Absolute Continuity of Periodic Schrödinger Operators with Potentials in the Kato Class

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Abstract. We consider the Schrödinger operator $-\Delta + V$ in $\mathbb{R}^d$ with periodic potential V in the Kato class. We show that, if $d = 2$ or 3, the spectrum of $-\Delta + V$ is purely absolutely continuous.

1991 Mathematics Subject Classification. Primary 35J10.

Key Words and Phrases. Schrödinger Operator, Periodic Potential, Absolute Continuous Spectrum, Kato Class.

1. Introduction

Let V be a real valued measurable function on $\mathbb{R}^d$, $d \geq 2$. V is said to belong to the Kato class K_d if

$$\lim_{r\to 0} \sup_{\mathbf{x}\in\mathbb{R}^d} \int_{|\mathbf{y}-\mathbf{x}|\leq r} \frac{|V(\mathbf{y})|d\,\mathbf{y}}{|\mathbf{y}-\mathbf{x}|^{d-2}} = 0, \qquad \text{for } d \geq 3, \tag{1.1}$$

$$\lim_{r\to 0} \sup_{\mathbf{x}\in\mathbb{R}^d} \int_{|\mathbf{y}-\mathbf{x}|\leq r} |V(\mathbf{y})| \ln\left\{|\mathbf{y}-\mathbf{x}|^{-1}\right\} d\mathbf{y} = 0, \qquad \text{for } d = 2. \tag{1.2}$$

It is well known that, if $V \in K_d$, then the quadratic form associated with $-\Delta + V$ defines a unique self-adjoint operator which we still denote by $-\Delta + V$ [7]. The purpose of this paper is to show that, if $d = 2$ or 3, and $V \in K_d$ is a real periodic function on $\mathbb{R}^d$, then the spectrum of $-\Delta + V$ is purely absolutely continuous.

Main Theorem. *Let $A = (a_{jk})_{d\times d}$ be a symmetric, positive-definite matrix with real constant entries. Let $V \in K_d$ be a real periodic function on $\mathbb{R}^d$. If $d = 2$ or*

[1]Research supported in part by the NSF grant DMS-9732894.

3, *then the spectrum of operator* $DAD^T + V$ *is purely absolutely continuous where* $D = -i\nabla$ *and* $DAD^T = \sum_{j,k} D_j a_{jk} D_k$.

A few remarks are in order.

Remark 1.3. For Schrödinger operator $-\Delta + V$ with periodic potential V, the absolute continuity of the spectrum was established first by L. Thomas [20] in $\mathbb{R}^3$ under the assumption $V \in L^2_{\text{loc}}(\mathbb{R}^3)$. Thomas' result was extended subsequently by M. Reed-B. Simon [13], L. Danilov [4], R. Hempel-I. Herbst [5,6], M. Birman-T. Suslina [1,2,3], A. Morame[12], and A. Sobolev[18]. In particular, the results in [2,3], pertaining to the case $-\Delta+V$, gives the absolute continuity for $V \in L^p_{\text{loc}}(\mathbb{R}^d)$ where $p > 1$ if $d = 2$, $p = d/2$ if $d = 3$ or 4, and $p = d-2$ if $d \geq 5$. In [15], the author established the absolute continuity of $-\Delta + V$ under the condition $V \in L^{d/2}_{\text{loc}}(\mathbb{R}^d)$, $d \geq 3$. This is best possible in the context of the L^p spaces. The case $V \in \text{weak} - L^{d/2}$ was also studied in [15].

Remark 1.4. In [16] the author investigated the periodic Schrödinger operator $-\Delta + V$ with potential V in the Morrey-Campanato class. It is shown that, if $d \geq 3$, $p \in ((d-1)/2, d/2]$, and

$$\limsup_{r\to 0} \sup_{\mathbf{x}\in\Omega} r^2 \left\{ \frac{1}{r^d} \int_{|\mathbf{y}-\mathbf{x}|\leq r} |V(\mathbf{y})|^p \, d\mathbf{y} \right\}^{1/p} < \varepsilon(p, d, \Omega), \tag{1.5}$$

where $\varepsilon(p,d,\Omega) > 0$ and Ω is a periodic cell for V, then $-\Delta+V$ has purely absolutely continuous spectrum. This improves the $L^{d/2}$ as well as the weak-$L^{d/2}$ results in [15]. We point out that the Kato class considered in this paper is not comparable with the Morrey-Campanato class for $d \geq 3$ and $p > 1$. Indeed one can construct a periodic potential V in $\mathbb{R}^3$ such that

$$|V(x)| \sim \frac{1}{|\mathbf{x}'|^2 \, |\ln(|\mathbf{x}'|)|^\delta} \qquad \text{as} \qquad |\mathbf{x}| \to 0 \tag{1.6}$$

where $\mathbf{x}' = (x_2, x_3)$. Then $V \in K_3$ if $\delta > 2$. But V does not satisfy (1.5) since $V \notin L^p_{\text{loc}}(\mathbb{R}^3)$ for any $p > 1$. On the other hand, if

$$|V(\mathbf{x})| \sim \frac{1}{|\mathbf{x}|^2 |\ln(|\mathbf{x}|)|^\delta} \qquad \text{as} \qquad |\mathbf{x}| \to 0, \tag{1.7}$$

then V satisfies (1.5) if $\delta > 0$. However $V \in K_3$ if and only of $\delta > 1$. Clearly in the two-dimensional case, our result improves the L^p $(p > 1)$ result in [2].

Remark 1.8. By a change of coordinates, we may assume that V is periodic with respect to the lattice $(2\pi\mathbb{Z})^d$.

Our main theorem is proved by using the approach of L. Thomas [20] and a new pointwise estimate on the kernel function of certain integral operator. To be more precise, let $\Omega = [0, 2\pi)^d \approx \mathbb{R}^d/(2\pi\mathbb{Z})^d = \mathbb{T}^d$. We consider a family of operators

$$\mathbb{H}_V(z\mathbf{a} + \mathbf{b}) = (\mathbf{D} + z\mathbf{a} + \mathbf{b})A(\mathbf{D} + z\mathbf{a} + \mathbf{b})^T + V, \qquad z \in \mathbb{C} \tag{1.9}$$

defined on $L^2(\mathbb{T}^d)$ with $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^d$ fixed. Using the Floquet decomposition and Thomas' argument, we may reduce the main theorem to the problem of showing that the family of operators $\{\mathbb{H}_V(z\mathbf{a}+\mathbf{b}) : z \in \mathbb{C}\}$ has no common eigenvalues. To this end, we will show that, for some appropriate chosen $\mathbf{a} \in \mathbb{R}^d$,

$$\| \{\mathbb{H}_V(z\mathbf{a} + \mathbf{b})\}^{-1} \|_{L^1(\mathbb{T}^d)\to L^1(\mathbb{T}^d)} \to 0 \qquad \text{as} \qquad \rho \to \infty \tag{1.10}$$

where $< \mathbf{b}, \mathbf{a} >= 0$, $z = \delta + i\rho$, and δ is some fixed number depending on $\mathbf{a}$ and $\mathbf{b}$.

