recurrence dimension*, a new topological invariant number, following the idea of a previous article [9], but with some modifications and improvements. We study some example of Toeplitz subshifts for which we can show that the recurrence dimension is a topologically invariant number different from the *topological entropy*.

1 Introduction

In [1] V. Afraimovich proposed to study the scaling properties of recurrence time by defining, through the Caratheodory construction, a dimension called either *recurrence dimension* or *Afraimovich-Pesin* (AP)*dimension* in [9]. This dimension is a variant of the Hausdorff dimension, with the diameter of a set replaced by some gauge functions of the smallest return time of the set into itself (Poincaré recurrence of the set). Afraimovich investigated mainly irrational rotations and used $1/t$ as a gauge function. In [9], a systematic study of the AP-dimension was carried out with the use of another gauge function, e^{-t} , and we remind here the principal result of this analysis:

1. It was proved that the AP-dimension is a topological invariant.

2. It was proved the equivalence of the AP-dimension with the topological entropy for a large class of dynamical systems (subshifts of finite type and Axiom-A systems), and a few examples were exhibited where the AP-dimension was different from the topological entropy.

3. It was proved a general lower bound for the AP-dimension in terms of the asymptotic distribution of the periodic points, $\limsup_{n\to\infty} \log \frac{Per(n)}{n}$.

4. The notion of the AP-dimension of a measure was introduced, as the infimum of the AP-dimensions of the sets of full measure ; it was proved that is was a metric invariant and that for aperiodic dynamical systems its value was always zero.

All these results were obtained by considering general open and closed covers in the Caratheodory construction, in the attempt to conform the theory to the usual construction of the Hausdorff measures. If, from one hand, this approach allowed to give general topological and metric characterizations like those quoted above, on the other hand it revealed difficult to compute the AP-dimension in particular examples and especially to answer the basic question adressed in [9], namely: is it possible to exhibit a system where the AP-dimension is different from the asymptotic distribution of periodic points? Translated to minimal dynamical systems, this would mean to find a system with positive AP-dimension: in this case the AP dimension will really impose as a new topological invariant to classify such systems and describe their recurrence properties.
A first step in this direction was obtained by Bruin [4]; he considered subshifts and he proved that the AP-dimension does not exceed its topological entropy; moreover he provided a non trivial example where the bound is strict.
A second improvement was given by Kurka and Maas [7]. Trying to answer the question adressed above and motivated by Bruin's result, these authors studied large classes of minimal subshifts with positive topological entropy (Toeplitz subshifts and Vershik systems defined by Bratelli diagrams): in all these cases they showed that the AP-dimension is zero. The conclusion of Kurka and Maas was that the class of open covers used in the definition of the AP-dimension was too large to get a distinguishing topological invariant. Consequently and still in subshifts, they modified the definition by replacing the open covers with dynamically defined partitions into cylinders of the same type of those used in the computation of topological entropy and pressure for non-compcat sets [10]. This new structure enjoyed the properties (1) and (3) quoted above and Bruin's general bound in terms of the topological entropy; moreover the associated AP-dimension, which Kurka and Maas called *recurrence dimension* and from now on we will conform to this definition, was found to be positive for a particular Toeplitz subshift.

In the present article we follow the same strategy with a few differences and novelties:

- We slightly modify the partition into cylinders (see Def. 1 below), with the aim to obtain a set function, arising from the Caratheodory construction, which is a *Borel measure*. This will allow us to get some interesting topological characterizations of the recurrence dimension, besides those quoted above which still persist, for example the specification of the recurrence dimension on the non-wandering set. Moreover our definition could be extended to non zero-dimensional dynamical systems; we discuss this shortly later although some problems remain to be solved to get all the desired properties in the general case.

- With our new partition, we are able to construct Toeplitz subshifts with positive recurrence dimension and Toeplitz subshifts where the recurrence dimension and the topological entropy differ.

- In section 3 we introduce the *polynomial recurrence dimension* by using the gauge function $1/t$ in the Caratheodory sum. This dimension will be related to what we call *polynomial entropy*, which is the exponent of the algebraic growth of the complexity function, whenever the topological entropy of the subshift is zero. The polynomial recurrence dimension is therefore computed for sturmian subshifts, thus completing and improving the previous work of Afraimovich on irrational rotations [1], and finally for some Toeplitz flows for which results of general type hold (Examples 5 and 6).

The Caratheodory structures used in this paper are close to the partition functions employed in the thermodynamic formalism. Recently Afraimovich and co-workers [2] constructed partition functions where, besides the exponential of negative Poincaré recurrence of a cylinder, there is some power of the measure of the cylinder. This would allow to get some sort of multifractal analysis of the systems through the definition of a whole spectrum of recurrence dimensions. The meaning of these "generalized recurrence dimensions" is not yet clear and their computation and interpretation in the context of the Toeplitz flows analysed in this paper is surely an interesting challenge.
We briefly spoked, at the beginning of this Introduction, of the recurrence dimension of a mesure; its value was always zero with the former definition of the AP-dimension. It would be interesting to reconsider this dimension in the framework develloped in this paper. There is no reason that it is again zero; on the contrary it could classify the invariant measures or, in the case of a uniquely ergodic systems, it could give a new metric invariant.
As a final remark, we would like to point out that the main interest of this paper was to study the recurrence dimension for minimal sets, just to show that it is a non-trivial topological invariant and we investigated in detail its relationship with other ergodic quantities like topological entropy and the algebraic growth rate of complexity. The variety of behaviors of this dimension shows that it captures fine combinatorial structures of the underlying symbolic systems. The role of this dimension for systems with higher chaotic behavior, partially disclosed in [9] and actually pursued by several people, is a very promising field of research.

2 Exponential dynamic recurrence dimension

2.1 Definition and properties

In this article, unless precised, we will consider a dynamical system to be a couple (X, f), where X is a compact metric space and $f : X \to X$ is a continuous mapping. We consider dynamical systems on zero-dimensional spaces. A space X is zero-dimensional, if for every distinct points $x, y \in X$ there exist disjoint clopen sets $U \ni x$, $V \ni y$. Every zero-dimensional space is homeomorphic to a subspace of $A^{\mathbb{N}}$, where A is a finite alphabet. A clopen partition of a zero-dimensional space X is a system $\mathcal{V} = \{V_a : \ a \in A\}$ of nonempty clopen sets

