ACTION OF $\sqrt{-\Delta}$ ON DISTRIBUTIONS

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Dedicated to Professor Yoshimi Satio on the occasion of his sixtieth birthday

Abstract. Action of $\sqrt{-\Delta}$ on distributions is examined in the context of Sobolev spaces, weighted L^2 spaces and weighted Sobolev spaces, respectively. The results obtained are as follows: Let k be a real number. (1) If f is in $H^k(\mathbf{R}^n)$, then $\sqrt{-\Delta}f$ is in $H^{k-1}(\mathbf{R}^n)$; (2) If f is in $\mathcal{S}(\mathbf{R}^n)$, the space of rapidly decreasing functions, then f is in $L^{2,s}(\mathbf{R}^n)$ for any $s < n/2+1$; (3) If f is in $H^{k,s}(\mathbf{R}^n)$ for some $s > -n/2-1$, then $\sqrt{-\Delta}f$ is in $\mathcal{S}'(\mathbf{R}^n)$, the space of tempered distributions. Also, it is shown in the one dimensional case that there exists a φ_0 in $\mathcal{S}(\mathbf{R})$ such that $\sqrt{-\Delta}\varphi_0$ does not belong to $L^{2,s}(\mathbf{R})$ for any $s \geq 3/2$.

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§1 Introduction.

Mathematically rigorous study of relativistic Schrödinger operators was initiated by Weder [8] in 1974. Since then, the relativistic Schrödinger operators have received a considerable attention, and there has been a substantial amount of literature for the relativistic Schrödinger operators from various view points. See a survey article [2] by Lieb and Umeda [7] and references therein.

The kinetic energy part of the relativistic Schrödinger operator is given by

$$\sqrt{-\Delta + m^2}\,,$$

where m is a nonnegative constant. If m is positive, then the operator $\sqrt{-\Delta + m^2}$ is a pseudodifferential operator of order 1 with smooth symbol. Actually, the symbol $\sqrt{|\xi|^2 + m^2}$ belongs to the class $S^1_{1,0}$. (See Section 2 for the definition of the symbol class.) It is straightforward that $\sqrt{-\Delta + m^2}$ maps $\mathcal{S}(\mathbf{R}^n)$, the space of rapidly decreasing functions, continuously into itself, hence by duality, $\sqrt{-\Delta + m^2}$ maps $\mathcal{S}'(\mathbf{R}^n)$, the space of tempered distributions, into itself.

The situation, however, changes dramatically for $m = 0$. The symbol of the operator $\sqrt{-\Delta}$ is given by $|\xi|$, which is singular at the origin. Therefore, $\sqrt{-\Delta}\,\varphi$ may not belong to $\mathcal{S}(\mathbf{R}^n)$ even if φ belongs to $\mathcal{D}(\mathbf{R}^n)$, the space of test functions, and the definition

$$\langle \sqrt{-\Delta} f,\, \varphi \rangle := \langle f,\, \overline{\sqrt{-\Delta}\,\overline{\varphi}} \rangle \qquad (\,\varphi \in \mathcal{D}(\mathbf{R}^n)\,)$$

does not make sense for a generic $f \in \mathcal{D}'(\mathbf{R}^n)$, the space of distributions. A remedy for this difficulty is found in Lieb and Loss [3, p. 172]: *If $f \in H^{1/2}(\mathbf{R}^n)$, the Sobolev space of order $1/2$, then $\sqrt{-\Delta} f \in \mathcal{D}'(\mathbf{R}^n)$.*

A few questions naturally arise.

Question 1. *Does $\sqrt{-\Delta} f$ belong to $H^{-1/2}(\mathbf{R}^n)$ when $f \in H^{1/2}(\mathbf{R}^n)$?*
Question 2. *Is it possible to define $\sqrt{-\Delta} f$ for f in a wider subspace of distributions?*
Question 3. *Can one find subspaces of distributions which are closed under the operation $\sqrt{-\Delta}$?*

The aims of this paper is twofold, one of which is to give answers to these questions. The answer to Question 1 is affirmative. Furthermore, we shall show that $\sqrt{-\Delta} f$ makes sense for any f in $H^{-\infty}(\mathbf{R}^n) = \cup_{k \in \mathbf{R}} H^k(\mathbf{R}^n)$, and that $\sqrt{-\Delta} f$ belongs $H^{k-1}(\mathbf{R}^n)$ if f belongs to $H^k(\mathbf{R}^n)$. This gives answers to Questions 2 and 3; $H^{-\infty}(\mathbf{R}^n)$ is closed under the operation $\sqrt{-\Delta}$. Therefore, the singularity of the symbol $|\xi|$ causes $\sqrt{-\Delta}$ to exhibit no difference in the context of Sobolev spaces from pseudodifferential operators with symbols in the class $S^{\mu}_{\rho,\,\delta}$ (e.g. Kumano-go [1, Theorem 2.7, p.124]). The singularity, however, does cause $\sqrt{-\Delta}$ to exhibit something different in the context of weighted L^2 spaces. In fact, we shall show that there exists a function φ_0 in $\mathcal{S}(\mathbf{R})$ such that $\sqrt{-\Delta}\varphi_0$ belongs to $L^{2,\,s}(\mathbf{R})$ for every $s < 3/2$, but does not for any $s \geq 3/2$. More precisely, we shall prove that the kth derivative of $\sqrt{-\Delta}\varphi_0$ belongs to $L^{2,\,s}(\mathbf{R})$ for all $s < k + 3/2$, but does not for any $s \geq k + 3/2$. In other words, $\sqrt{-\Delta}\varphi_0$ together with its all derivatives decays polynomially at infinity. The fact we can prove in general is that if f is in $\mathcal{S}(\mathbf{R}^n)$, then $\sqrt{-\Delta}f$ is in $L^{2,\,s}(\mathbf{R}^n)$ for all $s < n/2 + 1$.

