Remarks on the Ground State Energy of the Spin-Boson Model. An Application of the Wigner-Weisskopf Model.

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Abstract

For the ground state energy of the spin-boson (SB) model, we give a new upper bound in the case with infrared singularity condition (i.e., without infrared cutoff), and a new lower bound in the case of massless bosons with infrared regularity condition. We first investigate spectral properties of the Wigner-Weisskopf (WW) model, and apply them to SB model to achieve our purpose. Then, as an extra result of the spectral analysis for WW model, we show that a non-perturbative ground state appears, and its ground state energy is so low that we cannot conjecture it by using the regular perturbation theory.

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Contents

1 Introduction

The spin-boson (SB) model describes a two-level system coupled to a quantized Bose field. There are many papers treating it [LCDFGW, EG, Am, Ar1, Da2, FNV, Gé1, Gé2, HüS1, HüS2, Sp1, SSW, AH1, AHH, AH2, JP, KiMa, AKV, Ts]. Especially, for the ground state energy of this model, we know several approximate expression by, for instance, [EG, Ts]. Spohn and Dümcke analyzed a variational upper bound for the ground state energy of the spin-boson model through the Ising model [SD, §7]. Recently the author gave an explicit one in the way of [mH2], still he proved it in the case of massless bosons with infrared cutoff. In this paper, we give a new upper bound for the ground state energy of SB model *without infrared cutoff*, and a new lower bound for it in the case of *massless* bosons *with infrared regularity condition*, and argue how an effect by the spin appears in the ground state energy.

We take a Hilbert space of bosons to be

$$\mathcal{F}_b := \mathcal{F}\left(L^2\left(\mathbf{R}^d\right)\right) \equiv \bigoplus_{n=0}^{\infty}\left[\otimes_s^n L^2\left(\mathbf{R}^d\right)\right] \tag{1.1}$$

($d \in \mathbf{N}$) the symmetric Fock space over $L^2(\mathbf{R}^d)$ ($\otimes_s^n\mathcal{K}$ denotes the n-fold symmetric tensor product of a Hilbert space $\mathcal{K}$, $\otimes_s^0\mathcal{K} \equiv \mathbf{C}$). In this paper, we set both of $\hbar$ (the Planck constant divided by 2π) and c (the speed of light) one, i.e., $\hbar = c = 1$.

Let $\omega : \mathbf{R}^d \to [0, \infty)$ be a Borel measurable function such that $0 \le \omega(k) < \infty$ for all $k \in \mathbf{R}^d$ and $\omega(k) \ne 0$ for almost everywhere (a.e.) $k \in \mathbf{R}^d$ with respect to the d-dimensional Lebesgue measure. We here assume that

$$\inf_{k\in\mathbf{R}^d} \omega(k) = 0 \tag{1.2}$$

because we are interested in the case without infrared cutoff. Let $\hat{\omega}$ be the multiplication operator by the function ω, acting in $L^2(\mathbf{R}^\nu)$. We denote by $d\Gamma(\hat{\omega})$ the second quantization of $\hat{\omega}$ [RS2, §X.7] and set

$$H_b = d\Gamma(\hat{\omega}) = \int_{\mathbf{R}^d} dk\, \omega(k)a(k)^*a(k),$$

where $a(k)$ is the operator-valued distribution kernels of the smeared annihilation operator $a(f)$, so $a(k)^*$ is that of creation operator $a(f)^*$:

$$a(f) = \int_{\mathbf{R}^d} dk\, a(k)\overline{f(k)}, \tag{1.3}$$

$$a(f)^* = \int_{\mathbf{R}^d} dk\, a(k)^* f(k) \tag{1.4}$$

for every $f \in L^2(\mathbf{R}^d)$ on $\mathcal{F}_b$. Let Ω_0 be the Fock vacuum in $\mathcal{F}_b$:

$$\Omega_0 := \{1,\, 0,\, 0,\, \cdots\} \in \mathcal{F}_b. \tag{1.5}$$

The Segal field operator $\phi_S(f)$ $(f \in L^2(\mathbf{R}^d))$ is given by

$$\phi_S(f) := \frac{1}{\sqrt{2}} \left(a(f)^* + a(f) \right). \tag{1.6}$$

The inner product (resp. norm) of a Hilbert space $\mathcal{K}$ is denoted $(\,\cdot\,,\,\cdot\,)_{\mathcal{K}}$, complex linear in the second variable (resp. $\|\cdot\|_{\mathcal{K}}$). For each $s \in \mathbf{R}$, we define a Hilbert space

$$\mathcal{M}_s = \left\{ f : \mathbf{R}^d \to \mathbf{C}, \text{Borel measurable} \mid \omega^{s/2} f \in L^2(\mathbf{R}^\nu) \right\}$$

with inner product $(f, g)_s := (\omega^{s/2} f, \omega^{s/2} g)_{L^2(\mathbf{R}^\nu)}$ and norm

$$\|f\|_s := \|\omega^{s/2} f\|_{L^2(\mathbf{R}^d)}, \quad f \in \mathcal{M}_s.$$

We shall assume the following (A.1) to obtain upper bounds for the ground state energy:

(A.1) The function $\lambda(k)$ of $k \in \mathbf{R}^d$ satisfies that $\lambda \in \mathcal{M}_{-1} \cap \mathcal{M}_0$.

We call the following condition the *infrared singularity condition* (see [Pf1], [Da2, p153], [AH2])

$$\|\lambda\|_{-2} = \infty, \qquad \left(\text{i.e.,} \quad \lambda/\omega \notin L^2\left(\mathbf{R}^d\right) \right). \tag{1.7}$$

Conversely, we call the following condition the *infrared regularity condition*:

$$\lambda \in \mathcal{M}_{-2} \qquad \left(\text{i.e.,} \quad \lambda/\omega \in L^2\left(\mathbf{R}^d\right) \right). \tag{1.8}$$

The Hamiltonian of the spin-boson model is defined by

$$\begin{aligned} H_{\text{SB}} &:= \frac{\mu}{2} \sigma_3 \otimes I + I \otimes H_b + \sqrt{2} \alpha \sigma_1 \otimes \phi_S(\lambda) \qquad (1.9) \\ &= \begin{pmatrix} H_b + \frac{\mu}{2} & \sqrt{2}\alpha\phi_S(\lambda) \\ \sqrt{2}\alpha\phi_S(\lambda) & H_b - \frac{\mu}{2} \end{pmatrix} \end{aligned}$$

acting in the Hilbert space

$$\mathcal{F} := \mathbf{C}^2 \otimes \mathcal{F}_b = \mathcal{F}_b \oplus \mathcal{F}_b, \tag{1.10}$$

where $0 < \mu$ is a splitting energy, and σ_1, σ_3 the standard Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For simplicity, we denote the decoupled free Hamiltonian ($\alpha = 0$) by H_0:

$$\begin{aligned} H_0 &:= \frac{\mu}{2}\sigma_3 \otimes I + I \otimes H_b \\ &= \begin{pmatrix} H_b + \dfrac{\mu}{2} & 0 \\ 0 & H_b - \dfrac{\mu}{2} \end{pmatrix}. \end{aligned} \tag{1.11}$$

For the above H_{SB}, we temporally introduce an infrared cutoff $\nu > 0$ such that the infrared regularity condition, $\lambda/\omega_\nu \in L^2(\mathbf{R}^d)$, holds for $\nu > 0$, which raise the bottom of the frequency $\omega(k)$ of bosons (see [AH2]):

$$\omega_\nu(k) := \omega(k) + \nu, \qquad \nu > 0, \tag{1.12}$$

$$H_b(\nu) := d\Gamma(\widehat{\omega}_\nu), \tag{1.13}$$

$$H_{\text{SB}}(\nu) := \frac{\mu}{2}\sigma_3 \otimes I + I \otimes H_b(\nu) + \sqrt{2}\alpha\sigma_1 \otimes \phi_S(\lambda), \tag{1.14}$$

where ν means the lower bound of the frequency which we can observe precisely by an equipment. Of course, we shall remove 'ν' later by taking the limit $\nu \downarrow 0$ such as making the precision better.

For simplicity, we put

$$H_{\text{SB}}(0) := H_{\text{SB}}. \tag{1.15}$$

For a linear operator T on a Hilbert space, we denote its domain by $D(T)$. It is well-known that $H_{\text{SB}}(\nu)$ is self-adjoint on

$$D\left(H_{\text{SB}}\left(\nu\right)\right) = D\left(I \otimes H_b\left(\nu\right)\right), \tag{1.16}$$

$$\text{and bounded from below for all } \alpha \in \mathbf{R} \tag{1.17}$$

for every $\nu \geq 0$ by [AH1, Proposition 1.1(i)] since σ_1 is bounded now.

For a self-adjoint operator T bounded from below, we denote by $E_0(T)$ the infimum of the spectrum $\sigma(T)$ of T:

$$E_0(T) = \inf \sigma(T).$$

In this paper, when T is a Hamiltonian, we call $E_0(T)$ the *ground state energy* of T *even if T has no ground state.*

For $H_{\text{SB}}(\nu)$ ($\nu \geq 0$) we set

$$E_{\text{SB}}(\nu) := E_0\left(H_{\text{SB}}\left(\nu\right)\right).$$

It is well known that for $\nu > 0$

$$-\frac{\mu}{2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega_\nu}} \right\|_0^2 \leq E_{\text{SB}}(\nu) \leq -\frac{\mu}{2} e^{-2\alpha^2 \|\lambda/\omega_\nu\|_0^2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega_\nu}} \right\|_0^2 \tag{1.18}$$

by easy estimation and the variational principle ([Ar1, Theorem 2.4] and [Da2, p.161]).

In this paper, we give new upper and lower bounds for $E_{\rm SB}(0)$ in the massless case. The idea to obtain the bounds is very simple as follows: we know the model of a quantum harmonic oscillator (QHO) with the RWA-interaction is derived from the linear coupling model of the QHO coupled with a Bose field by neglecting the terms which were not observed by an equipment (see [mH1, Remark on RWA]). Such the approximation is called the *rotating wave approximation* (RWA). The Wigner-Weisskopf (WW) model has the same form as the model of QHO with the RWA-interaction, so we can regard WW model as the model derived from the spin-boson (SB) model through RWA as well as QHO with the RWA-interaction. But there is a difference between WW model and QHO with the RWA-interaction. Namely, WW and SB models describe models of the spin coupled with a Bose field. So, although QHO with the RWA-interaction cannot restore the neglected terms, WW can do it in the sense of (2.8) and (3.17) below. Thus, we prove Theorem 3.1 (upper bound) in terms of WW model by regarding WW as an approximation of SB model, and using the difference as mentioned above and the variational principle. Moreover, this idea with the simple means which Lieb and Yamazaki already used in [LY] for the polaron model brings Theorem 3.3 (lower bound).

2 Spectral Properties of Wigner-Weisskopf Model

To obtain the bounds for the ground state energy of SB model, we use the properties of WW model [WW, Pf2, Da1, HüS2, AH2]. So, in this section, we describe fundamental properties of the Wigner-Weisskopf model.

We define a matrix c by

$$c := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

And let

$$H_b(0) := H_b, \tag{2.2}$$

$$\omega_0(k) := \omega(k), \qquad k \in \mathbf{R}^d. \tag{2.3}$$

Then, for every $\mu_0 \in \mathbf{R}$ and $\nu \geq 0$, we define a Hamiltonian $H_\alpha(\mu_0\,;\,\nu)$ of the Wigner-Weisskopf model by

$$\begin{aligned} & H_\alpha(\mu_0\,;\,\nu) \\ := \; & \mu_0 c^* c \otimes I + I \otimes H_b(\nu) + \alpha\left(c^* \otimes a(\lambda) + c \otimes a(\lambda)^*\right) \\ = \; & \begin{pmatrix} H_b(\nu) + \mu_0 & \alpha a(\lambda) \\ \alpha a(\lambda)^* & H_b(\nu) \end{pmatrix}. \end{aligned} \tag{2.4}$$

We call $H_\alpha(\mu_0\,;\,\nu)$ the *Wigner-Weisskopf Hamiltonian.* We may put for $\nu = 0$

$$H_\alpha(\mu_0) := H_\alpha(\mu_0\,;\,0). \tag{2.5}$$

Remark 2.1 *The Wigner-Weisskopf model is one of several examples of the generalized spin-boson model. We know it if we put* $B_1 \equiv (c^* + c)/\sqrt{2}$, $B_2 \equiv i(c^* - c)/\sqrt{2}$; $\lambda_1 \equiv \lambda$ *and* $\lambda_2 \equiv i\lambda$.

