Generalized eigenfunctions for waves in inhomogeneous media

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Abstract

Many wave propagation phenomena in classical physics are governed by equations that can be recast in Schrödinger form. In this approach the classical wave equation (e.g., Maxwell's equations, acoustic equation, elastic equation) is rewritten in Schrödinger form, leading to the study of the spectral theory of its *classical wave operator*, a self-adjoint, partial differential operator on a Hilbert space of vector-valued, square integrable functions. Physically interesting inhomogeneous media give rise to nonsmooth coefficients. We construct a generalized eigenfunction expansion for classical wave operators with nonsmooth coefficients. Our construction yields polynomially bounded generalized eigenfunctions, the set of generalized eigenvalues forming a subset of the operator's spectrum with full spectral measure.

1 Introduction

A self-adjoint operator in a finite dimensional Hilbert spaces can always be diagonalized in an orthonormal basis of eigenvectors (i.e., it has a complete set of eigenvectors). In infinite dimensional Hilbert spaces only self-adjoint operators with pure point spectrum (e.g., compact operators) have this property, but this has not stopped physicists from writing generalized eigenfunction expansions for arbitrary self-adjoint operators (e.g., [6, 20]).

Generalized eigenfunction expansions have been developed and applied to elliptic partial differential operators with smooth coefficients by Gel'fand and Kostyucenko [11], Berezanskii [1], Browder [4], Kac [13], and others; we refer to Berezanskii's book [2] (see also [3]). These expansions were extended to Schrödinger operators with singular potentials (Simon [22] and references therein; see also [18, 17]).

These expansions construct polynomially bounded generalized eigenfunctions for a set of generalized eigenvalues with full spectral measure. These generalized eigenfunctions were first exploited by Pastur [16], followed by Martinelli and Scoppola [15], to prove that certain Schrödinger operators with random potentials have no absolutely continuous spectrum. They played a crucial role in the work by Fröhlich, Martinelli, Spencer and Scoppola [10] and by von Dreifus and Klein [7] on Anderson localization (pure point spectrum) of random Schrödinger operators; the exponential decay of finite volume Green's functions (obtained by a multiscale analysis) forces polynomially bounded generalized eigenfunctions to be bona fide eigenfunctions.

Many wave propagation phenomena in classical physics are governed by equations that can be recast in Schrödinger form [23, 9, 14]. These Schrödinger-like equations lead to the study of *classical wave operators* (as in [14]), a class of self-adjoint, partial differential operators on Hilbert spaces of vector-valued, square integrable functions on $\mathbb{R}^d$. In this approach the classical wave equation (e.g., Maxwell's equations, acoustic equation, elastic equation) is rewritten in Schrödinger form, leading to the study of the spectral theory of its classical wave operator. The method is particularly suitable for the study of phenomena normally associated with quantum mechanical electron waves, especially Anderson localization in random media [8, 9, 14].

In this article we study generalized eigenfunctions of classical wave operators with nonsmooth coefficients. Physically interesting inhomogeneous media give rise to nonsmooth coefficients in the classical wave equations, and hence in their classical wave operators. (E.g., a medium composed of two different homogeneous materials will be represented by piecewise constant coefficients.) We must also take into account that many classical wave operators are not elliptic operators (e.g., the Maxwell operator – see [9, 14]). We construct a generalized eigenfunction expansion for classical wave operators which yields polynomially bounded generalized eigenfunctions, the set of generalized eigenvalues forming a subset of the operator's spectrum with full spectral measure. These generalized eigenfunctions are used in the proofs of Anderson localization of classical waves in random media [8, 9, 14]. The generalized eigenfunction expansion itself plays a key role in recent work by Germinet and Klein [12], who show that strong dynamical localization follows from a multiscale analysis, both for classical waves in random media and random Schrödinger operators.

This article is organized as follows: Classical wave operators are introduced in Section 2. A trace estimate for partially elliptic classical wave operators, crucial for the construction of generalized eigenfunction expansions, is given in Theorem 2.13, and proven in Section 8. We define generalized eigenfunctions and eigenvalues of classical wave operators in Section 3; our main result on the generalized point spectrum of a classical wave operator is Theorem 3.4, with Corollary 3.6 being the result used in the proofs of Anderson localization for classical waves in random media. In Section 4 we rewrite the standard results on generalized eigenfunction expansions (as in Berezanskii [2]) in the framework of this article, making additions and modifications needed for classical wave operators; see Theorems 4.5 and 4.6. In Section 5 we introduce some mathematical machinery: we construct a chain of functional spaces and "extensions" and "restrictions" of classical wave operators. The key technical result is a generalization of interior estimates (Theorem 5.2). The main results about "extensions" and "restrictions" of second order classical wave operators are summarized in Theorem 5.4. We also prove that second order classical wave operators have an operator core consisting of functions with compact support (Corollary 5.5). In Section 6, the results of Sections 4 and 5 are combined to construct a generalized eigenfunction expansion for partially elliptic classical wave operators (Theorem 6.1), and prove Theorem 3.4. Section 7 analyzes the case when a second order classical wave operator is partially elliptic, but the corresponding first order classical wave operator is not. We show that we still have a generalized eigenfunction expansion for the corresponding classical waves; the main result is Theorem 7.1. Section 8 contains the proof of Theorem 2.13. A proof that 0 is at the bottom of the spectrum of proper classical wave operators is given in Appendix A. Appendix B contains a review of densely defined operators in locally convex spaces, which are used in Section 5. In Appendix C we summarize some results of the theory of Bochner integrals and vector valued Borel measures, including the Radon-Nikodym Theorem and Lebesgue's Differentiation Theorem.

2 Classical wave operators

Given an operator T on a Hilbert space $\mathcal{H}$, we denote its kernel by $\ker T$ and its range by $\operatorname{ran} T$; note $\ker T^*T = \ker T$. If T is self-adjoint, it leaves invariant the orthogonal complement of its kernel; *the restriction of T to $(\ker T)^\perp$ will be denoted by $T_\perp$*. Note that $T_\perp$ is a self-adjoint operator on the Hilbert space $(\ker T)^\perp = P_T^\perp \mathcal{H}$, where $P_T^\perp$ denotes the orthogonal projecction onto $(\ker T)^\perp$.

Let us set

$$\mathcal{H}^{(n)} = L^2(\mathbb{R}^d, dx; \mathbb{C}^n) \ . \tag{1}$$

Definition 2.1 *A constant coefficient, first order, partial differential operator D from $\mathcal{H}^{(n)}$ to $\mathcal{H}^{(m)}$ ($CPDO^{(1)}_{n,m}$) is of the form $D = D(-i\nabla)$,* *where*

$$D(k) = [D(k)_{r,s}]_{\substack{r=1,\dots,m \\ s=1,\dots,n}} \ , \quad \textit{with} \quad D(k)_{r,s} = a_{r,s} \cdot k \ , \quad a_{r,s} \in \mathbb{C}^d \ . \quad (2)$$

Defined on

$$\mathcal{D}(D) = \{\psi \in \mathcal{H}^{(n)} : \ D\psi \in \mathcal{H}^{(m)} \ \text{in distributional sense}\} \ , \qquad (3)$$

a $CPDO^{(1)}_{n,m}$ D is a closed, densely defined operator, and $C_0^\infty(\mathbb{R}^d;\mathbb{C}^n)$ (the space of infinitely differentiable functions with compact support) is an operator core for D. We will denote by D^* the $CPDO^{(1)}_{m,n}$ given by the formal adjoint of the matrix in (2).

For later use, we set D_+ to be the norm of the matrix $[|a_{r,s}|]_{\substack{r=1,\dots,m \\ s=1,\dots,n}}$, so $\|D(k)\| \leq D_+|k|$.

Definition 2.2 *A $CPDO^{(1)}_{n,m}$ D is said to be partially elliptic if there exists a $CPDO^{(1)}_{n,q}$ $D^\perp$ (for some q), satisfying the following two properties:*

$$\begin{aligned} D^\perp D^* \ &= \ 0 \ , & (4) \\ D^*D + (D^\perp)^* D^\perp \ &\geq \ \Upsilon \ [(-\Delta) \otimes I_n] \ , & (5) \end{aligned}$$

Υ being a positive constant . ($\Delta = \nabla \cdot \nabla$ is the Laplacian on $L^2(\mathbb{R}^d, dx)$; I_n denotes the $n \times n$ identity matrix.)

If D is partially elliptic, we have

$$\mathcal{H}^{(n)} = \ker D^\perp \oplus \ker D \ , \qquad (6)$$

and

$$D^*D + (D^\perp)^* D^\perp = (D^*D)_\perp \oplus ((D^\perp)^* D^\perp)_\perp \ . \qquad (7)$$

Note that D is elliptic if and only it is partially elliptic with $D^\perp = 0$.

Remark 2.3 *A $CPDO^{(1)}_{n,m}$ D may be partially elliptic with D^* not being partially elliptic; e.g., the two dimensional $CPDO^{(1)}_{2,3}$ given by*

$$D = \begin{bmatrix} i\frac{\partial}{\partial x_1} & 0 \\ 0 & i\frac{\partial}{\partial x_2} \\ i\frac{\partial}{\partial x_2} & i\frac{\partial}{\partial x_1} \end{bmatrix} \ . \qquad (8)$$

Definition 2.4 *A coefficient operator S on $\mathcal{H}^{(n)}$ (CO_n) is a bounded, invertible operator given by multiplication by an $n \times n$ matrix -valued measurable function $S(x)$, with*

$$0 < S_{-} I_n \leq S(x) \leq S_{+} I_n \ , \tag{9}$$

for some positive, finite constants $S_{\pm}$.

Definition 2.5 *A multiplicative coefficient, first order, partial differential operator from $\mathcal{H}^{(n)}$ to $\mathcal{H}^{(m)}$ ($MPDO^{(1)}_{n,m}$) is of the form*

$$A = RDS \quad on \quad \mathcal{D}(A) = S^{-1}\mathcal{D}(D) \ , \tag{10}$$

where D is a $CPDO^{(1)}_{n,m}$, S is a CO_n and R is a CO_m.

An $MPDO^{(1)}_{n,m}$ A is a closed, densely defined operator with $A^* = SD^*R$ an $MPDO^{(1)}_{m,n}$. Note that $S^{-1}C_0^\infty(\mathbb{R}^d; \mathbb{C}^n)$ is an operator core for A.

Definition 2.6 *A first order classical operator on $\mathcal{H}^{(n)}$ ($CWO^{(1)}_n$) is a $MPDO^{(1)}_{n,n}$ of the form SDS, with D a self-adjoint $CPDO^{(1)}_{n,n}$ and S a CO_n. If D is partially elliptic, the $CWO^{(1)}_n$ will also be said to be partially elliptic.*

To every $MPDO^{(1)}_{n,m}$ A we associate the $CWO^{(1)}_{n+m}$ $\mathbb{W}_A$ given by:

$$\mathbb{W}_A = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \quad \text{on} \quad \mathcal{H}^{(n+m)} \cong \mathcal{H}^{(n)} \oplus \mathcal{H}^{(m)} \ . \tag{11}$$

To see that $\mathbb{W}_A$ is indeed a $CWO^{(1)}_{n+m}$, we write $\mathbb{W}_A = K\mathbb{W}_D K$, where $K = S \oplus R$ is a CO_{n+m} and $\mathbb{W}_D$ is a self-adjoint $CPDO^{(1)}_{n+m,n+m}$. Note that

$$(\mathbb{W}_A)^2 = \begin{bmatrix} A^*A & 0 \\ 0 & AA^* \end{bmatrix} \ . \tag{12}$$

Remark 2.7 *$\mathbb{W}_A$ is partially elliptic if, and only if, both A^*A and AA^* are partially elliptic; i.e., $\mathbb{W}_D$ is partially elliptic if, and only if, both D and D^* are partially elliptic. To see this, first note that if D and D^* are partially elliptic, it is immediate that $\mathbb{W}_D$ is partially elliptic with*

$$(\mathbb{W}_D)^\perp = \begin{bmatrix} 0 & (D^*)^\perp \\ D^\perp & 0 \end{bmatrix} \ . \tag{13}$$

To prove the converse, suppose $\mathbb{W}_D$ is partially elliptic satisfying (4) and (5) for a $CPDO^{(1)}_{m+m,q}$ $(\mathbb{W}_D)^{\perp}$. Let us write $(\mathbb{W}_D)^{\perp} = [D_1\ D_2]$, where D_1 is a $CPDO^{(1)}_{n,q}$ and D_2 a $CPDO^{(1)}_{m,q}$. It follows that D and D^* satisfy (4) and (5) with $D^{\perp} = D_1$ and $(D^*)^{\perp} = D_2$.

The Schrödinger-like equations for classical waves are of the form [23, 9, 14]:

$$-i\frac{\partial}{\partial t}\mathbf{\Psi}_t = (\mathbb{W}_A)_{\perp}\mathbf{\Psi}_t \ , \quad \mathbf{\Psi}_t \in (\ker \mathbb{W}_A)^{\perp} = (\ker A)^{\perp} \oplus (\ker A^*)^{\perp}, \tag{14}$$

with $\mathbb{W}_A$ a $CWO^{(1)}_{n+m}$ as in (11). Examples include Maxwell's equations and the equations of acoustics and elasticity.

If $\mathbf{\Psi}_t = (\psi_t, \phi_t) \in \mathcal{H}^{(n)} \oplus \mathcal{H}^{(m)}$ is a solution of (14), it follows from (12) that its components satisfy the second order wave equations

$$\frac{\partial^2}{\partial t^2}\psi_t \;=\; -(A^*A)_{\perp}\psi_t \ , \ \text{ with } \ \psi_t \in (\ker A)^{\perp} \ , \tag{15}$$

$$\frac{\partial^2}{\partial t^2}\phi_t \;=\; -(AA^*)_{\perp}\phi_t \ , \ \text{ with } \ \phi_t \in (\ker A^*)^{\perp} \ . \tag{16}$$

It turns out that the operators $(A^*A)_{\perp}$ and $(AA^*)_{\perp}$ are unitarily equivalent. The operator U defined by

$$U\psi = A(A^*A)_{\perp}^{-\frac{1}{2}}\psi \ \text{ for } \ \psi \in \ \operatorname{ran}(A^*A)_{\perp}^{\frac{1}{2}}\,, \tag{17}$$

extends to a unitary operator from $(\ker A)^{\perp}$ to $(\ker A^*)^{\perp}$, and

$$(AA^*)_{\perp} = U(A^*A)_{\perp}U^* \ . \tag{18}$$

In addition, if

$$\mathbb{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} I_A & I_A \\ U & -U \end{bmatrix} \ , \ \text{ with } I_A \text{ the identity on } (\ker A)^{\perp} \ , \tag{19}$$

$\mathbb{U}$ is a unitary operator from $(\ker A)^{\perp}\oplus(\ker A)^{\perp}$ to $(\ker A)^{\perp}\oplus(\ker A^*)^{\perp}$, and we have the unitary equivalence:

$$\mathbb{U}^*(\mathbb{W}_A)_{\perp}\mathbb{U} = (A^*A)_{\perp}^{\frac{1}{2}} \oplus \left[-(A^*A)_{\perp}^{\frac{1}{2}}\right] \ . \tag{20}$$

Thus the operator $(A^*A)_{\perp}$ contains full information about the spectral theory of the operator $(\mathbb{W}_A)_{\perp}$ (e.g., [9, 14]). In particular

$$\sigma((\mathbb{W}_A)_{\perp}) = \sigma\left((A^*A)_{\perp}^{\frac{1}{2}}\right) \cup \left(-\sigma\left((A^*A)_{\perp}^{\frac{1}{2}}\right)\right) \ . \tag{21}$$

Definition 2.8 *A second order classical wave operator on $\mathcal{H}^{(n)}$ ($CWO_n^{(2)}$) is of the form A^*A, with A an $MPDO_{n,m}^{(1)}$ for some m. If D in (10) is partially elliptic, the $CWO_n^{(2)}$ will also be called partially elliptic.*

Remark 2.9 *If A is an $MPDO_{n,m}^{(1)}$, both A^*A and AA^* are second order classical wave operators. But A^*A may be partially elliptic with AA^* not partially elliptic, since a $CPDO_{n,m}^{(1)}$ D may be partially elliptic with D^* not being partially elliptic. (See Remark 2.3.)*

Remark 2.10 *If W is a $CWO_n^{(1)}$, then W^2 is a $CWO_n^{(2)}$. If W is partially elliptic so is W^2.*

Definition 2.11 *A classical wave operator (CWO) is either a $CWO_n^{(1)}$ or a $CWO_n^{(2)}$. If the operator W is a CWO, we call $W_\perp$ a proper CWO.*

Remark 2.12 *A proper CWO W has a trivial kernel by construction, so 0 is not an eigenvalue. However, using a dilation argument, one can show that 0 is in the spectrum of $W_\perp$ (Theorem .4), so $W_\perp$ and W have the same spectrum.*

Partially elliptic CWOs satisfy a trace estimate crucial for the construction of generalized eigenfunction expansions. In view of Remark 2.10 it suffices to consider second order classical wave operators.