Surprisingly enough, we shall find that there exists a function f_0, decaying polynomially at infinity and belonging to $H^\infty(\mathbf{R}^n) = \cap_{k\in\mathbf{R}} H^k(\mathbf{R}^n)$, such that $\sqrt{-\Delta}f_0$ together with its all derivatives have exactly the same asymptotic behaviors at infinity, apart from multiplicative constants, as $\sqrt{-\Delta}\varphi_0$ with φ_0 mentioned above.

The other aim of this paper is to investigate action of $\sqrt{-\Delta}$ on weighted Sobolev spaces. There are two reasons. One reason is that action of $\sqrt{-\Delta}$ on weighted Sobolev spaces seems to be much more interesting than that of $\sqrt{-\Delta}$ on Sobolev spaces, as suggested above. The other reason is that it natural to seek wider subspaces of distributions than $H^{-\infty}(\mathbf{R}^n)$, and that $\mathcal{S}'(\mathbf{R}^n)$ is expressed as a union of weighted Sobolev spaces. (This is an immediate consequence of Treves[6, Theorem 25.4, p. 272].)

The plan of the paper is as follows. In Section 2, we fix notation and prove a boundedness result of pseudodifferential operators on weighted Sobolev spaces, which will be used later in this paper. In Section 3, we discuss action of $\sqrt{-\Delta}$ on Sobolev spaces. In Section 4, we consider the one dimensional case and make explicit computations of $\sqrt{-\Delta}f$ for some specific functions f's. We examine action of $\sqrt{-\Delta}$ on weighted L^2 spaces in Section 5. The results in Section 5 will be applied in Section 6, where we discuss action of $\sqrt{-\Delta}$ in weighted Sobolev spaces.

§2 Preliminaries.

In this section, we introduce the notation which will be used throughout the paper, and prove a boundedness result on pseudodifferential operators in weighted Sobolev spaces, the boundedness result which we shall use in Section 6.

For $x \in \mathbf{R}^n$, $|x|$ denotes the Euclidean norm of x, and $\langle x\rangle = \sqrt{1+|x|^2}$. D stands for $-i\partial/\partial x$, and

$$\langle D\rangle = \sqrt{1+|D|^2} = \sqrt{1-\Delta}.$$

The seminorms of $\mathcal{S}(\mathbf{R}^n)$ are defined by

$$|\varphi|_{\ell,\,\mathcal{S}} = \max_{k+|\alpha|\leq\ell}\ \sup_x \{\, \langle x\rangle^k\, |D^\alpha\varphi(x)|\,\} \qquad (\,\ell = 0,\,1,\,2,\,\cdots).$$

For $k,\ \ s \in \mathbf{R}$, the weighted Sobolev space $H^{k,\,s}(\mathbf{R}^n)$ is defined by

$$H^{k,\,s}(\mathbf{R}^n) = \{f \in \mathcal{S}'(\mathbf{R}^n) \mid \langle x\rangle^s \langle D\rangle^k f \in L^2(\mathbf{R}^n)\}$$

with the associated norm

$$\|f\|_{k,s} = \|\langle x\rangle^s \langle D\rangle^k f\|_{L^2}.$$

The weighted Sobolev space $H^{0,\,s}(\mathbf{R}^n)$ turns out to be the weighted L^2 space

$$L^{2,\,s}(\mathbf{R}^n) = \{f \in \mathcal{S}'(\mathbf{R}^n) \mid \langle x\rangle^s f \in L^2(\mathbf{R}^n)\}$$

with the associated norm

$$\|f\|_s = \|\langle x\rangle^s f\|_{L^2},$$

and the weighted Sobolev space $H^{k,\,0}(\mathbf{R}^n)$ turns out to be the Sobolev space

$$H^k(\mathbf{R}^n) = \{f \in \mathcal{S}'(\mathbf{R}^n) \mid \langle D\rangle^k f \in L^2(\mathbf{R}^n)\}$$

with the associated norm

$$\|f\|_{k,\,0} = \|\langle D\rangle^k f\|_{L^2}.$$

Definition. *A C^∞-function $p(x,\,\xi)$ on $\mathbf{R}^n \times \mathbf{R}^n$ is said to be in the class $S^\mu_{\rho,\delta}$ ($\mu \in \mathbf{R},\ \ 0 \le \delta \le \rho \le 1,\ \ \delta < 1$) if for any pair α and β of multi-indices there exists a constant $C_{\alpha\beta}$ such that*

$$\Big|\Big(\frac{\partial}{\partial\xi}\Big)^\alpha \Big(\frac{\partial}{\partial x}\Big)^\beta p(x,\,\xi)\Big| \le C_{\alpha\beta}\langle\xi\rangle^{\mu+\delta|\beta|-\rho|\alpha|}.$$

It is well-known that if $p(x,\,\xi)$ belongs to $S^\mu_{\rho,\delta}$, then the pseudodifferential operator defined by

$$p(x,\,D)\varphi(x) = (2\pi)^{-n/2}\int e^{ix\cdot\xi}\, p(x,\,\xi)\hat{\varphi}(\xi)\,d\xi$$

maps $\mathcal{S}(\mathbf{R}^n)$ continuously into itself, and by duality, maps $\mathcal{S}'(\mathbf{R}^n)$ into itself (e.g. Kumano-go [1, Theorem 1.3, p. 58] and [1, Definition 1.1, p. 116]). Here

$$\hat{\varphi}(\xi) = (2\pi)^{-n/2}\int e^{-ix\cdot\xi}\,\varphi(x)\,dx.$$

With an abuse of notation, we write $p(x,\,D) \in S^\mu_{\rho,\delta}$ when $p(x,\,\xi) \in S^\mu_{\rho,\delta}$.