$V_a \subseteq X$, which are pairwise disjoint and their union is X. The diameter of a clopen partition is $\mathrm{diam}(\mathcal{V}) = \max\{\mathrm{diam}(V) : V \in \mathcal{V}\}$. Let $\mathcal{V} = \{V_a : a \in A\}$ be a clopen partition of X. We'll use the notation $A^* = \bigcup_{k>0} A^k$, i.e. the set of all word of finite length. For $u \in A^* \cup A^{\mathbb{N}}$ put

$$V_u = \{x \in X : \forall i < |u|, f^i(x) \in V_{u_i}\}$$

Then

$$\mathcal{V}^n = \{V_u : u \in A^n, V_u \neq \emptyset\}$$

is also a clopen partition. We say that $\mathcal{V}$ is generating, if $\mathrm{diam}(\mathcal{V}^n) \to 0$ as $n \to \infty$. The Poincaré recurrence time of a subset $Y \subseteq X$ is

$$\tau(Y) = \min\{k > 0 : f^k(Y) \cap Y \neq \emptyset\}$$

Definition 1. *Let (X, f) be a zero-dimensional dynamical system, $\mathcal{V}$ a clopen partition of X, $Y \subseteq X$, and $\alpha > 0$. Using upper and lower limits we define the (exponential) upper and lower recurrence dimensions $\overline{r}(Y, f)$ and $\underline{r}(Y, f)$ of Y as*

$$\begin{aligned}
\overline{M}(Y, f, \mathcal{V}, \alpha) &= \overline{\lim}_{n\to\infty} \sum_{V \in \mathcal{V}^n, V \cap Y \neq \emptyset} e^{-\tau(V)\alpha}, \\
\overline{R}(Y, f, \mathcal{V}) &= \sup\{\alpha > 0 : \overline{M}(Y, f, \mathcal{V}, \alpha) = \infty\}, \\
\overline{m}(Y, f, \alpha) &= \lim_{\epsilon\to 0} \sup\{\overline{M}(Y, f, \mathcal{V}, \alpha) : \mathrm{diam}(\mathcal{V}) < \epsilon, \mathcal{V} \textit{ clopen partition of } X\}, \\
\overline{r}(Y, f) &= \sup\{\alpha > 0 : \overline{m}(Y, f, \alpha) = \infty\}.
\end{aligned}$$

Clearly, the upper and lower recurrence dimensions are topologicaly invariants.

We now explain how this definition has been chosen :

- The set function m is defined similarly as in [9], except that we restrict ourself to a specific family of covering, namely a partition and its refinements.

- We then define M over all partitions in order to get rid of the arbitrary choice of initial partition. Taking partitions of diameter going to zero insure us that we get a topologicaly invariant number. The choice of the upper limit in ϵ rather than the lower limit is done to get the nice property (Proposition 5) that if we consider a generating partition, then the dimension R calculated with this partition will be equal to r. Taking the lower limit would not in general bring this property.

- Finally we define the critical exponent r as the supremum over all α such that $m(X, \alpha) = \infty$, which is of course always well defined. Taking the infimum over all α such that $m(X, \alpha) = 0$ gives a less interesting result since it would give us ∞ for every dynamical systems containing periodic points.

Remark. *One could generalize this definition to higher dimensions using homogeneous spaces and topological partitions. A metric space is homogeneous, if all its nonempty open set has the same topological dimension. A topological partition of a homogeneous space is a finite collection $\mathcal{V}$ of its open sets, such that for $V, V' \in \mathcal{V}$, $V \cap V' = \emptyset$, the (topological) dimension of $\overline{V} \cap \overline{V'}$ is strictly smaller than the dimension of X, and $\{\overline{V} : V \in \mathcal{V}\}$ covers X. In zero-dimensional spaces, topological partitions are just clopen partitions. In real interval, topological partitions consist of finite unions of open intervals.*

¿From now on, we write $r(Y, f)$ whenever the same arguments or statements apply for both $\overline{r}(Y, f)$ and $\underline{r}(Y, f)$, and similarly for m, M and R.

Proposition 1. *If (X, f) is minimal, then*

$$r(A, f) = \inf\{\alpha > 0 : \ m(A, f, \alpha) = 0\}.$$

The proof of this fact can be easily adapted from the original article of V. Afraimovich [1].

Proposition 2. *The set function $m(., f, \alpha)$ is a borelian measure.*

Proof. These are standard arguments for construction of this type. We first check that it is an outer measure. First it is monotonic, i.e. $Y \subset Z \Rightarrow m(Y, f, \alpha) \leq m(Z, f, \alpha)$. It is also $\sigma-$sub-additive, i.e. $m(\bigcup_k U_k, f, \alpha) \leq \sum_k m(U_k, f, \alpha)$. For details of such proof, see for instance [10]. To become a borelian measure, it just needs to verify the following property : $d(Y, Z) > 0 \Rightarrow m(Y \cup Z, f, \alpha) = m(Y, f, \alpha) + m(Z, f, \alpha)$. This follows immediately from the fact that in the definition of M, we take partitions $\mathcal{V}$ with diameter going to zero, thus Y and Z are covered by different sets of the same partitions. □

Proposition 3. *Let (X, f) be a dynamical system and $Y \subseteq X$ an invariant subset. Then*

$$m(Y, f_{|Y}, \alpha) \leq m(Y, f, \alpha) \leq m(X, f, \alpha)$$

$$r(Y, f_{|Y}) \leq r(Y, f) \leq r(X, f)$$

Proof. Let $\mathcal{V}$ be a clopen partition of Y, $\mathrm{diam}(\mathcal{V}) < \varepsilon$, $\eta > 0$, and $V \in \mathcal{V}$. Since V is open in Y, for every $x \in V$ there exists $0 < \delta_x < \eta$, such that $B_{\delta_x}(x) \cap Y \subseteq V$, and $B_{2\delta_x}(x) \cap V' = \emptyset$ for every $V \neq V' \in \mathcal{V}$. Let $K \subset V$ be a finite subset such that $\{B_{\delta_x}(x) : x \in K\}$ is a finite cover of V, and let $W_V = \bigcup_{x \in K} B_{\delta_x}(x)$. Since X is totally disconnected, the balls are clopen, so W_V is clopen too, $V = Y \cap W_V$, and $W_V \cap W_{V'} = \emptyset$, if $V \neq V'$. Let $\mathcal{W}$ be the clopen cover consisting of all $W_V, V \in \mathcal{V}$ and a clopen cover of the complement of $\bigcup_{V \in \mathcal{V}} W_V$. Then $\mathcal{W}$ is a clopen cover of X. We get $\mathcal{V}^n = \{Y \cap W : \ W \in \mathcal{W}^n\}$ and $\tau(Y \cap W) \geq \tau(W)$, so

$$M(Y, f_{|Y}, \mathcal{V}, \alpha) \leq M(Y, f, \mathcal{W}, \alpha)$$

Since $\mathrm{diam}(\mathcal{W}) \leq \varepsilon + 2\eta$ for arbitrary small η, $m(Y, f_{|Y}, \alpha) \leq m(Y, f, \alpha)$. From the preceding Proposition we get $m(Y, f, \alpha) \leq m(X, f, \alpha)$. □

¿From now on, we leave out f in the formulas for m, M, R, r.