Theorem 2.13 *Let W be a partially elliptic $CWO_n^{(2)}$, and let $P_W^\perp$ denote the orthogonal projection onto $(ker W)^\perp$. Then*

$$\operatorname{tr}(V^* P_W^\perp (W+I)^{-2r} V) \leq C_{d,n,S_\pm,R_\pm,D_+,D_+^\perp,\Upsilon} \|V\|_{\infty,2}^2 < \infty \quad , \tag{22}$$

for $r \geq \nu$, where ν is the smallest integer satisfying $\nu > \frac{d}{4}$. V is the bounded operator on $\mathcal{H}^{(n)}$ given by multiplication by an $n \times n$ matrix-valued measurable function $V(x)$, with

$$\|V\|_{\infty,2}^2 = \sum_{y \in \mathbb{Z}^d} \|\chi_{y,1}(x) V^*(x) V(x)\|_\infty < \infty \ . \tag{23}$$

($\chi_{y,L}$ denotes the characteristic function of a cube of side length L centered at y.) The constant $C_{d,n,S_\pm,R_\pm,D_+,D_+^\perp,\Upsilon}$ depends only on the fixed parameters $d, n, S_\pm, R_\pm, D_+, D_+^\perp, \Upsilon$.

Theorem 2.13 is proved in Section 8. It generalizes [9, Theorem 18].

3 Generalized eigenfunctions and eigenvalues

We now define generalized eigenfunctions and eigenvalues of classical wave operators.

A vector valued measurable function ψ is said to be *polynomially L^2-bounded*, if

$$\|(1+|x|^2)^{-p}\psi\|_2 < \infty \quad \text{for some } p > 0 \ . \tag{24}$$

Definition 3.1 *A non-zero, $\mathbb{C}^n$-valued polynomially L^2-bounded function ψ is said to be a generalized eigenfunction of a $CWO_n^{(1)}$ $W = SDS$, with generalized eigenvalue $\lambda \in \mathbb{C}$, if*

$$\langle SDS\phi, \psi\rangle = \lambda\langle\phi,\psi\rangle \quad \textit{for all} \quad \phi \in S^{-1}C_0^\infty(\mathbb{R}^d;\mathbb{C}^n)\,. \tag{25}$$

Definition 3.2 *A non-zero, $\mathbb{C}^n$-valued polynomially L^2-bounded function ψ is said to be a generalized eigenfunction of the $CWO_n^{(2)}$ $W = A^*A$, $A = RDS$, with generalized eigenvalue $\lambda \in \mathbb{C}$, if*

$$DS\psi \quad \textit{is a polynomially } L^2\textit{-bounded function}\ , \tag{26}$$

in the sense of distributions, and

$$\langle DS\phi, R^2 DS\psi\rangle = \lambda\langle\phi,\psi\rangle \quad \textit{for all} \quad \phi \in S^{-1}C_0^\infty(\mathbb{R}^d;\mathbb{C}^n)\,. \tag{27}$$

Property (26) is a regularity property needed for the formulation of the eigenvalue equation (27). It is explicitly used in [8, 9, 14].

Definition 3.2 may be recast in an analogous form to Definition 3.1, once we have an appropriate operator core for A^*A, as provided by Corollary 5.5. (See Proposition 6.2.)

Any eigenfunction of a CWO is easily seen to be also a generalized eigenfunction.

A $MPDO_{m,n}^{(1)}$ A gives raise to a $CWO_{n+m}^{(1)}$ $\mathbb{W}_A$ (as in (11)), and also to a $CWO_n^{(2)}$ A^*A and a $CWO_m^{(2)}$ AA^*, the diagonal entries in $(\mathbb{W}_A)^2$. It turns out that if $\mathbf{\Psi} = (\psi,\phi)$ is a generalized eigenfunction of $\mathbb{W}_A$ with generalized eigenvalue λ, then ψ is a generalized eigenfunction of A^*A and ϕ is a generalized eigenfunction of AA^*, both with generalized eigenvalue λ^2. Conversely, if ψ is a generalized eigenfunction of A^*A with generalized eigenvalue $\lambda^2 \neq 0$, we can find generalized eigenfunctions $\phi_\pm$ of AA^* also with generalized eigenvalue λ^2, such that $(\psi,\phi_\pm)$ is a generalized eigenfunction of $\mathbb{W}_A$ with generalized eigenvalue $\pm\lambda$. (See Proposition 6.3.)

Definition 3.3 *The generalized point spectrum of a classical wave operator W is the set of all its generalized eigenvalues; it wil be denoted by $\sigma_{gp}(W)$.*

Theorem 3.4 *The generalized point spectrum of a classical wave operator W is a subset of its spectrum. For a generalized eigenfunction ψ of W with generalized eigenvalue λ, we have:*

(i) *If ψ is an L^2-function, then ψ is an eigenfunction of W with eigenvalue λ (i.e., $\psi \in \mathcal{D}(W)$ and $W\psi = \lambda\psi$).*

(ii) *If ψ is not an L^2-function, then λ is in the essential spectrum of W.*

Moreover, if W is partially elliptic, or $W = \mathbb{W}_A$ as in (11) with A^*A partially elliptic, then the difference between its spectrum and generalized point spectrum has zero spectral measure.

We have two immediate corollaries.

Corollary 3.5 *Let W be a partially elliptic classical wave operator, or let $W = \mathbb{W}_A$ as in (11) with A^*A partially elliptic. Then its spectrum is the closure of its generalized point spectrum.*

Corollary 3.6 *Let W be a partially elliptic classical wave operator, or let $W = \mathbb{W}_A$ as in (11) with A^*A partially elliptic. Let Ω be a Borel subset of its spectrum. If every generalized eigenfunction with generalized eigenvalue $\lambda \in \Omega$ is an L^2-function, then W has pure point spectrum in Ω.*

Note that Corollary 3.6 is the result used in the proofs of Anderson localization for classical waves in random media [8, 9, 14], which also required the regularity property (26). (The analogous of Corollary 3.6 for Schrödinger operators is used in [10, 7].) In [8, 9], such a result was erroneously taken as following directly from the results in [2].

Simon [22] obtained the equivalent of Theorem 3.4 and its corollaries for Schrödinger operators with singular potentials. Simon's approach is based on the study of the Schrödinger semigroup e^{-tH}, H being the Schrödinger operator, via the Feynman-Kac formula. His methods allow the treatment of Schrödinger operators for which $C_0^\infty(\mathbb{R}^d)$ is only a form core; thus he also has a regularity condition in the definition of generalized eigenfunctions [22, Theorem C.1.1]. Simon's methods are not applicable to classical wave operators.

4 Generalized eigenfunction expansions: generalities

In this section we derive a generalized eigenfunction expansion in the framework of this article, in a form suitable for classical wave operators. These results strengthen and extend the standard generalized eigenfunction expansion, as in the books [2, 3].

Hypothesis I *T is a self-adjoint operator on a Hilbert space $\mathcal{H}$ with a bounded inverse T^{-1}.*

The domain of T, $\mathcal{D}(T)$, equipped with the norm $\|\phi\|_+ = \|T\phi\|$, is a Hilbert space, denoted by $\mathcal{H}_+$. The Hilbert space $\mathcal{H}_-$ is defined as the completion of $\mathcal{H}$ in the norm $\|\psi\|_- = \|T^{-1}\psi\|$. The sesquilinear form $\langle\ ,\ \rangle$ on $\mathcal{H}_+ \times \mathcal{H}_-$, defined as the inner product in $\mathcal{H}$ for $\phi \in \mathcal{H}_+, \psi \in \mathcal{H}$, turns the spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ into conjugate duals of each other. The adjoint of an operator O with respect to this duality will be denoted by $O^\dagger$, whereas the adjoint with respect to the usual Hilbert space self-duality is given by O^*.

By construction, $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$, and the natural injections $\imath_+ : \mathcal{H}_+ \to \mathcal{H}$ and $\imath_- : \mathcal{H} \to \mathcal{H}_-$ are continuous with dense range, with $\imath_+^\dagger = \imath_-$. The operators $T_+ : \mathcal{H}_+ \to \mathcal{H}$ and $T_- : \mathcal{H} \to \mathcal{H}_-$, defined by $T_+ = T\imath_+$, $T_- = \imath_- T$ on $\mathcal{D}(T)$, are unitary with $T_- = T_+^\dagger$.

Hypothesis II *H is a self-adjoint operator on $\mathcal{H}$, such that*

$$\mathcal{D}_+ = \{\phi \in \mathcal{D}(H) \cap \mathcal{H}_+ :\ H\phi \in \mathcal{H}_+\} \tag{28}$$

is dense in $\mathcal{H}_+$.

H_+, defined by $H_+\phi = H\phi$ for $\phi \in \mathcal{D}_+$, is a closed densely defined operator on $\mathcal{H}_+$. The operator H can be considered as an operator in $\mathcal{H}_-$. We have

Lemma 4.1 *The operator H is closable, densely defined as an operator on $\mathcal{H}_-$. If H_- denotes its closure in $\mathcal{H}_-$, we have*

$$H_- = H_+^\dagger\ . \tag{29}$$

Proof: H is closable in $\mathcal{H}_-$ since $H \subset H_+^\dagger$. To prove (29), let $\psi \in \mathcal{D}(H_+^\dagger)$ be such that, for all $\phi \in \mathcal{D}(H)$,

$$\begin{aligned} 0 &= \langle \phi, \psi\rangle_- + \langle H\phi, H_+^\dagger\psi\rangle_- & (30)\\ &= \langle \phi, T^{-2}\psi\rangle + \langle H\phi, T^{-2}H_+^\dagger\psi\rangle & (31)\\ &= \langle \phi, , T^{-2}\psi\rangle + \langle H\phi, H_+^* T^{-2}\psi\rangle\ , & (32)\end{aligned}$$

where we used $H_+^\dagger = T^2 H_+^* T^{-2}$. Thus $H_+^* T^{-2}\psi \in \mathcal{D}(H)$, and we have $HH_+^* T^{-2}\psi = -T^{-2}\psi$. We conclude that $HH_+^* T^{-2}\psi \in \mathcal{H}_+$, and hence that $H_+^* T^{-2}\psi \in \mathcal{D}_+$. Thus $H_+ H_+^* T^{-2}\psi = -T^{-2}\psi$. Since $H_+ H_+^*$ is a positive operator on $\mathcal{H}_+$, we must have $\psi = 0$.

Definition 4.2 *A generalized eigenfunction of H is defined as an eigenfunction of the operator H_- on $\mathcal{H}_-$: $\psi \in \mathcal{H}_-$ is a generalized eigenfunction of H with generalized eigenvalue λ, if and only if $\psi \in \mathcal{D}(H_-)$ and $H_-\psi = \lambda\psi$, i.e.,*

$$\langle H_+\phi, \psi\rangle = \lambda\langle \phi, \psi\rangle \quad \textit{for all} \quad \phi \in \mathcal{D}_+ . \tag{33}$$

An eigenfunction of H is always a generalized eigenfunction.

Hypothesis III *$\mathcal{D}_+$ is an operator core for H.*

Let $(H_-)_\mathcal{H}$ denote the restriction of H_- to $\mathcal{H}$, i.e., $(H_-)_\mathcal{H}\,\phi = H_-\phi$ on

$$\mathcal{D}\left((H_-)_\mathcal{H}\right) = \{\phi \in \mathcal{D}(H_-) \cap \mathcal{H} :\ (H_-)_\mathcal{H}\,\phi \in \mathcal{H}\} . \tag{34}$$

(Note that $H_+ = H_{\mathcal{H}_+}$.)

Lemma 4.3 *We have $H = (H_-)_\mathcal{H}$ if and only if $\mathcal{D}_+$ is an operator core for H.*

Proof: It follows from Lemma 4.1 that for $\psi \in \mathcal{D}((H_-)_\mathcal{H})$, $\phi \in \mathcal{D}_+$,

$$\langle \phi, H_-\psi\rangle = \langle H_+\phi, \psi\rangle = \langle H\phi, \psi\rangle \ . \tag{35}$$

If $\mathcal{D}_+$ is an operator core for H, we can conclude that $\psi \in \mathcal{D}(H)$ and $H\psi = H_-\psi$.

To prove the converse, let $H_{+,0}$ denote H_+ as an operator on $\mathcal{H}$. Let $\psi \in \mathcal{D}(H_{+,0}^*)$, we have

$$\langle H_+\phi, \psi\rangle = \langle \phi, H_{+,0}^*\,\psi\rangle \tag{36}$$

for all $\phi \in \mathcal{D}_+$. Again using Lemma 4.1, we obtain $\psi \in \mathcal{D}((H_-)_\mathcal{H})$, $(H_-)_\mathcal{H}\,\psi = H_{+,0}^*\,\psi$. Since $H = (H_-)_\mathcal{H}$, we conclude that $\mathcal{D}_+$ is an operator core for H.

This lemma has an important immediate consequence.

Proposition 4.4 *Assume Hypotheses I-III. Whenever a generalized eigenfunction ψ of H is in $\mathcal{H}$, it is also an eigenfunction of H: $\psi \in \mathcal{D}(H)$ and $H\psi = \lambda\psi$.*

By $E(\Omega)$ we denote the spectral projections of the self-adjoint operator H for Borel sets Ω. We use $\mathrm{tr}_{\mathcal{H}}$ to denote the usual trace in $\mathcal{H}$. If $1 \leq q < \infty$, $\mathcal{T}_q(\mathcal{H})$ denotes the Banach space of bounded operators Y on $\mathcal{H}$ with $\|Y\|_q = |\mathrm{tr}_{\mathcal{H}}|Y|^q|^{\frac{1}{q}} < \infty$; with $\mathcal{T}_{q,+}(\mathcal{H})$ being the subset of positive operators.