Lemma 2.1. *Let k and s be in $\mathbf{R}$, and let $p(\xi)$ be in $S^\mu_{0,\,0}$. Then $p(D)$ is a bounded operator from $H^{k,\,s}(\mathbf{R}^n)$ to $H^{k-\mu,\,s}(\mathbf{R}^n)$.*

Proof. To prove the lemma, it is sufficient to show that the operator T defined by

$$\begin{aligned} T :&= \langle x\rangle^{s}\langle D\rangle^{k-\mu}p(D)\langle D\rangle^{-k}\langle x\rangle^{-s} \\ &= \langle x\rangle^{s}p(D)\langle D\rangle^{-\mu}\langle x\rangle^{-s} \end{aligned}$$

is bounded in $L^2(\mathbf{R}^n)$ for $s > 0$. (The boundedness of T for $s < 0$ can be reduced to the boundedness of T^*.)

Introducing a new symbol $q(x, \xi)$ by

$$q(x,\xi) = \mathrm{Os}-\iint_{\mathbf{R}^{2n}} e^{-iy\cdot\eta}p(\xi+\eta)\langle\xi+\eta\rangle^{-\mu}\langle x+y\rangle^{-s}\,(2\pi)^{-n}\,dy\,d\eta\,, \tag{2.1}$$

we see that $T = \langle x\rangle^{s}q(x, D)$ (cf. [1, Theorem 2.1 (1), p. 74]). By differentiation under the oscillatory integral sign and integration by parts, we deduce that for any pair of multi-indices α and β

$$\Big|\Big(\frac{\partial}{\partial\xi}\Big)^{\alpha}\Big(\frac{\partial}{\partial x}\Big)^{\beta}q(x,\xi)\Big| \le C_{s\alpha\beta}\langle x\rangle^{-s}, \tag{2.2}$$

where $C_{s\alpha\beta}$ is a nonnegative constant. Therefore, the symbol of the operator T, which is given by $\langle x\rangle^{s}\,q(x, \xi)$, belongs to $S^{0}_{0,\,0}$. Thus, we can apply the Calderón-Vaillancourt theorem to conclude that T is a bounded operator in $L^2(\mathbf{R}^n)$. $\square$

§3 $\sqrt{-\Delta}$ on Sobolev spaces.

In this section we work in Sobolev spaces. We start with the formal definition of the operator $\sqrt{-\Delta}$:

$$\sqrt{-\Delta}\,f(x) = (2\pi)^{-n/2}\int e^{ix\cdot\xi}\,|\xi|\hat{f}(\xi)\,d\xi.$$

If $f \in \mathcal{S}(\mathbf{R}^n)$ and the support of $\hat{f}$ does not contain the origin in the ξ-space, then it follows that $\sqrt{-\Delta}f(x) \in \mathcal{S}(\mathbf{R}^n)$. This fact leads to a decomposition of $\sqrt{-\Delta}$ into a singular part and a regular part:

$$\sqrt{-\Delta} = \chi(D)\sqrt{-\Delta} + \big(1-\chi(D)\big)\sqrt{-\Delta}, \tag{3.1}$$

where χ is a C^∞-function satisfying

$$\chi(\xi) = \begin{cases} 1 & \text{if } |\xi| \le 1 \\ 0 & \text{if } |\xi| \ge 2. \end{cases} \tag{3.2}$$

This decomposition shall make the discussions in this section rather simple.

It is evident that $\big(1-\chi(D)\big)\sqrt{-\Delta}$ maps $\mathcal{S}(\mathbf{R}^n)$ continuously into itself. Hence $\big(1-\chi(D)\big)\sqrt{-\Delta}$ maps $\mathcal{S}'(\mathbf{R}^n)$ into itself. More precisely, we have the following result.

Lemma 3.1. *Let k be in $\mathbf{R}$. Then $\big(1-\chi(D)\big)\sqrt{-\Delta}$ is a bounded operator from $H^k(\mathbf{R}^n)$ to $H^{k-1}(\mathbf{R}^n)$.*

Proof. Let f be in $H^k(\mathbf{R}^n)$. We then see that

$$\begin{aligned}
&\|\big(1-\chi(D)\big)\sqrt{-\Delta}f\|_{k-1,\,0}\\
&\quad=\Big\{\int\big|\langle\xi\rangle^{k-1}\big(1-\chi(\xi)\big)\,|\xi|\hat{f}(\xi)\,\big|^2\,d\xi\Big\}^{1/2}\\
&\quad\le\Big\{\sup_\xi\big(\langle\xi\rangle^{-1}\big(1-\chi(\xi)\big)\,|\xi|\big)\Big\}\,\|f\|_{k,\,0}.
\end{aligned}$$

This inequality proves the lemma. $\square$

We next show a boundedness result of the singular part $\chi(D)\sqrt{-\Delta}$. Since the symbol $\chi(\xi)|\xi|$ is a bounded function, we readily see that $\chi(D)\sqrt{-\Delta}$ defines a bounded operator in $L^2(\mathbf{R}^n)$. Furthermore, we can prove that $\chi(D)\sqrt{-\Delta}$ defines a bounded operator from a Sobolev space of any order to another Sobolev space of any order.