To prove (1.10), the key step is to show that

$$\begin{aligned} &\|V\{\mathbb{H}_0(z\mathbf{a} + \mathbf{b})\}^{-1}\|_{L^1(\mathbb{T}^d)\to L^1(\mathbb{T}^d)} \\ &\leq \begin{cases} C \sup_{\mathbf{x}\in\Omega} \int_\Omega \frac{|V(\mathbf{y})|\,d\mathbf{y}}{|\mathbf{y}-\mathbf{x}|}, & \text{if } d = 3, \\ C \sup_{\mathbf{x}\in\Omega} \int_\Omega \{1 + \big|\ln|\mathbf{x} - \mathbf{y}|\big|\}\,|V(\mathbf{y})|\,d\mathbf{y}, & \text{if } d = 2, \end{cases} \end{aligned} \tag{1.11}$$

where $\mathbb{H}_0(z\mathbf{a}+\mathbf{b}) = (\mathbf{D}+z\mathbf{a}+\mathbf{b})A(\mathbf{D}+z\mathbf{a}+\mathbf{b})^T$. This will be done by establishing the following pointwise estimate on the kernel function $G_\rho(\mathbf{x}, \mathbf{y})$ of the operator $\{\mathbb{H}_0(z\mathbf{a} + \mathbf{b})\}^{-1}$,

$$|G_\rho(\mathbf{x}, \mathbf{y})| \leq \begin{cases} \frac{C}{|\mathbf{x}-\mathbf{y}|}, & \text{if } d = 3, \\ C\left\{1 + \big|\ln|\mathbf{x} - \mathbf{y}|\big|\right\}, & \text{if } d = 2. \end{cases} \tag{1.12}$$

The paper is organized as follows. In sections 2 and 3 we give the proof of the kernel function estimate (1.12). The main theorem will be proved in section 4.

Throughout the rest of this paper we assume that $d = 2$ or 3, $V \in K_d$ is periodic with respect to the lattice $(2\pi\mathbb{Z})^d$. We use $\|\cdot\|_p$ to denote the norm in $L^p(\mathbb{T}^d)$.

Finally we will use C and c to denote positive constants which may depend on the matrix A, which are not necessarily the same at each occurrence.

2. Some Preliminaries

We begin by choosing $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{R}^d$ such that

$$|\mathbf{a}| = 1, \quad \mathbf{a}A = (s_0, 0, \cdots, 0) \quad \text{and } s_0 > 0. \tag{2.1}$$

For $\mathbf{b} = (b_1, \cdots, b_d) \in \mathbb{R}^d$ with $< \mathbf{b}, \mathbf{a} >= 0$ and $|\mathbf{b}| \leq \sqrt{d}$, let

$$\delta = \frac{1}{a_1}\left(\frac{1}{2} - b_1\right). \tag{2.2}$$

Note that $a_1 > 0$ since $\mathbf{a}A\mathbf{a}^T = s_0 a_1 > 0$. We consider the operator

$$\mathbb{H}_0(\mathbf{k}) = (\mathbf{D} + \mathbf{k})A(\mathbf{D} + \mathbf{k})^T \tag{2.3}$$

defined on $L^2(\mathbb{T}^d)$, where

$$\mathbf{k} = (\delta + i\rho)\mathbf{a} + \mathbf{b} \quad \text{and} \quad \rho \geq 1. \tag{2.4}$$

For $\psi \in L^1(\Omega)$, let

$$\hat{\psi}(\mathbf{n}) = \frac{1}{(2\pi)^d} \int_\Omega e^{-i<\mathbf{n},\mathbf{y}>} \psi(\mathbf{y})\, d\mathbf{y}. \tag{2.5}$$

We may write

$$\{\mathbb{H}_0(\mathbf{k})\}^{-1} \psi(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} \frac{\hat{\psi}(\mathbf{n}) e^{i<\mathbf{n},\mathbf{x}>}}{(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T} \tag{2.6}$$

for $\psi \in C^\infty(\mathbb{T}^d)$. Using (2.4) and (2.1), it is easy to see that

$$\begin{aligned} &(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T \\ &= (\mathbf{n} + \delta\mathbf{a} + \mathbf{b})A(\mathbf{n} + \delta\mathbf{a} + \mathbf{b})^T - \rho^2 s_0 a_1 + 2i\rho s_0(n_1 + \delta a_1 + b_1). \end{aligned} \tag{2.7}$$

By (2.2) we have

$$\begin{aligned} |(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T| &\geq 2\rho s_0 |n_1 + \delta a_1 + b_1| \\ &= 2\rho s_0 |n_1 + \frac{1}{2}| \geq \rho s_0, \end{aligned}$$

since n_1 is an integer.

We now choose $\eta \in C^\infty(\mathbb{R}_+)$ such that $\eta(r) = 1$ if $r \geq s_0^2$, and $\eta(r) = 0$ if $0 < r < s_0^2/2$. Then

$$\eta\left(\frac{|(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T|^2}{\rho^2}\right) = 1 \qquad \text{for any } \mathbf{n} \in \mathbb{Z}^d.$$

It follows that

$$\begin{aligned}\{\mathbb{H}_0(\mathbf{k})\}^{-1}\psi(\mathbf{x}) &= \sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{e^{i<\mathbf{n},\mathbf{x}>}\hat{\psi}(\mathbf{n})\eta\left(\frac{|(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T|^2}{\rho^2}\right)}{(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T} \\ &= \int_\Omega G_\rho(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\,d\mathbf{y},\end{aligned} \tag{2.8}$$

where

$$G_\rho(\mathbf{x}) = \frac{1}{(2\pi)^d}\sum_{\mathbf{n}\in\mathbb{Z}^d}\frac{e^{i<\mathbf{n},\mathbf{x}>}\eta\left(\frac{|(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T|^2}{\rho^2}\right)}{(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T}. \tag{2.9}$$

Note that, by the Plancherel theorem, we have $G_\rho \in L^2(\Omega)$ if $d = 2$ or 3.

Let

$$\varphi(\xi,\rho) = \xi A\xi^T - \rho^2 s_0 a_1 + 2i\rho s_0\xi_1, \quad \text{where } \xi = (\xi_1,\ldots,\xi_d) \in \mathbb{R}^d. \tag{2.10}$$

Then

$$h_\rho(\xi) = \frac{\eta\left(\frac{|\varphi(\xi,\rho)|^2}{\rho^2}\right)}{\varphi(\xi,\rho)} \in L^2(\mathbb{R}^d). \tag{2.11}$$

We denote its inverse Fourier transform by $F_\rho(\mathbf{x})$, i.e.,

$$F_\rho(\mathbf{x}) = (h_\rho)^\vee(\mathbf{x}) = \int_{\mathbb{R}^d} e^{i<\mathbf{x},\xi>}h_\rho(\xi)\,d\xi. \tag{2.12}$$

Using the fact that $(-\mathbf{x})^\beta F_\rho(\mathbf{x})$ is the inverse Fourier transform of $\mathbf{D}^\beta h_\rho(\xi)$, one sees that

$$F_\rho(\mathbf{x}) = O\left(\frac{1}{|\mathbf{x}|^N}\right) \qquad \text{as } |\mathbf{x}| \to \infty$$

for any $N \geq 1$. It follows that

$$\frac{\eta\left(\frac{|(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T|^2}{\rho^2}\right)}{(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T} = \frac{\eta\left(\frac{|\varphi(\mathbf{n}+\delta\mathbf{a}+\mathbf{b},\rho)|^2}{\rho^2}\right)}{\varphi(\mathbf{n}+\delta\mathbf{a}+\mathbf{b},\rho)} = h_\rho(\mathbf{n}+\delta\mathbf{a}+\mathbf{b})$$
$$= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i<\mathbf{n}+\delta\mathbf{a}+\mathbf{b},\mathbf{x}>} F_\rho(\mathbf{x})\,d\mathbf{x}$$
$$= \frac{1}{(2\pi)^d}\sum_{\mathbf{m}\in\mathbb{Z}^d}\int_\Omega e^{-i<\mathbf{n}+\delta\mathbf{a}+\mathbf{b},\mathbf{x}+2\pi\mathbf{m}>} F_\rho(\mathbf{x}+2\pi\mathbf{m})\,d\mathbf{x}$$
$$= \frac{1}{(2\pi)^d}\int_\Omega e^{-i<\mathbf{n},\mathbf{x}>}\sum_{\mathbf{m}\in\mathbb{Z}^d} e^{-i<\delta\mathbf{a}+\mathbf{b},\mathbf{x}+2\pi\mathbf{m}>} F_\rho(\mathbf{x}+2\pi\mathbf{m})\,d\mathbf{x}.$$