It is easy to prove that $H_\alpha(\mu_0\,;\,\nu)$ is self-adjoint on

$$D\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right) = D\left(I \otimes H_b\left(\nu\right)\right), \tag{2.6}$$

and bounded from below (2.7)

for every $\nu \geq 0$ by [AH1, Proposition 1.1(i)] since each B_j is bounded, and

$$U_1^* \widetilde{H}_\alpha(\mu_0\,;\,\nu) U_1 = H_\alpha(\mu_0\,;\,\nu) \quad \text{for every } \nu \geq 0, \tag{2.8}$$

where the unitary operator U_1 is given by

$$U_1 := \sigma_1 \otimes I = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{2.9}$$

and $\widetilde{H}_\alpha(\mu_0\,;\,\nu)$ is given by

$$\begin{aligned} & \widetilde{H}_\alpha(\mu_0\,;\,\nu) \\ := \; & \mu_0 cc^* \otimes I + I \otimes H_b(\nu) + \alpha\left(c^* \otimes a\left(\lambda\right)^* + c \otimes a\left(\lambda\right)\right) \\ = \; & \begin{pmatrix} H_b(\nu) & \alpha a\left(\lambda\right)^* \\ \alpha a\left(\lambda\right) & H_b(\nu) + \mu_0 \end{pmatrix}. \end{aligned} \tag{2.10}$$

Remark 2.2 *For* $\mu_0 < 0$, *the above Wigner-Weisskopf Hamiltonian* $H_\alpha(\mu_0\,;\,\nu)$ *was treated in [AH2, Theorem 6.15]. On the other hand, for* $\mu_0 \geq 0$, $H_\alpha(\mu_0\,;\,\nu)$ *was treated in [HüS2, §6] with* $\nu > 0$, *and [AH2, Theorem 6.14] with* $\nu \geq 0$.

As we did in [AH2, §6.2], we introduce a function $D^\alpha_{\mu_0,\nu}$ for $\mu_0 \in \mathbf{R}$ and $\nu \geq 0$ by

$$D^\alpha_{\mu_0,\nu}(z) := -z + \varepsilon_0 - \alpha^2 \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega_\nu(k) - z}, \tag{2.11}$$

defined for all $z \in \mathbf{C}$ such that $|\lambda(k)|^2/|z - \omega_\nu(k)|$ is Lebesgue integrable on $\mathbf{R}^d$.

Remark 2.3 *It is well-known that the Wigner-Weisskopf model is the simplified Lee model [Le, KaMu, We] & [Ta, §5.2], and the solution of* $D^\alpha_{\varepsilon_0,\varepsilon_1,\nu}(z) = 0$ *gives the renormalized mass for the Lee model.*

As we mentioned in [AH2, §6.2], $D^\alpha_{\mu_0,\nu}(z)$ is defined in the cut plane

$$\mathbf{C}_\nu := \mathbf{C} \setminus [\nu\,,\,\infty)\,, \qquad \nu \geq 0 \tag{2.12}$$

and analytic there. It is easy to see that $D^\alpha_{\mu_0,\nu}(x)$ is monotone decreasing in $x < \nu$. Hence, the limit

$$\begin{aligned} d^\alpha_\nu(\mu_0) &:= \lim_{x\uparrow\nu} D^\alpha_{\mu_0,\nu}(x) \\ &= -\nu + \mu_0 - \alpha^2 \lim_{t\downarrow 0} \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega_\nu(k) - \nu + t} \end{aligned} \tag{2.13}$$

exists. Actually, for a.e. $k \in \mathbf{R}^d$,

$$0 < \frac{|\lambda(k)|^2}{\omega_\nu(k) - \nu + t} < \frac{|\lambda(k)|^2}{\omega(k)}, \quad t > 0 \quad \text{and} \quad \lim_{t\downarrow 0} \frac{|\lambda(k)|^2}{\omega_\nu(k) - \nu + t} = \frac{|\lambda(k)|^2}{\omega(k)} \qquad \text{(a.e.)},$$

and we assumed $\lambda/\sqrt{\omega} \in L^2(\mathbf{R}^d)$ in (A.1), moreover set $\omega_\nu(k) := \omega(k) + \nu$ $(\nu > 0$, $k \in \mathbf{R}^d)$. So, by the Lebesgue dominated convergence theorem, we have

$$d^\alpha_\nu(\mu_0) = -\nu + \mu_0 - \alpha^2 \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega(k)}. \tag{2.14}$$

We may put for $\nu = 0$

$$D^\alpha_{\mu_0}(z) := D^\alpha_{\mu_0,0}(z), \tag{2.15}$$

$$d^\alpha(\mu_0) := d^\alpha_0(\mu_0). \tag{2.16}$$

The Wigner-Weisskopf model has a conservation law for a kind of the particle number in the following sense:

We define

$$N_P := \frac{1+\sigma_3}{2} \otimes I + I \otimes N_b, \tag{2.17}$$

which was already introduced in [HüS2, §6], where N_b is the boson number operator,

$$N_b := d\Gamma(1) = \sum_{\ell=0} \ell P^{(\ell)}. \tag{2.18}$$

Here (2.18) is the spectral resolution of N_b, and $P^{(\ell)}$ is the orthogonal projection onto the ℓ-particle space in $\mathcal{F}_b$ for each $\ell \in \{0\} \cup \mathbf{N}$. The spectral resolution of N_P is given as

$$N_P = \sum_{\ell=0} \ell P_\ell, \tag{2.19}$$

where

$$P_\ell = \begin{cases} \dfrac{1-\sigma_3}{2} \otimes P^{(0)} & \text{if } \ell = 0, \\ \\ \dfrac{1+\sigma_3}{2} \otimes P^{(\ell-1)} + \dfrac{1-\sigma_3}{2} \otimes P^{(\ell)} & \text{if } \ell \in \mathbf{N}. \end{cases} \tag{2.20}$$

$H_\alpha(\mu_0\,;\,\nu)$ is reduced by $P_\ell\mathcal{F}$ for every $\alpha \in \mathbf{R}$ and each $\ell \in \{0\} \cup \mathbf{N}$, i.e.,

$$P_\ell H_\alpha(\mu_0\,;\,\nu) \subset H_\alpha(\mu_0\,;\,\nu) P_\ell, \tag{2.21}$$

which means that

$$\begin{aligned} &D\left(P_\ell H_\alpha\left(\mu_0\,;\,\nu\right)\right) \subset D\left(H_\alpha\left(\mu_0\,;\,\nu\right) P_\ell\right), \\ &P_\ell H_\alpha\left(\mu_0\,;\,\nu\right)\Psi = H_\alpha(\mu_0\,;\,\nu) P_\ell \Psi \quad \text{for } \Psi \in D\left(P_\ell H_\alpha\left(\mu_0\,;\,\nu\right)\right) \end{aligned}$$

(see [Ka, p.278]). So, for every $\alpha \in \mathbf{R}$, $H_\alpha(\mu_0\,;\,\nu)$ is decomposed to the direct sum of $H_{\ell,\alpha}(\mu_0\,;\,\nu)$'s as

$$H_\alpha(\mu_0\,;\,\nu) = \bigoplus_{\ell=0}^{\infty} H_{\ell,\alpha}(\mu_0\,;\,\nu), \tag{2.22}$$

where $H_{\ell,\alpha}(\mu_0\,;\,\nu)$ is self-adjoint on the closed subspace $\mathcal{F}_\ell$ defined by

$$\mathcal{F}_\ell := P_\ell \mathcal{F} \tag{2.23}$$

for each $\ell \in \{0\} \cup \mathbf{N}$ and

$$\mathcal{F} = \bigoplus_{\ell=0}^{\infty} \mathcal{F}_\ell. \tag{2.24}$$

The proof of the above statement is that, for instance, we have only to extend [Ka, Problem 3.29] to its infinite version by repeating [Ka, Problem 3.29] with the closedness of $H_\alpha(\mu_0\,;\,\nu)$.

We call $\mathcal{F}_\ell$ the *ℓ-sector*.

We define a vector $\Omega^0 \in \mathcal{F}_0$ by

$$\Omega^0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \Omega_0 = \begin{pmatrix} 0 \\ \Omega_0 \end{pmatrix}. \tag{2.25}$$

Then, we have

$$\|\Omega^0\| = 1. \tag{2.26}$$

For every $f \in D(\widehat{\omega}_\nu)$ $(\nu \geq 0)$, we define a vector $\Omega^1(f) \in \mathcal{F}_1$ by

$$\Omega^1(f) := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \Omega_0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes a(f)^*\Omega_0 = \begin{pmatrix} \Omega_0 \\ a\left(f\right)^* \Omega_0 \end{pmatrix}. \tag{2.27}$$

Then, we have

$$\|\Omega^1(f)\| = \left(1 + \|f\|_0^2\right)^{1/2}. \tag{2.28}$$

When a zero $E^{\alpha}_{\mu_0,\nu}$ of $D^{\alpha}_{\mu_0,\nu}(z)$ exists in $(-\infty\,,\,\nu)$, we define a function by

$$g^{\alpha}_{\mu_0,\nu}(k) := -\alpha\frac{\lambda(k)}{\omega_\nu(k)-E^{\alpha}_{\mu_0,\nu}} \in D(\widehat{\omega}_\nu), \qquad k\in\mathbf{R}^d. \tag{2.29}$$

Especially, we may put for $\nu=0$

$$E^{\alpha}_{\mu_0} := E^{\alpha}_{\mu_0,0}, \tag{2.30}$$
$$g^{\alpha}_{\mu_0} := g^{\alpha}_{\mu_0,0}. \tag{2.31}$$

For a self-adjoint operator T, we denote the set of all essential spectra of T by $\sigma_{ess}(T)$, and pure point spectra by $\sigma_{pp}(T)$.

By the definition (2.4) of the Hamiltonian $H_\alpha(\mu_0;\nu)$, the free Hamiltonian of the Wigner-Weisskopf model is $H_0(\mu_0;\nu)$ for every $\mu_0\in\mathbf{R}$ and $\nu\ge 0$. Then, it is clear that

$$\sigma_{pp}\left(H_0\left(\mu_0;\nu\right)\right)=\left\{0\,,\,\mu_0\right\}, \tag{2.32}$$
$$\sigma_{ess}\left(H_0\left(\mu_0;\nu\right)\right)=\left[\min\left\{0\,,\,\mu_0\right\}\,,\infty\right), \tag{2.33}$$
$$0 \text{ and } \mu_0 \text{ are simple,} \tag{2.34}$$
$$\text{the unique eigenvector of } 0 \text{ is } \Omega^0\in\mathcal{F}_0, \tag{2.35}$$
$$\text{and}\quad \text{the unique eigenvector of } \mu_0 \text{ is } \Omega^1(0)\in\mathcal{F}_1. \tag{2.36}$$

2.1 Review

The following theorem follows from [AH2, Proposition 6.13, Theorems 6.14 and 6.15] with a slight revision.

Theorem 2.1 **(a)** *Let $\nu, d^{\alpha}_{\nu}(\mu_0)\ge 0$. Then,*

$$0\in\sigma_{pp}\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right), \tag{2.37}$$
$$\sigma_{ess}\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right)=\left[\nu\,,\,\infty\right). \tag{2.38}$$

In particular, 0 is the ground state energy of $H_\alpha\left(\mu_0\,;\,\nu\right)$ with its unique ground state $\Omega^0\in\mathcal{F}_0$.