Lemma 3.2. *Let k and ℓ be in $\mathbf{R}$. Then $\chi(D)\sqrt{-\Delta}$ is bounded from $H^k(\mathbf{R}^n)$ to $H^\ell(\mathbf{R}^n)$.*

Proof. Let f be in $H^k(\mathbf{R}^n)$. Since the support of χ is compact, it follows that $\langle\xi\rangle^\ell\chi(\xi)|\xi|\hat{f}$ belongs to $L^2(\mathbf{R}^n_\xi)$, and that

$$\|\langle D\rangle^\ell\chi(D)\sqrt{-\Delta}f\|_0\le\Big\{\sup_\xi\big(\big|\langle\xi\rangle^\ell\chi(\xi)\,|\xi|\,\langle\xi\rangle^{-k}\big|\big)\Big\}\|f\|_{k,\,0}.$$

This implies the lemma. $\square$

Combining Lemmas 3.1 and 3.2, we obtain one of the main results.

Theorem 3.3. *Let k be in $\mathbf{R}$. Then $\sqrt{-\Delta}$ is bounded from $H^k(\mathbf{R}^n)$ to $H^{k-1}(\mathbf{R}^n)$.*

§4 **Examples.**

In this section, we deal with $\sqrt{-\Delta}$ on the line, i.e. $\sqrt{-d^2/dx^2}$, and make explicit computations of $\sqrt{-d^2/dx^2}f$ for $f(x)=\varphi_0(x):=\exp(-x^2/2)$ and $f(x)=f_0(x):=$

$1/(1+x^2)$. The results obtained below show that $\sqrt{-d^2/dx^2}\,\varphi_0$ and $\sqrt{-d^2/dx^2}\,f_0$, together with their all derivatives, have exactly the same asymptotic behaviors, apart from a multiplicative constant, as $|x| \to \infty$. This fact reveals that the operator $\sqrt{-\Delta}$ possesses a surprisingly interesting property, which neither differential operators nor pseudodifferential operators with smooth symbols possess.

Example 4.1. Consider the Gaussian function

$$\varphi_0(x) = \exp(-x^2/2),$$

which obviously belongs to $\mathcal{S}(\mathbf{R})$. Since the Fourier transform $\hat{\varphi}_0$ is given by $\exp(-\xi^2/2)$ (e.g. Strichartz [5, p. 41]) which is an even function, we see that

$$\sqrt{-d^2/dx^2}\,\varphi_0(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \xi\,\hat{\varphi}_0(\xi)\cos(x\xi)\,d\xi. \tag{4.1}$$

In view of the fact that $(d/d\xi)\hat{\varphi}_0 = -\xi\hat{\varphi}_0$, we can integrate the right hand side of (4.1), and obtain

$$\sqrt{-\frac{d^2}{dx^2}}\,\varphi_0(x) = \sqrt{\frac{2}{\pi}}\Big\{1 - x\int_0^\infty \exp\big(-\frac{\xi^2}{2}\big)\sin(x\xi)\,d\xi\Big\}. \tag{4.2}$$

To investigate the asymptotic behavior of $\sqrt{-d^2/dx^2}\,\varphi_0$ as $|x| \to \infty$, we put

$$g(x) = \int_0^\infty \exp\big(-\frac{\xi^2}{2}\big)\sin(x\xi)\,d\xi. \tag{4.3}$$

Then it is straightforward to verify that g is a C^∞-function, and satisfies the ordinary differential equation with an initial condition:

$$\begin{cases} \dfrac{dg}{dx} + xg = 1 \\ g(0) = 0. \end{cases} \tag{4.4}$$

(The solution to the equation (4.4) is given by

$$g(x) = \exp\big(-\frac{x^2}{2}\big)\int_0^x \exp\big(\frac{t^2}{2}\big)\,dt,$$

which is not an elementary function, and not a suitable expression for our purpose.) It then follows from (4.2) – (4.4) that

$$\sqrt{-\frac{d^2}{dx^2}}\,\varphi_0(x) = \sqrt{\frac{2}{\pi}}\,\frac{dg}{dx}(x).$$

Differentiating the right hand side of (4.3) under the integral sign, and integrating by parts twice yield

$$x^2\frac{dg}{dx}(x) = -1 - \int_0^\infty \frac{d^2}{d\xi^2}\Big\{\xi \exp\big(-\frac{\xi^2}{2}\big)\Big\} \cos(x\xi)\, d\xi.$$

The Riemann-Lebesgue theorem now implies that

$$x^2\frac{dg}{dx}(x) \to -1 \quad \text{as } \ |x| \to \infty.$$

Furthermore, we find, by induction, that

$$\begin{aligned} x^{k+1}\frac{d^k g}{dx^k}(x) &= (-1)^k\, k! \\ &\quad + \int_0^\infty \frac{d^{k+1}}{d\xi^{k+1}}\Big\{\xi^k \exp\big(-\frac{\xi^2}{2}\big)\Big\} \sin(x\xi + \frac{2k+1}{2}\pi)\, d\xi \\ &\qquad (k = 1,\, 2,\, 3, \cdots), \end{aligned}$$

from which it follows that

$$x^{k+2}\Big(\frac{d^k}{dx^k}\sqrt{-\frac{d^2}{dx^2}}\,\varphi_0\Big)(x) \to (-1)^{k+1}\sqrt{\frac{2}{\pi}}\,(k+1)! \tag{4.5}$$

as $|x| \to \infty$. Here we have used the Riemann-Lebesgue theorem again. Thus we have shown the following facts for all $k = 0,\, 1,\, 2, \cdots$:

(i) $\displaystyle\frac{d^k}{dx^k}\sqrt{-\frac{d^2}{dx^2}}\,\varphi_0$ belongs to $L^{2,\,s}(\mathbf{R})$ for all $s < k + \frac{3}{2}$;

(ii) $\displaystyle\frac{d^k}{dx^k}\sqrt{-\frac{d^2}{dx^2}}\,\varphi_0$ does not belong to $L^{2,\,s}(\mathbf{R})$ for any $s \geq k + \frac{3}{2}$.