In view of (2.9), this implies that

$$G_\rho(\mathbf{x}) = \frac{1}{(2\pi)^d}\sum_{\mathbf{m}\in\mathbb{Z}^d} e^{-i<\delta\mathbf{a}+\mathbf{b},\mathbf{x}+2\pi\mathbf{m}>} F_\rho(\mathbf{x}+2\pi\mathbf{m}), \tag{2.13}$$

which is a form of Poisson summation formula [19]. In particular,

$$|G_\rho(\mathbf{x})| \leq \frac{1}{(2\pi)^d}\sum_{\mathbf{n}\in\mathbb{Z}^d}|F_\rho(\mathbf{x}+2\pi\mathbf{n})|. \tag{2.14}$$

To estimate function $F_\rho(\mathbf{x})$, we first note that

$$\varphi(\xi,\rho) = \rho^2\varphi(\frac{\xi}{\rho},1)$$

and

$$h_\rho(\xi) = \frac{\eta\left(\rho^2|\varphi(\frac{\xi}{\rho},1)|^2\right)}{\rho^2\varphi(\frac{\xi}{\rho},1)}.$$

It follows that

$$F_\rho(\mathbf{x}) = (h_\rho)^\vee(\mathbf{x}) = \rho^{d-2}\left(\frac{\eta(\rho^2|\varphi(\cdot,1)|^2)}{\varphi(\cdot,1)}\right)^\vee(\rho\mathbf{x}).$$

Let

$$f_\rho(\mathbf{x}) = \left(\frac{\eta(\rho^2|\varphi(\cdot,1)|^2)}{\varphi(\cdot,1)}\right)^\vee(\mathbf{x}). \tag{2.15}$$

Then

$$F_\rho(\mathbf{x}) = \rho^{d-2}f_\rho(\rho\mathbf{x}). \tag{2.16}$$

Note that

$$
\begin{aligned}
\varphi(\xi,1) &= \sum_{j,k} a_{jk}\xi_j\xi_k - s_0 a_1 + 2is_0\xi_1, \\
|\varphi(\xi,1)|^2 &= \Big|\sum_{j,k} a_{jk}\xi_j\xi_k - s_0 a_1\Big|^2 + 4s_0^2\xi_1^2.
\end{aligned} \tag{2.17}
$$

A direct computation yields the following estimates:

$$
\Big|\frac{\partial^\ell}{\partial \xi_j^\ell}\left\{\frac{1}{\varphi(\xi,1)}\right\}\Big| \le \frac{C_\ell(1+|\xi|)^\ell}{|\varphi(\xi,1)|^{\ell+1}}, \tag{2.18}
$$

$$
\Big|\frac{\partial^\ell}{\partial \xi_j^\ell}\left\{\eta(\rho^2|\varphi(\xi,1)|^2)\right\}\Big| \le C_\ell \rho^\ell \tag{2.19}
$$

for $\ell \ge 0$ and $j = 1, \ldots, d$.

Lemma 2.20. *Let $f_\rho(\mathbf{x})$ be defined by (2.15). Then, for any $\mathbf{x} \in \mathbb{R}^d$,*

$$
|f_\rho(\mathbf{x})| \le \frac{C\rho^3}{|\mathbf{x}|^4}, \qquad \textit{if } \ d=3, \tag{2.21}
$$

$$
|f_\rho(\mathbf{x})| \le \frac{C\rho^4}{|\mathbf{x}|^4}, \qquad \textit{if } \ d=2. \tag{2.22}
$$

Proof. Since $x_j^4 f_\rho(\mathbf{x})$ is the inverse Fourier transform of

$$
\frac{\partial^4}{\partial \xi_j^4}\left\{\frac{\eta(\rho^2|\varphi(\xi,1)|^2)}{\varphi(\xi,1)}\right\},
$$

we have

$$
\begin{aligned}
|x_j^4 f_\rho(\mathbf{x})| &\le \int_{\mathbb{R}^d} \Big|\frac{\partial^4}{\partial \xi_j^4}\left\{\frac{\eta(\rho^2|\varphi(\xi,1)|^2)}{\varphi(\xi,1)}\right\}\Big|\, d\xi \\
&\le C\int_{\mathbb{R}^d} \sum_{\ell=0}^{4} \Big|\frac{\partial^\ell}{\partial \xi_j^\ell}\left\{\frac{1}{\varphi(\xi,1)}\right\}\Big| \cdot \Big|\frac{\partial^{4-\ell}}{\partial \xi_j^{4-\ell}}\{\eta(\rho^2|\varphi(\xi,1)|^2)\}\Big|\, d\xi \\
&\le C\int_{|\varphi(\xi,1)|\ge \frac{c}{\rho}} \frac{(1+|\xi|)^4}{|\varphi(\xi,1)|^5}\, d\xi \\
&\qquad\qquad + C\int_{|\varphi(\xi,1)|\sim\frac{1}{\rho}} \sum_{\ell=0}^{3} \frac{(1+|\xi|)^\ell}{|\varphi(\xi,1)|^{\ell+1}} \cdot \rho^{4-\ell}\, d\xi \\
&\le C\int_{|\varphi(\xi,1)|\ge \frac{c}{\rho}} \frac{(1+|\xi|)^4}{|\varphi(\xi,1)|^5}\, d\xi \\
&\le C\int_{\mathbb{R}^d} \frac{(1+|\xi|)^4}{\left\{|\varphi(\xi,1)| + \frac{1}{\rho}\right\}}\, d\xi.
\end{aligned}
$$

Note that

$$|\varphi(\xi,1)| \sim |\xi_1| + |\xi A\xi^T - \mathbf{a}A\mathbf{a}^T| = |\xi_1| + \big\{|\xi B| + |\mathbf{a}B|\big\}\big||\xi B| - |\mathbf{a}B|\big| \tag{2.23}$$

where $B = \sqrt{A} \geq 0$. With this it is not hard to see that

$$I_1 := \int_{|\xi B|\geq 2|\mathbf{a}B|} \frac{(1+|\xi|)^4}{\left\{|\varphi(\xi,1)| + \frac{1}{\rho}\right\}^5}\, d\xi \leq C\int_{|\xi|\geq c} \frac{|\xi|^4}{|\xi|^{10}}\, d\xi \leq C.$$

Also,

$$\begin{aligned}
I_2 &:= \int_{|\xi B|\leq 2|\mathbf{a}B|} \frac{(1+|\xi|)^4}{\left\{|\varphi(\xi,1)| + \frac{1}{\rho}\right\}^5}\, d\xi \\
&\leq C\int_{|\xi B|\leq 2|\mathbf{a}B|} \frac{d\xi}{\left\{|\xi_1| + \big||\xi B| - |\mathbf{a}B|\big| + \frac{1}{\rho}\right\}^5} \\
&\leq C\int_{|\xi|\leq 2|\mathbf{a}B|} \frac{d\xi}{\left\{|(\xi B^{-1})_1| + \big||\xi| - |\mathbf{a}B|\big| + \frac{1}{\rho}\right\}^5} \\
&\leq C\int_{|\xi|\leq 2|\mathbf{a}B|} \frac{d\xi}{\left\{|\xi_1| + \big||\xi| - |\mathbf{a}B|\big| + \frac{1}{\rho}\right\}^5}
\end{aligned}$$

where the last inequality follows by a rotation.