Hypothesis IV *There exist a Borel set Δ and a bounded, continuous function f, strictly positive on the spectrum of H, such that*

$$tr_{\mathcal{H}}\, T^{-1}E(\Delta)f(H)T^{-1} < \infty \ . \tag{37}$$

This implies

$$\mu(\Omega) = \mathrm{tr}_{\mathcal{H}}\, T^{-1}E(\Omega\cap\Delta)T^{-1} < \infty \ , \tag{38}$$

for all bounded Borel sets Ω. Thus $\Omega \mapsto T^{-1}E(\Omega\cap\Delta)T^{-1}$ is a $\mathcal{T}_{1,+}(\mathcal{H})$-valued Borel measure, and μ is a spectral measure for the restriction of H to the Hilbert space $E(\Delta)\mathcal{H}$. (See Appendix C.)

The map $\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}_+, \mathcal{H}_-)$, with $\tau(C) = T_- C T_+$, is a Banach space isomorphism, as $T_\pm$ are unitary operators. If $1 \leq q < \infty$, we define

$$\mathcal{T}_q(\mathcal{H}_+,\mathcal{H}_-) = \tau\left(\mathcal{T}_q(\mathcal{H})\right) \ ; \ \ \mathcal{T}_{q,+}(\mathcal{H}_+,\mathcal{H}_-) = \tau\left(\mathcal{T}_{q,+}(\mathcal{H})\right) \ . \tag{39}$$

By construction, $\mathcal{T}_q(\mathcal{H}_+,\mathcal{H}_-)$, equipped with the norm $\|B\,\|_q = \|\tau^{-1}(B)\|_q$, is a Banach space isomorphic to $\mathcal{T}_q(\mathcal{H})$, a separable dual Banach space. $\mathcal{T}_{q,+}(\mathcal{H}_+,\mathcal{H}_-)$ is the subset of positive operators with respect to the pairing between $\mathcal{H}_+$ and $\mathcal{H}_-$: $\langle\phi, B\phi\rangle \geq 0$ for $\phi \in \mathcal{H}_+$.

For $B \in \mathcal{T}_1(\mathcal{H}_+,\mathcal{H}_-)$, we set

$$\mathrm{tr}\, B \;\; = \;\; \mathrm{tr}_{\mathcal{H}}\ \tau^{-1}(B) = \mathrm{tr}_{\mathcal{H}}\ T_-^{-1}BT_+^{-1} \ . \tag{40}$$

If $\{\phi_n\}$ is an orthonormal basis of $\mathcal{H}_+$, we have

$$\mathrm{tr}\, B = \sum_n \langle\phi_n, B\phi_n\rangle \ . \tag{41}$$

Moreover, for $B \in \mathcal{T}_2(\mathcal{H}_+,\mathcal{H}_-)$, $\|B\|_2$ is the usual Hilbert-Schmidt norm:

$$\|B\|_2^2 = \sum_n \|B\phi_n\|_-^2 \ . \tag{42}$$

It follows that $\Omega \mapsto \imath_- E(\Omega\cap\Delta)\imath_+$ is a $\mathcal{T}_{1,+}(\mathcal{H}_+,\mathcal{H}_-)$-valued measure, with $\mu(\Omega) = \mathrm{tr}\,\imath_- E(\Omega\cap\Delta)\imath_+$, as

$$\mathrm{tr}\ \imath_- C \imath_+ = \mathrm{tr}_{\mathcal{H}}\ T^{-1}CT^{-1} , \ \text{ if } \ C \in \mathcal{B}(\mathcal{H}) \, , \ T^{-1}CT^{-1} \in \mathcal{T}_1(\mathcal{H}) \ . \tag{43}$$

Theorem 4.5 *Assume Hypotheses I-IV. There exits a μ-locally integrable function $P(\lambda)$ from the real line into $\mathcal{T}_{1,+}(\mathcal{H}_+,\mathcal{H}_-)$, such that*

$$\imath_- E(\Omega \cap \Delta)\imath_+ = \int_\Omega P(\lambda)\, d\mu(\lambda) \quad \textit{for bounded Borel sets } \Omega\ , \tag{44}$$

where the integral is the Bochner integral of $\mathcal{T}_1(\mathcal{H}_+,\mathcal{H}_-)$-valued functions. The function $P(\lambda)$ satisfies

$$tr\, P(\lambda) = 1 \quad \text{for } \mu-\text{a.e. } \lambda\ , \tag{45}$$

$$P(\lambda) = \lim_{\delta\downarrow 0} \frac{\imath_- E(I_{\lambda,\delta}\cap\Delta)\imath_+}{\mu(I_{\lambda,\delta})} \quad \textit{in } \mathcal{T}_1(\mathcal{H}_+,\mathcal{H}_-)\ ,\ \textit{for } \mu-a.e.\ \lambda\ , \tag{46}$$

where $I_{\lambda,\delta}$ is the interval $[\lambda-\delta,\lambda+\delta]$. In addition, we have

$$\imath_- E(\Delta) g(H)\, \imath_+ = \int g(\lambda) P(\lambda)\, d\mu(\lambda)\ , \tag{47}$$

for all μ-integrable, bounded, scalar Borel functions g.

Proof: In view of (38), it is a consequence of the Radon-Nikodym Theorem (Theorem .9), that there exits a locally integrable function $P(\lambda)$ from the real line into $\mathcal{T}_{1,+}(\mathcal{H}_+,\mathcal{H}_-)$, such that (44) holds. According to (150) and (151), the function $P(\lambda)$ satisfies (45) and (46). Using (44), and taking limits of Borel step functions, we obtain (47).

We will also use the following variant of Hypothesis IV.

Hypothesis IV$_2$ *There exist a Borel set Δ and a spectral measure μ for the restriction of H to the Hilbert space $E(\Delta)\mathcal{H}$, such that $\Omega \mapsto \imath_- E(\Omega\cap\Delta)\imath_+$ is a $\mathcal{T}_{2,+}(\mathcal{H}_+,\mathcal{H}_-)$-valued Borel measure, with the property that for each compact interval I we have*

$$\|\imath_- E(\Omega\cap\Delta)\imath_+\|_2 \le C_I\, \mu(\Omega) \quad \textit{for all Borel sets } \ \Omega \subset I\ , \tag{48}$$

with $C_I < \infty$.

Theorem 4.6 *Assume Hypotheses I-III and IV$_2$. There exists a μ-locally integrable function $P(\lambda)$ from the real line into $\mathcal{T}_{2,+}(\mathcal{H}_+,\mathcal{H}_-)$, such that*

$$\imath_- E(\Omega \cap \Delta)\imath_+ = \int_\Omega P(\lambda)\, d\mu(\lambda) \quad \textit{for bounded Borel sets } \Omega\ , \tag{49}$$

where the integral is the Bochner integral of $\mathcal{T}_2(\mathcal{H}_+,\mathcal{H}_-)$-valued functions. The function $P(\lambda)$ is is μ-locally bounded and given by

$$P(\lambda) = \lim_{\delta \downarrow 0} \frac{\imath_- E(I_{\lambda,\delta} \cap \Delta)\imath_+}{\mu(I_{\lambda,\delta})} \quad \text{in } \mathcal{T}_2(\mathcal{H}_+,\mathcal{H}_-)\,,\ \text{for } \mu - a.e.\ \lambda\ , \tag{50}$$

where $I_{\lambda,\delta}$ is the interval $[\lambda-\delta, \lambda+\delta]$. In addition, we have

$$\imath_- E(\Delta) g(H)\, \imath_+ = \int g(\lambda) P(\lambda)\, d\mu(\lambda)\ , \tag{51}$$

for all bounded, scalar Borel functions g with $g(\lambda)\|P(\lambda)\|_2$ μ-integrable.

Proof: The proof proceeds as in the proof of Theorem 4.5, using (48).

It follows from either Theorem 4.5 or Theorem 4.6 that the values of the operator-valued Radon-Nikodym derivative $P(\lambda)$ are generalized eigenprojectors, in a sense made precise in the following corollary.

Corollary 4.7 *Assume either Hypotheses I-IV or Hypotheses I-III and IV$_2$. Then, for almost every $\lambda \in \Delta$ with respect to the spectral measure for H, we have $H_- P(\lambda) = \lambda P(\lambda)$, and $P(\lambda)\phi \in \mathcal{H}_-$ is a generalized eigenfunction of H with generalized eigenvalue λ for any $\phi \in \mathcal{H}_+$.*

Proof: Using (46) or (50), Lemma 4.3, (47) or (51), and Lebesgue's Differentiation Theorem (Theorem .8), we obtain

$$\lim_{\delta \downarrow 0} H_- \frac{\imath_- E(I_{\lambda,\delta} \cap \Delta)\imath_+}{\mu(I_{\lambda,\delta})} = \lim_{\delta \downarrow 0} \imath_- H \frac{E(I_{\lambda,\delta} \cap \Delta)}{\mu(I_{\lambda,\delta})}\imath_+ = \lambda P(\lambda) \tag{52}$$

in either $\mathcal{T}_1(\mathcal{H}_+,\mathcal{H}_-)$ or $\mathcal{T}_2(\mathcal{H}_+,\mathcal{H}_-)$, for almost every $\lambda \in \Delta$ with respect to the spectral measure for H. Since H_- is closed by Lemma 4.1, it follows from either (46) or (50) and (52), that $\operatorname{ran} P(\lambda) \subset \mathcal{D}(H_-)$ and $H_- P(\lambda) = \lambda P(\lambda)$ for almost every $\lambda \in \Delta$ with respect to the spectral measure for H.

Corollary 4.8 *Assume either Hypotheses I-IV or Hypotheses I-III and IV$_2$. Then, almost every $\lambda \in \Delta$ with respect to the spectral measure for H is a generalized eigenvalue of H.*

Corollary 4.8 is an immediate consequence of Corollary 4.7 and either (44) or (49) .

5 Spaces and operators

In this section we construct "extensions" and "restrictions" of classical wave operators. Along the way we prove that a second order classical wave operator has an operator core consisting of functions with compact support.

We start by introducing the following functional spaces:

• $\mathcal{H}_\tau^{(n)} = L^2(\mathbb{R}^d, \tau(x)^2 dx; \mathbb{C}^n)$, with $\tau(x) \geq 0$ a locally bounded measurable function; a Hilbert space whose norm and inner product will be denoted by $\| \;\|_\tau$ and $\langle \;,\; \rangle_\tau$.

• $\mathcal{H}_{\max}^{(n)} = L^2_{loc}(\mathbb{R}^d, dx; \mathbb{C}^n)$, the space of $\mathbb{C}^n$-valued, locally L^2-functions with the locally convex topology given by L^2 convergence on bounded sets; a Fréchet space (e.g., [19, Section V.2]).

• $\mathcal{H}_{\min}^{(n)} = L^2_0(\mathbb{R}^d, dx; \mathbb{C}^n)$, the space of $\mathbb{C}^n$-valued, L^2-functions with compact support; a complete locally convex space as the strict inductive limit of the Hilbert spaces $L^2(B_r, dx; \mathbb{C}^n)$, $B_r \subset \mathbb{R}^d$ being the ball of radius r centered at the origin (e.g., [19, Section V.4]).

The spaces $\mathcal{H}_{\max}^{(n)}$ and $\mathcal{H}_{\min}^{(n)}$ are used to prove the existence of an operator core of compactly supported functions for a $CWO_n^{(2)}$. To study generalized eigenfunctions we use spaces $\mathcal{H}_\alpha^{(n)}$ and $\mathcal{H}_{1/\alpha}^{(n)}$, where

$$\alpha \in C^1(\mathbb{R}^d), \quad \text{with} \quad \alpha(x) \geq 1 \quad \text{and} \quad \frac{\nabla \alpha}{\alpha} \quad \text{bounded} \,. \tag{53}$$

(In Section 6 we will take $\alpha(x) = (1 + |x|^2)^p$, $p > 0$.) Note that $\frac{\nabla(1/\alpha)}{1/\alpha} = -\frac{\nabla \alpha}{\alpha}$, so it is also bounded. The sesquilinear form

$$\langle \phi, \psi \rangle = \int \overline{\phi(x)} \cdot \psi(x) \, dx \,, \tag{54}$$

defined for either $\phi \in \mathcal{H}_{\min}^{(n)}, \psi \in \mathcal{H}_{\max}^{(n)}$ or $\phi \in \mathcal{H}_\alpha^{(n)}, \psi \in \mathcal{H}_{1/\alpha}^{(n)}$, turns $\mathcal{H}_{\min}^{(n)}$ and $\mathcal{H}_{\max}^{(n)}$, and $\mathcal{H}_\alpha^{(n)}$ and $\mathcal{H}_{1/\alpha}^{(n)}$, into conjugate duals of each other. We denote the adjoint of an operator O with respect to this duality by $O^\dagger$, whereas the adjoint with respect to the usual Hilbert space self-duality in $\mathcal{H}^{(n)}$ is denoted by O^*. (Unbounded operators on locally convex spaces are discussed in Appendix B.) It is easy to see that

$$\mathcal{H}_{\min}^{(n)} \subset \mathcal{H}_\alpha^{(n)} \subset \mathcal{H}^{(n)} \subset \mathcal{H}_{1/\alpha}^{(n)} \subset \mathcal{H}_{\max}^{(n)} \,,$$

the natural injections being continuous with dense range. Note $\mathcal{H}^{(n)} = \mathcal{H}_\alpha^{(n)}$ for $\alpha(x) \equiv 1$.

The following notation will be useful: Let $\mathcal{H}_i, \mathcal{K}_i$, $i=1,2$, be locally convex spaces, with $\mathcal{H}_2 \subset \mathcal{H}_1$ and $\mathcal{K}_2 \subset \mathcal{K}_1$, the natural injections being continuous. Given an operator C from $\mathcal{H}_1$ to $\mathcal{K}_1$, we will denote by $C_{\mathcal{H}_2,\mathcal{K}_2}$ its restriction to an operator from $\mathcal{H}_2$ to $\mathcal{K}_2$, i.e.,

$$C_{\mathcal{H}_2,\mathcal{K}_2}\phi = C\phi \ \text{ on } \ \mathcal{D}\left(C_{\mathcal{H}_2,\mathcal{K}_2}\right) = \left\{\phi \in \mathcal{D}(C) \cap \mathcal{H}_2 : \ C\phi \in \mathcal{K}_2\right\}. \quad (55)$$

Note that if C is a closed operator so is $C_{\mathcal{H}_2,\mathcal{K}_2}$. If $\mathcal{H}_i, = \mathcal{K}_i$, $i=1,2$, we write $C_{\mathcal{H}_2}$ for $C_{\mathcal{H}_2,\mathcal{H}_2}$.