(b) *Let $d^{\alpha}_{\nu}(\mu_0) < 0 < \nu$ and $\alpha^2\|\lambda/\sqrt{\omega_\nu}\|_0^2 \le \mu_0$. Then,*

$$\left\{0\,,\, E^{\alpha}_{\mu_0,\nu}\right\} \subset \sigma_{pp}\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right), \tag{2.39}$$

$$\sigma_{ess}\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right) = \left[\nu\,,\,\infty\right), \tag{2.40}$$

with $0 \le E^{\alpha}_{\mu_0,\nu} < \nu$. In particular, 0 is the ground state energy of $H_\alpha(\mu_0\,;\,\nu)$. Moreover,

$$\begin{aligned}&\text{if } \alpha^2\|\lambda/\sqrt{\omega_\nu}\|_0^2 < \mu_0, \text{ then } 0 < E^{\alpha}_{\mu_0,\nu};\ 0 \text{ is simple,}\\ &\text{and } \Omega^0 \in \mathcal{F}_0 \text{ is the unique ground state of } H_\alpha(\mu_0\,;\,\nu),\end{aligned} \tag{2.41}$$

$$\begin{aligned}&\text{if } \alpha^2\|\lambda/\sqrt{\omega_\nu}\|_0^2 = \mu_0, \text{ then } 0 = E^{\alpha}_{\mu_0,\nu};\ \ \Omega^0 \in \mathcal{F}_0 \text{ and } \Omega^1(g^{\alpha}_{\mu_0,\nu}) \in \mathcal{F}_1\\ &\text{are the degenerate ground states of } H_\alpha(\mu_0\,;\,\nu).\end{aligned} \tag{2.42}$$

(c) *Let $d^{\alpha}_{\nu}(\mu_0) < 0 < \nu$ and $\mu_0 < \alpha^2\|\lambda/\sqrt{\omega_\nu}\|_0^2$. Suppose that*

$$2\nu - \mu_0 > \alpha^2\left(\left\|\frac{\lambda}{\sqrt{\omega_\nu}}\right\|_0^2 - M\left(\alpha, \mu_0, \omega_\nu\right)\right) + \frac{\|\lambda\|_0^2}{M\left(\alpha, \mu_0, \omega_\nu\right)}, \tag{2.43}$$

where

$$M\left(\alpha, \mu_0, \omega_\nu\right) := \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega_\nu(k) - \mu_0 + \alpha^2\|\lambda/\sqrt{\omega_\nu}\|_0^2}. \tag{2.44}$$

Then,

$$\left\{E^{\alpha}_{\mu_0,\nu}\,,\,0\right\} \subset \sigma_{pp}\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right), \tag{2.45}$$

$$\sigma_{ess}\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right) = \left[E^{\alpha}_{\mu_0,\nu} + \nu\,,\,\infty\right), \tag{2.46}$$

with $E^{\alpha}_{\mu_0,\nu} < 0$. In particular, $E^{\alpha}_{\mu_0,\nu}$ is the ground state energy of $H_\alpha(\mu_0\,;\,\nu)$ with its ground state $\Omega^1(g^{\alpha}_{\mu_0,\nu}) \in \mathcal{F}_1$.

Proof. Parts (a) and (b) follows from [AH2, Proposition 6.13, Theorem 6.14]. For (2.41), we did not state the simpleness of 0 in [AH2, Theorem 6.14(ii)]. But, if we use $\alpha^2\|\lambda/\sqrt{\omega_\nu}\|_0^2 < \mu_0$ instead of [AH2, (6.60)], then we get $\alpha^2 a(\lambda)^* a(\lambda) \le \mu_0 H_b(\nu)$. So, we obtain the simpleness in the same way as the proof of that in [AH2, Proposition 6.13(i)] by using the argument in [HüS2] which derives [HüS2, (6.14)] from [HüS2, (6.3)]. In order to prove part (c), we have only to modify [AH2, Theorem 6.15]. Here we note the following. By applying [AH2, Proposition 6.13(ii)] to our assumption, we have $E_0\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right) \le E^{\alpha}_{\mu_0,\nu} < \mu_0$, which is essential in [AH2, (6.72)] in the proof of [AH2, Theorem 6.15], not $\mu_0 < 0$. ∎

Remark 2.4 *We are also interested in the case for the large absolute value of the coupling constant(i.e., $|\alpha| \gg 1$). Fix μ_0 and make $|\alpha|$ so large. Then, we have $d_\nu^\alpha(\mu_0) < 0$. Thus, we have to investigate the case for $d_\nu^\alpha(\mu_0) < 0$ to investigate the case for large $|\alpha|$. See Theorem 2.3 below.*

Remark 2.5 *In $[\nu\,,\,\infty)$ for $\nu \geq 0$, we can make a different eigenvalue from both of $E_{\mu_0,\nu}^\alpha$ and 0 by adding some conditions to $\omega(k)$ and $\lambda(k)$ as we mentioned in [AH2, Remark 6.4]. Namely, as an effect of the scalar Bose field, a new eigenvalue appears in $(\nu\,,\,\infty)$.*

Remark 2.6 *It is easy to check that*

$$\left\| \frac{\lambda}{\sqrt{\omega_\nu}} \right\|_0^2 - M\left(\alpha, \mu_0, \omega_\nu\right) > 0. \tag{2.47}$$

Let $\mu_0 \geq 0$. Then, if $\nu = 0$, then (2.43) does not hold by (2.47). Let $\mu_0 < 0$. Then, by the definition (2.44), we get

$$M\left(\alpha, \mu_0, \omega_\nu\right) < \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{-\mu_0} = \frac{\|\lambda\|_0^2}{-\mu_0}$$

since $\mu_0 < 0$ now, which implies that

$$\text{the left hand side of (2.43)} > -\mu_0$$

since $\mu_0 < 0 < M\left(\alpha, \mu_0, \omega_\nu\right)$. Thus, (2.43) is meaningful for the case of massive bosons only.

In Theorem 2.1(c), we cannot show the ground state energy of $H_\alpha(\mu_0)$ for the massless bosons under the condition $d^\alpha(\mu_0) < 0$ alone as we remarked in Remark 2.6, but *if we add the condition (S)*, then we can determine the pure point spectra of $H_\alpha(\mu_0)$ completely for the massless bosons by using Skibsted's result [Sk, Theorem 3.1]. So, we need the following condition:

(S) $\lambda^{(1)} \in \mathcal{M}_0 \bigcap \mathcal{M}_{-2}$, where

$$\lambda^{(1)}(k) := \frac{\partial}{\partial|k|}\lambda(k) + \frac{(d-1)\lambda(k)}{2|k|}, \qquad k \in \mathbf{R}^d \tag{2.48}$$

considered as a distribution on $C_0^\infty(\mathbf{R}^d \setminus \{0\})$.

Remark 2.7 *Assuming (S) practically amounts to assuming the infrared regularity condition, namely not the infrared singularity condition (1.8).*

Proposition 2.2 *Assume (A.1), (S) and (1.8). Let $\omega(k) = |k|$ and $d^\alpha(\mu_0) < 0$. Then,*

$$\sigma_{pp}\left(H_\alpha\left(\mu_0\right)\right) = \left\{E^\alpha_{\mu_0}\,,\,0\right\}, \tag{2.49}$$

$$\sigma_{ess}\left(H_\alpha\left(\mu_0\right)\right) = \left[E^\alpha_{\mu_0}\,,\,\infty\right) \tag{2.50}$$

for all $\alpha \in \mathbf{R}$ with

$$\alpha^2 < \frac{1}{4\|\lambda^{(1)}\|_0^2}. \tag{2.51}$$

Especially, $E^\alpha_{\mu_0}$ is the simple ground state energy with its unique ground state $\Omega^1(g^\alpha_{\mu_0}) \in \mathcal{F}_1$, and 0 is the simple first excited state energy with its unique first excited state $\Omega^0 \in \mathcal{F}_0$.

Proof. First, we had already known that $E^\alpha_{\mu_0} \in \sigma_{pp}\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right)$ with its eigenvector $\Omega^1(g^\alpha_{\mu_0})$, and $0 \in \sigma_{pp}\left(H_\alpha\left(\mu_0\right)\right)$ with its eigenvector Ω^0. We note here that, if $d^\alpha(\mu_0) < 0$, then

$$\mu_0 < \alpha^2 \lim_{t\downarrow 0}\int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega(k)+t} \le \alpha^2 \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega(k)} \tag{2.52}$$

since

$$\int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega(k)+t} < \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega(k)}$$

for all $t > 0$. Moreover, by [AH2, Lemma 6.5(ii)] and (2.52), we get $E^\alpha_{\mu_0} < 0$. The existence of a ground state follows from [Gé2, Theorem 1] which is an improvement of [AH1, Theorem 1.3]. On the other hand, set $\beta := \alpha^2\|\lambda^{(1)}\|_0^2$. Then, we get $2(1-\beta)^{-1} < 3$ by (2.51). Thus, by [Sk, Theorem 3.7] which is an improvement of [HüS2], we know that the total number of all eigenvectors of $H_\alpha\left(\mu_0;\nu\right)$ is less that 3, namely 2. Therefore, we obtain (2.49), parts (b) and (c). (2.50) follows from [Ar2, Theorem 3.6]. ∎

2.2 Existence of a Non-Perturbative Ground State

We use the following condition in the main theorem below:

(A.2) The functions, $\omega(k)$, $\lambda(k)$, are continuous, and

$$\lim_{|k|\to\infty} \omega(k) = \infty. \tag{2.53}$$

Moreover, there exist constants $\gamma_\omega > 0$ and $C_\omega > 0$ such that

$$|\omega(k) - \omega(k')| \le C_\omega |k-k'|^{\gamma_\omega}\left(1 + \omega\left(k\right) + \omega\left(k'\right)\right), \qquad k, k' \in \mathbf{R}^d. \tag{2.54}$$

Theorem 2.3 *Let $\nu \geq 0$. Assume (A.1). Then,*

(a) *there exists $\alpha_{\mathrm{WW}}(\nu) > 0$ such that*

$$\left\{E^{\alpha}_{\mu_0,\nu}\,,\,0\right\} \subset \sigma_{pp}\left(H_\alpha\left(\mu_0;\nu\right)\right) \tag{2.55}$$

$$\text{with } E_0\left(H_\alpha\left(\mu_0;\nu\right)\right) < \min\left\{E^{\alpha}_{\mu_0,\nu}\,,\,0\right\}, \tag{2.56}$$

$$\sigma_{ess}\left(H_\alpha\left(\mu_0;\nu\right)\right) = \left[\,E_0\left(H_\alpha\left(\mu_0;\nu\right)\right) + \nu\,,\,\infty\right) \tag{2.57}$$

for every $\alpha \in \mathbf{R}$ with $|\alpha| > \alpha_{\mathrm{WW}}(\nu)$.

(b) *Let $\nu > 0$ (massive bosons). Assume (A.2) in addition. Then, there exists a ground state $\Psi_{\mathrm{WW}} \in \mathcal{F}$ of $H_\alpha\left(\mu_0;\nu\right)$, namely*

$$H_\alpha\left(\mu_0;\nu\right)\Psi_{\mathrm{WW}} = E_0\left(H_\alpha\left(\mu_0;\nu\right)\right)\Psi_{\mathrm{WW}},$$

such that

$$\left\{E_0\left(H_\alpha\left(\mu_0;\nu\right)\right)\,,\,E^{\alpha}_{\mu_0,\nu}\,,\,0\right\} \subset \sigma_{pp}\left(H_\alpha\left(\mu_0;\nu\right)\right), \tag{2.58}$$

with (2.56)

$$\Psi_{\mathrm{WW}} \notin \mathcal{F}_0 \cup \mathcal{F}_1 \tag{2.59}$$

for every $\alpha \in \mathbf{R}$ with $|\alpha| > \alpha_{\mathrm{WW}}(\nu)$.