Now suppose $d = 3$. Then, using spherical coordinates with $\xi_1 = r\cos\theta$, we have

$$\begin{aligned}
I_2 &\leq C\int_0^{2|\mathbf{a}B|} r^2\left\{\int_0^{\pi/2} \frac{\sin\theta\, d\theta}{\left\{r\cos\theta + \big|r - |\mathbf{a}B|\big| + \frac{1}{\rho}\right\}^5}\right\} dr \\
&\leq C\int_0^{2|\mathbf{a}B|} \frac{dr}{\left\{\big|r - |\mathbf{a}B|\big| + \frac{1}{\rho}\right\}^4} \\
&\leq C\int_0^{|\mathbf{a}B|} \frac{dr}{\left\{r + \frac{1}{\rho}\right\}^4} \\
&\leq C\rho^3.
\end{aligned}$$

Similarly, if $d = 2$,

$$\begin{aligned}
I_2 &\leq C\int_0^{2|\mathbf{a}B|} r\left\{\int_0^{\pi/2} \frac{d\theta}{\left\{r\cos\theta + \big|r - |\mathbf{a}B|\big| + \frac{1}{\rho}\right\}^5}\right\} dr \\
&\leq C\int_0^{2|\mathbf{a}B|} \frac{dr}{\left\{\big|r - |\mathbf{a}B|\big| + \frac{1}{\rho}\right\}^5} \\
&\leq C\int_0^{|\mathbf{a}B|} \frac{dr}{\left\{r + \frac{1}{\rho}\right\}^5} \\
&\leq C\rho^4.
\end{aligned}$$

Thus we have proved that, for $j = 1, \cdots, d$,

$$|x_j^4 f_\rho(\mathbf{x})| \le C\{I_1 + I_2\} \le \begin{cases} C\rho^3, & \text{if } d = 3, \\ C\rho^4, & \text{if } d = 2. \end{cases}$$

Estimates (2.21)-(2.22) then follow.

It follows from (2.16) and Lemma 2.20 that, for any $\mathbf{x} \in \mathbb{R}^d$,

$$|F_\rho(\mathbf{x})| \le \frac{C}{|\mathbf{x}|^4} \qquad \text{for } d = 2 \text{ or } 3. \tag{2.24}$$

This will be used to estimate the terms in the right hand side of (2.14) where $|\mathbf{x} + 2\pi\mathbf{n}| \ge 1/2$.

3. Pointwise Estimate of the Kernel Function $G_\rho(\mathbf{x})$

In this section we will show that, if $|\mathbf{x}| \le 1/2$, then

$$|F_\rho(\mathbf{x})| \le \begin{cases} \frac{C}{|\mathbf{x}|}, & \text{if } d = 3, \\ C \ln \frac{1}{|\mathbf{x}|}, & \text{if } d = 2. \end{cases} \tag{3.1}$$

This, together with (2.24) and (2.14), would imply that

$$|G_\rho(\mathbf{x})| \le \begin{cases} C\{1 + \sum_{|\mathbf{x}+2\pi\mathbf{n}| \le 1/2} \frac{1}{|\mathbf{x}+2\pi\mathbf{n}|}\}, & \text{if } d = 3, \\ C\{1 + \sum_{|\mathbf{x}+2\pi\mathbf{n}| \le 1/2} \ln \frac{1}{|\mathbf{x}+2\pi\mathbf{n}|}\}, & \text{if } d = 2. \end{cases} \tag{3.2}$$

To this end, we recall that $F_\rho(\mathbf{x}) = \rho^{d-2} f_\rho(\rho\mathbf{x})$ and write

$$f_\rho(\mathbf{x}) = \left\{ \frac{1}{\varphi(\cdot, 1)} \right\}^\vee (\mathbf{x}) + \left\{ \frac{\eta(\rho^2 |\varphi(\cdot, 1)|^2) - 1}{\varphi(\cdot, 1)} \right\}^\vee (\mathbf{x}). \tag{3.3}$$

Lemma 3.4. *We have*

$$\int_{\mathbb{R}^d} \left| \frac{\eta(\rho^2 |\varphi(\xi, 1)|^2) - 1}{\varphi(\xi, 1)} \right| d\xi \le \frac{C}{\rho}.$$

Proof. Recall that $\eta(r) = 1$ for $r \ge s_0^2$. Thus, as in the proof of Lemma 2.20, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \Big| \frac{\eta(\rho^2|\varphi(\xi,1)|^2)-1}{\varphi(\xi,1)} \Big| \, d\xi &\le C \int_{|\varphi(\xi,1)|\le \frac{c}{\rho}} \frac{d\xi}{|\varphi(\xi,1)|} \\
&\le C \int_{|\xi_1|+\big||\xi|-|\mathbf{a}B|\big|\le \frac{c}{\rho}} \frac{d\xi}{|\xi_1|+\big||\xi|-|\mathbf{a}B|\big|} \\
&\le C \int_{\substack{|\xi_1|\le \frac{c}{\rho} \\ \big||\xi'|-|\mathbf{a}B|\big|\le \frac{c}{\rho}}} \frac{d\xi}{|\xi_1|+\big||\xi'|-|\mathbf{a}B|\big|} \quad \text{where } \ \xi=(\xi_1,\xi') \\
&\le C \int_{\big|r-|\mathbf{a}B|\big|\le \frac{c}{\rho}} \int_{|\xi_1|\le \frac{c}{\rho}} \frac{d\xi_1 dr}{|\xi_1|+\big|r-|\mathbf{a}B|\big|} \\
&\le \frac{C}{\rho} \int_{0<r<c} \int_{|\xi_1|\le c} \frac{d\xi_1 dr}{|\xi_1|+r} \\
&\le \frac{C}{\rho}.
\end{aligned}$$

It follows from Lemma 3.4 that

$$\Big| \left\{ \frac{\eta(\rho^2|\varphi(\cdot,1)|^2)-1}{\varphi(\cdot,1)} \right\}^{\vee} (\mathbf{x}) \Big| \le \frac{C}{\rho}. \tag{3.5}$$

To estimate the first term on the right hand side of (3.3), we first note that, by several changes of variables, we have

$$\left\{ \frac{1}{\varphi(\cdot,1)} \right\}^{\vee} (\mathbf{x}) = \frac{|\mathbf{a}B|^{d-2}}{det(B)} \left\{ \frac{1}{|\xi|^2-1+2i\xi_1} \right\}^{\vee} (|\mathbf{a}B|\mathbf{x}B^{-1}O^{-1}) \tag{3.6}$$

where O is a $d\times d$ orthogonal matrix such that $\mathbf{a}BO^{-1}=(|\mathbf{a}B|,0,\cdots,0)$.

Lemma 3.7. *Let $u(\mathbf{x})$ denote the inverse Fourier transform of $\big\{|\xi|^2-1+2i\xi_1\big\}^{-1}$ in $\mathbb{R}^d$, $d=2$ or 3. Let $\mathbf{x}=(x_1,\mathbf{x}')\in\mathbb{R}^d$. Then*

$$u(\mathbf{x}) = 2\pi \int_0^\infty J_0(|\mathbf{x}'|r)v(r,x_1)r\,dr, \qquad \text{if } \ d=3, \tag{3.8}$$

$$u(\mathbf{x}) = 2 \int_0^\infty \cos(|x_2|r)v(r,x_1)\,dr, \qquad \text{if } \ d=2, \tag{3.9}$$

where

$$v(r,x_1) = \int_{\mathbb{R}} \frac{e^{ix_1\xi_1}}{r^2+\xi_1^2-1+2i\xi_1}\, d\xi_1 \tag{3.10}$$

and

$$J_0(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{it\cos\omega} d\omega \tag{3.11}$$

is the Bessel function of the first kind of order 0.