Given a $CPDO^{(1)}_{n,m}$ D, we define $D_{\max}$ as the closure of $D|_{C_0^\infty(\mathbb{R}^d,\mathbb{C}^n)}$ as an operator from $\mathcal{H}^{(n)}_{\max}$ into $\mathcal{H}^{(m)}_{\max}$; its domain is given by

$$\mathcal{D}(D_{\max}) = \{\psi \in \mathcal{H}^{(n)}_{\max} : \ D\psi \in \mathcal{H}^{(m)}_{\max} \text{ in distributional sense}\}\ . \quad (56)$$

From (3) and (56) we immediately see that $D = (D_{\max})_{\mathcal{H}^{(n)},\mathcal{H}^{(m)}}$. We define $D_\sigma = (D_{\max})_{\mathcal{H}^{(n)}_\sigma,\mathcal{H}^{(m)}_\sigma}$, for $\sigma = \min, \alpha, 1/\alpha$; they all have $C_0^\infty(\mathbb{R}^d,\mathbb{C}^n)$ as an operator core. We similarly define D^*_σ, for $\sigma = \min, \alpha, 1/\alpha, \max$.

A CO_n S, defined by multiplication by the matrix-valued measurable function $S(x)$ as in (9), is easily seen to be a bounded operator with a bounded inverse in $\mathcal{H}^{(n)}_\sigma$, for all $\sigma = \min, \alpha, 1/\alpha, \max$.

We now consider an $MPDO^{(1)}_{n,m}$ $A = RDS$, and introduce closed densely defined operators A_σ from $\mathcal{H}^{(n)}_\sigma$ into $\mathcal{H}^{(m)}_\sigma$, $\sigma = \min, \alpha, 1/\alpha, \max$, by

$$A_\sigma = RD_\sigma S \ \text{ on } \ \mathcal{D}(A_\sigma) = S^{-1}\mathcal{D}(D_\sigma)\ ; \quad (57)$$

note that $S^{-1}C_0^\infty(\mathbb{R}^d,\mathbb{C}^n)$ is an operator core for A_σ. The same considerations apply to $A^*_\sigma = SD^*_\sigma R$.

We use the notation $1/\min = \max, 1/\max = \min$, and the ordering $\min < \alpha < 1 < 1/\alpha < \max$. We have, for $\sigma, \sigma' \in \{\min, \alpha, 1, 1/\alpha, \max\}$, with $\sigma < \sigma'$,

$$\begin{aligned} A_\sigma &= (A_{\sigma'})_{\mathcal{H}^{(n)}_\sigma,\mathcal{H}^{(m)}_\sigma}\ , && (58)\\ A^\dagger_{1/\sigma} &= A^*_\sigma = (A^*_{\sigma'})_{\mathcal{H}^{(m)}_\sigma,\mathcal{H}^{(n)}_\sigma} = (A^\dagger_{1/\sigma'})_{\mathcal{H}^{(m)}_\sigma,\mathcal{H}^{(n)}_\sigma}\ . && (59)\end{aligned}$$

We also have, for $\sigma \in \{\min, \alpha, 1, 1/\alpha, \max\}$,

$$\begin{aligned} \mathcal{D}(A_{\min}) &= \mathcal{D}(A_\sigma) \cap \mathcal{H}^{(n)}_{\min}\ , && (60)\\ \mathcal{D}(A^\dagger_{\max}) &= \mathcal{D}(A^\dagger_{1/\sigma}) \cap \mathcal{H}^{(m)}_{\min}\ . && (61)\end{aligned}$$

The following lemma is easily verifiable.

Lemma 5.1 *Let $\sigma \in \{\min, \alpha, 1, 1/\alpha, \max\}$, then if $\rho \in C^1(\mathbb{R}^d)$, with ρ and $\nabla\rho$ bounded, we have $\rho\psi \in \mathcal{D}(A_\sigma)$ for any $\psi \in \mathcal{D}(A_\sigma)$, and*

$$A_\sigma \rho\psi = \rho A_\sigma \psi + R\, D[\rho] S\psi \ , \tag{62}$$

where $D[\rho]$, given by multiplication by the matrix valued function $D(-i\nabla\rho(x))$, is a bounded operator from $\mathcal{H}_\sigma^{(n)}$ to $\mathcal{H}_\sigma^{(m)}$, with norm bounded by $D_+\|\nabla\rho\|_\infty$.

We now turn our attention to the operators $A^\dagger_{1/\sigma}A_\sigma = A^*_\sigma A_\sigma$ on $\mathcal{H}_\sigma^{(n)}$, defined on their natural domains:

$$D(A^\dagger_{1/\sigma}A_\sigma) = \{\phi \in \mathcal{H}_\sigma^{(n)} :\ \phi \in \mathcal{D}(A_\sigma) \text{ and } A_\sigma\phi \in \mathcal{D}(A^\dagger_{1/\sigma})\} \ . \tag{63}$$

The following result is a generalization of interior estimates (e.g., [8, 9]).

Theorem 5.2 *Let $\rho \in C^1(\mathbb{R}^d)$ and $\tau \in L^\infty_{loc}(\mathbb{R}^d, dx)$, with $0 \le \rho(x) \le \tau(x)$ and $|\nabla\rho(x)| \le c\tau(x)$ a.e., where c is a finite constant. Then for any $a > 0$ we can find $b < \infty$, depending only on a and c, such that*

$$\|A_{\max}\psi\|_\rho \le a\|A^\dagger_{\min}A_{\max}\psi\|_\tau + b\|\psi\|_\tau \ , \tag{64}$$

for any $\psi \in \mathcal{D}(A^\dagger_{\min}A_{\max})$.

Proof: Let χ_L denote the characteristic function of the cube of side L centered at the origin. We can find a sequence $\rho_L \in C^1(\mathbb{R}^d)$, with $\chi_L(x)\rho(x) \le \rho_L(x) \le \chi_{L+4d}(x)\rho(x)$ and $|\nabla\rho_L(x)| \le (c+1)\tau(x)$.

Now let $\psi \in \mathcal{D}(A^\dagger_{\min}A_{\max})$; using Lemma 5.1 we have $\rho_L\psi \in \mathcal{D}(A_{\max})$ and, since $\rho_L A_{\max}\psi \in \mathcal{H}^{(n)}_{\min}$,

$$\begin{aligned}
\|A_{\max}\psi\|^2_{\rho_L} &= \langle \rho_L^2 A_{\max}\psi, A_{\max}\psi\rangle && (65)\\
&= \langle A_{\min}\rho_L^2\psi, A_{\max}\psi\rangle - 2\langle \rho_L R\, D[\rho_L]S\psi, A_{\max}\psi\rangle && (66)\\
&= \langle \rho_L^2\psi, A^\dagger_{\min}A_{\max}\psi\rangle - 2\langle R\, D[\rho_L]S\psi, \rho_L A_{\max}\psi\rangle && (67)\\
&\le \|\psi\|_\tau\|A^\dagger_{\min}A_{\max}\psi\|_\tau + c_1\|\psi\|_\tau\|A_{\max}\psi\|_{\rho_L} \ , && (68)
\end{aligned}$$

where c_1 is a constant depending only on c. We may now let $L \to \infty$; (64) follows from the resulting inequality plus the fact that $2rs \le tr^2 + \frac{1}{t}s^2$ for $r, s, t > 0$.

Let α be as in (53); applying Theorem 5.2 with $\rho = \tau = \sigma$, where $\sigma = \alpha, 1/\alpha$, we can find a constant $c < \infty$, such that

$$\|A_{\max}\psi\|_\sigma \le \|A^\dagger_{\min}A_{\max}\psi\|_\sigma + c\|\psi\|_\sigma \ , \tag{69}$$

for any $\psi \in \mathcal{D}(A^\dagger_{\min} A_{\max})$. Thus we obtain the second inclusion in

$$\mathcal{D}(A^\dagger_{1/\sigma} A_\sigma) \subset \mathcal{D}((A^\dagger_{\min} A_{\max})_{\mathcal{H}^{(n)}_\sigma}) \subset \mathcal{D}(A_\sigma) \ , \tag{70}$$

the first inclusion being evident. Hence we must have equality, so $A^\dagger_{1/\sigma} A_\sigma = (A^\dagger_{\min} A_{\max})_{\mathcal{H}^{(n)}_\sigma}$. Thus we have

$$A^\dagger_{1/\sigma} A_\sigma = (A^\dagger_{1/\sigma'} A_{\sigma'})_{\mathcal{H}^{(n)}_\sigma} \quad \text{for} \quad \sigma < \sigma' \in \{\min, \alpha, 1, 1/\alpha, \max\} \ , \tag{71}$$

since the case $\sigma = \min$ follows immediately from (60) and (61).

Applying again Theorem 5.2, this time with $\rho = \rho_L \sigma$ and $\tau = \chi_{L+1}\sigma$, where $\rho_L \in C^1_0(\mathbb{R}^d)$, with $\chi_L(x) \leq \rho_L(x) \leq \chi_{L+1}(x)$ and $\nabla \rho_L$ uniformly bounded in L, we get that for any $a > 0$ we can find $b < \infty$, independent of L, such that

$$\|\chi_L A_{\max} \psi\|_\sigma \leq a \|\chi_{L+1} A^\dagger_{\min} A_{\max} \psi\|_\sigma + b \|\chi_{L+1} \psi\|_\sigma \ , \tag{72}$$

for any $\psi \in \mathcal{D}(A^\dagger_{\min} A_{\max})$. From (72) with $\sigma = 1$ and the fact that $A_{\max}$ and $A^\dagger_{\min}$ are closed operators, we can conclude that $A^\dagger_{\min} A_{\max}$ is a closed operator. Using (71), we have that $A^\dagger_{1/\sigma} A_\sigma$ is a closed operator for $\sigma = \min, \alpha, 1, 1/\alpha, \max$.

We now show that

$$(A^\dagger_{1/\sigma} A_\sigma)^\dagger = A^\dagger_\sigma A_{1/\sigma} \quad \text{for} \quad \sigma \in \{\min, \alpha, 1, 1/\alpha, \max\} \ . \tag{73}$$

It is immediate that $A^\dagger_\sigma A_{1/\sigma} \subset (A^\dagger_{1/\sigma} A_\sigma)^\dagger$. (For operators C_1, C_2, $C_1 \subset C_2$ means that $\mathcal{D}(C_1) \subset \mathcal{D}(C_2)$ and $C_1 = C_2$ on $\mathcal{D}(C_1)$.) To prove the converse, note that since $A^* A = (A^\dagger_{1/\sigma} A_\sigma)_{\mathcal{H}^{(n)}}$ for $\sigma = 1/\alpha, \max$ by (71), we have

$$(A^\dagger_{1/\sigma} A_\sigma)^\dagger \subset ((A^* A)^\dagger)_{\mathcal{H}^{(n)}_{1/\sigma}} = (A^* A)_{\mathcal{H}^{(n)}_{1/\sigma}} = A^\dagger_\sigma A_{1/\sigma} \ , \tag{74}$$

so $(A^\dagger_{1/\sigma} A_\sigma)^\dagger = A^\dagger_\sigma A_{1/\sigma}$ for $\sigma = 1/\alpha, \max$. Since $((A^\dagger_{1/\sigma} A_\sigma)^\dagger)^\dagger = A^\dagger_{1/\sigma} A_\sigma$, we also have (73) for $\sigma = \min, \alpha$.

Since $A^* A$ is densely defined, it follows from (71) and (73) that $A^\dagger_{1/\sigma} A_\sigma$ is densely defined for $\sigma = \min, \alpha, 1, 1/\alpha, \max$. In fact, we have more.

Lemma 5.3 *The domain of $A^\dagger_{\max} A_{\min}$ is an operator core for $A^\dagger_{1/\sigma} A_\sigma$, for $\sigma = \alpha, 1, 1/\alpha$.*

Proof: Let $H_\sigma = A^\dagger_{1/\sigma} A_\sigma$. If $\sigma = \alpha, 1, 1/\alpha$, then $\mathcal{H}^{(n)}_\sigma$ is a Hilbert space, and $H_\sigma = H^\dagger_{1/\sigma} = \frac{1}{\sigma^2} H^*_{1/\sigma}\,\sigma^2$, where $H^*_{1/\sigma}$ is the adjoint of the closed, densely defined operator $H_{1/\sigma}$ in the Hilbert space $\mathcal{H}^{(n)}_{1/\sigma}$.

To show that $\mathcal{D}(H_{\min})$ is an operator core for H_σ, we need to show that $\Gamma(H_\sigma)$ is the closure of $\Gamma(H_{\min})$ in $\mathcal{H}^{(n)}_\sigma \times \mathcal{H}^{(n)}_\sigma$. So let $\psi \in \mathcal{D}(H_\sigma)$ be such that, for all $\phi \in \mathcal{D}(H_{\min})$,

$$
\begin{aligned}
0 &= \langle \phi, \psi \rangle_\sigma + \langle H_{\min}\phi, H_\sigma \psi \rangle_\sigma && (75)\\
&= \langle \phi, \sigma^2 \psi \rangle + \langle H_{\min}\phi, \sigma^2 H_\sigma \psi \rangle && (76)\\
&= \langle \phi, \sigma^2 \psi \rangle + \langle H_{\min}\phi, H^*_{1/\sigma}\,\sigma^2 \psi \rangle\ . && (77)
\end{aligned}
$$

Thus $H^*_{1/\sigma}\,\sigma^2\psi \in \mathcal{D}(H^\dagger_{\min})$, and, recalling $H_{\max} = H^\dagger_{\min}$, we have $H_{\max} H^*_{1/\sigma}\,\sigma^2\psi = -\sigma^2\psi$. But, since $\sigma^2\psi$, $H^*_{1/\sigma}\,\sigma^2\psi \in \mathcal{H}^{(n)}_{1/\sigma}$, we have $H^*_{1/\sigma}\,\sigma^2\psi \in \mathcal{D}(H_{1/\sigma})$ and $H_{1/\sigma} H^*_{1/\sigma}\,\sigma^2\psi = -\sigma^2\psi$. Since $H_{1/\sigma} H^*_{1/\sigma}$ is a positive operator on $\mathcal{H}^{(n)}_{1/\sigma}$, we must have $\psi = 0$.

We summarize our results in the following theorem:

Theorem 5.4 *$A^\dagger_{1/\sigma} A_\sigma$ is a closed, densely defined operator on $\mathcal{H}^{(n)}_\sigma$, for each $\sigma = \min, \alpha, 1, 1/\alpha, \max$. We have $A^\dagger_{1/\sigma} A_\sigma = (A^\dagger_{1/\sigma'} A_{\sigma'})_{\mathcal{H}_\sigma}$ for $\sigma < \sigma'$, and $(A^\dagger_{1/\sigma} A_\sigma)^\dagger = A^\dagger_\sigma A_{1/\sigma}$. In addition, the domain of $A^\dagger_{\max} A_{\min}$ is an operator core for $A^\dagger_\sigma A_\sigma$, for $\sigma = \alpha, 1, 1/\alpha$.*

The following corollary is a nontrivial statement when A^*A has nonsmooth coefficients.

Corollary 5.5 *The functions in the domain of A^*A with compact support form an operator core.*

Note that A^*A is a local operator, in the sense that if $\psi \in \mathcal{D}(A^*A)$ and $\psi = 0$ in some open set Ω, then we must also have $A^*A\psi = 0$ in Ω. This follows from the same property for A and A^*.

6 Generalized eigenfunction expansions for classical wave operators

In this section we prove Theorem 3.4.