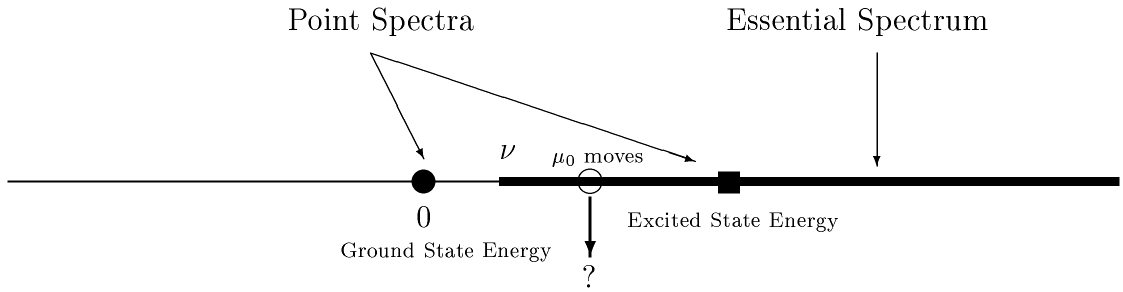
(c) *Let $\nu = 0$ (massless bosons). Assume (A.2), $\nabla\omega \in L^\infty(\mathbf{R}^d)$ and (1.8) in addition. Then, there exists a ground state $\Psi_{\mathrm{WW}} \in \mathcal{F}$ of $H_\alpha\left(\mu_0;\nu\right)$ such that (2.56), (2.58) and (2.59) hold for every $\alpha \in \mathbf{R}$ with $|\alpha| > \alpha_{\mathrm{WW}}(0)$.*

Remark 2.8 *When the case of massive bosons ($\nu > 0$), we can apply the regular perturbation theory to WW model for sufficiently small absolute value of the coupling constant α, and then Theorem 3.1 says that we get either $E^{\alpha}_{\mu_0,\nu}$ or 0 as the ground state energy. Theorem 2.3 means that, for sufficiently large absolute value of the coupling constant, a non-perturbative ground state appears as an influence of the scalar Bose field, and its ground state energy is so low that we cannot conjecture it by the regular perturbation theory for sufficiently small absolute value of the coupling constant. We will clarify the other non-perturbative properties of Ψ_{WW} in [AH3]. For other models, the similar phenomenon were investigated by Hiroshima and Spohn [fHS].*

The following Figures 3-6 show the spectra which we had found, so not all:

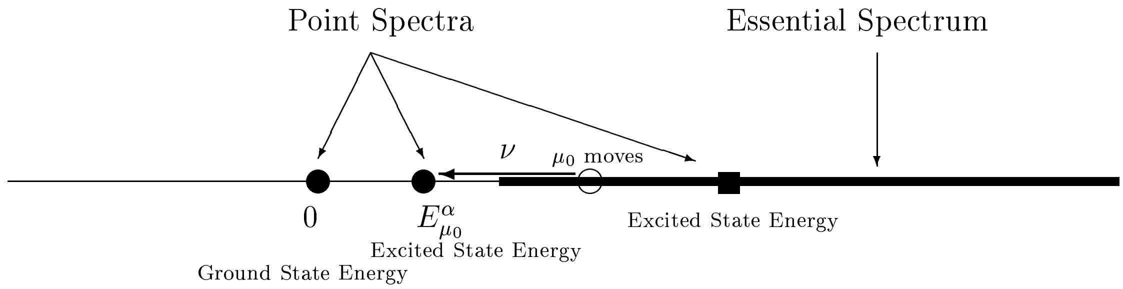
(I) For $|\alpha| < \alpha_{\mathrm{WW}}(\nu)$:

(I-a) Let $d_\nu^\alpha(\mu_0) \geq 0$. Then

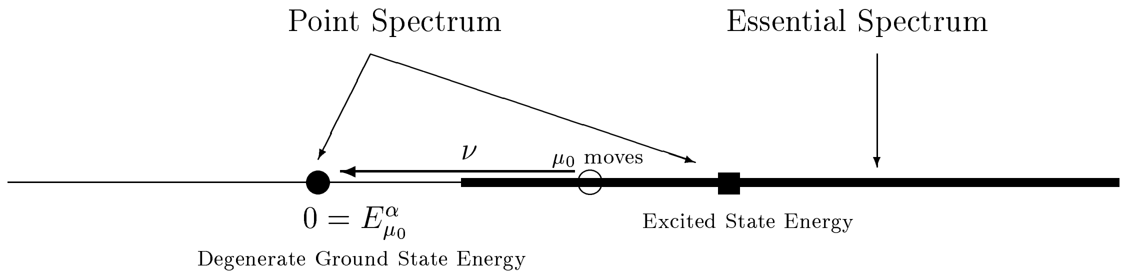


(I-b) Let $d_\nu^\alpha(\mu_0) < 0$.

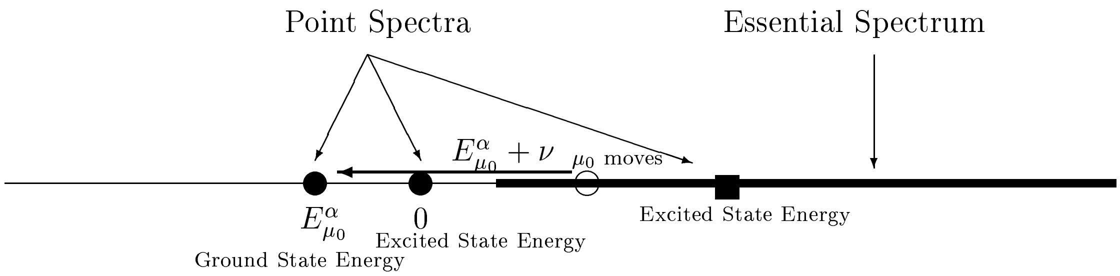
(I-b-1) If $\mu_0 > \alpha^2 \|\lambda/\sqrt{\omega_\nu}\|_0^2$, then



(I-b-2) If $\mu_0 = \alpha^2 \|\lambda/\sqrt{\omega_\nu}\|_0^2$, then



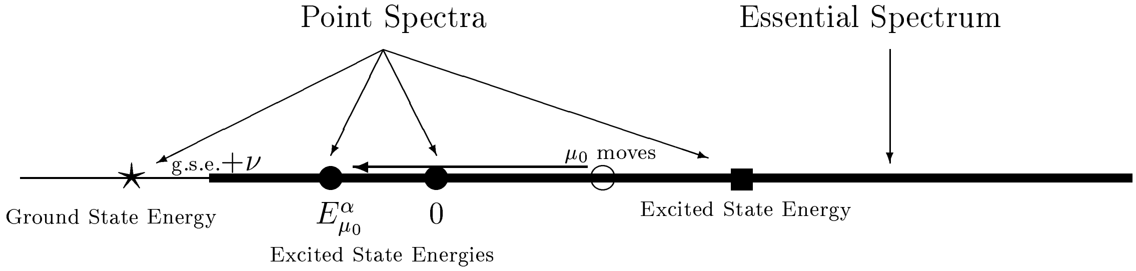
(I-b-3) If $\mu_0 < \alpha^2 \|\lambda/\sqrt{\omega_\nu}\|_0^2$, and all other hypothese in Theorem 2.1(c) hold, then



Appearance or disappearance of ■ depends on the condition for λ by an effect of the scalar Bose field as non-purterbative eigenvalue.

Figure 1: Spectra We Had Found for WW Model (I) for $\nu > 0$

(II) For $|\alpha| > \alpha_{\mathrm{WW}}(\nu)$: If all hypotheses in Theorem 2.3 (b) hold, then

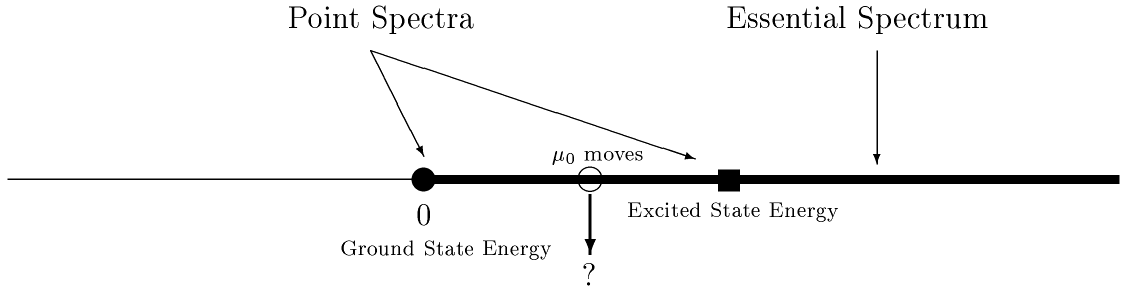


Appearance or disappearance of ■ depends on the condition for λ, and ★ appears by an effect of the scalar Bose field. Both of ★ and ■ are non-perturbative eigenvalues.

Figure 2: Spectra We Had Found for WW Model (II) for $\nu > 0$

(I) For $|\alpha| < \alpha_{\mathrm{WW}}(0)$:

(I-a) If $d^{\alpha}(\mu_0) \geq 0$, then



Appearance or disappearance of ■ depends on the condition for λ by an effect of the scalar Bose field as non-perturbative eigenvalue.

(I-b) If all hypotheses in Proposition 2.2 hold, then

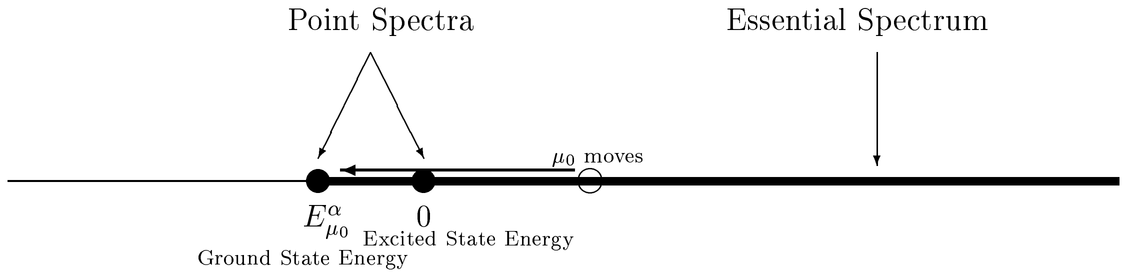
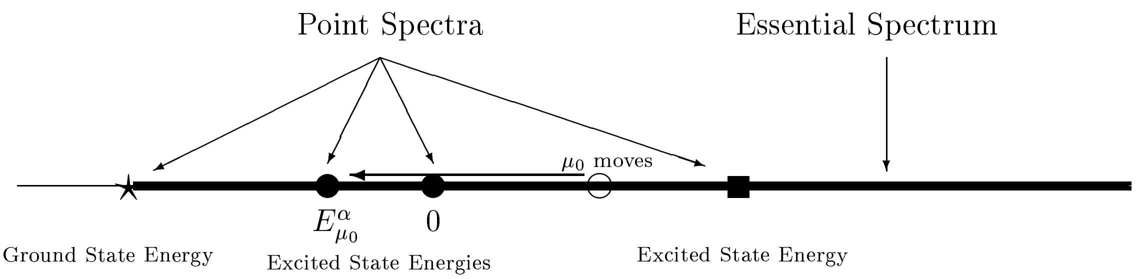


Figure 3: Spectra We Had Found for WW Model (I) for $\nu = 0$

(II) For $|\alpha| > \alpha_{\mathrm{WW}}(0)$: If all hypotheses in Theorem 2.3 (c) hold, then



Appearance or disappearance of ■ depends on the condition for λ, and ★ appears by an effect of the scalar Bose field. Both of ★ and ■ are non-perturbative eigenvalues.