Proposition 4. *Let (X,f) be a zero-dimensional dynamical system and*

$$Q(n) = \mathrm{card}\{x \in X : \ f^n(x) = x\}.$$

Then

$$\overline{r}(X) \geq \limsup_{n\to\infty} \frac{\ln Q(n)}{n}, \quad \underline{r}(X) \geq \liminf_{n\to\infty} \frac{\ln Q(n)}{n}$$

Proof. Suppose that for every n, $Q(n)$ is finite. For every n there exists $\varepsilon > 0$, such that if $f^n(x) = x$, $f^n(y) = y$, then $d(x,y) > \varepsilon$. If $\mathrm{diam}(\mathcal{V}) < \varepsilon$, then for every k

$$\sum_{V\in\mathcal{V}^k} e^{-\tau(V)\alpha} \geq Q(n)e^{-n\alpha}$$

If $\alpha < \beta < \limsup_{n\to\infty} \frac{\ln Q(n)}{n}$, then for an infinite number of n, $Q(n)e^{-n\alpha} \geq e^{n(\beta-\alpha)}$, so $\overline{M}(X,\mathcal{V},\alpha) = \infty$, $\overline{m}(X,\alpha) = \infty$ and $\alpha < \overline{r}(X)$. Similarly for lower limits.

□

Proposition 5. *Let $\mathcal{V}$ be a generating clopen partition. Then $r(X) = R(X,\mathcal{V})$.*

We will use the following lemma :

Lemma 1. *Let $\mathcal{U}$ and $\mathcal{V}$ two clopen* generating *partitions of X, then one can find a constant $0 < C < \infty$ independently of α such that*

$$\frac{1}{C}M(X,\mathcal{V},\alpha) \leq M(X,\mathcal{U},\alpha) \leq CM(X,\mathcal{V},\alpha).$$

Proof of the Lemma. We have $\mathrm{diam}(\mathcal{U}^n) \to 0$, $\mathrm{diam}(\mathcal{V}^n) \to 0$ as $n \to \infty$. Thus there exist positive integers m, n such that $\mathcal{U}^{m+1} \succeq \mathcal{V}$ and $\mathcal{V}^{n+1} \succeq \mathcal{U}$, so for every $k > n$, $\mathcal{U}^{k+m} \succeq \mathcal{V}^k \succeq \mathcal{U}^{k-n}$. Every $V \in \mathcal{V}^k$ is a subset of some set $U \in \mathcal{U}^{k-n}$ and a union $V = U_1 \cup \cdots \cup U_p$, of at most $p \leq |\mathcal{U}|^{n+m}$ sets $U_i \in \mathcal{U}^{k+m}$. We get $\tau(V) \leq \tau(U_i)$, so

$$p \cdot e^{-\tau(V)\alpha} \geq e^{-\tau(U_1)\alpha} + \cdots + e^{-\tau(U_p)\alpha}$$

$$\sum_{V\in\mathcal{V}^k} e^{-\tau(V)\alpha} \geq |\mathcal{U}|^{-n-m} \sum_{U\in\mathcal{U}^{k+m}} e^{-\tau(U)\alpha}.$$

Thus $M(X,\mathcal{V},\alpha) \geq |\mathcal{U}|^{-m-n} \cdot M(X,\mathcal{U},\alpha)$. Interchanging $\mathcal{U}$ and $\mathcal{V}$, we obtain the result. □

Proof of the Proposition. It is well known that if there exists a generator, then there exists an $\epsilon > 0$ such that any clopen partition of diameter less than ϵ is also generating. For a given α, let us choose a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ of clopen partitions with diameter going to zero such that $M(X,\mathcal{V}_n,\alpha)$ decreases and converges to $m(X,\alpha)$. Thus, for n big enough, $\mathcal{V}_n$ is generating, and we can apply Lemma 1 with $\mathcal{V}$ and $\mathcal{V}_n$ for each n. If $\alpha < R(X, \mathcal{V})$, then by Lemma 1, $CM(X, \mathcal{V}_n, \alpha) \geq M(X, \mathcal{V}, \alpha) = \infty$, thus since this apply for any n large enough and any $\alpha < R(X, \mathcal{V})$, this shows $m(X, \alpha) = \infty$ and $r(X) \geq R(X, \mathcal{V})$. If $\alpha > R(X, \mathcal{V})$, then for some k large enough, $M(X, \mathcal{V}_k, \alpha) \leq CM(X, \mathcal{V}, \alpha) < \infty$. Then, since the sequence $M(X, \mathcal{V}_n, \alpha)$ is decreasing, this shows that $m(X, \alpha) < \infty$. This is true for any $\alpha > R(X, \mathcal{V})$, thus $r(X) \leq R(X, \mathcal{V})$. □

Recall that a point $x \in X$ is non-wandering, if every its neighborhood has finite recurrence time. It is a known fact that the topological entropy of a dynamical system is equal to the topological entropy of the system restricted to the set of non-wandering points. Here we get a similar result, not as strong because in Proposition 3 we have only an inequality in the first part of the statement.

Proposition 6. *Let (X, f) a zero dimensional dynamical system, and Ω the set of non-wandering points, then $r(X) = r(\Omega)$.*

Proof. If we denote $W = \Omega^c$ the complement of Ω, then for any point $x \in W$, one can find an open set $U(x)$ such that $\tau(U(x)) = \infty$. Since W is separable, the cover $\{U(x) : x \in W\}$ of W has a countable sub-cover $\mathcal{F}$. By σ-additivity, we have $m(W, \alpha) \leq \sum_{F \in \mathcal{F}} m(F, \alpha) = 0$, $m(\Omega, \alpha) = m(X, \alpha) - m(W, \alpha) = m(X, \alpha)$, and $r(X) = r(\Omega)$. □

For a finite alphabet A denote by $A^{\mathbb{N}}$ the space of one-sided sequences of letters of A and $(A^{\mathbb{N}}, \sigma)$ the shift dynamical system. A sub-shift is any subset $\Sigma \subseteq A^{\mathbb{N}}$ which is closed and σ-invariant. We denote by $\mathcal{L}(\Sigma) \subseteq A^*$ the language of words appearing in Σ and $\mathcal{L}^n(\Sigma) = \mathcal{L}(\Sigma) \cap A^n$. The size of a finite set A is denoted by $|A|$, and the length of a word $u \in A^n$ is denoted by $|u| = n$. Denote by $P(n) = |\mathcal{L}^n(\Sigma)|$ the number of cylinders of length n. The topological entropy of a sub-shift is $h(\Sigma) = \lim_{n \to \infty} \frac{\ln P(n)}{n}$.

The following result has first been proved by H.Bruin [4], in a slightly different context.

Proposition 7. *The recurrence dimension of a sub-shift is less or equal to its topological entropy.*

Proof. Let $P(n)$ be the number of cylinders of length n. If $u \in \mathcal{L}^n(\Sigma)$ and $\tau[u] = k < n$, then $u_{k+i} = u_i$ for all $i \in [0, |u|)$, so the number of cylinders of length n with the return time k is at most the number of cylinders of length k. For $\tau[u] = k \geq n$ we use $e^{-\tau[u]} \leq e^{-n}$ to obtain

$$\begin{aligned} \sum_{u \in \mathcal{L}^n(\Sigma)} e^{-\tau[u]\alpha} &\leq \sum_{k=1}^{n} P(k) \cdot e^{-k\alpha} \\ M(\Sigma, \alpha) &\leq \sum_{k=1}^{\infty} P(k) \cdot e^{-k\alpha} \end{aligned}$$

If $\lim_{k\to\infty} \frac{\ln P(k)}{k} < \beta < \alpha$, then there exists k_0, such that for all $k \geq k_0$, $P(k)e^{-k\alpha} \leq e^{k(\beta-\alpha)}$, so $M(\Sigma, \alpha) < \infty$, and $r(\Sigma) \leq \alpha$. Thus we get

$$r(\Sigma) \leq \lim_{k\to\infty} \frac{\ln P(k)}{k} = h(\Sigma)$$

□

Combining Propositions 4 and 7 we get that the recurrence dimension of a sofic sub-shift is equal to its topological entropy.