Proof. One may verify that, for $R>0$,

$$\left\{|\xi|^2-1+2i\xi_1\right\}^{-1}\chi_{\{\xi\in\mathbb{R}^d:|\xi'|\le R\}}\in L^1(\mathbb{R}^d)$$

where $\xi=(\xi_1,\xi')$. Since $\left\{|\xi|^2-1+2i\xi_1\right\}^{-1}\in L^p(\mathbb{R}^d)$ for $3/2<p<2$, by the inequality of Hausdorff-Young [19],

$$u(\mathbf{x})=\lim_{R\to\infty}\int_{\substack{\xi\in\mathbb{R}^d\\|\xi'|\le R}}\frac{e^{i<\mathbf{x},\xi>}}{|\xi|^2-1+2i\xi_1}\,d\xi$$

where the limit is taken in the $L^{p'}$ space. From this (3.8)-(3.9) follow by using Fubini's theorem and polar coordinates. We omit the details.

Lemma 3.12. *Let $v(r,x_1)$ be the function defined by (3.10). Then*

$$v(r,x_1)=\begin{cases}\frac{\pi}{r}e^{x_1-r|x_1|}, & \text{if } r>1,\\ \frac{\pi}{r}\left\{e^{x_1-r|x_1|}-e^{(1-r)x_1}\right\}, & \text{if } 0<r<1.\end{cases}$$

Proof. First we write

$$v(r,x_1)=e^{x_1}\int_{\mathbb{R}}\frac{e^{ix_1(\xi_1+i)}}{r^2+(\xi_1+i)^2}\,d\xi_1.$$

Using Cauchy's integral theorem for function

$$w(z)=\frac{e^{ix_1z}}{r^2+z^2}=\frac{e^{ix_1z}}{(z+ri)(z-ri)},$$

we obtain

$$v(r,x_1)=\begin{cases}e^{x_1}\int_{\mathbb{R}}\frac{e^{ix_1y}}{r^2+y^2}\,dy, & \text{if } r>1,\\ e^{x_1}\int_{\mathbb{R}}\frac{e^{ix_1y}}{r^2+y^2}\,dy-\frac{\pi}{r}e^{(1-r)x_1}, & \text{if } 0<r<1.\end{cases}\tag{3.13}$$

Next we note that

$$\begin{aligned}\int_{\mathbb{R}}\frac{e^{ix_1y}}{r^2+y^2}\,dy&=\frac{1}{r}\int_{\mathbb{R}}\frac{e^{irx_1y}}{1+y^2}\,dy=\frac{1}{r}\int_{\mathbb{R}}e^{irx_1y}\left\{\int_0^\infty e^{-(1+y^2)t}dt\right\}dy\\&=\frac{1}{r}\int_0^\infty e^{-t}\frac{dt}{\sqrt{t}}\left\{\int_{\mathbb{R}}e^{-y^2+\frac{irx_1y}{\sqrt{t}}}dy\right\}=\frac{\sqrt{\pi}}{r}\int_0^\infty e^{-t-\frac{r^2x_1^2}{4t}}\frac{dt}{\sqrt{t}}\\&=\frac{\sqrt{\pi}}{r}\cdot\frac{\sqrt{2}}{(r|x_1|)^{-1/2}}\cdot K_{-1/2}(r|x_1|)\end{aligned}$$

where

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-t-\frac{z^2}{4t}} \frac{dt}{t^{1+\nu}}$$

is the Bessel function of the third kind of order ν [11].

Since

$$K_\nu(z) = K_{-\nu}(z) \quad \text{and} \quad K_{1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z},$$

we obtain

$$\int_{\mathbb{R}} \frac{e^{ix_1 y}}{r^2+y^2}\, dy = \frac{\pi}{r} e^{-r|x_1|}. \tag{3.14}$$

This, together with (3.13), yields the lemma.

Lemma 3.15. *Let $v(r, x_1)$ be the function defined by (3.10). Then*

$$|v(r,x_1)| \le \begin{cases} \frac{\pi}{r} e^{(1-r)\cdot|x_1|}, & \text{if } r > 1, \\ C\left\{e^{(r-1)|x_1|} + |x_1| e^{-\frac{|x_1|}{2}}\right\}, & \text{if } 0 < r < 1. \end{cases}$$

and

$$\left|\frac{\partial v}{\partial r}(r,x_1)\right| \le \begin{cases} \frac{C}{r^2}(1+r|x_1|)e^{(1-r)|x_1|}, & \text{if } r > 1, \\ C\left\{|x_1|^2 e^{-\frac{|x_1|}{2}} + (1+|x_1|)e^{(r-1)|x_1|}\right\}, & \text{if } 0 < r < 1. \end{cases}$$

Proof. We give the proof of the estimate for $|\frac{\partial v}{\partial r}|$ using Lemma 3.12. The estimate for $|v|$ is easier.

If $r > 1$,

$$\frac{\partial v}{\partial r} = -\frac{\pi}{r^2} e^{x_1 - r|x_1|} + \frac{\pi}{r} e^{x_1 - r|x_1|}(-|x_1|).$$

Hence,

$$\left|\frac{\partial v}{\partial r}\right| = \frac{\pi(1+r|x_1|)}{r^2} e^{x_1 - r|x_1|} \le \frac{\pi(1+r|x_1|)}{r^2} e^{(1-r)|x_1|}.$$

Next we suppose $0 < r < 1$. We may assume $x_1 < 0$ since $v(r, x_1) = 0$ if $0 < r < 1$ and $x_1 \ge 0$. Note that, in this case, we have

$$\frac{\partial v}{\partial r} = -\frac{\pi}{r^2}\left\{e^{(1+r)x_1} - e^{(1-r)x_1}\right\} + \frac{\pi x_1}{r}\left\{e^{(1+r)x_1} + e^{(1-r)x_1}\right\}.$$

If $1/2 \le r < 1$, it is easy to see that

$$\begin{aligned} \left|\frac{\partial v}{\partial r}\right| &\le C\left|e^{(1+r)x_1} - e^{(1-r)x_1}\right| + C|x_1|\left\{e^{(1+r)x_1} + e^{(1-r)x_1}\right\} \\ &\le C\left\{1 + |x_1|\right\} e^{(r-1)|x_1|}. \end{aligned}$$

Also, if $0 < r < 1/2$ and $|rx_1| \geq 1$, then $|x_1| \geq 1/r$. It follows that

$$\begin{aligned}|\frac{\partial v}{\partial r}| &\leq C|x_1|^2 \left\{ e^{(1+r)x_1} + e^{(1-r)x_1} \right\} \\ &\leq C|x_1|^2 e^{-\frac{|x_1|}{2}}.\end{aligned}$$

Finally, if $0 < r < 1/2$ and $|rx_1| < 1$, we use $e^t = 1 + t + O(t^2)$ for $|t| < 1$ to obtain

$$\begin{aligned}\frac{\partial v}{\partial r} &= -\frac{\pi}{r^2} e^{x_1} \left\{ 2rx_1 + O((rx_1)^2) \right\} + \frac{\pi x_1}{r} e^{x_1} \left\{ 2 + O((rx_1)^2) \right\} \\ &= \frac{\pi}{r^2} e^{x_1} O((rx_1)^2) + \frac{\pi x_1}{r} e^{x_1} O((rx_1)^2).\end{aligned}$$

It follows that

$$\left| \frac{\partial v}{\partial r} \right| \leq C \left\{ |x_1|^2 e^{x_1} + r|x_1|^3 e^{x_1} \right\} \leq C|x_1|^2 e^{-|x_1|}.$$

The proof is now complete.