Let us set the weight $\alpha(x)$ in (53) to be

$$\alpha_p(x) = (1+|x|^2)^p \ , \quad \text{with} \ \ p > 0 \ , \tag{78}$$

and simplify the notation by replacing α_p and $1/\alpha_p$ in subscripts by $+$ and $-$, respectively. Thus

$$\mathcal{H}_{\pm}^{(n)} = L^2(\mathbb{R}^d, (1+|x|^2)^{\pm 2p} dx; \mathbb{C}^n) \ . \tag{79}$$

(These spaces depend on our choice of $p > 0$, but we omit p from the notation.) Note that $\mathcal{H}_{-}^{(n)}$ is the space of $\mathbb{C}^n$-valued polynomially L^2-bounded functions that satisfy (24) with the same p as in (79). We use $\| \ \|_{\pm}$, $\langle \ , \ \rangle_{\pm}$ for the corresponding norms and inner products.

Theorem 6.1 *Hypotheses I-III (Section 4) are satisfied for classical wave operators W, with T being the operator given by multiplication by $(1+|x|^2)^p$, $p > 0$. If W is a first order classical wave operator, $W_{\pm}$ are as in (57). If $W = A^*A$ is a second order classical wave operator, we have $W_+ = A_-^{\dagger} A_+$ and $W_- = A_+^{\dagger} A_-$.*

If the classical wave operator W is partially elliptic and $p > \frac{d}{4}$, Hypothesis IV is also satisfied with $\Delta = \mathbb{R}\backslash\{0\}$, so W has the generalized eigenfunction expansion of Theorem 4.5.

Proof: If $W = SDS$ is a first order classical wave operator, Hypotheses I-III are satisfied, since $S^{-1}C_0^{\infty}(\mathbb{R}^d;\mathbb{C}^n) \subset \mathcal{D}_+$ and is an operator core for W, and $W_{\pm}$ are as in (57). For a second order classical wave operator $W = A^*A$, it follows from (71) and Theorem 5.4 that $W_+ = A_-^{\dagger} A_+$, a densely defined operator , whose domain is an operator core for W. Thus Hypotheses I-III of Section 4 are also satisfied in this case, and $W_- = A_+^{\dagger} A_-$ by (73) and Lemma 4.1.

If the classical wave operator W is partially elliptic and $p > \frac{d}{4}$, Hypothesis IV is a direct consequence of Theorem 2.13

It turns out that Definitions 3.1 and 3.2 are equivalent to Definition 4.2, for an appropriate choice of $p > 0$.

Proposition 6.2 *A non-zero, $\mathbb{C}^n$-valued polynomially L^2-bounded function ψ is a generalized eigenfunction of a classical wave operator W with generalized eigenvalue λ if, and only if, ψ is an eigenfunction of W_- with eigenvalue λ for some $p > 0$, i.e., $\psi \in \mathcal{D}(W_-)$ and $W_-\psi = \lambda\psi$.*

Proof: If $W = SDS$ is first order, (25) is equivalent to (33) for an appropriate choice of $p > 0$, since $S^{-1}C_0^\infty(\mathbb{R}^d;\mathbb{C}^n) \subset \mathcal{D}_+$ and is an operator core for W.

If $W = A^*A$ is second order, it is easy to see that, for an appropriate choice of $p > 0$, (26) is equivalent to $\psi \in \mathcal{D}(A_-)$, and (27) is equivalent to $A_-\psi \in \mathcal{D}(A_+^\dagger)$ with $A_+^\dagger A_-\psi = \lambda\psi$.

Proposition 6.3 *Let A be a $MPDO_{m,n}^{(1)}$; fix $p > 0$. If $\mathbf{\Psi} = (\psi, \phi) \in \mathcal{H}_-^{(n+m)} \cong \mathcal{H}_-^{(n)} \oplus \mathcal{H}_-^{(m)}$ is a generalized eigenfunction of $\mathbb{W}_A$ with generalized eigenvalue λ, then ψ is a generalized eigenfunction of A^*A and ϕ is a generalized eigenfunction of AA^*, both with generalized eigenvalue λ^2. Conversely, if $\lambda \neq 0$ and $\psi \in \mathcal{H}_-^{(n)}$ is a generalized eigenfunction of A^*A with generalized eigenvalue λ^2, we can find generalized eigenfunctions $\phi_\pm \in \mathcal{H}_-^{(m)}$ of AA^* also with generalized eigenvalue λ^2, such that $(\psi, \phi_\pm)$ is a generalized eigenfunction of $\mathbb{W}_A$ with generalized eigenvalue $\pm\lambda$. Moreover, if $\psi \in \mathcal{H}^{(n)}$, we also have $\phi_\pm \in \mathcal{H}^{(m)}$, so $(\psi, \phi_\pm) \in \mathcal{H}^{(n+m)}$.*

We have the following relation for the generalized point spectra:

$$\sigma_{gp}(\mathbb{W}_A) = \left\{ \lambda \in \mathbb{C} : \ \lambda^2 \in \sigma_{gp}(A^*A) \right\} \ . \tag{80}$$

Proof: We use Proposition 6.2. It is easy to see that

$$(\mathbb{W}_A)_- = \begin{bmatrix} 0 & A_-^* \\ A_- & 0 \end{bmatrix} \ , \quad (\mathbb{W}_A)_-^2 = \begin{bmatrix} A_-^*A_- & 0 \\ 0 & A_-A_-^* \end{bmatrix} \ . \tag{81}$$

If $\mathbf{\Psi} = (\psi, \phi) \in \mathcal{D}((\mathbb{W}_A)_-)$, with $(\mathbb{W}_A)_-\Psi = \lambda\Psi$, we have $\Psi \in \mathcal{D}((\mathbb{W}_A)_-^2)$, $(\mathbb{W}_A)_-^2\Psi = \lambda^2\Psi$, i.e., $\psi \in \mathcal{D}(A_-^*A_-)$, $A_-^*A_-\psi = \lambda^2\psi$, and $\phi \in \mathcal{D}(A_-A_-^*)$, $A_-A_-^*\phi = \lambda^2\phi$.

Conversely, let $\lambda \neq 0$ and $\psi \in \mathcal{D}(A_-^*A_-)$, with $A_-^*A_-\psi = \lambda^2\psi$. Set $\phi_\pm = \pm\frac{1}{\lambda}A_-\psi$. Then $\phi_\pm \in \mathcal{D}(A_-^*)$, so $\Psi_\pm = (\psi, \phi_\pm) \in \mathcal{D}((\mathbb{W}_A)_-)$ and we can check that $(\mathbb{W}_A)_-\Psi_\pm = \pm\lambda\Psi_\pm$. Moreover, if $\psi \in \mathcal{H}^{(n)}$, we have $\psi \in \mathcal{D}(A^*A)$ by (71), so $\psi \in \mathcal{D}(A)$. Thus $\phi_\pm \in \mathcal{H}^{(m)}$ by (58).

The relation (80) follows.

We are now ready to prove Theorem 3.4. In view of Theorem 6.1 and Proposition 6.2, Theorem 3.4(i) follows from Proposition 4.4.

To prove Theorem 3.4(ii): it follows from Proposition 6.2 and Theorem .4 that it suffices to show that, if $\lambda \neq 0$ is an eigenvalue of W_- with a corresponding eigenfunction $\psi \notin \mathcal{H}^{(n)}$, then λ is in the essential spectrum of W. In view of Proposition 6.3, it is enough to consider the case when W is a first order classical wave operator, say $W = SDS$.

So let $\psi \in \mathcal{D}(W_-)$, $W_-\psi = \lambda\psi$. We pick a sequence $\rho_L \in C_0^\infty(\mathbb{R}^d)$, such that $\chi_L(x) \leq \rho_L(x) \leq \chi_{L+1}(x)$ (χ_L denotes the characteristic function of the cube of side L centered at the origin) and $\sup_L \|\nabla\rho_L\|_\infty < \infty$, and set $\psi_L = \rho_L\psi$; it follows from Lemma 5.1 and (58) that $\psi_L \in \mathcal{D}(W)$, and

$$W\psi_L = \rho_L W_-\psi + SD[\rho_L]S\psi = \lambda\psi_L + SD[\rho_L]S\psi \ . \tag{82}$$

Since $\|\psi_L\| \geq \|\chi_L\psi\|$, it follows from (82) that

$$\frac{\|(W-\lambda)\psi_L\|}{\|\psi_L\|} \leq C\frac{\|(\chi_{L+1}-\chi_L)\psi\|}{\|\chi_L\psi\|} = C\left(\frac{\|\chi_{L+1}\psi\|^2}{\|\chi_L\psi\|^2} - 1\right)^{\frac{1}{2}} \ , \tag{83}$$

where C is a finite constant.

Using an argument of Sch'nol (see [22, Theorem C.4.1]), we can show that the polynomial boundedness of ψ implies

$$\liminf_{L\to\infty} \frac{\|\chi_{L+1}\psi\|}{\|\chi_L\psi\|} = 1 \ . \tag{84}$$

Theorem 3.4(ii) now follows from Weyl's criterion.

It follows from (i) and (ii) that the generalized point spectrum of a classical wave operator is a subset of its spectrum: $\sigma_{gp}(W) \subset \sigma(W)$.

If the classical wave operator W is partially elliptic, we can use Corollary 4.8 to conclude that $\sigma(W)\backslash\sigma_{gp}(W)$ has zero spectral measure.

If $W = \mathbb{W}_A$ as in (11) with A^*A partially elliptic, we also conclude that $\sigma(W)\backslash\sigma_{gp}(W)$ has zero spectral measure from (80), (21), (20), and the result for A^*A.

Theorem 3.4 is proved.

7 Generalized eigenfunction expansions of classical waves

Finite energy classical waves are the solutions of the classical wave equation (14) in the Hilbert space $(\ker \mathbb{W}_A)^\perp = P^\perp_{\mathbb{W}_A}\mathcal{H}^{(n+m)}$; they are of the form

$$\mathbf{\Psi}_t = \mathrm{e}^{it\mathbb{W}_A} P^\perp_{\mathbb{W}_A}\mathbf{\Phi}_0 \ , \quad \mathbf{\Phi}_0 \in \mathcal{H}^{(n+m)} \ . \tag{85}$$

If the first order classical wave operator $\mathbb{W}_A$ is partially elliptic, the second order classical wave operators $(\mathbb{W}_A)^2$, A^*A and AA^* are also partially elliptic, so they all have generalized eigenfunction expansions given by Theorems 6.1 and 4.5. In particular, if $p > \frac{d}{4}$, $\mu_{\mathbb{W}_A}$ given by (38) is a spectral measure for $(\mathbb{W}_A)_\perp$, we have generalized eigenprojectors $P_{\mathbb{W}_A}(\lambda)$ as in Theorem 4.5 and Corollary 4.7, and it follows from (47) that a finite energy classical wave Ψ_t, as in (85), has an expansion in generalized eigenfunctions:

$$\Psi_t = \lim_{j\to\infty} \int_{(-j,j)} \mathrm{e}^{it\lambda} P_{\mathbb{W}_A}(\lambda)\Phi_0 \, d\mu_{\mathbb{W}_A}(\lambda) \quad \text{in} \quad \mathcal{H}_-^{(n+m)} \; . \tag{86}$$

It may happen that A^*A is partially elliptic with $\mathbb{W}_A$ and AA^* not being partially elliptic, as discussed in Remark 2.9. It turns out that we still have generalized eigenfunction expansions for $\mathbb{W}_A$ and AA^*, and an expansion similar to (86).

Theorem 7.1 *Let A be a $MPDO^{(1)}_{n,m}$ with A^*A partially elliptic, and let μ_{A^*A} be the spectral measure for $(A^*A)_\perp$ given by (38) with $\Delta = \mathbb{R}\backslash\{0\}$. Then Hypothesis IV_2 (Section 4) is satisfied by AA^* and $\mathbb{W}_A$ with $\Delta = \mathbb{R}\backslash(-\varepsilon, \varepsilon)$, for any $\varepsilon > 0$, and spectral measure $\mu = \mu_{A^*A}$ for $(AA^*)_\perp$ and $\mu(\Omega) = \mu_{A^*A}(\Omega^2)$ for $(\mathbb{W}_A)_\perp$, with T being the operator given by multiplication by $(1 + |x|^2)^p$, $p > \frac{d}{4}$. Thus AA^* and $\mathbb{W}_A$ have the generalized eigenfunction expansion of Theorem 4.6. In particular, if $P_{A^*A}(\lambda)$ is as in Theorem 4.5, we have*

$$\imath_- E_{AA^*}(\Omega)\imath_+ = \int_\Omega \tfrac{1}{\lambda} A_- P_{A^*A}(\lambda) A_+^* \, d\mu_{A^*A}(\lambda) \; , \tag{87}$$

and

$$\imath_- E_{\mathbb{W}_A}(\Omega)\imath_+ = \tag{88}$$
$$\frac{1}{2}\int_\Omega \begin{bmatrix} P_{A^*A}(\lambda^2) & \frac{1}{\lambda} P_{A^*A}(\lambda^2) A_+^* \\ \frac{1}{\lambda} A_- P_{A^*A}(\lambda^2) & \frac{1}{\lambda^2} A_- P_{A^*A}(\lambda^2) A_+^* \end{bmatrix} d\mu_{A^*A}(\lambda^2) \; ,$$

for all bounded Borel sets Ω with $0 \notin \bar{\Omega}$, as Bochner integrals in $\mathcal{T}_2(\mathcal{H}_+^{(m)}, \mathcal{H}_-^{(m)})$ and $\mathcal{T}_2(\mathcal{H}_+^{(n+m)}, \mathcal{H}_-^{(n+m)})$.