Example 4.2. Let f_0 be a function defined by

$$f_0(x) = \frac{1}{x^2+1}.$$

It is well-known (cf. Stricharts [5, p.49]) that

$$\hat{f}_0(\xi) = \sqrt{\frac{\pi}{2}}e^{-|\xi|}.$$

By a similar computation as in Example 5.1, we see that

$$h(x) := \sqrt{-\frac{d^2}{dx^2}}f_0(x) = \int_0^\infty \xi\, e^{-\xi}\cos(x\xi)\, d\xi.$$

We then find, by integration by parts, that

$$x^2 h(x) = -1 - h(x) + 2\int_0^\infty e^{-\xi}\cos(x\xi)\, d\xi,$$

which implies that

$$h(x) = -\frac{1}{x^2+1} + \frac{2}{(x^2+1)^2}. \tag{4.6}$$

It follows from (4.6) that for $k = 0, 1, 2, \cdots$

$$x^{k+2}\Big(\frac{d^k}{dx^k}\sqrt{-\frac{d^2}{dx^2}}\, f_0\Big)(x) \to (-1)^{k+1}\,(k+1)!$$

as $|x| \to \infty$. This is exactly the same as (4.5), apart from a multiplicative constant.

§5 $\sqrt{-\Delta}$ in weighted L^2-spaces.

The examples in the previous section suggest that the operator $\sqrt{-\Delta}$ should be considered in weighted L^2-spaces and weighted Sobolev spaces. In this section, we consider action of $\sqrt{-\Delta}$ in weighted L^2-spaces. As was mentioned in Section 3, the regular part $\big(1-\chi(D)\big)\sqrt{-\Delta}$ maps $\mathcal{S}(\mathbf{R}^n)$ continuously into itself, so it maps $\mathcal{S}(\mathbf{R}^n)$ continuously into $L^{2,\,s}(\mathbf{R}^n)$ for any s in $\mathbf{R}$. So the main task in this section is to examine whether the singular part $\chi(D)\sqrt{-\Delta}$ maps $\mathcal{S}(\mathbf{R}^n)$ into $L^{2,\,s}(\mathbf{R}^n)$ for some $s > 0$. To this end, we first write $\chi(D)\sqrt{-\Delta}$ as an integral operator:

$$\begin{aligned}
\chi(D)\sqrt{-\Delta}f(x) &= (2\pi)^{-n/2}\!\int e^{ix\cdot\xi}\,\chi(\xi)\,|\xi|\,\Big\{(2\pi)^{-n/2}\!\int e^{-iy\cdot\xi} f(y)\,dy\Big\}\,d\xi \\
&= \int K(x-y)f(y)\,dy, \\
&= K * f(x)
\end{aligned} \tag{5.1}$$

where

$$K(y) = (2\pi)^{-n}\!\int e^{iy\cdot\xi}\,\chi(\xi)\,|\xi|\,d\xi.$$

We note that the integrals in (5.1) are finite at least for $f \in \mathcal{S}(\mathbf{R}^n)$. Also, by inspection, we see that K is a C^∞-function, and that K together with its all derivatives is bounded on $\mathbf{R}^n$. More generally, we introduce an integral kernel

$$K_\alpha(y) = (2\pi)^{-n}\!\int e^{iy\cdot\xi}\,\chi(\xi)\,|\xi|\,\xi^\alpha\,d\xi$$

for each multi-index α. It is true again that K_α is a C^∞-function bounded together with its all derivatives. Note that

$$\chi(D)\sqrt{-\Delta}D^\alpha f = K_\alpha * f. \tag{5.2}$$

Also, note that $K_0 = K$.

Lemma 5.1. *For each $0 < \sigma < 1$, there exists a positive constant C_σ, depending also on α and n, such that*

$$|K_\alpha(y)| \le C_\sigma \langle y\rangle^{\sigma-n-1-|\alpha|}. \tag{5.3}$$

Proof. (Following the idea of Nagase and Umeda [4, Lemma 2.2].) Since K_α is a bounded function, it is sufficient to show that for each $0 < \sigma < 1$, there exists a positive constant C_σ such that

$$|K_\alpha(y)| \le C_\sigma |y|^{\sigma-n-1-|\alpha|}. \tag{5.4}$$

To show (5.4), we need to introduce a cut-off function at the origin in the ξ-space: Let ψ be a C^∞-function such that

$$\psi(\xi) = \begin{cases} 0 & \text{if } |\xi| \le 1 \\ 1 & \text{if } |\xi| \ge 2. \end{cases}$$