Lemma 3.16. *Let $u(\mathbf{x})$ be the inverse Fourier transform of $\{|\xi|^2 - 1 + 2i\xi_1\}^{-1}$ in $\mathbb{R}^d$. Then, if $d = 3$,*

$$|u(\mathbf{x})| \leq \frac{C}{|\mathbf{x}|}$$

and, if $d = 2$,

$$|u(\mathbf{x})| \leq \begin{cases} C \ln \frac{1}{|\mathbf{x}|}, & \text{if } \ |\mathbf{x}| \leq \frac{1}{2}, \\ \frac{C}{|\mathbf{x}|}, & \text{if } \ |\mathbf{x}| > \frac{1}{2}. \end{cases}$$

Proof. We first consider the case $d = 3$.

It follows from (3.8) that

$$\begin{aligned}u(\mathbf{x}) &= 2\pi \int_0^{\frac{1}{|\mathbf{x}'|}} J_0(|\mathbf{x}'|r) v(r, x_1) r \, dr + 2\pi \int_{\frac{1}{|\mathbf{x}'|}}^{\infty} J_0(|\mathbf{x}'|r) v(r, x_1) r \, dr \\ &= I_1 + I_2.\end{aligned}$$

By Lemma 3.15, $|v(r, x_1)|r \leq C$. This, together with the observation $|J_0(t)| \leq 1$, gives

$$|I_1| \leq 2\pi \int_0^{\frac{1}{|\mathbf{x}'|}} |v(r, x_1)| r \, dr \leq \frac{C}{|\mathbf{x}'|}.$$

To estimate I_2, we first assume that $|\mathbf{x}'| \leq 1$. Since

$$rJ_0(r) = \frac{d}{dr} \left\{ rJ_1(r) \right\} \tag{3.17}$$

where $J_\nu(r)$ denotes the Bessel function of the first kind of order ν [11], we may use integration by parts to obtain

$$\begin{aligned} I_2 &= \frac{2\pi}{|\mathbf{x}'|^2}\int_1^\infty r J_0(r) v\big(\frac{r}{|\mathbf{x}'|}, x_1\big)\, dr \\ &= -\frac{2\pi}{|\mathbf{x}'|^2} J_1(1) v\big(\frac{1}{|\mathbf{x}'|}, x_1\big) - \frac{2\pi}{|\mathbf{x}'|^3}\int_1^\infty r J_1(r) \frac{\partial v}{\partial r}\big(\frac{r}{|\mathbf{x}'|}, x_1\big)\, dr. \end{aligned}$$

It then follows from the estimate

$$|J_1(r)| \le \frac{C}{r^{1/2}}, \qquad \text{for } \; r \ge 1 \tag{3.18}$$

[11] and Lemma 3.15 that

$$\begin{aligned} |I_2| &\le \frac{C}{|\mathbf{x}'|} + \frac{C}{|\mathbf{x}'|^3}\int_1^\infty r^{1/2} \big|\frac{\partial v}{\partial r}\big(\frac{r}{|\mathbf{x}'|}, x_1\big)\big|\, dr \\ &\le \frac{C}{|\mathbf{x}'|} + \frac{C}{|\mathbf{x}'|^{3/2}}\int_{\frac{1}{|\mathbf{x}'|}}^\infty r^{1/2} \big|\frac{\partial v}{\partial r}(r, x_1)\big|\, dr \\ &\le \frac{C}{|\mathbf{x}'|} + \frac{C}{|\mathbf{x}'|^{3/2}}\int_{\frac{1}{|\mathbf{x}'|}}^\infty r^{1/2} \cdot \frac{1}{r^2} \cdot \{1 + r|x_1|\}\, e^{(r-1)|x_1|}\, dr \\ &\le \frac{C}{|\mathbf{x}'|}. \end{aligned}$$

If $|\mathbf{x}'| \ge 1$, we write

$$\begin{aligned} I_2 &= \frac{2\pi}{|\mathbf{x}'|^2}\int_1^{|\mathbf{x}'|} r J_0(r) v\big(\frac{r}{|\mathbf{x}'|}, x_1\big)\, dr + \frac{2\pi}{|\mathbf{x}'|^2}\int_{|\mathbf{x}'|}^\infty r J_0(r) v\big(\frac{r}{|\mathbf{x}'|}, x_1\big)\, dr \\ &= I_{21} + I_{22}. \end{aligned}$$

Note that, using (3.17), integration by parts and (3.18), we have

$$\begin{aligned} |I_{21}| &\le \frac{C}{|\mathbf{x}'|^{3/2}} + \frac{C}{|\mathbf{x}'|^3}\int_1^{|\mathbf{x}'|} r^{1/2} \big|\frac{\partial v}{\partial r}\big(\frac{r}{|\mathbf{x}'|}, x_1\big)\big|\, dr \\ &\le \frac{C}{|\mathbf{x}'|^{3/2}} + \frac{C}{|\mathbf{x}'|^{3/2}}\int_{\frac{1}{|\mathbf{x}'|}}^1 r^{1/2} \big|\frac{\partial v}{\partial r}(r, x_1)\big|\, dr \\ &\le \frac{C}{|\mathbf{x}'|^{3/2}} + \frac{C}{|\mathbf{x}'|^{3/2}}\int_0^1 |x_1| e^{(r-1)|x_1|}\, dr \\ &\le \frac{C}{|\mathbf{x}'|^{3/2}} \le \frac{C}{|\mathbf{x}'|}. \end{aligned}$$

Similarly,

$$\begin{aligned}
|I_{22}| &= \frac{2\pi}{|\mathbf{x}'|^2}\Big|\int_{|\mathbf{x}'|}^{\infty}\frac{d}{dr}\,\{rJ_1(r)\}\,v\big(\frac{r}{|\mathbf{x}'|},x_1\big)\,dr\Big| \\
&\le \frac{C}{|\mathbf{x}'|^{3/2}}+\frac{C}{|\mathbf{x}'|^3}\int_{|\mathbf{x}'|}^{\infty} r^{1/2}\Big|\frac{\partial v}{\partial r}\big(\frac{r}{|\mathbf{x}'|},x_1\big)\Big|\,dr \\
&= \frac{C}{|\mathbf{x}'|^{3/2}}+\frac{C}{|\mathbf{x}'|^{3/2}}\int_{1}^{\infty} r^{1/2}\Big|\frac{\partial v}{\partial r}(r,x_1)\Big|\,dr \\
&\le \frac{C}{|\mathbf{x}'|^{3/2}} \le \frac{C}{|\mathbf{x}'|}.
\end{aligned}$$

Thus we have proved that, for any $\mathbf{x}\in\mathbb{R}^3$,

$$|u(\mathbf{x})| \le \frac{C}{|\mathbf{x}'|}. \tag{3.19}$$

To finish the case $d=3$, we still need to show that

$$|u(\mathbf{x})| \le \frac{C}{|x_1|}, \qquad \text{for any } \ \mathbf{x}\in\mathbb{R}^3. \tag{3.20}$$

Clearly, (3.19)-(3.20) implies that $|u(\mathbf{x})|\le C/|\mathbf{x}|$ for any $\mathbf{x}\in\mathbb{R}^3$.

To see (3.20), we use Lemma 3.15 to obtain

$$\begin{aligned}
|u(\mathbf{x})| &\le 2\pi\int_0^{\infty}|v(r,x_1)|r\,dr \\
&\le C\int_0^1\Big\{e^{(r-1)|x_1|}+|x_1|e^{-\frac{|x_1|}{2}}\Big\}\,dr + C\int_1^{\infty}e^{(1-r)|x_1|}\,dr \\
&\le \frac{C}{|x_1|}.
\end{aligned}$$

We now consider the case $d=2$.