Proof: If H is a self-adjoint operator, we denote by $E_H^\perp(\Omega)$ the spectral projections of $H_\perp$ on $(\ker H)^\perp$. It follows from (18) that

$$E_{AA^*}^\perp(\Omega) = A (A^*A)_\perp^{-\frac{1}{2}} E_{A^*A}^\perp(\Omega) (A^*A)_\perp^{-\frac{1}{2}} A^* \; , \tag{89}$$

on an the domain of A^*. Similarly, it follows from (20) that

$$E^{\perp}_{\mathbb{W}_A}(\Omega) = \mathbb{U}\Big\{E^{\perp}_{\sqrt{A^*A}}(\Omega_+) \oplus E^{\perp}_{\sqrt{A^*A}}(\Omega_-)\Big\}\,\mathbb{U}^* = \tag{90}$$
$$\frac{1}{2}\begin{bmatrix} \Big\{E^{\perp}_{\sqrt{A^*A}}(\Omega_+) + E^{\perp}_{\sqrt{A^*A}}(\Omega_-)\Big\} & \Big\{E^{\perp}_{\sqrt{A^*A}}(\Omega_+) - E^{\perp}_{\sqrt{A^*A}}(\Omega_-)\Big\}U^* \\ U\Big\{E^{\perp}_{\sqrt{A^*A}}(\Omega_+) - E^{\perp}_{\sqrt{A^*A}}(\Omega_-)\Big\} & U\Big\{E^{\perp}_{\sqrt{A^*A}}(\Omega_+) + E^{\perp}_{\sqrt{A^*A}}(\Omega_-)\Big\}U^* \end{bmatrix},$$

where $\Omega_\pm = (\pm\Omega) \cap (0,\infty)$. Note that

$$E^{\perp}_{\sqrt{A^*A}}(\Omega_\pm) = E^{\perp}_{A^*A}(\Omega_\pm^2) \tag{91}$$

In addition, we have

$$UE^{\perp}_{\sqrt{A^*A}}(\Omega_\pm) = A(A^*A)_\perp^{-\frac{1}{2}} E^{\perp}_{A^*A}(\Omega_\pm^2)\ . \tag{92}$$

Since A^*A is partially elliptic, we have $\imath_- E_{A^*A}(\Omega\backslash\{0\})\imath_+ \in \mathcal{T}_1(\mathcal{H}_+, \mathcal{H}_-)$ for any bounded Borel set Ω. Let $\Omega \subset [\theta, \Theta]$ with $0 < \theta \leq \Theta < \infty$. We have, using the estimate (69) and the definition of the Hilbert-Schmidt norm, that

$$\begin{aligned} \|\imath_- A(A^*A)^{-\frac{1}{2}} E_{A^*A}(\Omega)\imath_+\|_2 &= \|A_-\imath_-(A^*A)^{-\frac{1}{2}} E_{A^*A}(\Omega)\imath_+\|_2 \\ &\leq (\Theta^{\frac{1}{2}} + c\theta^{-\frac{1}{2}})\|\imath_- E_{A^*A}(\Omega)\imath_+\|_2 \\ &\leq (\Theta^{\frac{1}{2}} + c\theta^{-\frac{1}{2}})\|\imath_- E_{A^*A}(\Omega)\imath_+\|_1 \\ &= (\Theta^{\frac{1}{2}} + c\theta^{-\frac{1}{2}})\mu_{A^*A}(\Omega)\ , \end{aligned} \tag{93}$$

so we can conclude that $\imath_- A(A^*A)^{-\frac{1}{2}} E_{A^*A}(\Omega)\imath_+ \in \mathcal{T}_2(\mathcal{H}_+^{(n)}, \mathcal{H}_-^{(m)})$. Moreover, the same argument shows that

$$\begin{aligned} &\|\imath_- A(A^*A)^{-\frac{1}{2}} E_{A^*A}(\Omega)(A^*A)^{-\frac{1}{2}} A^*\imath_+\|_2 \\ &\quad\leq (\Theta^{\frac{1}{2}} + c\theta^{-\frac{1}{2}})\|\imath_- E_{A^*A}(\Omega)(A^*A)^{-\frac{1}{2}} A^*\imath_+\|_2 \\ &\quad= (\Theta^{\frac{1}{2}} + c\theta^{-\frac{1}{2}})\|\imath_- A(A^*A)^{-\frac{1}{2}} E_{A^*A}(\Omega)\imath_+\|_2 \\ &\quad\leq (\Theta^{\frac{1}{2}} + c\theta^{-\frac{1}{2}})^2\|\imath_- E_{A^*A}(\Omega)\imath_+\|_1 \\ &\quad= (\Theta^{\frac{1}{2}} + c\theta^{-\frac{1}{2}})^2\mu_{A^*A}(\Omega)\ , \end{aligned} \tag{94}$$

so we also conclude that

$$\imath_- A(A^*A)^{-\frac{1}{2}} E_{A^*A}(\Omega)(A^*A)^{-\frac{1}{2}} A^*\imath_+ \in \mathcal{T}_2(\mathcal{H}_+^{(m)}, \mathcal{H}_-^{(m)})\ .$$

Since any spectral measure for $(A^*A)_\perp$ is also a spectral measure for $(AA^*)_\perp$ by (18), μ_{A^*A} is a spectral measure for $(AA^*)_\perp$. It now follows from (89) and (94) that Hypothesis IV$_2$ is satisfied by AA^* with $\Delta = \mathbb{R}\backslash(-\varepsilon, \varepsilon)$, for any $\varepsilon > 0$, and spectral measure μ_{A^*A}. We can thus apply Theorem 4.6 to obtain (87).

We now turn to the first order wave operator $\mathbb{W}_A$. Using (90)-(94), we get that $\imath_- E_{\mathbb{W}_A}(\Omega)\imath_+ \in \mathcal{T}_2(\mathcal{H}_+^{(n+m)}, \mathcal{H}_-^{(n+m)})$ for all bounded Borel sets Ω with $0 \notin \bar{\Omega}$. Moreover, the measure $\nu_{\sqrt{A^*A}}$, given by

$$d\nu_{\sqrt{A^*A}}(\lambda) = \begin{cases} d\mu_{A^*A}(\lambda^2) & if\ \lambda > 0 \\ 0 & if\ \lambda \leq 0 \end{cases}, \tag{95}$$

is a spectral measure for $(\sqrt{A^*A})_\perp$. It then follows from (20) that the measure $\nu_{\mathbb{W}_A}$, defined by $d\nu_{\mathbb{W}_A}(\lambda) = d\mu_{A^*A}(\lambda^2)$, is a spectral measure for $(\mathbb{W}_A)_\perp$. Thus Hypothesis IV$_2$ is satisfied by $\mathbb{W}_A$ with $\Delta = \mathbb{R}\backslash(-\varepsilon, \varepsilon)$, for any $\varepsilon > 0$, and spectral measure $\nu_{\mathbb{W}_A}$. Applying Theorem 4.6, we get (88).

Remark 7.2 *We have explicit bounds on the $\mathcal{T}_2$- valued Radon-Nikodym derivatives in (87) and (88). Proceeding as in (93), and using Corollary 4.7, we get*

$$\|A_- P_{A^*A}(\lambda)\|_2 \leq (|\lambda| + c)\|P_{A^*A}(\lambda)\|_2 \leq (|\lambda| + c)\ . \tag{96}$$

Similarly, as in (94)., we obtain

$$\begin{aligned} &\|A_- P_{A^*A}(\lambda)A_+^*\|_2 \leq (|\lambda| + c)\|P_{A^*A}(\lambda)A_+^*\|_2 \\ &\quad = (|\lambda| + c)\|A_- P_{A^*A}(\lambda)\|_2 \leq (|\lambda| + c)^2\|P_{A^*A}(\lambda)\|_2 \leq (|\lambda| + c)^2\ . \end{aligned} \tag{97}$$

Remark 7.3 *If A^*A is partially elliptic, a finite energy classical wave Ψ_t, as in (85), has an expansion in generalized eigenfunctions:*

$$\Psi_t = \lim_{j\to\infty} \int_{(-j,j)} \tfrac{1}{2}\mathrm{e}^{it\lambda} \begin{bmatrix} P_{A^*A}(\lambda^2) & \frac{1}{\lambda}P_{A^*A}(\lambda^2)A_+^* \\ \frac{1}{\lambda}A_- P_{A^*A}(\lambda^2) & \frac{1}{\lambda^2}A_- P_{A^*A}(\lambda^2)A_+^* \end{bmatrix} \Phi_0\, d\mu_{A^*A}(\lambda^2) \tag{98}$$

in $\mathcal{H}_-^{(n+m)}$.

Remark 7.4 *If both A^*A and AA^* are partially elliptic, or equivalently, if the first order classical wave operator $\mathbb{W}_A$ is partially elliptic (see Reamark 2.7), the operators AA^* and $\mathbb{W}_A$ have generalized eigenfunction expansions given by Theorem 4.5. Since A^*A is partially elliptic, they also have the expansions given in Theorem 7.1. Comparing these expansions, we conclude that*

$$P_{AA^*}(\lambda) = \frac{1}{\lambda}\left(\frac{d\mu_{AA^*}}{d\mu_{A^*A}}(\lambda)\right)^{-1} A_- P_{A^*A}(\lambda)A_+^* \tag{99}$$

for μ_{A^*A}*-a.e.* λ*, and*

$$P_{\mathbb{W}_A}(\lambda) = \tag{100}$$

$$\left(1+\frac{d\mu_{AA^*}}{d\mu_{A^*A}}(\lambda^2)\right)^{-1}\begin{bmatrix} P_{A^*A}(\lambda^2) & \frac{1}{\lambda}P_{A^*A}(\lambda^2)A_+^* \\ \frac{1}{\lambda}A_-P_{A^*A}(\lambda^2) & \frac{1}{\lambda^2}A_-P_{A^*A}(\lambda^2)A_+^* \end{bmatrix}$$

for $\mu^{(2)}_{A^*A}$*-a.e.* λ*, with* $d\mu^{(2)}_{A^*A}(\lambda) = d\mu_{A^*A}(\lambda^2)$*.*

8 Proof of the trace estimate

In this section we prove Theorem 2.13. Note that it suffices to prove (22) for $r=\nu$ and, since

$$\operatorname{tr}(V^*P_W^\perp(W+I)^{-2r}V) = \operatorname{tr}(P_W^\perp(W+I)^{-r}VV^*P_W^\perp(W+I)^{-r})\ , \tag{101}$$

and the constant on the right-hand-side of (22) is tranlation invariant, we need only to consider the case when V is multiplication by the scalar function $\chi_\ell \equiv \chi_{0,\ell}$ with $\ell = 1$.

Tha partially elliptic $CWO_n^{(2)}$ W is of the form A^*A, where $A = RDS$ is an $MPDO^{(1)}_{n,m}$ with D partially elliptic. Thus there exists a $CPDO^{(1)}_{n,q}$ $D^\perp$, as in Definition 2.2. Let us define the operators

$$B = D^\perp S^{-1} \quad , \quad H = A^*A + B^*B \quad , \quad R = (H+I)^{-1} \quad . \tag{102}$$

Note that, because of (6) and (7),

$$\mathcal{H}^{(n)} = \ker A \oplus \ker B \quad , \quad H = (A^*A)_\perp \oplus (B^*B)_\perp \quad , \tag{103}$$

so H is a self-adjoint operator. In addition,

$$P_W^\perp(W+I)^{-2r} = P_W^\perp(H+I)^{-2r} \quad . \tag{104}$$

Thus, if we establish an estimate on the Hilbert-Schmidt norm of the form

$$\|R^\nu\chi_1\|_2 \leq C_{d,n,S_\pm,R_\pm,D_+,D_+^\perp,\Upsilon} < \infty\ , \tag{105}$$

the theorem follows. Here and on what follows $C_{\alpha,\beta,\ldots,}$ denotes a constant depending only on $\alpha,\beta,\ldots$.

We first prove the analogous result on the torus $\Lambda_\ell = \left[-\frac{\ell}{2},\frac{\ell}{2}\right]^d$ and then carry it over to $\mathbb{R}^d$. We set $\mathcal{H}_\ell^{(n)} = L^2(\Lambda_\ell, dx; \mathbb{C}^n)$. A

$CPDO^{(1)}_{n,m}$ D defines a closed densely defined operator D_ℓ from $\mathcal{H}^{(n)}_\ell$ to $\mathcal{H}^{(m)}_\ell$ with periodic boundary condition; an operator core is given by $C^\infty_{\rm per}(\Lambda_\ell, \mathbb{C}^n)$, the infinitely differentiable, periodic $\mathbb{C}^n$-valued functions on Λ_ℓ. The restriction of a CO_n S to Λ_ℓ gives the bounded, invertible operator S_ℓ on $\mathcal{H}^{(n)}_\ell$.

Thus we define the restrictions $A_\ell = R_\ell D_\ell S_\ell$, $B_\ell = D^\perp_\ell S^{-1}_\ell$ of the operators A, B to the torus, and set

$$H_\ell = A^*_\ell A_\ell + B^*_\ell B_\ell \quad , \quad R_\ell = (H_\ell + I)^{-1} \ . \tag{106}$$

As before, H_ℓ is a self-adjoint operator on $L^2(\Lambda_\ell, dx; \mathbb{C}^n)$, since the kernels of A_ℓ and B_ℓ are mutually orthogonal.

Lemma 8.1 *If $p > \frac{d}{2}$, we have*

$$\mathrm{tr}\, R^p_\ell \le C_{d,n,R_-,S_\pm,\Upsilon,\ell,p} < \infty \ . \tag{107}$$

Proof: First we notice that the lemma is valid for $-\Delta_\ell$, the Laplacian on the torus with periodic boundary conditions. Namely, using the Fourier series representation for $L^2(\Lambda_\ell)$, we find

$$\mathrm{tr}((-\Delta_\ell + I)^{-p}) = \sum_{k \in \frac{2\pi}{\ell}\mathbb{Z}^d} (1 + |k|^2)^{-p} < \infty \ . \tag{108}$$

In what follows,we write $F \simeq G$ if the positive self-adjoint operators F and G are unitarily equivalent, and we write $F \preceq G$ if $F \simeq J$ for some positive self-adjoint operator $J \le G$.

Recall that if C is a closed, densely defined operator on a Hilbert space, we always have $(C^*C)_\perp \simeq (CC^*)_\perp$ (as in (18)). Thus,

$$\begin{aligned}
(H_\ell)_\perp &= (S_\ell D^*_\ell R^2_\ell D_\ell S_\ell)_\perp \oplus (S^{-1}_\ell D^{\perp *}_\ell D^\perp_\ell S^{-1}_\ell)_\perp \\
&\ge (R^2_- S_\ell D^*_\ell D_\ell S_\ell)_\perp \oplus (S^{-1}_\ell D^{\perp *}_\ell D^\perp_\ell S^{-1}_\ell)_\perp \\
&\simeq (R^2_- D_\ell S^2_\ell D^*_\ell)_\perp \oplus (D^\perp_\ell S^{-2}_\ell D^{\perp *}_\ell)_\perp \\
&\ge (S^2_- R^2_- D_\ell D^*_\ell)_\perp \oplus (S^{-2}_+ D^\perp_\ell D^{\perp *}_\ell)_\perp \\
&\simeq (S^2_- R^2_- D^*_\ell D_\ell)_\perp \oplus (S^{-2}_+ D^{\perp *}_\ell D^\perp_\ell)_\perp \\
&\ge \min(S^2_- R^2_-, S^{-2}_+) \ \left[(D^*_\ell D_\ell)_\perp \oplus (D^{\perp *}_\ell D^\perp_\ell)_\perp\right] \\
&\ge \Upsilon \min(S^2_- R^2_-, S^{-2}_+) \ (-\Delta_\ell \otimes I_n)_\perp \quad ,
\end{aligned} \tag{109}$$

where we used (5) to obtain the last inequality.

If $\mathcal{E}$ and $\mathcal{F}$ are (not necessarily closed) subspaces of some Hilbert space with $\mathcal{E} \subset \mathcal{F}$, by $\operatorname{codim}_{\mathcal{F}}(\mathcal{E})$ we denote the dimension of the orthogonal complement of $\mathcal{E}$ in $\overline{\mathcal{F}}$. Again using (5), we obtain

$$\begin{aligned}
\dim(\ker H_l) &= \operatorname{codim}_{\ker A_l^* A_l}(\operatorname{ran} B_l^* B_l) \\
&= \operatorname{codim}_{S_\ell^{-1}\ker D_\ell}(S_\ell^{-1}(\operatorname{ran} D_\ell^{\perp *})) = \operatorname{codim}_{\ker D_\ell}(\operatorname{ran} D_\ell^{\perp *}) \\
&= \operatorname{codim}_{\ker D_\ell^* D_\ell}(\operatorname{ran} D_\ell^{\perp *} D_\ell^{\perp}) = \dim(\ker(D_\ell^* D_\ell + D_\ell^{\perp *} D_\ell^{\perp})) \\
&= \dim(\ker(-\Delta_\ell \otimes I_n)) = n \quad . \qquad (110)
\end{aligned}$$

Combining (109) and (110), we obtain

$$H_l \succeq C_{R_-,S_\pm,\Upsilon}\,(-\Delta_\ell \otimes I_n) \quad , \quad C_{R_-,S_\pm,\Upsilon} > 0 \quad . \qquad (111)$$

If $0 \le F \preceq G$, then $\operatorname{tr} f(G) \le \operatorname{tr} f(F)$, for any positive, decreasing function f on $[0,\infty)$. Thus, the lemma follows from (108) and (111).