Proposition 8. *Let $\Sigma \subseteq A^{\mathbb{N}}$ be a sub-shift and $a, b > 0$. Then*

$$\begin{array}{rcl}
\forall n_0, \exists n \geq n_0, \forall u \in \mathcal{L}^n(\Sigma), \tau[u] \leq b|u| & \Rightarrow & \overline{r}(\Sigma) \geq h(\Sigma)/b \\
\exists n_0, \forall n \geq n_0, \forall u \in \mathcal{L}^n(\Sigma), \tau[u] \leq b|u| & \Rightarrow & \underline{r}(\Sigma) \geq h(\Sigma)/b \\
\forall n_0, \exists n \geq n_0, \forall u \in \mathcal{L}^n(\Sigma), \tau[u] \geq a|u| & \Rightarrow & \underline{r}(\Sigma) \leq h(\Sigma)/a \\
\exists n_0, \forall n \geq n_0, \forall u \in \mathcal{L}^n(\Sigma), \tau[u] \geq a|u| & \Rightarrow & \overline{r}(\Sigma) \leq h(\Sigma)/a
\end{array}$$

Proof. 1. Let $\alpha < h(\Sigma)/b$, and choose β with $\alpha < \beta < h(\Sigma)/b$. There exists n_0 such that for all $n \geq n_0$, $P(n) > e^{bn\beta}$. If for all $u \in \mathcal{L}^n(\Sigma)$, $\tau[u] \leq b|u|$, then

$$\sum_{u\in\mathcal{L}^n(\Sigma)} e^{-\tau[u]\alpha} \geq P(n)e^{-bn\alpha} > e^{bn(\beta-\alpha)} \to \infty$$

If this holds for infinite number of n, then $\overline{M}(\Sigma, \mathcal{A}, \alpha) = \infty$ and $\overline{r}(\Sigma) \geq \alpha$, so $\overline{r}(\Sigma) \geq h(\Sigma)/b$.
2. If this holds for all sufficiently large n, then $\underline{M}(\Sigma, \mathcal{A}, \alpha) = \infty$ and $\underline{r}(\Sigma) \geq \alpha$, so $\underline{r}(\Sigma) \geq h(\Sigma)/b$.
3. Let $\alpha > h(\Sigma)/a$, and choose β with $\alpha > \beta > h(\Sigma)/a$. There exists n_1 such that for all $n \geq n_1$, $P(n) < e^{an\beta}$. For an infinite number of n we have $a|u| \leq \tau[u]$ for all $u \in \mathcal{L}^n(\Sigma)$, and

$$\sum_{u\in\mathcal{L}^n(\Sigma)} e^{-\tau[u]\alpha} \leq P(n)e^{-an\alpha} < e^{an(\beta-\alpha)} \to 0$$

Thus $\underline{M}(\Sigma, \mathcal{A}, \alpha) = 0$ and $\underline{r}(\Sigma) \leq \alpha$.
4. If this holds for all sufficiently large n, then $\overline{M}(\Sigma, \mathcal{A}, \alpha) = 0$ and $\overline{r}(\Sigma) \leq \alpha$, so $\overline{r}(\Sigma) \leq h(\Sigma)/a$. It follows $\underline{r}(\Sigma) \leq h(\Sigma)/a$. □

2.2 Examples

We construct several examples of Toeplitz sub-shifts. For a point $x \in A^{\mathbb{N}}$, $p > 0$ put

$$\mathrm{Per}_p(x) = \{k \in \mathbb{N} : \forall n \in \mathbb{N}, x_{k+np} = x_k\}$$

The p-skeleton $y = S_p(x) \in (A \cup \{*\})^{\mathbb{N}}$ of x is obtained from x by replacing x_i by $*$ for every $i \notin \mathrm{Per}_p(x)$, so $y_i = x_i$ if $i \in \mathrm{Per}_p(x)$, and $y_i = *$ otherwise.

A sequence $x \in A^{\mathbb{N}}$ is a Toeplitz sequence, if the union of all $\mathrm{Per}_p(x)$ is $\mathbb{N}$. A Toeplitz sub-shift is the orbit closure of a Toeplitz sequence $\Sigma_x = \overline{\mathfrak{o}(x)}$. If x is a Toeplitz sequence, then $\sigma^p(S_p(x)) = S_p(x)$. An integer $p > 1$ is an essential period of x, if $S_p(x)$ is not periodic with any smaller period than p. A periodic structure for an aperiodic Toeplitz sequence $x \in A^{\mathbb{N}}$ is a sequence $p = (p_i)_{i\geq 0}$ such that every p_i is an essential period of x, p_i divides p_{i+1} and the union of all $\mathrm{Per}_{p_i}(x)$ is $\mathbb{N}$.

Example 1. *There exists a Toeplitz sub-shift Σ with $\underline{r}(\Sigma) \geq h(\Sigma)/2 > 0$.*

Proof. Define sequences $(p_i)_{i\geq 0}$, $(q_i)_{i\geq 0}$ by

$$p_0 = 3,\ q_0 = 2,\ p_{n+1} = p_n \cdot 2^{q_n},\ q_{n+1} = q_n \cdot (2^{q_n} - 1)$$

so $p_1 = 12$ and $q_1 = 6$. Construct a sequence of words $v_n \in \{0, 1, *\}^{p_n}$, $w_n \in \mathbf{2}^{p_n}$.

$$\begin{aligned} v_0 &= \mathtt{0{*}{*}}, & v_1 &= \mathtt{0000{*}{*}0{*}{*}0{*}{*}} \\ w_0 &= \mathtt{000}, & w_1 &= \mathtt{000001010011} \end{aligned}$$

Suppose we have constructed words v_n, w_n, with $|v_n| = |w_n| = p_n$, $|v_n|_* = q_n$. This means that the number of stars in v_n is q_n. Put $v_{n+1} = w_n(v_n)^{2^{q_n}-1}$ and obtain w_{n+1} from v_{n+1} by replacing the stars in every v_n by a different combination of binary digits, different from the combination used in w_n. There are exactly 2^{q_n} possibilities to do this, so every combination is used exactly ones. Since w_n is an initial word of w_{n+1}, their limit $w = \lim\limits_{n\to\infty} w_n$ is a Toeplitz sequence with periodic structure $(p_n)_{n\geq 0}$ and skeletons $(v_n)^\infty$. We show that for the sub-shift $\Sigma = \overline{\mathfrak{o}(w)}$, $\tau[u] \leq 2|u|$ for every $u \in \mathcal{L}(\Sigma)$ with $|u| > 6$. There exist $k > 0$ and $i \geq 1$ such that

$$\frac{p_k}{2} < |u| \leq \frac{p_{k+1}}{2},\ (i-1)p_k \leq |u| < ip_k$$

There exists $v \in \mathcal{L}(\Sigma)^{2ip_k}$ such that $v_{[0,|u|)} = u$ and $v_{[ip_k, ip_k+|u|)} = u$, so $\tau[u] \leq ip_k$. We distinguish two possibilities

$$\begin{aligned} i = 1 \quad &\Rightarrow \quad \tau[u] \leq p_k \leq 2|u| \\ i > 1 \quad &\Rightarrow \quad \tau[u] \leq \frac{i}{i-1}|u| \leq 2|u| \end{aligned}$$