Figure 4: Spectra We Had Found for WW Model (II) for $\nu = 0$

2.3 Proof of Theorem 2.3

Here, we set the following condition

(D)$_\nu$ The function $\dfrac{|\lambda(k)|^2}{|\omega_\nu(k) - x|}$ is not Lebesgue integrable for all $x \in (\nu\,,\,\infty)$,

and we prove the following lemma:

Lemma 2.4 *Let $\nu \geq 0$, and $\Psi \in \mathcal{F}$ be an eigenvector of $H_\alpha(\mu_0;\nu)$.*

(a) *If $\Psi \in \mathcal{F}_0$, then $\Psi = c\Omega^0$ for some $c \in \mathbf{C} \setminus \{0\}$.*

(b) *If $\Psi \in \mathcal{F}_1$ is a ground state and $d^\alpha_\nu(\mu_0) \neq 0$, then $\Psi = c\Omega^1(g^\alpha_{\mu_0,\nu})$ for some $c \in \mathbf{C}\setminus\{0\}$, and $d^\alpha_\nu(\mu_0) < 0$.*

(c) *If $\Psi \in \mathcal{F}_1$ with eigenvalue below ν, then $\Psi = c\Omega^1(g^\alpha_{\mu_0,\nu})$ for some $c \in \mathbf{C} \setminus \{0\}$, and $d^\alpha_\nu(\mu_0) < 0$.*

(d) *Fix $\nu \geq 0$ arbitrarily, and assume $d^\alpha_\nu(\mu_0) \neq 0$ and (D)$_\nu$. If $\Psi \in \mathcal{F}_1$, then $\Psi = c\Omega^1(g^\alpha_{\mu_0,\nu})$ for some $c \in \mathbf{C} \setminus \{0\}$, and $d^\alpha_\nu(\mu_0) < 0$.*

Proof. It is easy to see that there exists just one eigenvector Ω^0 in $\mathcal{F}_0$ but the constant-times Ω^0. So, we have part (a). In the case of parts (b), (c) and (d), we can write as

$$\Psi =: \begin{pmatrix} \Omega_0 \\ a(g)^*\Omega_0 \end{pmatrix} \in \mathcal{F}_1, \qquad g \in D(\widehat{\omega}_\nu), \tag{2.60}$$

and

$$H_\alpha(\mu_0;\nu)\Psi = E\Psi. \tag{2.61}$$

If Ψ is a ground state and $d_\nu^\alpha(\mu_0) \neq 0$, then $E \leq 0$ since $0 \in \sigma_{pp}\left(H_\alpha\left(\mu_0;\nu\right)\right)$ by [AH2, Proposition 6.13], and $E \neq 0$ since $d_\nu^\alpha(\mu_0) \neq 0$. So especially $E < \nu$. By (2.61), we get

$$\left(\mu_0 - E + \alpha\left(\lambda\,,\,g\right)_0\right)\Omega_0 = 0, \tag{2.62}$$

$$a\left(\{\alpha\lambda + (\omega_\nu - E)\,g\}\right)^*\Omega_0 = 0. \tag{2.63}$$

Considering $(\Omega_0\,,\,\text{L.H.S. of (2.62)})_{\mathcal{F}_b} = 0$, we get

$$\mu_0 - E + \alpha\left(\lambda\,,\,g\right)_0 = 0. \tag{2.64}$$

Considering $(a\,(f)^*\,\Omega_0\,,\,\text{L.H.S. of (2.63)})_{\mathcal{F}_b} = 0$ for every $f \in L^2(\mathbf{R}^d)$, we get

$$\left(f\,,\,\alpha\lambda + (\omega_\nu - E)\,g\right)_0 = 0,$$

so that

$$\alpha\lambda + (\omega_\nu - E)\,g = 0$$

in $L^2(\mathbf{R}^d)$, which implies that

$$g(k) = -\alpha\frac{\lambda(k)}{\omega_\nu(k) - E} \qquad \text{a.e. } k \in \mathbf{R}^d. \tag{2.65}$$

Inserting (2.65) into (2.64), we have

$$\mu_0 - E - \alpha^2\int_{\mathbf{R}^d} dk\,\frac{|\lambda(k)|^2}{\omega_\nu(k) - E} = 0.$$

So, E is a zero of $D_{\mu_0,\nu}^\alpha(z)$ below ν. Suppose that $d_\nu^\alpha(\mu_0) \geq 0$. Then $D_{\mu_0,\nu}^\alpha(z)$ has no zero in $\mathbf{C}_\nu$ by [AH2, Lemma 6.5(i)], which is a contradiction. Thus, $d_\nu^\alpha(\mu_0) < 0$. So, by [AH2, Lemma 6.5(ii)], $D_{\mu_0,\nu}^\alpha(z)$ $(z \in \mathbf{C}\setminus[\nu\,,\,\infty))$ has a unique simple zero. Thus, $E = E_{\mu_0,\nu}^\alpha$, and $g = cg_{\mu_0,\nu}^\alpha$ for some $c \in \mathbf{C}\setminus\{0\}$. Thus, we have part (b), and the proof of part (b) includes that of part (c). Under assumptions in part (d), $D_{\mu_0,\nu}^\alpha(z)$ does not have a zero in $[\nu\,,\,\infty)$. Because $(\mathrm{D})_\nu$ implies that $D_{\mu_0,\nu}^\alpha(z)$ does not have a zero in $(\nu\,,\,\infty)$ by the similar reason in [AH2, Remark 6.4]. And $d_\nu^\alpha(\mu_0) \neq 0$ implies that $E < \nu$ in the same way above. Therefore, part (d) can be proved in the same way as part (b). ∎

To show the existence of a non-perturbative ground state for sufficiently large $|\alpha|$, we prove the following lemma.

Lemma 2.5 *Fix μ_0 arbitrarily. Let $\nu \geq 0$. Then, even under infrared singularity condition (1.7), we have*

$$\min\{0\,,\,\mu_0\} - \alpha^2\left\|\frac{\lambda}{\sqrt{\omega_\nu}}\right\|_0^2 \leq E_0\left(H_\alpha\left(\mu_0;\nu\right)\right) \leq \frac{\mu_0}{2} - \frac{\alpha^2}{4}\left\|\frac{\lambda}{\sqrt{\omega_\nu}}\right\|_0^2. \tag{2.66}$$

Thus, for any parameter $\alpha = \alpha_t$ $(t > 0)$ and sequence $\alpha = \alpha_n$ $(n \in \mathbf{N})$ with $|\alpha| \to \infty$ as $t \uparrow \infty$ and $n \uparrow \infty$,

$$\lim_{|\alpha|\to\infty} E_0\left(H_\alpha\left(\mu_0;\nu\right)\right) = -\infty, \tag{2.67}$$

$$-\left\|\frac{\lambda}{\sqrt{\omega_\nu}}\right\|_0^2 \leq \limsup_{|\alpha|\to\infty} \frac{E_0\left(H_\alpha\left(\mu_0;\nu\right)\right)}{\alpha^2} \leq -\frac{1}{4}\left\|\frac{\lambda}{\sqrt{\omega_\nu}}\right\|_0^2. \tag{2.68}$$

Remark 2.9 *For (2.68), we cannot apply [AH1, Proposition 1.4] to $E_0\left(H_\alpha\left(\mu_0;\nu\right)\right)$ because [AH1, (A.6)] dose not hold now, namely $B_1 \equiv (c^* + c)/\sqrt{2}$ and $B_2 \equiv i(c^* - c)/\sqrt{2}$ are not commutative.*

Proof of Lemma 2.5. We can define a self-adjoint operator $H_{test}(\nu)$ acting in $\mathcal{F}$ for $\nu \geq 0$ by

$$\begin{aligned} H_{test}\left(\nu\right) &:= \frac{\mu_0}{2} I \otimes I + I \otimes H_b(\nu) + \frac{\alpha}{\sqrt{2}}\sigma_1 \otimes \phi_S(\lambda) \\ &= \begin{pmatrix} H_b(\nu) + \frac{\mu_0}{2} & \frac{\alpha}{\sqrt{2}}\phi_S(\lambda) \\ \frac{\alpha}{\sqrt{2}}\phi_S(\lambda) & H_b(\nu) + \frac{\mu_0}{2} \end{pmatrix} \end{aligned} \tag{2.69}$$

with $D\left(H_{test}\left(\nu\right)\right) = D\left(I \otimes H_b\left(\nu\right)\right)$. Then we get

$$H_{test}\left(\nu\right) = \frac{1}{2}\left\{H_\alpha(\mu_0\,;\,\nu) + \widetilde{H}_\alpha(\mu_0\,;\,\nu)\right\}$$

on $D\left(I \otimes H_b\left(\nu\right)\right)$, which implies that, for all $\Psi \in D\left(I \otimes H_b\left(\nu\right)\right) = D\left(H_{test}\left(\nu\right)\right) = D\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right) = D\left(\widetilde{H}_\alpha\left(\mu_0\,;\,\nu\right)\right)$ with $\|\Psi\|_{\mathcal{F}} = 1$,

$$\begin{aligned} \left(\Psi\,,\,H_{test}\left(\nu\right)\Psi\right)_{\mathcal{F}} &= \frac{1}{2}\left\{\left(\Psi\,,\,H_\alpha\left(\mu_0\,;\,\nu\right)\Psi\right)_{\mathcal{F}} + \left(\Psi\,,\,\widetilde{H}_\alpha\left(\mu_0\,;\,\nu\right)\Psi\right)_{\mathcal{F}}\right\} \\ &\geq \frac{1}{2}\left\{E_0\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right) + E_0\left(\widetilde{H}_\alpha\left(\mu_0\,;\,\nu\right)\right)\right\}. \end{aligned}$$

Since $E_0\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right) = E_0\left(\widetilde{H}_\alpha\left(\mu_0\,;\,\nu\right)\right)$ by (2.8), we have

$$E_0\left(H_{test}\left(\nu\right)\right) \geq E_0\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right) \quad \text{for every } \nu \geq 0. \tag{2.70}$$

By [AH1, Proposition 1.4], we get

$$\frac{\mu_0}{2} - \frac{\alpha^2}{4}\|\lambda/\sqrt{\omega_\nu}\|_0^2 \leq E_0\left(H_{test}\left(\nu\right)\right). \tag{2.71}$$

Let $\Psi_{_{VH}}$ be the ground state of the van Hove model $H_b(\nu) + \dfrac{\alpha}{\sqrt{2}}\phi_{_S}(\lambda)$ for $\nu > 0$. Then $\begin{pmatrix} \Psi_{_{VH}} \\ \Psi_{_{VH}} \end{pmatrix}$ is the eigenvector of $H_{test}(\nu)$ with its eigenvalue $\dfrac{\mu_0}{2} - \dfrac{\alpha^2}{4}\|\lambda/\sqrt{\omega_\nu}\|_0^2$. So, by (2.71), we have

$$E_0\left(H_{test}\left(\nu\right)\right) = \frac{\mu_0}{2} - \frac{\alpha^2}{4}\|\lambda/\sqrt{\omega_\nu}\|_0^2$$

for $\nu > 0$. At last, by [AH2, Proposition 3.2(iii)], we have

$$E_0\left(H_{test}\left(\nu\right)\right) = \frac{\mu_0}{2} - \frac{\alpha^2}{4}\|\lambda/\sqrt{\omega_\nu}\|_0^2 \text{ for } \nu \geq 0. \tag{2.72}$$

By (2.70) and (2.72), we obtain the right inequality of (2.66). We can obtain the left inequality of (2.66) by [AH2, (2.21)]. The equalities (2.67) and (2.68) follows from (2.66) directly. ■

Proof of Theorem 2.3

We have only to consider large coupling constants α's such that $d_\nu^\alpha(\mu_0) < 0$ by Remark 2.4 (so, $E_{\mu_0,\nu}^\alpha < 0$ by [AH2, Proposition 6.13(ii)]). Then, (2.55) follows from [AH2, Proposition 6.13(ii)]. Fix $\nu \geq 0$ arbitrarily. We use the reductive absurdity. Suppose that there is no such an $\alpha_{\mathrm{WW}}(\nu)$ that (2.56) holds for every $\alpha \in \mathbf{R}$ with $|\alpha| > \alpha_{\mathrm{WW}}(\nu)$. Then, there exists a sequence $\{\alpha_n\}_{n\in\mathbf{N}}$ such that $E_0\left(H_{\alpha_n}\left(\mu_0;\nu\right)\right) = E_{\mu_0,\nu}^{\alpha_n}\ \ (<0)$ and $|\alpha_n| \to \infty$ as $n \to \infty$. Since $E_{\mu_0,\nu}^{\alpha_n}$ is the zero of $D_{\mu_0,\nu}^{\alpha_n}(z)$, we get

$$E_{\mu_0,\nu}^{\alpha_n} = \mu_0 - \alpha_n^2 \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega_\nu(k) - E_{\mu_0,\nu}^{\alpha_n}}.$$