To make the connection between the torus Λ_ℓ and $\mathbb{R}^d$, for any positive integer q, let $\imath_\ell : L^2(\Lambda_\ell, dx; \mathbb{C}^q) \to L^2(\mathbb{R}^d, dx; \mathbb{C}^q)$ be the isometry that extends functions from Λ_ℓ to $\mathbb{R}^d$ by setting them equal to zero outside Λ_ℓ. (We omit q from the notation.) Its adjoint, $\imath_\ell^*$, is the partial isometry that restricts square integrable functions on $\mathbb{R}^d$ to Λ_ℓ, i.e.,

$$\imath_\ell^* \imath_\ell = I \quad , \quad \imath_\ell \imath_\ell^* = \chi_\ell \quad . \qquad (112)$$

Given a real-valued function $\phi \in C_0^\infty(\mathbb{R}^d)$, wth $\operatorname{supp}\phi \subset \Lambda_{\ell'}$, $\ell' < \ell$, we do not distinguish in the notation between ϕ as a multiplication operator on the torus Λ_ℓ and on the whole space. If D is a $CPDO_{n,m}^{(1)}$, we can verify that

$$D \imath_\ell \phi = \imath_\ell D_\ell \phi \quad \text{on} \quad \mathcal{D}(D_\ell) \ . \qquad (113)$$

It follows that, if Y is a $MPDO_{n,m}^{(1)}$, we have

$$Y \imath_\ell \phi = \imath_\ell Y_\ell \phi \quad \text{on} \quad \mathcal{D}(Y_\ell) \ . \qquad (114)$$

We let

$$A(\phi) = RD[\phi]S \quad , \quad B(\phi) = D^\perp[\phi]S^{-1} \quad , \qquad (115)$$

where $D[\phi]$, $D^\perp[\phi]$ are as in Lemma 5.1. Thus $A(\phi)$ and $B(\phi)$ are bounded operators given by multiplication by matrix-valued, measurable functions, with

$$\|A(\phi)\| \le D_+ R_+ S_+ \|\nabla\phi\|_\infty \quad , \quad \|B(\phi)\| \le D_+^\perp S_-^{-1} \|\nabla\phi\|_\infty \ . \qquad (116)$$

We denote by $A_\ell(\phi)$ and $B_\ell(\phi)$ the corresponding restrictions to the torus Λ_ℓ; they also satisfy the bounds (116). Furthermore, we set, with X standing for either A or B,

$$\begin{aligned} F_\ell^X(\phi) &= \sqrt{R}X(\phi)^* \imath_\ell (X_\ell \sqrt{R_\ell}) - (X\sqrt{R})^* \imath_\ell X_\ell(\phi)\sqrt{R_\ell} \ , \\ \widehat{F}_\ell^X(\phi) &= \sqrt{R_\ell} X_\ell(\phi)^* (X_\ell \sqrt{R_\ell}) - (X_\ell \sqrt{R_\ell})^* X_\ell(\phi)\sqrt{R_\ell} \ , \end{aligned}$$

$$F_\ell(\phi) = F_\ell^A(\phi) + F_\ell^B(\phi) \ , \ \widehat{F}_\ell(\phi) = \widehat{F}_\ell^A(\phi) + \widehat{F}_\ell^B(\phi) \ . \tag{117}$$

It is easy to see that

$$\|X\sqrt{R}\| \le 1 \ , \ \|X_\ell \sqrt{R_\ell}\| \le 1 \ , \tag{118}$$

so

$$\|F_\ell^X(\phi)\| \le C_{R_+, S_\pm, D_+, D_+^\perp} \|\nabla\phi\|_\infty \ . \tag{119}$$

In addition, it follows from Lemma 8.1 that if $p > d$ we have

$$\|\widehat{F}_\ell^X(\phi)\|_p \le C_{d,n,R_\pm,S_\pm,\Upsilon,\ell,p} \|\nabla\phi\|_\infty \ . \tag{120}$$

Lemma 8.2 *Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be real-valued with* $\operatorname{supp}\phi \subset \Lambda_{\ell'}$*, $\ell' < \ell$. Then*

$$R\phi = \imath_\ell \phi R_\ell \imath_\ell^* + \sqrt{R} F_\ell(\phi) \sqrt{R_\ell} \imath_\ell^* \ . \tag{121}$$

Proof: Because of (112) and $\phi\chi_\ell = \phi$, equation (121) is the same as

$$R\phi\imath_\ell = \phi\imath_\ell R_\ell + \sqrt{R} F_\ell(\phi)\sqrt{R_\ell} \ . \tag{122}$$

To prove (122), let $u \in L^2(\mathbb{R}^d, dx; \mathbb{C}^n)$ and $v \in L^2(\Lambda_\ell, dx; \mathbb{C}^n)$. We compute, as in [8, Lemma 24],

$$\begin{aligned} \langle u, (R\phi\imath_\ell - \phi\imath_\ell R_\ell)v\rangle &= \langle \phi R u, \imath_\ell (H_\ell + I) R_\ell v\rangle - \langle (H+I)Ru, \phi\imath_\ell R_\ell v\rangle \\ &= \langle \phi R u, \imath_\ell H_\ell R_\ell v\rangle - \langle HRu, \phi\imath_\ell R_\ell v\rangle \ . \end{aligned} \tag{123}$$

For $X = A$ or B, we may use Lemma 5.1 (which is valid also on the torus) and (114) twice to obtain

$$\begin{aligned} &\langle \phi R u, \imath_\ell X_\ell^* X_\ell R_\ell v\rangle = \langle Ru, \imath_\ell X_\ell^* \phi X_\ell R_\ell v\rangle + \langle Ru, \imath_\ell X_\ell(\phi)^* X_\ell R_\ell v\rangle \\ &\quad = \langle XRu, \imath_\ell \phi X_\ell R_\ell v\rangle + \langle Ru, X(\phi)^* \imath_\ell X_\ell R_\ell v\rangle \\ &\quad = \langle XRu, \imath_\ell X_\ell \phi R_\ell v\rangle - \langle XRu, \imath_\ell X(\phi) R_\ell v\rangle + \langle Ru, X(\phi)^* \imath_\ell X_\ell R_\ell u\rangle \\ &\quad = \langle X^* X R u, \phi\imath_\ell R_\ell v\rangle + \langle u, \sqrt{R} F_\ell^X(\phi)\sqrt{R_\ell} v\rangle \ . \end{aligned} \tag{124}$$

The identity (122) now follows from (123) and (124).

Lemma 8.3 *Let $\phi, \psi \in C_0^\infty(\mathbb{R}^d)$ be real-valued with* $\operatorname{supp}\phi$, $\operatorname{supp}\psi \subset \Lambda_{\ell'}$, *$\ell' < \ell$, with $\psi = 1$ on* $\operatorname{supp}\phi$. *Then*

$$\sqrt{R}F_\ell(\phi) = \imath_\ell \psi \sqrt{R_\ell}\widehat{F}_\ell(\phi) + \sqrt{R}F_\ell(\psi)\widehat{F}_\ell(\phi) \ . \tag{125}$$

Proof: We observe that

$$D(R_\ell^{-\frac{1}{2}}) = D(A_\ell) \cap D(B_\ell) \supset \imath_\ell^* \psi \left(D(A) \cap D(B) \right) = \imath_\ell^* \psi \operatorname{ran} R^{\frac{1}{2}} \ .$$

Thus, $R_\ell^{-1/2} \imath_\ell^* \psi R^{1/2}$ is a closed, everywhere defined operator, hence bounded, and so is its adjoint $R^{1/2}\psi \imath_\ell R_\ell^{-1/2}$. Furthermore, we note that $\operatorname{supp}\phi \cap \operatorname{supp}\nabla\psi = \emptyset$ and $\operatorname{supp} X(\phi) \subset \operatorname{supp}\nabla\phi \subset \operatorname{supp}\phi$, for $X = A$ or B. From this we conclude

$$\psi X(\phi) = X(\phi) \ , \quad X(\phi)^* X \psi = X(\phi)^* X \ . \tag{126}$$

Using the definitions (117), we get

$$\begin{aligned} F_\ell^X(\phi) &= R^{\frac{1}{2}} \psi X(\phi)^* \imath_\ell (X_\ell R_\ell^{\frac{1}{2}}) - (X(\phi)^* X \psi R^{\frac{1}{2}})^* \imath_\ell R_\ell^{\frac{1}{2}} \\ &= (R^{\frac{1}{2}} \psi \imath_\ell R_\ell^{-\frac{1}{2}}) \widehat{F}_\ell(\phi) \ . \end{aligned} \tag{127}$$

We now apply Lemma 8.2 in the form (122), obtaining

$$R^{\frac{1}{2}} F_\ell(\phi) = \left(R \psi \imath_\ell R_\ell^{-\frac{1}{2}} \right) \widehat{F}_\ell(\phi) = \left(\psi \imath_\ell R_\ell^{\frac{1}{2}} + R^{\frac{1}{2}} F_\ell(\psi) \right) \widehat{F}_\ell(\phi) \ , \tag{128}$$

which proves proves the lemma.

We may now finish the proof of Theorem 2.13. We pick $\rho \in C_0^\infty(\mathbb{R}^d)$, positive with $\operatorname{supp}\rho \subset \Lambda_{\frac{1}{2}}$ and $\int \rho(x)\,dx = 1$, and set $\phi_j = \rho * \chi_{j+\frac{1}{2}}$ for $j > 0$; note that $\phi_j \in C_0^\infty(\mathbb{R}^d)$ and $\chi_j \le \phi_j \le \chi_{j+1}$.

Let $0 < \ell + \tau < L$, where τ is a positive integer. Applying (121) and then iterating (125) $\tau - 1$ times, we get

$$R\chi_\ell = [R\phi_\ell]\chi_\ell = \imath_L \phi_\ell R_L \imath_L^* \chi_\ell + \left[\sqrt{R} F_L(\phi_\ell) \right] \sqrt{R_L} \imath_L^* \chi_\ell \tag{129}$$

$$\begin{aligned} &= \imath_L \phi_\ell R_L \imath_L^* \chi_\ell + \imath_L \phi_{\ell+1} \sqrt{R_L} \widehat{F}_L(\phi_\ell) \sqrt{R_L} \imath_L^* \chi_\ell \\ &\qquad\qquad + \left[\sqrt{R} F_L(\phi_{\ell+1}) \right] \widehat{F}_L(\phi_\ell) \sqrt{R_L} \imath_L^* \chi_\ell \end{aligned} \tag{130}$$

$$\vdots \qquad\qquad \vdots$$

$$\begin{aligned} &= \left\{ \imath_L \phi_\ell R_L \imath_L^* \chi_\ell + \sum_{k=1}^{\tau-1} \imath_L \phi_{\ell+k} \sqrt{R_L} \prod_{j=k-1}^{0} \widehat{F}_L(\phi_{\ell+j}) \sqrt{R_L} \imath_L^* \chi_\ell \right\} \\ &\qquad + \sqrt{R} F_L(\phi_{\ell+\tau-1}) \widehat{F}_L(\phi_{\ell+\tau-2}) \ldots \widehat{F}_L(\phi_\ell) \sqrt{R_L} \imath_L^* \chi_\ell \end{aligned} \tag{131}$$

$$\equiv \Gamma_\ell^{(\tau,L)} + \Xi_\ell^{(\tau,L)} \ . \tag{132}$$

The last line defines the operators $\Gamma_\ell^{(\tau,L)}$ and $\Xi_\ell^{(\tau,L)}$ on $L^2(\mathbb{R}^d, dx; \mathbb{C}^n)$. Note that

$$\Gamma_\ell^{(\tau,L)} = \chi_{\ell+\tau}\Gamma_\ell^{(\tau,L)} \ . \tag{133}$$

Moreover, it follows from (107), (119), and (120) that

$$\|\Gamma_\ell^{(\tau,L)}\|_p \leq C_{d,n,R_\pm,S_\pm,D_+,D_+^\perp,\Upsilon,L,\ell,\tau,p} < \infty \ , \quad \text{if} \ \ p > \frac{d}{2} \ , \tag{134}$$

$$\| \ \Xi_\ell^{(\tau,L)}\|_q \leq C_{d,n,R_\pm,S_\pm,D_+,D_+^\perp,\Upsilon,L,\ell,\tau,q} < \infty \ , \quad \text{if} \ \ q > \frac{d}{\tau} \ . \tag{135}$$

These constants also depend on our choice of the function ρ, but we fixed ρ once for all, so we omit the dependence.

We now let $0 < \ell + \theta\tau < L$, where τ and θ are positive integers. Applying (132) and (133) successively θ times, we obtain

$$\begin{aligned} R^\theta\chi_\ell &= R^{\theta-1}\left(\Xi_\ell^{(\tau,L)} + \Gamma_\ell^{(\tau,L)}\right) = R^{\theta-1}\Xi_\ell^{(\tau,L)} + R^{\theta-2}\left[R\chi_{\ell+\tau}\right]\Gamma_\ell^{(\tau,L)} \\ &= R^{\theta-1}\Xi_\ell^{(\tau,L)} + R^{\theta-2}\Xi_{\ell+\tau}^{(\tau,L)}\Gamma_\ell^{(\tau,L)} + R^{\theta-2}\Gamma_{\ell+\tau}^{(\tau,L)}\Gamma_\ell^{(\tau,L)} \\ &\vdots \qquad\qquad\qquad \vdots \\ &= R^{\theta-1}\Xi_\ell^{(\tau,L)} + \sum_{j=2}^{\theta} R^{\theta-j}\,\Xi_{\ell+(j-1)\tau}^{(\tau,L)} \prod_{r=j-2}^{0} \Gamma_{\ell+r\tau}^{(\tau,L)} \\ &\qquad\qquad + \Gamma_{\ell+(\theta-1)\tau}^{(\tau,L)}\Gamma_{\ell+(\theta-2)\tau}^{(\tau,L)} \cdots \Gamma_\ell^{(\tau,L)} \ . \end{aligned} \tag{136}$$

Let us choose $\tau = [[\frac{d}{2}]]$, where $[[x]]$ denotes the the smallest integer bigger than x, and $\theta = \nu \equiv [[\frac{d}{4}]]$. It then follows from (134), (135) and (136) that

$$\|R^\nu\chi_\ell\|_2 \leq C_{d,n,R_\pm,S_\pm,D_+,D_+^\perp,\Upsilon,L,\ell} < \infty \ . \tag{137}$$

If we now take $\ell = 1$, $L = [[\frac{d}{2}]]\,[[\frac{d}{4}]] + 2$, the desired estimate (105) follows from (137). This concludes the proof of Theorem 2.13.