We then put

$$K_\alpha(y;\varepsilon) = (2\pi)^{-n}\int e^{iy\cdot\xi}\,\psi(\varepsilon^{-1}\xi)\ \chi(\xi)\,|\xi|\,\xi^\alpha\,d\xi \tag{5.5}$$

for every $\varepsilon > 0$. The Lebesgue dominated convergence theorem implies that $K_\alpha(y;\varepsilon)$ converges pointwise to $K_\alpha(y)$ as $\varepsilon \to 0$. Hence, in order to obtain (5.4), it is sufficient to show that for each $0 < \sigma < 1$, there exists a positive constant C_σ, independent of $\varepsilon > 0$, such that

$$|K_\alpha(y;\varepsilon)| \le C_\sigma |y|^{\sigma-n-1-|\alpha|}. \tag{5.6}$$

Since the integrand of the right hand side of (5.5) is a C^∞-function of ξ with compact support, we see, by repeated use of integration by parts, that

$$y^\beta K_\alpha(y;\varepsilon) = (2\pi)^{-n}\int e^{iy\cdot\xi}\left(i\frac{\partial}{\partial\xi}\right)^\beta \left\{\psi(\varepsilon^{-1}\xi)\ \chi(\xi)\,|\xi|\,\xi^\alpha\right\}d\xi \tag{5.7}$$

for any multi-index β. In particular, if $\beta \ne 0$, taking $y = 0$ in (5.7) gives

$$0 = (2\pi)^{-n}\int \left(i\frac{\partial}{\partial\xi}\right)^\beta \left\{\psi(\varepsilon^{-1}\xi)\ \chi(\xi)\,|\xi|\,\xi^\alpha\right\}d\xi. \tag{5.8}$$

Subtracting (5.8) from (5.7), we get

$$|y^\beta K_\alpha(y;\varepsilon)| \le (2\pi)^{-n}\int \Big|(e^{iy\cdot\xi}-1)\Big(i\frac{\partial}{\partial\xi}\Big)^\beta \big\{\psi(\varepsilon^{-1}\xi)\ \chi(\xi)\,|\xi|\,\xi^\alpha\big\}\Big|\,d\xi \tag{5.9}$$

for any $\beta \neq 0$. Here we note that the support of every derivative of order ≥ 1 of $\psi(\varepsilon^{-1}\xi)$ is contained in the annulus $\varepsilon \le |\xi| \le 2\varepsilon$. Using this fact, we deduce that

$$\Big|\Big(i\frac{\partial}{\partial\xi}\Big)^\beta \big\{\psi(\varepsilon^{-1}\xi)\,\chi(\xi)\,|\xi|\,\xi^\alpha\big\}\Big| \le C_{\alpha\beta}|\xi|^{1+|\alpha|-|\beta|}, \tag{5.10}$$

where $C_{\alpha\beta}$ is a constant independent of $\varepsilon > 0$. On noting the inequalities $|e^{iy\cdot\xi}-1| \le 2$ and $|e^{iy\cdot\xi}-1| \le |y|\,|\xi|$, we see, by interpolation, that

$$|e^{iy\cdot\xi}-1| \le 2|y|^\sigma\,|\xi|^\sigma \tag{5.11}$$

for any $0 < \sigma < 1$. Combining (5.9) – (5.11), we now obtain

$$|y^\beta K_\alpha(y;\varepsilon)| \le (2\pi)^{-n}\,C_{\alpha\beta}\,2|y|^\sigma \int_{|\xi|\le 2} |\xi|^{\sigma+|\alpha|+1-|\beta|}\,d\xi \tag{5.12}$$

for any $\beta \neq 0$ and any $0 < \sigma < 1$, where $C_{\alpha\beta}$ is the same constant as in (5.10). The integral on the right hand side of (5.12) is finite for all β satisfying $|\beta| \le |\alpha| + 1 + n$. This fact implies (5.6) $\square$

Remark 5.2. The inverse Fourier transform of $|\xi|$ is formally given by const.$|x|^{-n-1}$, of which singularity is very severe. This fact illustrates that it is not quite appropriate to handle $\sqrt{-\Delta}$ as an integral operator. As was shown in Lemma 5.1, the decomposition of $\sqrt{-\Delta}$ into the singular part and the regular part enables us to avoid the singularity of the integral kernel at the cost of an arbitrarily small loss of the decay rate at infinity.

Lemma 5.3. *If $s < n/2+1$, then $\chi(D)\sqrt{-\Delta}$ maps $\mathcal{S}(\mathbf{R}^n)$ continuously into $L^{2,s}(\mathbf{R}^n)$.*

Proof. Let φ be in $\mathcal{S}(\mathbf{R}^n)$, and choose, if necessary, $0 < \sigma < 1$ so that

$$\sigma < \frac{n}{2} + 1 - s. \tag{5.13}$$

It follows from (5.1) and Lemma 5.1 with $\alpha = 0$ that

$$|\chi(D)\sqrt{-\Delta}\varphi(x)| \le C_\sigma \int \langle x-y\rangle^{\sigma-n-1}|\varphi(y)|\,dy. \tag{5.14}$$

Appealing to the inequality

$$\langle x-y\rangle^{-1} \le 2\langle x\rangle^{-1}\langle y\rangle, \tag{5.15}$$

we get

$$\begin{aligned}|\chi(D)\sqrt{-\Delta}\varphi(x)| &\le C'_\sigma \langle x\rangle^{\sigma-n-1}\int \langle y\rangle^{-\sigma+n+1}|\varphi(y)|\,dy\\ &\le C''_\sigma \langle x\rangle^{\sigma-n-1}\,|\varphi|_{2n+1,\,\mathcal{S}},\end{aligned}$$

which implies that

$$\|\chi(D)\sqrt{-\Delta}\varphi\|_s^2 \le \big(C''_\sigma|\varphi|_{2n+1,\,\mathcal{S}}\big)^2\int \langle x\rangle^{2(s+\sigma-n-1)}\,dx. \tag{5.16}$$

Since $2(s+\sigma-n-1) < -n$ by (5.13), the assertion of the lemma follows from (5.16). $\square$

Theorem 5.4. *If $s < n/2+1$, then $\sqrt{-\Delta}$ maps $\mathcal{S}(\mathbf{R}^n)$ continuously into $L^{2,\,s}(\mathbf{R}^n)$.*

Proof. Lemma 5.3, together with the fact that the regular part $\big(1-\chi(D)\big)\sqrt{-\Delta}$ maps $\mathcal{S}(\mathbf{R}^n)$ continuously into $L^{2,\,s}(\mathbf{R}^n)$ for any s, gives the lemma. $\square$

Remark 5.5. When $n=1$, Theorem 5.4 is optimal in the following sense: there exists a function $\varphi_0 \in \mathcal{S}(\mathbf{R})$ such that $\sqrt{-d^2/dx^2}\,\varphi_0$ belongs to $L^{2,\,s}(\mathbf{R})$ for all $s < 3/2$, but does not belong to $L^{2,\,s}(\mathbf{R})$ for any $s \ge 3/2$ (see Example 4.1 in Section 4).