Thus $\underline{r}(\Sigma) \geq h(\Sigma)/2$. To see that the entropy of Σ is positive, note that $P(p_n) \geq 2^{q_n}$ and

$$\frac{\ln 2^{q_{n+1}}}{p_{n+1}} = \ln 2 \cdot \frac{q_{n+1}}{p_{n+1}} = \ln 2 \cdot \frac{q_n}{p_n} \cdot \frac{2^{q_n} - 1}{2^{q_n}}$$

Since $q_n > n$, we have

$$\prod_{n=1}^{\infty} \frac{2^{q_n} - 1}{2^{q_n}} > \prod_{n=1}^{\infty} \frac{2^n - 1}{2^n} > 0$$

so $h(\Sigma) > \limsup_{n\to\infty} \frac{\ln 2^{q_n}}{p_n} > 0$. □

Example 2. *There exists a Toeplitz sub-shift with lower recurrence dimension zero and topological entropy positive.*

Proof. We construct a Toeplitz sub-shift in alphabet $A = \{0,1,2,3\}$. Put

$$n_1 = 4,\ p_1 = 4,\ n_{k+1} = \frac{(n_k-k)!}{(n_k-2k)!} \cdot (n_k-2k)^{2^{k+1}-2k},\ p_k = 2^2 \cdots 2^{k+1}$$

Put $A_k = \{0,1,\dots,n_k-1\}$, so $A_1 = A$, $A_2 = \{0,\dots,11\}$. Construct injective substitutions $\nu_k : A_{k+1} \to (A_k)^{2^{k+1}}$ such that for every $a \in A_{k+1}$, $\nu_k(a)$ starts with $01\dots k-1$, does not contain $0,1,\dots k-1$ elsewhere, and if $\nu_k(a)_i = \nu_k(a)_j$, then $|i-j| \geq k+1$. Thus for every $u \in A_{k+1}^*$, every letter in $\nu_k(u)$ differs from k preceding and k succeeding letters. At position k we have $n_k - k$ possibilities, at position $k+i$ we have $n_k - k - i$ possibilities for $0 \leq i \leq k$ and $n_k - 2k$ for $i \geq k$. Thus there are exactly

$$n_{k+1} = (n_k-k)\cdots(n_k-2k+1)(n_k-2k)^{2^{k+1}-2k}$$

possibilities to construct $\nu_k(a)$. For example, if we choose the words of $\nu_k(a)$ in lexicographic order, we get

$$\begin{aligned} \nu_1 : A_2 &\mapsto \{0121, 0123, 0131, 0132, 0212, 0213, 0231, 0232, \dots, 0323\} \\ \nu_2 : A_3 &\mapsto \{01234234, 01234235, 01234236, 01234237, \dots\} \end{aligned}$$

For every $k \geq 1$, $\Sigma_k = \{\sigma^i \nu_1 \dots \nu_k(u) :\ u \in (A_{k+1})^{\mathbb{N}}, i \geq 0\} \subseteq A^{\mathbb{N}}$ is a sub-shift of finite type, and $\Sigma_{k+1} \subseteq \Sigma_k$. The intersection $\Sigma = \cap_k \Sigma_k$ is a Toeplitz sub-shift with periodic structure $(p_{k+1})_{k \geq 0}$. It is generated by a Toeplitz sequence with skeletons

```
0***0***0***0***0***0***0***0***0***0***0***0***0***0***0*** ...
012101230***0***0***0***0***0***0***0***0***0***012101230*** ...
```

For every $a \in A_2$, $\tau[\nu_1(a)] = 12 = 3|\nu_1(a)|$, and for every $a \in A_{k+1}$,

$$\tau[\nu_1 \dots \nu_k(a)] = (k+2)p_k = (k+2)|\nu_1 \dots \nu_k(a)|$$

If $u \in \mathcal{L}(\Sigma)$, and $|u| = 2p_k$ for some k, then there exists $a \in A_{k+1}$ with $\nu_1 \dots \nu_k(a) \sqsubseteq u$, so

$$\tau[u] \geq \tau[\nu_1 \dots \nu_k(a)] = (k+2)p_k = (\tfrac{k}{2}+1)|u|$$

By Proposition 8, $\underline{r}(\Sigma) \leq \frac{2h(\Sigma)}{k+2}$, so $\underline{r}(\Sigma) = 0$. We have $P(p_k) \geq n_{k+1}$, $n_{k+1} > (n_k - 2k)^{2^{k+1}-k}$, so

$$\frac{\ln n_{k+1}}{p_k} \geq \frac{2^{k+1}-k}{2^{k+1}} \cdot \frac{\ln(n_k-2k)}{\ln n_k} \cdot \frac{\ln n_k}{p_{k-1}}$$

Using l'Hospital rule, we get that the product $\prod_k \frac{2^{k+1}-k}{2^{k+1}}$ is positive. Since $n_k > 2^{k+1}$ and the function $f(x) = \frac{\ln(x-2k)}{\ln x}$ is increasing,

$$\prod_k \frac{\ln(n_k - 2k)}{\ln n_k} \geq \prod_k \frac{2^{k+1} - k}{2^{k+1}} > 0$$

is positive too and

$$h(\Sigma) = \lim_{k\to\infty} \frac{\ln P(p_k)}{p_k} \geq \lim_{k\to\infty} \frac{\ln n_{k+1}}{p_k} > 0$$

□

Example 3. *There exists a Toeplitz sub-shift with lower recurrence dimension zero and upper recurrence dimension positive.*

Proof. We modify Example 2, applying formulas for n_{k+1} and ν_k only for k even. For k odd, put $n_{k+1} = n_k^{2^{k+1}}$ and construct a bijective substitution $\nu_k : A_{k+1} \to (A_k)^{2^{k+1}}$. Thus for k even and $u \in \mathcal{L}(\Sigma)$ with $|u| = 2p_k$ we get again $\tau[u] \geq (\frac{k}{2} + 1)|u|$ and $\underline{r}(\Sigma) = 0$. For k odd, however, $\tau[u] \leq |u|$, so $\overline{r}(\Sigma) = h(\Sigma) > 0$. □

3 Polynomial entropy and dimension

3.1 Definition and properties

When the topological entropy of a sub-shift is zero, its complexity function $P(n)$ grows slower than any exponential. Its growth may be then polynomial $P(n) \approx n^a$. We call $a \approx \frac{\ln P(n)}{\ln n}$ the polynomial entropy. In contrast to the (exponential) entropy, the limit in question need not exist, so we get lower and upper polynomial entropies. They could be defined in arbitrary dynamical system using open covers. For an open cover $\mathcal{V}$ of a compact metric space denote by $|\mathcal{V}|$ the number of elements in its smallest finite sub-cover.