The bottom of the spectrum

Theorem .4 *Let W be a classical wave operator. The bottom of the spectrum of $W_\perp$ is at 0, so $W_\perp$ and W have the same spectrum.*

Lemma .5 *Let H_0 be a self-adjoint operator with a gap $(0,\gamma)$ in its spectrum, and let $H = QH_0Q$, where Q is a strictly positive, bounded operator, say $0 < Q_- \leq Q \leq Q_+$ for some constants $Q_\pm$. Then $(0, \gamma Q_-^2)$ is a gap in the spectrum of H.*

Proof: Let $\lambda > 0$, $\mu \in (0, \gamma)$. Then

$$\begin{aligned} H - \lambda &= Q(H_0 - \mu)Q + (\mu Q^2 - \lambda) \\ &= Q\left(I + \left(\mu - \lambda Q^{-2}\right)(H_0 - \mu)^{-1}\right)(H_0 - \mu)Q \ . \end{aligned} \tag{138}$$

We fix $\mu = \frac{\gamma}{2}$, so $\|(H_0 - \frac{\gamma}{2})^{-1}\| \le \frac{2}{\gamma}$. Since Q is invertible with a bounded inversee, it suffices to show that

$$\|\left(\tfrac{\gamma}{2} - \lambda Q^{-2}\right)(H_0 - \tfrac{\gamma}{2})^{-1}\| < 1 \tag{139}$$

for $\lambda \in (0, \gamma Q_-^2)$. But for such λ we have

$$-\tfrac{\gamma}{2} < \tfrac{\gamma}{2} - \lambda Q_-^{-2} \le \tfrac{\gamma}{2} - \lambda Q^{-2} \le \tfrac{\gamma}{2} - \lambda Q_+^{-2} < \tfrac{\gamma}{2} \ , \tag{140}$$

so (139) follows.

We may now prove Theorem .4. In view of (12) and (18), it suffices to consider the case of a first order classical wave operator $W = SDS$. In this case 0 is in the spectrum of $W_\perp$ if and only if W has no spectral gap of the form $(-\gamma, \gamma)$, and it follows from Lemma .5 that we can take $W = D$.

Thus it suffices to show that 0 is in the spectrum of $D_\perp$. For $r > 0$, let us define

$$\varphi_r(x) = r^{d/2}\varphi(rx) \tag{141}$$

for any vector-valued measurable function $\varphi(x)$ on $\mathbb{R}^d$. Note that $\|\varphi_r\|_2 = \|\varphi\|_2$. Let us pick $\psi \in \mathcal{D}(D) \cap (\ker D)^\perp$, with $\|\psi\| = 1$. It is easy to see that $\psi_r \in \mathcal{D}(D) \cap (\ker D)^\perp$, $\|\psi_r\| = 1$, and $D\psi_r = r(D\psi)_r$. Thus $\lim_{r\to 0} D\psi_r = 0$ in $\mathcal{H}^{(m)}$, so we can use Weyl's criterion to conclude that that 0 is in the spectrum of $D_\perp$.

Theorem .4 is proved.

Operators on locally convex spaces

In this appendix we review some basic facts about unbounded operators on locally convex spaces.

Let $\mathcal{E}$ and $\mathcal{F}$ be locally convex spaces. An operator C from $\mathcal{E}$ to $\mathcal{F}$ is a linear mapping from $\mathcal{D}(C)$ to $\mathcal{F}$, the domain $\mathcal{D}(C)$ being a linear subspace of $\mathcal{E}$. C is densely defined if $\mathcal{D}(C)$ is dense in $\mathcal{E}$. C is closed if its graph, $\Gamma(C) = \{(\phi, C\phi) : \ \phi \in \mathcal{D}(C)\}$, is closed in $\mathcal{E} \times \mathcal{F}$. C is closable if $\overline{\Gamma(C)} = \Gamma(\overline{C})$ for a (necessarily unique) operator $\overline{C}$ from $\mathcal{E}$ to $\mathcal{F}$, the closure of C, i.e., if $\overline{\Gamma(C)} \cap \{0\} \times \mathcal{F} = \{(0,0)\}$.

We set $J_{\mathcal{E},\mathcal{F}} : \mathcal{E} \times \mathcal{F} \to \mathcal{F} \times \mathcal{E}$ to be the isomorphism given by $J_{\mathcal{E},\mathcal{F}}(\phi, \psi) = (-\psi, \phi)$; note $J_{\mathcal{E},\mathcal{F}} J_{\mathcal{F},\mathcal{E}} = -I_{\mathcal{E}\times\mathcal{F}}$.

We say that two locally convex spaces $\mathcal{E}_\pm$ form a dual pair, if there exists a sesquilinear form $\langle\ ,\ \rangle$ on $\mathcal{E}_+ \times \mathcal{E}_-$, which turns $\mathcal{E}_\pm$ into the duals of each other. (Example: $\mathcal{E}_+ = L^2_0(\mathbb{R}^d, dx; \mathbb{C}^q)$ and $\mathcal{E}_- = L^2_{loc}(\mathbb{R}^d, dx; \mathbb{C}^q)$ as in Section 5, with the sesquilinear form given in (54).)

Let $\mathcal{E}_\pm$ be a dual pair. Given $S \subset \mathcal{E}_+$ we set

$$S^\perp = \{\psi \in \mathcal{E}_- :\ \langle \phi, \psi \rangle = 0 \ \text{ for all } \ \phi \in S\}\ . \tag{142}$$

It is easy to see that $S^\perp$ is closed and $S^\perp = \bar{S}^\perp$.

Proposition .6 *Let V be a linear subspace of $\mathcal{E}_+$, then $\bar{V} = (V^\perp)^\perp$.*

Proof: We can assume that V is closed. Obviously, $V \subset (V^\perp)^\perp$. To see the converse, let $\phi \in \mathcal{E}_+$, $\phi \notin V$. According to the Hahn-Banach Theorem, there exists $\psi \in \mathcal{E}_-$, such that $\langle \phi, \psi \rangle \neq 0$ but $\langle \phi', \psi \rangle = 0$ for all $\phi' \in V$. Thus $\psi \in V^\perp$, hence $\phi \notin (V^\perp)^\perp$.

Let us consider dual pairs $\mathcal{E}_\pm$ and $\mathcal{F}_\pm$; note that $\mathcal{E}_\pm \times \mathcal{F}_\pm$ also forms a dual pair. Let C be a densely defined operator from $\mathcal{E}_-$ to $\mathcal{F}_-$.Its adjoint, $C^\dagger$, is defined as the operator from $\mathcal{F}_+$ to $\mathcal{E}_+$ with domain

$$\mathcal{D}(C^\dagger) = \{\psi \in \mathcal{F}_+ :\ \phi \in \mathcal{D}(C) \mapsto \langle \psi, C\phi \rangle \ \text{ is continuous}\}\ , \tag{143}$$

with $C^\dagger\psi$, $\phi \in \mathcal{D}(C^\dagger)$, being the unique element of $\mathcal{E}_+$ such that $\langle C^\dagger\psi, \phi \rangle = \langle \psi, C\phi \rangle$ for all $\phi \in \mathcal{D}(C)$. It follows that

$$\Gamma(C^\dagger) = (J_{\mathcal{E}_-,\mathcal{F}_-}\Gamma(C))^\perp = J_{\mathcal{E}_+,\mathcal{F}_+}\Gamma(C)^\perp \tag{144}$$

.

The following theorem is identical to the familiar result for operators in Hilbert spaces (e.g., [19, Theorem VIII.1]).

Theorem .7 *Let $\mathcal{E}_\pm$ and $\mathcal{F}_\pm$ be dual pairs of locally convex spaces, and let C be a densely defined operator from $\mathcal{E}_-$ to $\mathcal{F}_-$. Then:*

(i) *$C^\dagger$ is closed.*

(ii) *C is closable if and only if $C^\dagger$ is densely defined, in which case $\bar{C} = (C^\dagger)^\dagger$.*

(iii) *If C is closable, then $\bar{C}^\dagger = C^\dagger$.*

Proof: (i) follows immediately from (144).

To prove (ii), note that if C is not closable, we can find $\phi \in \mathcal{F}_-$, $\phi \neq 0$, such that $(0, \phi) \in \overline{\Gamma(C)}$, so $(\phi, 0) \in \Gamma(C^\dagger)^\perp$ by (144) and Proposition .6. Thus $\langle \eta, \phi \rangle = 0$ for all $\eta \in \mathcal{D}(C^\dagger)$, hence $C^\dagger$ is not densely defined.

On the other hand, if $C^\dagger$ is not densely defined, it follows from the Hahn-Banach Theorem that there is $\phi \in \mathcal{F}_-$, $\phi \neq 0$, such that $\langle \eta, \phi \rangle = 0$ for all $\eta \in \mathcal{D}(C^\dagger)$. Thus $(\phi, 0) \in \Gamma(C^\dagger)^\perp$, so $(0, \phi) \in \overline{\Gamma(C)}$ by (144) and Proposition .6.

Thus, if C is closable, we get from (144) that

$$\begin{aligned} \Gamma((C^\dagger)^\dagger) &= (J_{\mathcal{F}_+, \mathcal{E}_+} \Gamma(C^\dagger))^\perp = (J_{\mathcal{F}_+, \mathcal{E}_+} J_{\mathcal{E}_+, \mathcal{F}_+} \Gamma(C)^\perp)^\perp & (145) \\ &= (\Gamma(C)^\perp)^\perp = \overline{\Gamma(C)} = \Gamma(\bar{C}) \ . & (146) \end{aligned}$$

To finish the proof, it suffices to note that (iii) follows from (ii).

Bochner integrals and vector valued measures

In this appendix, we summarize some results of the theory of Bochner integrals and vector valued Borel measures [5]. We briefly discuss the Radon-Nikodym Theorem and Lebesgue's Differentiation Theorem. All results are true for Borel measures in $\mathbb{R}^d$. For the sake of simplicity, everything is stated for the real line, $\mathbb{R}$.

Let $\mathcal{B}$ be a separable Banach space. A function $f(\lambda) : \mathbb{R} \to \mathcal{B}$ is said to be weakly Borel measurable if $l(f(\lambda))$ is Borel measurable, for any $l \in \mathcal{B}^*$. f is said to be strongly Borel measurable if it is the pointwise limit of a sequence of finitely valued Borel step functions. Since $\mathcal{B}$ is separable, strong measurability and weak measurability are equivalent. Hence, such a function is simply called measurable. If $f(\lambda)$ is measurable, then so is $\|f(\lambda)\|$.

Let μ be a positive Borel measure on $\mathbb{R}$. A Borel measurable, $\mathcal{B}$-valued function $f(\lambda)$ is called integrable if $\|f(\lambda)\|$ is integrable. Then, there exists a sequence $f_n(\lambda)$ of finitely valued Borel step functions such that

$$\lim_{n\to\infty} \int \|f(\lambda) - f_n(\lambda)\| \, d\mu(\lambda) \ = \ 0 \ .$$

The norm limit

$$\int f(\lambda)\, d\mu(\lambda) = \lim_{n\to\infty} \int f_n(\lambda)\, d\mu(\lambda) \qquad (147)$$

exists and is independent of the sequence f_n; it is called the Bochner integral. A Borel measurable, $\mathcal{B}$-valued function $f(\lambda)$ is said to be locally integrable if $\|f(\lambda)\|$ is locally integrable.

Let $I_{\lambda,\delta}$ denote the closed interval $[\lambda - \delta, \lambda + \delta]$. The separability of $\mathcal{B}$ allows to prove Lebesgue's Differentiation Theorem for vector valued functions using the same result for scalar valued functions (e.g., [5, Theorem 9 on page 49]):

Theorem .8 (Lebesgue's Differentiation Theorem) *If $f(\lambda)$ is a $\mathcal{B}$-valued, μ-locally integrable function, then*

$$\lim_{\delta \downarrow 0} \frac{1}{\mu(I_{\lambda,\delta})} \int_{I_{\lambda,\delta}} \|f(\lambda') - f(\lambda)\| \, d\mu(\lambda') \;=\; 0 \quad \textit{for } \mu - \textit{a.e. } \lambda \; . \tag{148}$$

A $\mathcal{B}$-valued Borel measure on $\mathbb{R}$ is a countably additive set function $E(\Omega)$, defined on the family of bounded Borel subsets of $\mathbb{R}$ and taking its values in $\mathcal{B}$. $E(\Omega)$ is said to be of local bounded variation if the positive set function $\|E(\Omega)\|$ is of bounded variation in any bounded Borel subset of $\mathbb{R}$. $E(\Omega)$ is said to be absolutely continuous with respect to a Borel measure μ if $E(\Omega) = 0$ whenever $\mu(\Omega) = 0$. If $\mathcal{B}$ is a separable dual space, one has the Radon-Nikodym Theorem [5, Theorem 1 on page 79].

If $\mathcal{H}$ is a separable Hilbert space, then $\mathcal{T}_p(\mathcal{H})$, the Banach space of bounded operators S on $\mathcal{H}$ with $\|S\|_p = (\mathrm{tr}\,|S|^p)^{\frac{1}{p}} < \infty$, is a separable dual space for $1 \le p < \infty$. Let $\mathcal{T}_{p,+}(\mathcal{H})$ denote the positive operators in $\mathcal{T}_p(\mathcal{H})$. If $p = 1$ and $E(\Omega)$ is a $\mathcal{T}_{1,+}(\mathcal{H})$-valued Borel measure on $\mathbb{R}$, then $E(\Omega)$ is of local bounded variation, $\mu(\Omega) = \mathrm{tr}\, E(\Omega)$ is a positive Borel measure, and E is absolutely continuous with respect to μ. For $\mathcal{T}_{p,+}(\mathcal{H})$-valued Borel measures we can state the Radon-Nikodym theorem as follows (see also [2, Theorem 1.1 on page 321], [3, Theorem 1.1 on page 187]):

Theorem .9 *Let $\mathcal{H}$ be a separable Hilbert space and $1 \le p < \infty$. If $E(\Omega)$ is a $\mathcal{T}_{p,+}(\mathcal{H})$-valued Borel measure on $\mathbb{R}$, of local bounded variation, and absolutely continuous with respect to a positive Borel measure μ on $\mathbb{R}$, there exists a μ-locally integrable, $\mathcal{T}_{p,+}(\mathcal{H})$-valued function $\frac{dE}{d\mu}$ on $\mathbb{R}$, the Radon-Nikodym derivative, such that*

$$E(\Omega) = \int_{\Omega} \frac{dE}{d\mu}(\lambda) \, d\mu(\lambda) \quad \textit{for any bounded Borel set } \Omega \; . \tag{149}$$

If $p = 1$, $E(\Omega)$ is always of local bounded varation and, if $\mu(\Omega) \;=\; \mathrm{tr}\, E(\Omega)$, the Radon-Nikodym derivative satisfies

$$\mathrm{tr}\, \frac{dE}{d\mu}(\lambda) = 1 \quad \textit{for } \mu - \text{a.e. } \lambda \; . \tag{150}$$

Using Theorems .8 and .9, we obtain

Corollary .10 *Under the hypotheses of Theorem .9, the Radon-Nikodym derivative is given by*

$$\frac{dE}{d\mu}(\lambda) = \lim_{\delta \downarrow 0} \frac{E(I_{\lambda,\delta})}{\mu(I_{\lambda,\delta})} \quad \textit{for } \mu - \text{a.e. } \lambda \ , \tag{151}$$

where the limit is in $\mathcal{T}_p(\mathcal{H})$.

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