Thus, we get

$$\frac{E_{\mu_0,\nu}^{\alpha_n}}{\alpha_n^2} = \frac{\mu_0}{\alpha_n^2} - \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega_\nu(k) - E_{\mu_0,\nu}^{\alpha_n}}. \tag{2.73}$$

By (2.67) in Lemma 2.5 and Lebesgue's dominated convergence theorem, we have

$$\liminf_{n\to\infty} \left(\text{RHS of (2.73)}\right) = \lim_{n\to\infty} \left(\text{RHS of (2.73)}\right) = 0. \tag{2.74}$$

By (2.68) in Lemma 2.5, we have

$$-\|\lambda/\sqrt{\omega_\nu}\|_0^2 \leq \limsup_{n\to\infty} \left(\text{LHS of (2.73)}\right) \leq -\frac{1}{4}\|\lambda/\sqrt{\omega_\nu}\|_0^2,$$

which contradicts to (2.74). Thus, we obtain (2.56). (2.57) follows from [Ar3, Theorem 3.3]. Let $\nu > 0$ now. Then, [AH1, (A.1)] is satisfied because the operator A in [AH1] is $c^* c$

now, and we assumed [AH1, (A.2), (A.4) and (A.5)] in our (A.1) and (A.2). The condition [AH1, (A.3)] is satisfied for all α because $B_1 \equiv (c^* + c)/\sqrt{2}$ and $B_2 \equiv i(c^* - c)/\sqrt{2}$ are bounded operators, namely all a_j's in [AH1] are zero now. Thus, by [AH1, Theorem 1.2], $H_\alpha(\mu_0;\nu)$ has a ground state Ψ_{WW} for $\nu > 0$, which implies (2.58). The existence of a ground state Ψ_{WW} for the case of massless bosons (i.e., $\nu = 0$) is due to [Gé2, Theorem 1] which is an improvement of [AH1, Theorem 1.3]. (2.59) follows from Lemma 2.4.

3 Ground State Energy of Spin-Boson Model

3.1 New Upper and Lower Bounds for the Ground State Energy

In this subsection, we give new upper and lower bounds for the ground state energy in the case of massless bosons.

Theorem 3.1 *(upper bound without infrared cutoff) Assume (A.1). For the Hamiltonian H_{SB} of the spin-boson model without infrared cutoff (i.e., even under the infrared singularity condition (1.7)), an upper bound is given as follows:*

$$E_{\mathrm{SB}}(0) \le -\frac{\mu}{2} + \inf_{f \in D(\widehat{\omega})} \frac{2\alpha\Re(f\,,\,\lambda)_0 + (f\,,\,\omega f)_0 + \mu\|f\|_0^2}{1 + \|f\|_0^2}. \tag{3.1}$$

Fix arbitrarily δ with

$$0 < \delta < 1/3. \tag{3.2}$$

(A.3) The splitting energy μ and the coupling constant α satisfy

$$\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\left(\omega\,(k) + \frac{\mu}{2}\right)^2} < \frac{1-3\delta}{\delta^2} =: \gamma_\delta. \tag{3.3}$$

Corollary 3.2 *Assume (A.1) and (A.3). If $\mu\alpha \neq 0$, then*

$$E_{\mathrm{SB}}(0) < -\frac{\mu}{2} - \delta\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k) + \frac{\mu}{2}}. \tag{3.4}$$

Remark 3.1 *By Corollary 3.2, we know that*

$$E_{\mathrm{SB}}(0) < E_0(H_0). \tag{3.5}$$

So, considering the diamagnetic inequality by Hiroshima [fH, Theorem 5.1], (3.5) means that there is a difference between the spin-boson model and the Pauli-Fierz model as far as concerning the ground state energy though the spin-boson model is regarded as an approximation of the Pauli-Fierz model in physics.

(A.4) The splitting energy μ and the coupling constant α satisfy

$$4\alpha^2 \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega(k)} < \mu. \tag{3.6}$$

Theorem 3.3 *(lower bound in the case of massless bosons with infrared regularity condition) Let $\omega(k) = |k|$. Assume (A.1), (A.4), (S), and (1.8). Then, for all $\alpha \in \mathbf{R}$ with*

$$\alpha^2 < \frac{1}{12\|\lambda^{(1)}\|_0^2}, \tag{3.7}$$

a lower bound is given as

$$E_{\mathrm{SB}}(0) > -\frac{\mu}{2} - 2\alpha^2 \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega(k) + \frac{\mu}{2}} \tag{3.8}$$

3.2 Proofs of Theorem 3.1 and Corollary 3.2

We prove Theorem 3.1 by decomposing the Hamiltonian, H_{SB}, of the spin-boson model into the sum of two Hamiltonians of the Wigner-Weisskopf model. That is all we have to do to prove our Theorem 3.1.

We define two vectors, $\overrightarrow{\varepsilon_0}$ and $\overrightarrow{\varepsilon_1}$, in $\mathbf{R}^2$ by $\overrightarrow{\varepsilon_0} = (\varepsilon_0^-, \varepsilon_0^+)$ and $\overrightarrow{\varepsilon_1} = (\varepsilon_1^-, \varepsilon_1^+)$, respectively. Set

$$\mathcal{E} := \left\{ (\overrightarrow{\varepsilon_0}, \overrightarrow{\varepsilon_1}) \in \mathbf{R}^4 \,|\, (\overrightarrow{\varepsilon_0}, \overrightarrow{\varepsilon_1}) \text{ satisfies the following (3.9)} \right\}.$$

$$\varepsilon_0^+ + \varepsilon_1^- = \mu, \quad \varepsilon_0^- + \varepsilon_1^+ = -\mu. \tag{3.9}$$

For instance, put

$$\varepsilon_0^+ = \varepsilon_1^\pm = \frac{\mu}{2} \text{ and } \varepsilon_0^- = -\frac{3}{2}\mu. \tag{3.10}$$

Then, $(\varepsilon_0, \varepsilon_1) \in \mathcal{E}$.

In this section, we assume that $(\overrightarrow{\varepsilon_0}, \overrightarrow{\varepsilon_1}) \in \mathcal{E}$.

Proof of Theorem 3.1:

We put

$$
\begin{aligned}
H^{+}_{\mathrm{WW}} &:= H_{2\alpha}(\varepsilon_0^{+}) + \varepsilon_1^{+} cc^* \otimes I, && (3.11)\\
H^{-}_{\mathrm{WW}} &:= \widetilde{H}_{2\alpha}(\varepsilon_0^{-}) + \varepsilon_1^{-} c^* c \otimes I. && (3.12)
\end{aligned}
$$

We rewrite Ω^0 and $\Omega^1(f)$ as

$$
\begin{aligned}
&\Omega^0_{+} := \Omega^0, && (3.13)\\
&\Omega^1_{+}(f) := \Omega^1(f), \qquad f \in D(\widehat{\omega}_\nu), \qquad \nu \geq 0. && (3.14)
\end{aligned}
$$

We introduce new vectors as follows:

$$
\Omega^0_{-} := U_1 \Omega^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \Omega_0 = \begin{pmatrix} \Omega_0 \\ 0 \end{pmatrix}, \tag{3.15}
$$

$$
\begin{aligned}
\Omega^1_{-}(f) := U_1 \Omega^1(f) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes a(f)^* \Omega_0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \Omega_0 \\
&= \begin{pmatrix} a(f)^* \Omega_0 \\ \Omega_0 \end{pmatrix}, \qquad f \in D(\widehat{\omega}_\nu), \qquad \nu \geq 0, \qquad (3.16)
\end{aligned}
$$

where U_1 is defined by (2.9).

By (1.16) and (2.6), we get

$$
H_{\mathrm{SB}} = \frac{1}{2}\left(H^{+}_{\mathrm{WW}} + H^{-}_{\mathrm{WW}} \right) \qquad \text{on } D(I \otimes H_b). \tag{3.17}
$$

By (3.17), we get

$$
\frac{1}{2}\left\{ \left(\Omega^0_{+}\,,\, H^{+}_{\mathrm{WW}} \Omega^0_{+} \right)_{\mathcal{F}} + \left(\Omega^0_{+}\,,\, H^{-}_{\mathrm{WW}} \Omega^0_{+} \right)_{\mathcal{F}} \right\} = \left(\Omega^0_{+}\,,\, H_{\mathrm{SB}} \Omega^0_{+} \right)_{\mathcal{F}} \geq E_{\mathrm{SB}}(0) \|\Omega^0_{+}\|^2.
$$

On the other hand, we have

$$
\left(\Omega^0_{+}\,,\, H^{+}_{\mathrm{WW}} \Omega^0_{+} \right)_{\mathcal{F}} = \varepsilon_1^{+}, \qquad \left(\Omega^0_{+}\,,\, H^{-}_{\mathrm{WW}} \Omega^0_{+} \right)_{\mathcal{F}} = \varepsilon_0^{-}.
$$

Thus, we have

$$
E_{\mathrm{SB}}(0) \leq \frac{1}{2}\left(\varepsilon_0^{-} + \varepsilon_1^{+} \right) = -\frac{\mu}{2}. \tag{3.18}
$$

For every $f \in D(\widehat{\omega})$, we get

$$
\begin{aligned}
&\left(\Omega^1_{+}(f)\,,\, H^{+}_{\mathrm{WW}} \Omega^1_{+}(f) \right)_{\mathcal{F}} \\
&\quad = \varepsilon_0^{+} + 2\alpha \left\{ (f\,,\, \lambda)_0 + (\lambda\,,\, f)_0 \right\} + (f\,,\, \omega f)_0 + \varepsilon_1^{+} \|f\|_0^2, \qquad (3.19)
\end{aligned}
$$

$$
\left(\Omega^1_{+}(f)\,,\, H^{-}_{\mathrm{WW}} \Omega^1_{+}(f) \right)_{\mathcal{F}} = \varepsilon_1^{-} + (f\,,\, \omega f)_0 + \varepsilon_0^{-} \|f\|_0^2. \tag{3.20}
$$

By (3.17), (3.19), and (3.20), we have

$$
\begin{aligned}
& \left(\varepsilon_0^+ + \varepsilon_1^-\right) + 4\alpha\Re\left(f\,,\,\lambda\right)_0 + 2\left(f\,,\,\omega f\right)_0 + \left(\varepsilon_0^- + \varepsilon_1^+\right)\|f\|_0^2 \\
= {} & \left(\Omega_+^1\left(f\right)\,,\,H_{\mathrm{WW}}^+\Omega_+^1\left(f\right)\right)_{\mathcal{F}} + \left(\Omega_+^1\left(f\right)\,,\,H_{\mathrm{WW}}^-\Omega_+^1\left(f\right)\right)_{\mathcal{F}} \\
= {} & 2\left(\Omega_+^1(f)\,,\,H_{\mathrm{SB}}\Omega_+^1(f)\right)_{\mathcal{F}} \\
\geq {} & 2E_{\mathrm{SB}}(0)\|\Omega_+^1(f)\|^2 = 2E_{\mathrm{SB}}(0)\|\Omega^1(f)\|^2.
\end{aligned}
\tag{3.21}
$$

Similarly, for every $f \in D(\widehat{\omega})$, we get

$$
\left(\Omega_-^1(f)\,,\,H_{\mathrm{WW}}^+\Omega_-^1(f)\right)_{\mathcal{F}} = \varepsilon_1^+ + \left(f\,,\,\omega f\right)_0 + \varepsilon_0^+\|f\|_0^2,
\tag{3.22}
$$

$$
\begin{aligned}
& \left(\Omega_-^1(f)\,,\,H_{\mathrm{WW}}^-\Omega_-^1(f)\right)_{\mathcal{F}} \\
& \quad = \varepsilon_0^- + 2\alpha\left\{\left(f\,,\,\lambda\right)_0 + \left(\lambda\,,\,f\right)_0\right\} + \left(f\,,\,\omega f\right)_0 + \varepsilon_1^-\|f\|_0^2.
\end{aligned}
\tag{3.23}
$$