Theorem 5.4 allows us to define $\sqrt{-\Delta}f$ as a tempered distribution for some f which does not belong to $H^{-\infty}(\mathbf{R}^n)$.

Theorem 5.6. Let f be in $L^{2,\,-s}(\mathbf{R}^n)$ for some $s < n/2+1$. Then $\sqrt{-\Delta}f$ is in $\mathcal{S}'(\mathbf{R}^n)$.

Proof. It follows from the definition

$$\big\langle \sqrt{-\Delta}f,\,\varphi\big\rangle = \big\langle f,\,\overline{\sqrt{-\Delta}\,\overline{\varphi}}\,\big\rangle \qquad (\,\varphi \in \mathcal{S}(\mathbf{R}^n)\,)$$

and Theorem 5.4 that

$$\begin{aligned}\big|\big\langle \sqrt{-\Delta}f,\,\varphi\big\rangle\big| &\le \|f\|_{-s}\|\sqrt{-\Delta}\overline{\varphi}\|_s\\ &\le C\|f\|_{-s}\,|\varphi|_{\ell,\,\mathcal{S}},\end{aligned}$$

where C is a positive constant and ℓ is a positive integer. $\square$

§6 $\sqrt{-\Delta}$ on weighted Sobolev spaces.

In this section we work in weighted Sobolev spaces. We exploit the decomposition (3.1) again. As to the regular part $\big(1-\chi(D)\big)\sqrt{-\Delta}$, Lemma 2.1 immediately gives a boundedness result in weighted Sobolev spaces.

Lemma 6.1. *Let k and s be in $\mathbf{R}$. Then $\big(1-\chi(D)\big)\sqrt{-\Delta}$ is a bounded operator from $H^{k,\,s}(\mathbf{R}^n)$ to $H^{k-1,\,s}(\mathbf{R}^n)$.*

To investigate action of the singular part $\chi(D)\sqrt{-\Delta}$ in weighted Sobolev spaces, we need some preliminary lemmas, all of which are based on Lemma 5.1.

Lemma 6.2. *If $0 \le s < n/2+1$ and $t < s-n/2$, then*

$$\|\chi(D)\sqrt{-\Delta}\varphi\|_t \le C\|\varphi\|_s \qquad \big(\varphi\in\mathcal{S}(\mathbf{R}^n)\big), \tag{6.1}$$

where C is a constant depending on s and t.

Proof. Let φ be in $\mathcal{S}(\mathbf{R}^n)$, and choose $0<\sigma<1$ so that

$$\sigma < \frac{n}{2}+1-s. \tag{6.2}$$

Then it follows from (5.1) and Lemma 5.1, with $\alpha=0$, that

$$\begin{aligned}\|\chi(D)\sqrt{-\Delta}\varphi\|_t^2 &\le C_\sigma^2\int\langle x\rangle^{2t}\Big\{\int\langle x-y\rangle^{\sigma-n-1}|\varphi(y)|\,dy\Big\}^2dx\\ &\le C_\sigma^2\,\|\varphi\|_s^2\int\langle x\rangle^{2t}\Big\{\int\langle x-y\rangle^{2(\sigma-n-1)}\langle y\rangle^{-2s}\,dy\Big\}dx,\end{aligned}$$

where the Schwarz inequality was used in the second inequality. In view of the inequality (5.15) and the fact that $s\ge 0$, we have

$$\begin{aligned}\langle x-y\rangle^{2(\sigma-n-1)} &= \langle x-y\rangle^{2(\sigma-n-1+s)}\langle x-y\rangle^{-2s}\\ &\le 2^{2s}\langle x-y\rangle^{2(\sigma-n-1+s)}\langle x\rangle^{-2s}\langle y\rangle^{2s}.\end{aligned}$$

Hence we get

$$\|\chi(D)\sqrt{-\Delta}\varphi\|_t^2 \le C_\sigma^2\,2^{2s}\,\|\varphi\|_s^2\int\langle x\rangle^{2(t-s)}\,dx\int\langle x-y\rangle^{2(\sigma-n-1+s)}dy.$$

This implies (6.1), since $2(t-s)<-n$, by assumption of the lemma, and since $2(\sigma-n-1+s)<-n$ by (6.2). $\square$

Lemma 6.3. *If $s\ge n/2+1$ and $t<1$, then the inequality (6.1) holds.*

Proof. By assumption it is possible to choose $\tilde{s} \geq 0$ so that

$$\frac{n}{2} + t < \tilde{s} < \frac{n}{2} + 1. \tag{6.3}$$

Then Lemma 6.2 yields

$$\|\chi(D)\sqrt{-\Delta}\varphi\|_t \leq C\|\varphi\|_{\tilde{s}} \qquad \big(\varphi \in \mathcal{S}(\mathbf{R}^n)\big). \tag{6.4}$$

Since $\tilde{s} < s$ by (6.3) and the assumption of the lemma, it follows from (6.4) that (6.1) holds with the same constant C as in (6.1). $\square$

Lemma 6.4. *If $-n/2 - 1 - |\alpha| < s < 0$ and $t < s - n/2$, then*

$$\|\chi(D)\sqrt{-\Delta}\, D^\alpha \varphi\|_t \leq C\|\varphi\|_s \qquad \big(\varphi \in \mathcal{S}(\mathbf{R}^n)\big),$$

where C is a constant depending on s, t and α.