By Lemmas 3.7 and 3.5,

$$\begin{aligned}
|u(\mathbf{x})| &= 2\Big|\int_0^{\infty}\cos(|x_2|r)v(r,x_1)\,dr\Big| \\
&\le 2\int_0^{\infty}|v(r,x_1)|\,dr \\
&\le C\int_0^1\Big\{e^{(r-1)|x_1|}+|x_1|e^{-\frac{|x_1|}{2}}\Big\}\,dr + C\int_1^{\infty}e^{(1-r)|x_1|}\frac{dr}{r} \\
&\le C|x_1|e^{-\frac{|x_1|}{2}} + C\int_0^1 e^{-r|x_1|}\,dr + C\int_0^{\infty}e^{-r|x_1|}\frac{dr}{r+1}.
\end{aligned}$$

From this it is not hard to see that

$$|u(\mathbf{x})| \le \begin{cases} C\ln\frac{1}{|x_1|}, & \text{if } \ |x_1|\le\frac12, \\ \frac{C}{|x_1|}, & \text{if } \ |x_1|>\frac12. \end{cases} \tag{3.21}$$

Finally we will show that

$$|u(\mathbf{x})| \le \begin{cases} C \ln \frac{1}{|x_2|}, & \text{if } |x_2| \le \frac{1}{2}, \\ \frac{C}{|x_2|}, & \text{if } |x_2| > \frac{1}{2}. \end{cases} \tag{3.22}$$

The desired estimate for $|u(\mathbf{x})|$ follows easily from (3.21)-(3.22).

To see (3.22), we write

$$\begin{aligned} u(\mathbf{x}) &= 2\int_0^{\frac{1}{|x_2|}} \cos(|x_2|r)v(r,x_1)\,dr + 2\int_{\frac{1}{|x_2|}}^{\infty} \cos(|x_2|r)v(r,x_1)\,dr \\ &= I_3 + I_4 \end{aligned}$$

as in the case of $d = 3$. If $|x_2| > 1/2$, by Lemma 3.15, we have

$$|I_3| \le C\int_0^{\frac{1}{|x_2|}} |v(r,x_1)|\,dr \le C\int_0^{\frac{1}{|x_2|}} dr \le \frac{C}{|x_2|}.$$

Similarly, if $|x_2| \le 1/2$,

$$|I_3| \le 2\int_0^1 |v(r,x_1)|\,dr + 2\int_1^{\frac{1}{|x_2|}} |v(r,x_1)|\,dr \le C + C\int_1^{\frac{1}{|x_2|}} \frac{dr}{r} \le C\ln\frac{1}{|x_2|}.$$

To estimate I_4, we use integration by parts. Suppose $|x_2| \le 1/2$.

$$\begin{aligned} |I_4| &= \frac{2}{|x_2|}\Big|\int_{\frac{1}{|x_2|}}^{\infty} \frac{\partial}{\partial r}\{\sin(|x_2|r)\}\,v(r,x_1)\,dr\Big| \\ &\le C + \frac{C}{|x_2|}\int_{\frac{1}{|x_2|}}^{\infty} \Big|\frac{\partial v}{\partial r}(r,x_1)\Big|\,dr \\ &\le C + \frac{C}{|x_2|}\int_{\frac{1}{|x_2|}}^{\infty} \frac{1}{r^2}\{1 + r|x_1|\}\,e^{(1-r)|x_1|}\,dr \\ &\le C \le C\ln\frac{1}{|x_2|}. \end{aligned}$$

If $|x_2| > 1/2$, then

$$\begin{aligned} |I_4| &\le \frac{2}{|x_2|}\Big|\int_{\frac{1}{|x_2|}}^{1} \frac{\partial}{\partial r}\{\sin(|x_2|r)\}\cdot v(r,x_1)\,dr\Big| \\ &\quad + \frac{2}{|x_2|}\Big|\int_1^{\infty} \frac{\partial}{\partial r}\{\sin(|x_2|r)\}\cdot v(r,x_1)\,dr\Big| \\ &\le \frac{C}{|x_2|} + \frac{C}{|x_2|}\int_{\frac{1}{|x_2|}}^{1} \Big|\frac{\partial v}{\partial r}(r,x_1)\Big|\,dr + \frac{C}{|x_2|}\int_1^{\infty} \Big|\frac{\partial v}{\partial r}(r,x_1)\Big|\,dr \\ &\le \frac{C}{|x_2|}. \end{aligned}$$

(3.22) is proved. The proof of Lemma 3.16 is finally complete.

It follows from Lemma 3.16 and (3.6) that

$$|\left\{\frac{1}{\varphi(\cdot,1)}\right\}^{\vee}(\mathbf{x})| \leq \begin{cases} \frac{C}{|\mathbf{x}|}, & \text{if } d=3, \\ C\ln(1+\frac{1}{|\mathbf{x}|}), & \text{if } d=2. \end{cases} \tag{3.23}$$

This, together with (3.3) and (3.5), implies that

$$|f_\rho(\mathbf{x})| \leq \begin{cases} C\left\{\frac{1}{\rho}+\frac{1}{|\mathbf{x}|}\right\}, & \text{if } d=3, \\ C\left\{\frac{1}{\rho}+\ln(1+\frac{1}{|\mathbf{x}|})\right\}, & \text{if } d=2. \end{cases} \tag{3.24}$$

Thus, by (2.16), for any $\mathbf{x}\in\mathbb{R}^d$,

$$|F_\rho(\mathbf{x})| = \rho^{d-2}|f_\rho(\rho\mathbf{x})| \leq \begin{cases} C\left\{1+\frac{1}{|\mathbf{x}|}\right\}, & \text{if } d=3, \\ C\left\{\frac{1}{\rho}+\ln(1+\frac{1}{\rho|\mathbf{x}|})\right\}, & \text{if } d=2. \end{cases} \tag{3.25}$$

(3.1) now follows from (3.25). Estimate (3.2) is then proved.

4. The Proof of Main Theorem

Suppose $V\in K_d$. It is well known that, for any $\varepsilon>0$, there exists a constant $C_{\varepsilon,V}>0$ such that

$$\int_{\mathbb{R}^d}|g|^2|V|\,d\mathbf{x} \leq \varepsilon\int_{\mathbb{R}^d}|\nabla g|^2\,d\mathbf{x} + C_{\varepsilon,V}\int_{\mathbb{R}^d}|g|^2\,d\mathbf{x} \tag{4.1}$$

for any $g\in H^1(\mathbb{R}^d)$ [7,17]. It follows from (4.1) that the quadratic form associated with $\mathbf{D}A\mathbf{D}^T+V$ generates a unique self-adjoint operator on $L^2(\mathbb{R}^d)$, which we still denote by $\mathbf{D}A\mathbf{D}^T+V$.