Definition 2. *Let (X, f) be a dynamical system and $\mathcal{V}$ an open cover of X. The lower ($\underline{h}(X, f)$) and upper ($\overline{h}(X, f)$) polynomial entropies are defined by*

$$\begin{aligned} \overline{\underline{H}}_p(X, f, \mathcal{V}) &= \overline{\underline{\lim}}_{n\to\infty} \frac{\ln |\mathcal{V}^n|}{\ln n} \\ \overline{\underline{h}}_p(X, f) &= \lim_{\varepsilon\to\infty} \sup\{\overline{\underline{H}}_p(X, f, \mathcal{V}) : \ \mathrm{diam}(\mathcal{V}) < \varepsilon, \ \mathcal{V} \text{ open cover of } X\} \end{aligned}$$

Similarly as for the (exponential) topological entropy we get

$$\overline{\underline{h}}_p(X, f) = \lim_{n\to\infty} \overline{\underline{h}}_p(X, f, \mathcal{V}_n)$$

whenever $(\mathcal{V}_n)_{n\geq 0}$ is a sequence of open covers with $\mathcal{V}_{n+1} \geq \mathcal{V}_n$ and $\mathrm{diam}(\mathcal{V}_n) \to 0$ as $n \to \infty$. In particular for sub-shifts we get

$$\underline{h}(\Sigma) = \liminf_{n\to\infty} \frac{\ln P(n)}{\ln n}, \quad \overline{h}(\Sigma) = \limsup_{n\to\infty} \frac{\ln P(n)}{\ln n}$$

Definition 3. *Let (X,f) be a zero-dimensional dynamical system, $\mathcal{V}$ a clopen partition of X, $Y \subseteq X$, and $\alpha > 0$. The lower polynomial recurrence dimension $\underline{r}_p(Y,f)$ and the upper polynomial dimension $\overline{r}_p(Y,f)$ of Y is defined by*

$$\begin{aligned}
\overline{m}_p(Y,f,\mathcal{V},\alpha) &= \overline{\lim}_{n\to\infty} \sum_{V\in\mathcal{V}^n, V\cap Y\neq\emptyset} \frac{1}{\tau(V)^\alpha}, \\
\overline{R}_p(Y,f,\mathcal{V}) &= \sup\{\alpha > 0 : \overline{m}_p(Y,f,\mathcal{V},\alpha) = \infty\}, \\
\overline{\underline{M}}_p(Y,f,\alpha) &= \lim_{\epsilon\to 0} \sup\{\overline{m}_p(Y,f,\mathcal{V},\alpha) : \operatorname{diam}(\mathcal{V}) < \epsilon, \mathcal{V} \text{ clopen partition of } X\}, \\
\overline{\underline{r}}_p(Y,f) &= \sup\{\alpha > 0 : \overline{\underline{M}}_p(Y,f,\alpha) = \infty\}.
\end{aligned}$$

Clearly, the polynomial recurrence dimension and polynomial entropies are topologicaly invariants. From now on we write $r_p(X,f)$ whenever the same arguments apply for both $\underline{r}_p(X,f)$ and $\overline{r}_p(X,f)$.

Proposition 9. *For every zero dimensional, minimal dynamical system (X,f), $r_p(X,f) \geq 1$.*

Proof. There exists an ergodic measure μ, which is positive on open sets. By Lemma of Kac,

$$\tau(U) \leq \int_U \tau_U(x) \frac{d\,\mu(x)}{\mu(U)} = \frac{1}{\mu(U)}$$

Here $\tau_U(x) = \min\{k > 0 : f^k(x) \in U\}$. For every clopen cover

$$\sum_{U\in\mathcal{U}} \tau(U)^{-1} \geq \sum_{U\in\mathcal{U}} \mu(U) \geq 1$$

so $m_p(X,f,\mathcal{U},1) > 0$. Since the system is minimal, by Proposition 1, the transition is sharp, thus $r_p(X,f,\mathcal{U}) \geq 1$. □

Proposition 10. *If $\mathcal{V}$ is a generating clopen partition, then $r_p(X,f) = R_p(X,f,\mathcal{V})$.*

The proof is the same as for the exponential recurrence dimension case.

Proposition 11. *For every sub-shift $\Sigma \subseteq A^{\mathbb{N}}$, $r_p(\Sigma) \leq \overline{h}_p(\Sigma) + 1$*

Proof. If $\overline{h}_p(\Sigma) + 1 < \alpha$, pick some β with $\overline{h}_p(\Sigma) < \beta < \alpha - 1$. There exists k_0, such that for all $k \geq k_0$, $P(k) \leq e^{k\beta}$. If $u \in \mathcal{L}^n(\Sigma)$ and $\tau[u] = k < n$, then $u_{k+i} = u_i$ for all $i \in [0,|u|)$, so the number of cylinders of length n with the return time k is at most the number of cylinders of length k. For $\tau[u] = k \geq n$ we use $e^{-\tau[u]} \leq e^{-n}$ to obtain

$$\begin{aligned}
\sum_{u\in\mathcal{L}^n(\Sigma)} \tau[u]^{-\alpha} &\leq \sum_{k=1}^{n} P(k)\cdot k^{-\alpha} \leq \sum_{k=1}^{n} k^{\beta-\alpha} \\
m_p(\Sigma,\alpha) &\leq \sum_{k=1}^{\infty} k^{\beta-\alpha} < \infty
\end{aligned}$$

so $m_p(\Sigma,\alpha) < \infty$, and $r_p(\Sigma) \leq \alpha$. □

Proposition 12. *Let $\Sigma \subseteq A^{\mathbb{N}}$ be a sub-shift and $a, b > 0$.*

$$\begin{aligned}
\forall n_0, \exists n \geq n_0, \forall u \in \mathcal{L}^n(\Sigma), a|u| \leq \tau[u] \quad &\Rightarrow \quad \underline{r}_p(\Sigma) \leq \overline{h}_p(\Sigma) \\
\exists n_0, \forall n \geq n_0, \forall u \in \mathcal{L}^n(\Sigma), a|u| \leq \tau[u] \quad &\Rightarrow \quad \underline{r}_p(\Sigma) \leq \underline{h}_p(\Sigma) \\
\exists n_0, \forall n \geq n_0, \forall u \in \mathcal{L}^n(\Sigma), \tau[u] \leq b|u| \quad &\Rightarrow \quad \underline{h}_p(\Sigma) \leq \overline{r}_p(\Sigma)
\end{aligned}$$

Proof. 1. Let $\overline{h}_p(\Sigma) < \beta < \alpha$ so for every $n \geq n_0$, $P(n) < n^\beta$. There exists an infinite number of n such that $a|u| \leq \tau[u]$ whenever $|u| = n$. We get

$$\sum_{u \in \mathcal{L}^n(\Sigma)} \tau[u]^{-\alpha} \leq P(n)(an)^{-\alpha} \leq a^{-\alpha} \cdot n^{\beta-\alpha} \to 0$$