By (3.17), (3.22), and (3.23), we have

$$
\begin{aligned}
& \left(\varepsilon_0^- + \varepsilon_1^+\right) + 4\alpha\Re\left(f\,,\,\lambda\right)_0 + 2\left(f\,,\,\omega f\right)_0 + \left(\varepsilon_0^+ + \varepsilon_1^-\right)\|f\|_0^2 \\
= {} & \left(\Omega_-^1\left(f\right)\,,\,H_{\mathrm{WW}}^+\Omega_-^1\left(f\right)\right)_{\mathcal{F}} + \left(\Omega_-^1\left(f\right)\,,\,H_{\mathrm{WW}}^-\Omega_-^1\left(f\right)\right)_{\mathcal{F}} \\
\geq {} & 2\left(\Omega_-^1(f)\,,\,H_{\mathrm{SB}}\Omega_-^1(f)\right)_{\mathcal{F}} \\
\geq {} & 2E_{\mathrm{SB}}(0)\|\Omega_-^1(f)\|^2 = 2E_{\mathrm{SB}}(0)\|\Omega^1(f)\|^2.
\end{aligned}
\tag{3.24}
$$

By (2.28), (3.9) and (3.24), we obtain

$$
\begin{aligned}
2(1+\|f\|_0^2)E_{\mathrm{SB}}(0) & \leq -\mu + 4\Re(f\,,\,\lambda)_0 + 2(f\,,\,\omega f)_0 + \mu\|f\|_0^2 \\
& = -\mu(1+\|f\|_0^2) + 4\Re(f\,,\,\lambda)_0 + 2(f\,,\,\omega f)_0 + 2\mu\|f\|_0^2,
\end{aligned}
$$

so we get

$$
E_{\mathrm{SB}}(0) \leq -\frac{\mu}{2} + \frac{1}{1+\|f\|_0^2}\left\{2\alpha\Re\left(f\,,\,\lambda\right)_0 + \left(f\,,\,\omega f\right)_0 + \mu\|f\|_0^2\right\}
\tag{3.25}
$$

for all $f \in D(\widehat{\omega})$, which implies Theorem 3.1.

Proof of Corollary 3.2:

To prove Corollary 3.2, we introduce a parameter $c \in \mathbf{R}$ in the upper bound in Theorem 3.1:

For every $c \in \mathbf{R}$, we define a function $g_c \in D(\widehat{\omega})$ by

$$g_c(k) := -c\alpha \frac{\lambda(k)}{\omega(k) + \frac{\mu}{2}}, \qquad k \in \mathbf{R}^d. \tag{3.26}$$

By Theorem 3.1 and (3.26), we have

$$\begin{aligned} E_{\mathrm{SB}}(0) &\leq -\frac{\mu}{2} + \inf_{f \in D(\widehat{\omega})} \frac{2\alpha \Re\,(f\,,\,\lambda)_0 + (f\,,\,\omega f)_0 + \mu \|f\|_0^2}{1 + \|f\|_0^2} \\ &\leq -\frac{\mu}{2} + \frac{2\alpha \Re\,(g_c\,,\,\lambda)_0 + (g_c\,,\,\omega g_c)_0 + \mu \|g_c\|_0^2}{1 + \|g_c\|_0^2} \\ &= -\frac{\mu}{2} + \frac{1}{1 + \|g_c\|_0^2} \Bigg(-2c\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k) + \frac{\mu}{2}} + c^2\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{\omega(k)|\lambda(k)|^2}{\left(\omega(k) + \frac{\mu}{2}\right)^2} \\ &\qquad + \mu c^2 \alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\left(\omega(k) + \frac{\mu}{2}\right)^2} \Bigg). \end{aligned}$$

Since

$$\frac{\omega(k)}{\omega(k) + \frac{\mu}{2}} < 1 \quad \text{and} \quad \frac{1}{\omega(k) + \frac{\mu}{2}} \leq \frac{2}{\mu}, \tag{3.27}$$

we have

$$\begin{aligned} E_{\mathrm{SB}}(0) &\leq -\frac{\mu}{2} + \frac{1}{1 + \|g_c\|_0^2} \Bigg(-2c\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k) + \frac{\mu}{2}} + c^2\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k) + \frac{\mu}{2}} \\ &\qquad + \mu c^2 \alpha^2 \frac{2}{\mu} \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k) + \frac{\mu}{2}} \Bigg) \\ &= -\frac{\mu}{2} + F(c)\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k) + \frac{\mu}{2}} \end{aligned} \tag{3.28}$$

for all $c \in \mathbf{R}$, where

$$F(c) := \frac{3c^2 - 2c}{1 + \|g_c\|_0^2} = \frac{3c^2 - 2c}{\gamma(\mu)\, c^2 + 1}, \tag{3.29}$$

$$\gamma(\mu) := \alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\left(\omega(k) + \frac{\mu}{2}\right)^2} > 0. \tag{3.30}$$

It is easy to show that

$$-c_+(\mu) = F\left(c_+\left(\mu\right)\right) \leq F(c) \tag{3.31}$$

for every $c \in \mathbf{R}$, where

$$c_+(\mu) := \frac{-3 + \sqrt{9 + 4\gamma\left(\mu\right)}}{2\gamma\left(\mu\right)} > 0 \tag{3.32}$$

since

$$\frac{d}{dc}F(c) = \frac{2\left(\gamma\left(\mu\right)c^2 + 3c - 1\right)}{\left(\gamma\left(\mu\right)c^2 + 1\right)^2}.$$

We define a function $h(\gamma)$ by

$$h(\gamma) := \frac{3 - \sqrt{9 + 4\gamma}}{2\gamma} \tag{3.33}$$

for $\gamma > 0$. Then, we have easily the following:

$$h\left(\gamma\left(\mu\right)\right) = -c_+(\mu) \tag{3.34}$$

by the definition (3.32) of $c_+(\mu)$.

Fix arbitrarily δ with

$$0 < \delta < 1/3$$

as in (3.2). Let γ_δ be

$$\gamma_\delta := \frac{1 - 3\delta}{\delta^2}$$

as in (3.3). Then, we get

$$h(\gamma_\delta) = -\delta \tag{3.35}$$

since a unique solution of $h(\gamma) = -\delta$ with $\gamma > 0$ is given by γ_δ, or we can calculate directly in (3.3). Moreover, we get easily that

$$\text{if } \gamma < \gamma', \quad \text{then } h(\gamma) < h(\gamma') \tag{3.36}$$

since

$$\begin{aligned} \frac{d}{d\gamma}h(\gamma) &= \frac{2\gamma + 9 - 3\sqrt{9 + 4\gamma}}{2\gamma^2\sqrt{9 + 4\gamma}} \\ &= \frac{2}{\sqrt{9 + 4\gamma}(2\gamma + 9 + 3\sqrt{9 + 4\gamma})} > 0. \end{aligned}$$

By (3.34)-(3.36), we get

$$\text{if } \gamma(\mu) < \gamma_\delta, \text{ then } -c_+(\mu) < -\delta. \tag{3.37}$$

Thus, by (3.28), (3.31) and (3.37), we obtain

$$\begin{aligned} E_{\mathrm{SB}}(0) &\leq -\frac{\mu}{2} + F(c_+(\mu))\,\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k) + \frac{\mu}{2}} \\ &= -\frac{\mu}{2} - \alpha^2 c_+(\mu) \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k) + \frac{\mu}{2}} \\ &< -\frac{\mu}{2} - \delta\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k) + \frac{\mu}{2}} \end{aligned}$$

since we assumed (3.3) in (A.3).

3.3 Proof of Theorem 3.3

As we mentioned in §1 Introduction, we employ (3.17) and the means which Lieb and Yamazaki already used in [LY] for the polaron model to get the lower bound.

We set

$$H_{\mathrm{WW}}(+\mu) := H_{2\alpha}(\mu), \tag{3.38}$$
$$\widetilde{H}_{\mathrm{WW}}(-\mu) := \widetilde{H}_{2\alpha}(-\mu). \tag{3.39}$$

Namely, $\varepsilon_0^+ = \mu$, $\varepsilon_1^+ = 0$ in (3.11), and $\varepsilon_0^- = -\mu$, $\varepsilon_1^- = 0$ in (3.12). Then, by (2.8) we get

$$H_{\mathrm{WW}}(-\mu) = H_{2\alpha}(-\mu) = U_1^* \widetilde{H}_{\mathrm{WW}}(-\mu) U_1. \tag{3.40}$$

By (3.17), for all $\Psi \in D(I \otimes H_b)$ with $\|\Psi\|_{\mathcal{F}} = 1$

$$\begin{aligned} 2(\Psi\,,\, H_{\mathrm{SB}}\Psi)_{\mathcal{F}} &= (\Psi\,,\, H_{\mathrm{WW}}(+\mu)\Psi)_{\mathcal{F}} + (\Psi\,,\, \widetilde{H}_{\mathrm{WW}}(-\mu)\Psi)_{\mathcal{F}} \\ &\geq E_0(H_{\mathrm{WW}}(+\mu)) + E_0\left(\widetilde{H}_{\mathrm{WW}}(-\mu)\right) \\ &= E_0(H_{\mathrm{WW}}(+\mu)) + E_0(H_{\mathrm{WW}}(-\mu)). \end{aligned}$$

So, we have

$$E_{\mathrm{SB}}(0) \geq \frac{1}{2}\left\{E_0(H_{\mathrm{WW}}(+\mu)) + E_0(H_{\mathrm{WW}}(-\mu))\right\}. \tag{3.41}$$

For $H_{\mathrm{WW}}(+\mu)$, using the original notation,

$$H_{\mathrm{WW}}(+\mu) = H_{2\alpha}(\mu),$$

namely $\mu_0 = \mu$. Thus, we get by (3.6) in (A.4)

$$\begin{aligned} d^{\alpha}(\mu) &= \mu - \lim_{x\uparrow 0} 4\alpha^2 \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega(k) - x} && (3.42) \\ &\geq \mu - 4\alpha^2 \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega(k)} && (3.43) \\ &> 0 && (3.44) \end{aligned}$$

since

$$\int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega(k) - x} < \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega(k)}, \qquad x < 0.$$

By Theorem 2.1(a) and (3.44), we have

$$E_0\left(H_{\mathrm{WW}}(+\mu)\right) = E_0\left(H_{2\alpha}(\mu)\right) = 0. \tag{3.45}$$

On the other hand, for $H_{\mathrm{WW}}(-\mu)$, using the original notation,

$$H_{\mathrm{WW}}(-\mu) = H_{2\alpha}(-\mu),$$

namely $\mu_0 = -\mu$. Thus, by Proposition 2.2, we have

$$E_0\left(H_{\mathrm{WW}}(-\mu)\right) = E_0\left(H_{2\alpha}(-\mu)\right) = E_{-\mu}^{2\alpha} < 0 \tag{3.46}$$

for $\alpha \in \mathbf{R}$ with (3.7), where $E_{-\mu}^{2\alpha}$ is the zero of $D_{-\mu}^{2\alpha}(z)$. Namely,

$$E_{-\mu}^{2\alpha} = -\mu - 4\alpha^2 \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega(k) - E_{-\mu}^{2\alpha}}. \tag{3.47}$$

And we have

$$D_{-\mu}^{2\alpha}\left(-\frac{\mu}{2}\right) = -\frac{\mu}{2} - 4\alpha^2 \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega(k) + \frac{\mu}{2}} < 0. \tag{3.48}$$

Since $D_{-\mu}^{2\alpha}(x)$ is monotone decreasing in $x < 0$ as mentioned above, we have

$$E_{-\mu}^{2\alpha} < -\frac{\mu}{2}$$

by (3.48), which implies that

$$\int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega(k) - E_{-\mu}^{2\alpha}} < \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega(k) + \frac{\mu}{2}}$$

by the direct computation. Thus, we have

$$-\mu - 4\alpha^2 \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega(k) - E_{-\mu}^{2\alpha}} > -\mu - 4\alpha^2 \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega(k) + \frac{\mu}{2}}. \tag{3.49}$$

The lower bound (3.8) follows from (3.41), (3.45) and (3.49).