Proof. Choose $0 < \sigma < 1$ so that

$$\sigma < \frac{n}{2} + 1 + |\alpha| + s. \tag{6.5}$$

It then follows from (5.2) and Lemma 5.1 that

$$\begin{aligned} \|\chi(D)\sqrt{-\Delta}\, D^\alpha \varphi\|_t^2 &\leq C_\sigma^2 \int \langle x\rangle^{2t} \Big\{ \int \langle x-y\rangle^{\sigma-n-1-|\alpha|} |\varphi(y)|\, dy \Big\}^2 dx \\ &\leq C_\sigma^2\, \|\varphi\|_s^2 \int \langle x\rangle^{2t} \Big\{ \int \langle x-y\rangle^{2(\sigma-n-1-|\alpha|)} \langle y\rangle^{-2s}\, dy \Big\} dx. \end{aligned} \tag{6.6}$$

Because $-2s > 0$, the inequality $\langle y\rangle \leq 2\langle x-y\rangle\langle x\rangle$ yields

$$\langle y\rangle^{-2s} \leq 2^{-2s} \langle x-y\rangle^{-2s} \langle x\rangle^{-2s},$$

so that (6.6) implies that

$$\begin{aligned} \|\chi(D)\sqrt{-\Delta}\, D^\alpha \varphi\|_t^2 &\leq C_\sigma^2\, 2^{-2s}\, \|\varphi\|_s^2 \times \\ &\qquad \int \langle x\rangle^{2(t-s)}\, dx \int \langle x-y\rangle^{2(\sigma-n-1-|\alpha|-s)}\, dy. \end{aligned} \tag{6.7}$$

The integrals on the right hand side of (6.7) are finite, since $2(t-s) < -n$ and $2(\sigma - n - 1 - |\alpha| - s) < -n$. $\square$

As a direct consequence of Lemma 6.4, we have the following estimates.

Lemma 6.5. *If $-n/2-1<s<0$ and $t<s-n/2$, then*

$$\|\chi(D)\sqrt{-\Delta}\varphi\|_t \le C\|\varphi\|_s \qquad \bigl(\varphi\in\mathcal{S}(\mathbf{R}^n)\bigr).$$

We now notice the following two facts:

Fact 1. *$\mathcal{S}(\mathbf{R}^n)$ is dense in $H^{k,\,s}(\mathbf{R}^n)$ for all k and s in $\mathbf{R}$.*

Fact 2. *If φ is in $\mathcal{S}(\mathbf{R}^n)$, then $\langle D\rangle^k\varphi$ is also in $\mathcal{S}(\mathbf{R}^n)$ for all k in $\mathbf{R}$ and $\|\langle D\rangle^k\varphi\|_s = \|\varphi\|_{k,\,s}$ for all s in $\mathbf{R}$.*

Combining these two facts with Lemmas 6.2 – 6.5, we immediately get results on the boundedness of the singular part in weighted Sobolev spaces.

Lemma 6.6. *Let k be in $\mathbf{R}$.*

(i) *If $s\ge 0$ and $t<\min\{1,\,s-n/2\}$, then $\chi(D)\sqrt{-\Delta}$ can be extended to a bounded operator from $H^{k,\,s}(\mathbf{R}^n)$ to $H^{k,\,t}(\mathbf{R}^n)$.*

(ii) *If $-n/2-1<s<0$ and $t<s-n/2$, then the same conclusion as in (i) is valid.*

Lemma 6.7. *Let k be in $\mathbf{R}$. Under the same assumption as in Lemma 6.4, $\chi(D)\sqrt{-\Delta}D^\alpha$ can be extended to a bounded operator from $H^{k,\,s}(\mathbf{R}^n)$ to $H^{k,\,t}(\mathbf{R}^n)$.*

Finally we can state the main results on action of $\sqrt{-\Delta}$ on weighted Sobolev spaces.

Theorem 6.8. *Let k be in $\mathbf{R}$.*

(i) *If $s\ge 0$ and $t<\min\{1,\,s-n/2\}$, then $\sqrt{-\Delta}$ can be extended to a bounded operator from $H^{k,\,s}(\mathbf{R}^n)$ to $H^{k-1,\,t}(\mathbf{R}^n)$.*

(ii) *If $-n/2-1<s<0$ and $t<s-n/2$, then the same conclusion as in (i) is valid.*

Proof. Conclusions (i) and (ii) follow immediately from Lemmas 6.6 and 6.1. □

Theorem 6.9. *Let k be in $\mathbf{R}$. If $-n/2-1-|\alpha|<s<0$ and $t<s-n/2$, then $\sqrt{-\Delta}D^\alpha$ can be extended to a bounded operator from $H^{k,\,s}(\mathbf{R}^n)$ to $H^{k-1-|\alpha|,\,t}(\mathbf{R}^n)$.*

Proof. It follows from Lemma 2.1 that $\bigl(1-\chi(D)\bigr)\sqrt{-\Delta}D^\alpha$ is a bounded operator from $H^{k,\,s}(\mathbf{R}^n)$ to $H^{k-1-|\alpha|,\,s}(\mathbf{R}^n)$. This fact, together with Lemma 6.7, gives the conclusion of the theorem. □

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