Thus $\underline{m}_p(\Sigma, \mathcal{A}, \alpha) = 0$, so $\underline{r}_p(\Sigma) \leq \alpha$ and $\underline{r}_p(\Sigma) \leq \overline{h}_p(\Sigma)$.
2.Let $\underline{h}_p(\Sigma) < \beta < \alpha$ so there exists an infinite number of n for which $P(n) < n^\beta$, and $a|u| \leq \tau[u]$ for all u with $|u| = n$. We get again $\underline{m}_p(\Sigma, \mathcal{A}, \alpha) = 0$, so $\underline{r}_p(\Sigma) \leq \alpha$ and $\underline{r}_p(\Sigma) \leq \underline{h}_p(\Sigma)$.
3. Let $\underline{h}_p(\Sigma) > \beta > \alpha$ so for every $n \geq n_0$, $P(n) > n^\beta$ and

$$\sum_{u \in \mathcal{L}^n(\Sigma)} \tau[u]^{-\alpha} \geq P(n)(bn)^{-\alpha} \geq b^{-\alpha} \cdot n^{\beta-\alpha} \to \infty$$

Thus $\overline{m}_p(\Sigma, \mathcal{A}, \alpha) = \infty$, so $\overline{r}_p(\Sigma) \geq \alpha$ and $\overline{r}_p(\Sigma) \geq \underline{h}_p(\Sigma)$. □

3.2 Sturmian sub-shifts

A sturmian sub-shift is a coding of a one dimensional irrational rotation. More precisely, let $R_\theta : \mathcal{S}^1 \to \mathcal{S}^1$, such that $R_\theta x = x + \theta \mod 1$, with θ irrational. Let $\mathcal{V} = \{[0, 1-\theta), [1-\theta, 1)\} = \{V_0, V_1\}$ and $\Omega_\theta = \{\{x_i\}_{i \geq 0} \in \Sigma_2 : \ \exists x \in [0,1)$ s.t. $\forall i \geq 0, R_\theta^i x \in V_{x_i}\}$. Then the dynamical system (Ω_θ, σ) is a sturmian sub-shift.

Example 4. *For any sturmian sub-shift, upper and lower polynomial recurrence dimensions are equal to 1.*

Proof. There is a conjugation ϕ between S_θ and Ω_θ such that any cylinder $[x_0 \dots x_{n-1}] \xrightarrow{\phi} V \in \mathcal{V}^n$. Thus, $\tau([x_0 \dots x_{n-1}]) = \tau(V)$. We will use the three distances theorem and three gaps theorem (see for example [3] for an excellent review) to compute the return times of cylinders in a sturmian sub-shift.

The three distance theorem states that the number and the length of intervals in the dynamic partition $\mathcal{V}^n$ are

- $n + 1 - q_k$ intervals of length $l_1 = \eta_k$,
- $r + 1$ intervals of length $l_2 = \eta_{k-1} - m\eta_k$,
- $q_k - (r+1)$ intervals of length $l_3 = \eta_{k-1} - (m-1)\eta_k$,

where $\frac{p_k}{q_k}$ and a_k are respectively the convergents and quotients associated to θ, $\eta_k = (-1)^k(q_k\theta - p_k)$, and k, m and r is the unique solution of

$$n = mq_k + q_{k-1} + r, \quad 1 \leq m \leq a_{k+1}, \quad 0 \leq r < q_k.$$

The three gaps theorem tells us that the three possible return times of points of an interval of length β into itself are $t_1 = q_{k'}$, $t_2 = q_{k'+1} - m'q_{k'}$ and $t_3 = q_{k'+1} - (m'-1)q_{k'}$, where k', m' (and ϕ) is the unique solution of

$$\beta = m'\eta_{k'} + \eta_{k'+1} + \phi, \quad 0 < \phi \leq \eta_{k'}, \quad 1 \leq m \leq a_{k'+1}.$$

The third return time t_3 is always the biggest and does not interest us here. $t_1 < t_2$ except when $m' = a_{k'+1}$, in which case $t_2 = q_{k'-1} < t_1$ by the equality $q_{k'+1} = a_{k'+1}q_k + q_{k'-1}$.

We will always choose $\phi = \eta_{k'}$, i.e. the greatest value it can take. We will use the equality that holds for any k : $\eta_{k-1} = a_{k+1}\eta_k + \eta_{k+1}$. If we take

- $\beta = l_1 = \eta_k$: there are two cases
  - $a_{k+2} = 1$: we set $k' = k+2, m' = a_{k'+1}$, thus the return time of the interval of length l_1 will be $t_2 = q_{k+1}$.
  - $a_{k+2} > 1$: we set $k' = k+1, m' = a_{k'+1} - 1$, thus the return time will be $t_1 = q_{k+1}$ again.
- $\beta = l_2 = \eta_{k-1} - m\eta_k$: there are three cases
  - $m < a_{k+1} - 1$: we set $k' = k, m' = a_{k'+1} - m - 1$, thus the return time of the interval of length l_2 will be $t_1 = q_k$.
  - $m = a_{k+1}$: we set $k' = k+2, m' = a_{k'+1} - 1$, thus the return time will be $t_1 = q_{k+2}$.
  - $m = a_{k+1} - 1$: we set $k' = k+1, m' = a_{k'+1}$, thus the return time will be $t_2 = q_k$.
- $\beta = l_3 = \eta_{k-1} - (m-1)\eta_k$: there are two cases
  - $m < a_{k+1}$: we set $k' = k, m' = a_{k'+1} - m$, thus the return time of the interval of length l_3 will be $t_1 = q_k$.
  - $m = a_{k+1}$: we set $k' = k+1, m' = a_{k'+1}$, thus the return time will be $t_2 = q_k$.

For a fixed n, either $m < a_{k+1}$, and then

$$m_p(\Omega_\theta, \mathcal{V}^n, 1) = \frac{(m-1)q_k + q_{k-1} + (r+1)}{q_{k+1}} + \frac{q_k}{q_k},$$

which takes values between 1 and 2. Either $m = a_{k+1}$, and then

$$m_p(\Omega_\theta, \mathcal{V}^n, 1) = \frac{(a_{k+1}-1)q_k + q_{k-1} + (r+1)}{q_{k+1}} + \frac{r+1}{q_{k+2}} + \frac{q_k - (r+1)}{q_k},$$

which is less than 2 again.