4 Discussion

4.1 Wigner-Weisskopf Model

We knew in Theorem 2.3 that there exists a non-perturbative ground state Ψ_{WW} in $\mathcal{F}$, and Ψ_{WW} does not belong to the 0-sector or 1-sector. But we have not yet known which sector Ψ_{WW} belongs to. This is an open problem. Remember the switch, appearing in Theorem 2.1, of the ground states between the 0-sector and the 1-sector. As $|\alpha|$ grows, are there such switches between the n-sector and the $(n+1)$-sector for $n = 0, 1, 2, \cdots$ in sequence or not?

In Theorem 2.3 we assumed the infrared regularity condition, $\lambda/\omega \in L^2(\mathbf{R}^d)$. The next open problem is whether the ground state Ψ_{WW} appears in the standard state space $\mathcal{F}$ under the infrared singularity condition, $\lambda/\omega \notin L^2(\mathbf{R}^d)$, or not.

Although the author cannot mention what the non-perturbative ground state in Theorem 2.3 (b) & (c) means in physics exactly, such the non-perturbative ground state seems to be called *superradiant* ground state [Pr1, Pr2] in physics from the point of view of superradiance of soft photons, and its existence was shown by Preparata with the path-integral method and confirmed by Enz [En] with another means. This notion of the superradiant ground state is an extension of the notion of superradiant states which first appeared in Dicke model [Di, Ha], which was investigated mathematically by Scharf [Sc1, Sc2] and by Hepp and Lieb [HL1, HL2, HL3]. Especially, Hepp and Lieb gave the ground state of the Dicke model.

We will show in [AH3] that Ψ_{WW} derives the other non-perturbative properties.

4.2 Spin-Boson Model

By (1.18) we have for every $\nu > 0$

$$E_{\mathrm{SB}}(\nu) = -\frac{\mu}{2} e^{-2\alpha^2 \|\lambda/\omega_\nu\|_0^2 G_\nu} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega_\nu}} \right\|_0^2 \tag{4.1}$$

for some $G_\nu \in [0\,,\,1]$. Under a condition we know a concrete expression of G_ν [mH2, Theorems 1.5 and 1.6]. We can prove that

$$-\frac{\mu}{2} - \alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k)} \le \lim_{\nu\downarrow 0} E_{\mathrm{SB}}(\nu) = E_{\mathrm{SB}}(0) \le -\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k)} \tag{4.2}$$

by (1.18) even under the infrared singularity condition (1.7) (see [AH2, Proposition 3.2(iii)]). Under (1.7) we have the infrared divergence

$$\lim_{\nu\downarrow 0} \left\| \frac{\lambda}{\omega_\nu} \right\|_0 = \infty \tag{4.3}$$

appearing in the van Hove model. On the other hand, we have

$$0 \le G_\nu \le 1, \qquad \nu > 0. \tag{4.4}$$

Then, the problem of expressing the $E_{\rm SB}(0)$ in the case without infrared cutoff is as follows: although $\lim_{\nu\downarrow 0}\|\lambda/\omega_\nu\|_0^2 G_\nu$ is apparently infinite (except for the fortunate case $\lim_{\nu\downarrow 0} G_\nu = 0$) and the term of μ is seemingly removed as $E_{\rm SB}(0) = -\alpha^2\|\lambda/\sqrt{\omega}\|_0^2$ under the limit $\nu \downarrow 0$, *how does μ from the effect by the spin survive in $E_{\rm SB}(0)$*? In more concrete terms, *how does μ from the effect by the spin influence $-\alpha^2\|\lambda/\sqrt{\omega}\|_0^2$ (the ground state energy of the van Hove model) from the effect by the Bose scalar field*? The point the author would like to know best is this. Thus, we so found upper and lower bounds that we can clarify the role of μ in $E_{\rm SB}$.

Let $\mu\alpha \neq 0$. Assume (A.1) and (A.3). Then, there exists $\delta < c_{\mu,\alpha}$ such that

$$\begin{aligned} E_{\rm SB}(0) &= -\frac{\mu}{2} - c_{\mu,\alpha}\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k) + \dfrac{\mu}{2}} && (4.5) \\ &< -\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k)}, && (4.6) \end{aligned}$$

and G_ν in (4.1) renormalizes the infrared divergence (4.3) in the following sense:

$$\begin{aligned} \lim_{\nu\downarrow 0}\left\|\frac{\lambda}{\omega_\nu}\right\|_0^2 G_\nu &= -\frac{1}{2\alpha^2}\ln\left\{1 + \frac{2\alpha^2}{\mu}\left(c_{\mu,\alpha} - 1\right)\int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k) + \dfrac{\mu}{2}} \right. \\ &\qquad \left. -\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k)\left(\omega(k) + \dfrac{\mu}{2}\right)}\right\} < \infty. \qquad (4.7) \end{aligned}$$

Moreover, let $\omega(k) = |k|$, and assume (A.4), (S), and (1.8) in addition to (A.1) and (A.3). Then $c_{\mu,\alpha}$ in (4.5) satisfies, for all α with (3.7),

$$c_{\mu,\alpha} \in (\delta\, ,\, 2)\,. \qquad (4.8)$$

We should not overlook that $\mu/2$ appears in the denominator of the integrand in RHS of (4.5) as $\omega(k) + \frac{\mu}{2}$, though the denominators of the integrands in (4.2) had $\omega(k)$ only, and $\mu/2$ was not in the denominators. Namely, under (A.1) and (A.3), *the half of the splitting energy μ plays a role of the lower bound of frequency (like a mass) of bosons* in $E_{SB}(0)$ for all $\alpha \in \mathbf{R}\setminus\{0\}$. This may be meaningful for our attempt [AH2] to know the existence of the ground state of the generalized spin-boson Hamiltonian $H_{\rm GSB}(0)$ by evaluating the right differential of the ground state energy $E_0\left(H_{\rm GSB}\left(\nu\right)\right)$ at $\nu = 0$. For the (standard) spin-boson model, the effect of $\mu/2$ from the spin may prevent us from the infrared divergence as the van Hove model considering the right differential $E'_{\rm SB}(0+) \equiv \lim_{\nu\downarrow 0}\nu^{-1}\left(E_{\rm SB}\left(\nu\right) - E_{\rm SB}\left(0\right)\right)$ for the argument in [AH2]. Namely, the estimation as (4.2) is not suitable to evaluate $E'_{\rm SB}(0+)$. Because (4.2) is obtained by regarding $H_{\rm SB}(\nu)$ as the van Hove model $H_{\rm VH}(\nu)$ perturbed by bounded operator ($\nu \geq 0$):

$$U_2^* H_{\rm SB}(\nu) U_2 = H_{\rm VH}(\nu) - \frac{\mu}{2}\sigma_1, \qquad (4.9)$$

where

$$H_{\mathrm{VH}}(\nu) = I \otimes H_b(\nu) + \sqrt{2}\alpha\sigma_3 \otimes \phi_s(\lambda) = \begin{pmatrix} H_b(\nu) + \sqrt{2}\alpha\phi_S(\lambda) & 0 \\ 0 & H_b(\nu) - \sqrt{2}\alpha\phi_S(\lambda) \end{pmatrix},$$

$$U_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}.$$

And, under the infrared singularity condition (1.7), the right differential of the ground state energy $E_{\mathrm{VH}}(\nu) = -\alpha^2\|\lambda/\sqrt{\omega_\nu}\|_0^2$ $(\nu \geq 0)$ of $H_{\mathrm{VH}}(\nu)$ is infinite [AH2, §6.1], i.e.,

$$E'_{\mathrm{VH}}(0+) = \alpha^2 \left\| \frac{\lambda}{\omega} \right\|_0^2 = \infty. \tag{4.10}$$

So, the appearance $\mu/2$ in (4.5) may have the possibility of preventing the divergence (4.10) though we had left the problem of clarifying $c_{\mu,\alpha}$. This is an open problem.

Because of our purpose as mentioned above, the upper bound (3.4) in Corollary 3.2 and the lower bound (3.8) in Theorem 3.3 are not necessarily the best. But there is a possibility for our methods to get better bounds than those in this paper. For the upper bound, of course, we must seek the proper function $f \in D(\widehat{\omega})$ in (3.1) in Theorem 3.1 to minimize the upper bound. For the lower bound, remember (3.17). Then we have

$$E_{\mathrm{SB}}(0) \geq \frac{1}{2}\left\{E_0\left(H_{\mathrm{WW}}^{+}\right) + E_0\left(H_{\mathrm{WW}}^{-}\right)\right\}$$

for arbitrary $(\overrightarrow{\varepsilon_0}, \overrightarrow{\varepsilon_1}) \in \mathcal{E}$ in the same way as (3.41). So, to maximize the lower bound, we must obtain $E_0\left(H_{\mathrm{WW}}^{\pm}\right)$ for all of the parameters $(\overrightarrow{\varepsilon_0}, \overrightarrow{\varepsilon_1}) \in \mathcal{E}$. And then we shall be able to clarify the properties of the lower bound under several conditions concerning μ and α by controlling $(\overrightarrow{\varepsilon_0}, \overrightarrow{\varepsilon_1}) \in \mathcal{E}$ in the same way as Lieb and Yamazaki considered the parameter for the lower bound of the ground state energy of the polaron model in [LY]. Thus, it is important to clarify the ground state energy of the Wigner-Weisskopf model under more general conditions than ours in this paper. The problem of investigating the lower bound as mentioned above is also open, which includes the problem of clarifying $c_{\mu,\alpha}$.

Proofs of (4.6), (4.7), and (4.8):

By the left inequality in (4.2), we have

$$E_{\mathrm{SB}}(0) \geq -\frac{\mu}{2} - c_{max}\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k) + \frac{\mu}{2}}, \tag{4.11}$$

where

$$c_{max} := \left(\int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k)}\right)\left(\int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k) + \frac{\mu}{2}}\right)^{-1} > 1. \tag{4.12}$$

By Corollary 3.2 and (4.11), we have the existence of $\delta < c_{\mu,\alpha}$ in (4.5).

Conditions (3.3) in (A.3) and (3.6) in (A.4) imply that

$$
\begin{aligned}
2(1-\delta)\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k)+\frac{\mu}{2}} &< 2(1-\delta)\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k)} \\
&< 2\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k)} \qquad \text{by (3.2)} \\
&< \mu - 2\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k)} \qquad \text{by (3.6) in (A.4)} \\
&= \mu \left(1 - \alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k)\frac{\mu}{2}} \right) \\
&\le \mu \left(1 - \alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k)\left(\omega(k)+\frac{\mu}{2}\right)} \right) \\
&\qquad\qquad \text{by the second of (3.27).}
\end{aligned}
$$

So, we get

$$
\begin{aligned}
\mu &> \alpha^2 \int_{\mathbf{R}^d} dk\, |\lambda(k)|^2 \left\{ \frac{\mu}{\omega(k)\left(\omega(k)+\frac{\mu}{2}\right)} + \frac{2(1-\delta)}{\omega(k)+\frac{\mu}{2}} \right\} \\
&= 2\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k)} - 2\delta\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k)+\frac{\mu}{2}},
\end{aligned}
$$

which implies that

$$
-\frac{\mu}{2} - \delta\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k)+\frac{\mu}{2}} < -\alpha^2 \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{\omega(k)}. \tag{4.13}
$$

Inequality (4.6) follows from (3.4) and (4.13). The limit (4.7) follows from (4.1), (4.2) and (4.6).

We obtain (4.8) by (3.4) in Corollary 3.2 and (3.8) in Theorem 3.3.

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