This shows that $\overline{r}_p \leq 1$. Using the global lower bound for minimal dynamical systems (Proposition 9), we get the result. $\square$

3.3 Examples with Toeplitz sub-shifts

Example 5. *For every $\alpha \geq 1$ there exists a Toeplitz sub-shift such that*

$$\underline{h}_p(\Sigma) = r_p(\Sigma) = \overline{h}_p(\Sigma) = \alpha.$$

Proof. Let $(n_k)_{k\geq 1}$ be a sequence satisfying $3 \leq n_k \leq n_{k+1} \leq (n_k - 1)^2$. Put $A_k = \{0, 1, \ldots, n_k - 1\}$ and construct injective substitutions $\nu_k : A_{k+1} \to A_k^5$, so that for every $a \in A_{k+1}$, $\nu_k(a) = 00b0c$ for some $b, c \in A_k \setminus \{0\}$ and for every $b \in A_k \setminus \{0\}$, there exists $a \in A_{k+1}$ such that $\nu_k(a) = 00b0b$. For example, if $n_k = k + 2$, we have $A_1 = \{0, 1, 2\}$ and construct

$$\begin{aligned} \nu_1 : \{0,1,2,3\} &\mapsto \{00101, 00202, 00102, 00201\} \\ \nu_2 : \{0,1,2,3,4\} &\mapsto \{00101, 00202, 00303, 00102, 00103\} \end{aligned}$$

Put

$$\Sigma = \bigcap_{k=1}^{\infty} \{\sigma^i \nu_1 \ldots \nu_{k-1}(u) : \ u \in A_k, \ i \geq 0\} \subseteq A_1^{\mathbb{N}}$$

If $u \in \mathcal{L}^6(\Sigma)$, then there exist $a, b \in A_2$ and a phase $0 \leq m < 5$ such that $u_i = \nu_1(ab)_{m+i}$ for all $i < |u|$. For different phases m, u contains zeros in different positions, so $\tau[u] \geq 5$. Moreover, since either $a = 0$ or $b = 0$, and $0a0a0 \in \mathcal{L}(\Sigma)$ for all $a \in A_1$, we have $\tau[u] \leq 10$. If $u \in \mathcal{L}(\Sigma)$ and $|u| = 5^k$ for some $k > 1$, then there exists $v \in A_k^6$ and a phase $0 \leq m < 5^k$ such that $u_i = \nu_1 \ldots \nu_{k-1}(v)_{m+i}$. Since $5 \leq \tau[v] \leq 10$, we get $5^k \leq \tau[u] \leq 2 \cdot 5^k$. In general, if $5^{k-1} < |u| \leq 5^k$, then

$$\frac{|u|}{5} \leq 5^{k-1} \leq \tau[u] \leq 2 \cdot 5^k \leq 10|u|$$

We estimate now the complexity of Σ. If $|u| = 5^{k-1}$, then u is a sub-word of $\nu_1 \ldots \nu_{k-1}(ab)$ for some $a, b \in A_k$ and either $a = 0$ or $b = 0$. Thus we have at most $2n_k$ possibilities for ab and 5^{k-1} possibilities for the phase. On the other hand there are at least n_k different words of the form $\nu_1 \ldots n_{k-1}(a)$, and at least $2 \cdot 5^{k-2}$ different phases for them, in which the distinguishing part $b0c$ of $\nu_1(a) = 00b0c$ occurs. Thus we get

$$2 \cdot 5^{k-2} \cdot n_k \leq P(5^{k-1}) \leq 2 \cdot 5^{k-1} \cdot n_k$$

In general, if $5^{k-1} \leq m < 5^k$, then

$$2 \cdot 5^{k-2} \cdot n_k \leq P(m) \leq 2 \cdot 5^k \cdot n_{k+1}$$

To construct a sub-shift with polynomial entropies $\alpha = 1$, put $n_k = 3$ for all k and get

$$\underline{h}_p(\Sigma) = r_p(\Sigma) = \overline{h}_p(\Sigma) = 1$$

Let $\alpha > 1$, and let k_0 be the first integer with $5^{k_0(\alpha-1)} \geq 3$. For $k \geq k_0$ define n_k by

$$5^{k(\alpha-1)} \leq n_k < 5^{k(\alpha-1)} + 1.$$

Using the sequence $(n_k)_{k\geq k_0}$ we construct a sub-shift in alphabet $A = A_{k_0} = \{0,\ldots,n_{k_0}-1\}$. We verify easily $n_k \leq n_{k+1} \leq (n_k-1)^2$. If $5^{k-1} < m \leq 5^k$, then

$$\frac{\ln(2\cdot 5^{k-2+k(\alpha-1)}+1)}{k\ln 5} \leq \frac{\ln P(m)}{\ln m} \leq \frac{\ln(2\cdot 5^{k+(k+1)(\alpha-1)}+1)}{(k-1)\ln 5}$$

Both sides of this inequality tend to α as $k\to\infty$, so $\underline{h}_p(\Sigma) = \overline{h}_p(\Sigma) = \alpha$. By Proposition 12, $\overline{r}_p(\Sigma) = \underline{r}_p(\Sigma) = \alpha$. □

Example 6. *For every $\alpha > 2$ there exists a Toeplitz sub-shift with*

$$\underline{r}_p(\Sigma) \leq \underline{h}_p(\Sigma) = \alpha,\ \overline{h}_p(\Sigma) = \alpha + 1.$$

Proof. We use the construction of the preceding example, but we do no more require $n_k \leq n_{k+1}$, so not all words $00a0a$ may be present. We loose the upper estimate for $\tau[u]$, but we still have $\tau[u] \geq |u|$. Thus we put (for large enough k)

$$n_{2k} = \lceil 5^{2k(\alpha-1)}\rceil,\ n_{2k+1} = \lceil 5^{(2k+1)\alpha}\rceil$$

and we get $\underline{h}_p(\Sigma) = \alpha$, $\overline{h}_p(\Sigma) = \alpha+1$, and $\underline{r}_p(\Sigma) \leq \underline{h}_p(\Sigma)$. □

References

[1] V. Afraimovich, *Pesin's dimension for Poincaré recurrence*, Chaos **7** (1997)

[2] V. Afraimovich, private communication

[3] P. Alessandri, V.Berthé, *Three distance theorems and combinatorics of words*, L'enseignement Mathématique, **44**, 1998, 103-132.

[4] H.Bruin, *Dimensions of recurrence times and minimal sub-shifts*, proceeding of the conference "Dynamical Systems : From crystal to Chaos", J.-M. Gambaudo, P. Hubert, S. Vaienti eds, to be published in World Scientific.

[5] M.Denker, Ch.Grillenberger, K.Sigmund: Ergodic Theory on Compact Spaces. Lecture Notes in Mathematics vol. 527, Springer, Berlin 1976.

[6] H.Herman, I.F.Putnam, Ch.F.Skau, *Ordered Bratelli diagrams*, dimensions groups and topological dynamics, manuscript.

[7] P.Kůrka, A.Maass, *Recurrence dimension in Toeplitz sub-shifts*, proceeding of the conference "Dynamical Systems : From crystal to Chaos", J.-M. Gambaudo, P. Hubert, S. Vaienti eds, to be published in World Scientific.

[8] D.Lind, B.Marcus: An Introduction to Symbolic Dynamics and Coding, Cambridge University Press, Cambridge 1995.

[9] V.Penné, B.Saussol, S.Vaienti: *Dimensions for recurrence times: topological and dynamical properties*, Discrete and Continuous Dynamical Systems, vol. 4, no. 4, 1998.

[10] Y.B.Pesin, Dimension Theory in Dynamical Systems, The University of Chicago Press, Chicago 1997.

[11] S.Williams, *Toeplitz minimal flows which are not uniquely ergodic,* Z. Wahrschainlichkeitstheorie verw. Gebiete 67, 95-107 (1984)