Let $\psi\in H^1(\mathbb{T}^d)$ where

$$H^1(\mathbb{T}^d)=\left\{\phi\in L^2(\mathbb{T}^d):\ \phi(\mathbf{x})=\sum_{\mathbf{n}\in\mathbb{Z}^d}a_{\mathbf{n}}e^{i<\mathbf{x},\mathbf{n}>}\ \text{ and }\ \sum_{\mathbf{n}\in\mathbb{Z}^d}|\mathbf{n}|^2|a_{\mathbf{n}}|^2<\infty\right\}.$$

Applying (4.1) to $\psi\widetilde{\eta}$ where $\widetilde{\eta}$ is a C^∞ cut-off function such that $\widetilde{\eta}=1$ on Ω, one obtains

$$\int_\Omega|\psi|^2|V|\,d\mathbf{x} \leq \varepsilon\int_\Omega|\nabla\psi|^2\,d\mathbf{x} + \widetilde{C}_{\varepsilon,V}\int_\Omega|\psi|^2\,d\mathbf{x} \tag{4.2}$$

for any $\varepsilon > 0$. This implies that, for any $\mathbf{k} \in \mathbb{C}^d$, the quadratic form associated with $(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T + V$ on $\mathbb{T}^d$ defines a unique closed operator on $L^2(\mathbb{T}^d)$, which we denote by $\mathbb{H}_V(\mathbf{k})$. Moreover,

$$\text{(4.3)} \quad \text{Domain}\big(\mathbb{H}_V(\mathbf{k})\big) = \{\psi \in H^1(\mathbb{T}^d) : \ \ \mathbb{H}_V(0)\psi = (\mathbf{D}A\mathbf{D}^T + V)\psi \in L^2(\mathbb{T}^d)\,\}\,.$$

Let $\mathbf{a} \in \mathbb{R}^d$ be a vector satisfying (2.1) and

$$L = \left\{ \mathbf{b} \in \mathbb{R}^d : < \mathbf{b}, \mathbf{a} >= 0 \ \text{ and } \ |\mathbf{b}| \leq \sqrt{d} \right\}. \tag{4.4}$$

Proposition 4.5. *If, for every* $\mathbf{b} \in L$, *the family of operators* $\{\mathbb{H}_V(z\mathbf{a}+\mathbf{b}) : \ z \in \mathbb{C}\}$ *has no common eigenvalue, then the spectrum of the operator* $\mathbf{D}A\mathbf{D}^T+V$ *on* $L^2(\mathbb{R}^d)$ *is purely absolutely continuous.*

Proof. See [13] and [15].

Fix $\mathbf{b} \in L$. Let

$$\delta = \frac{1}{a_1}\left(\frac{1}{2} - b_1\right),$$

as in (2.2). We will show that the family of operators $\{\mathbb{H}_V((\delta+i\rho)\mathbf{a}+\mathbf{b}) : \ \rho \geq 1\}$ has no common eigenvalue under the assumption of our main theorem.

We need the following estimate on the norm of $\{\mathbb{H}_0((\delta+i\rho)\mathbf{a}+\mathbf{b})\}^{-1}$ on $L^1(\mathbb{T}^d)$.

Theorem 4.6. *There exists a constant* $C > 0$ *such that*

$$\|\,\{\mathbb{H}_0((\delta+i\rho)\mathbf{a}+\mathbf{b})\}^{-1}\,\|_{L^1(\mathbb{T}^d)\to L^1(\mathbb{T}^d)} \leq \begin{cases} \frac{C\ln(\rho+1)}{\rho^{1/2}}, & \text{if } \ d=3, \\ \frac{C}{\rho^{1/2}}, & \text{if } \ d=2. \end{cases}$$

Proof. In view of (2.8), it suffices to show that

$$\int_\Omega |G_\rho(\mathbf{x})|\,d\mathbf{x} \leq \begin{cases} \frac{C\ln(\rho+1)}{\rho^{1/2}}, & \text{if } \ d=3, \\ \frac{C}{\rho^{1/2}}, & \text{if } \ d=2. \end{cases} \tag{4.7}$$

To this end, note that, by Hölder's inequality, (2.9) and the Plancherel theorem, we have

$$\int_\Omega |G_\rho(\mathbf{x})|\,d\mathbf{x} \le |\Omega|^{1/2}\left\{\int_\Omega |G_\rho(\mathbf{x})|^2 d\mathbf{x}\right\}^{1/2}$$
$$= C\left\{\sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{1}{|(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T|^2}\right\}^{1/2}$$
$$\le C\left\{\sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{1}{\left\{|(\mathbf{n}+\mathbf{b})A(\mathbf{n}+\mathbf{b})^T - \rho^2 a_1 s_0| + \rho|n_1+\frac{1}{2}|\right\}^2}\right\}^{1/2}.$$

The desired estimate (4.7) follows from the proof of [Lemma 3.2, 15]. We omit the details. See the estimate of (3.11) in [15].

The next theorem is a consequence of the pointwise estimate (3.2) of the kernel function G_ρ.

Theorem 4.8. *There exists a constant $C > 0$ such that*

$$\|V\left\{\mathbb{H}_0((\delta+i\rho)\mathbf{a}+\mathbf{b})\right\}^{-1}\|_{L^1(\mathbb{T}^d)\to L^1(\mathbb{T}^d)}$$
$$\le \begin{cases} C\sup_{\mathbf{x}\in\Omega}\int_\Omega \frac{|V(\mathbf{y})|}{|\mathbf{y}-\mathbf{x}|}\,d\mathbf{y}, & \text{if } d=3,\\ C\sup_{\mathbf{x}\in\Omega}\int_\Omega |V(\mathbf{y})|\left\{1+\big|\ln|\mathbf{y}-\mathbf{x}|\big|\right\}d\mathbf{y}, & \text{if } d=2.\end{cases}$$

Proof. Recall that, if $\psi \in C^\infty(\mathbb{T}^d)$,

$$\left\{\mathbb{H}_0((\delta+i\rho)\mathbf{a}+\mathbf{b})\right\}^{-1}\psi(\mathbf{x}) = \int_\Omega G_\rho(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\,d\mathbf{y}.$$

It follows that

$$\|V\left\{\mathbb{H}_0((\delta+i\rho)\mathbf{a}+\mathbf{b})\right\}^{-1}\psi\|_1 \le \int_\Omega |V(\mathbf{x})|\left\{\int_\Omega |G_\rho(\mathbf{x}-\mathbf{y})|\,|\psi(\mathbf{y})|\,d\mathbf{y}\right\}d\mathbf{x}$$
$$\le \sup_{\mathbf{y}\in\Omega}\int_\Omega |V(\mathbf{x})|\,|G_\rho(\mathbf{x}-\mathbf{y})|\,d\mathbf{x}\,\|\psi\|_1.$$

The desired estimate now follows easily from (3.2).

We are now in a position to give the

Proof of Main Theorem. We give the proof for the case $d = 3$. The two dimensional case may be handled in the same manner.

To show that $\{\mathbb{H}_V((\delta+i\rho)\mathbf{a}+\mathbf{b}):\ \rho\geq 1\}$ has no common eigenvalue, one argues by contradiction. Suppose that there exists $E\in\mathbb{R}$ such that, for every $\rho\geq 1$, there exists $\psi_\rho\in \text{Domain}(\mathbb{H}_V((\delta+i\rho)\mathbf{a}+\mathbf{b}))$ such that $\|\psi_\rho\|_2=1$ and

$$\mathbb{H}_V((\delta+i\rho)\mathbf{a}+\mathbf{b})\psi_\rho=E\psi_\rho.$$

Since $\psi_\rho\in H^1(\mathbb{T}^d)$, by Cauchy inequality and (4.2),

$$\int_\Omega |\psi_\rho|\,|V|\,d\mathbf{x}\leq\left\{\int_\Omega |V|\,d\mathbf{x}\right\}^{1/2}\left\{\int_\Omega |\psi_\rho|^2|V|\,d\mathbf{x}\right\}^{1/2}<\infty.$$

It follows that $(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T\psi_\rho=E\psi_\rho-V\psi_\rho\in L^1(\mathbb{T}^d)$.

Let

$$V_N(\mathbf{x})=\begin{cases} V(\mathbf{x}), & \text{if } |V(\mathbf{x})|>N,\\ 0, & \text{if } |V(\mathbf{x})|\leq N.\end{cases}$$

Then

$$\|(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T\psi_\rho\|_1\leq\{|E|+N\}\,\|\psi_\rho\|_1+\|V_N\psi_\rho\|_1. \tag{4.9}$$

By Theorem 4.8,

$$\|V_N\psi_\rho\|_1\leq C\sup_{\mathbf{x}\in\Omega}\int_\Omega\frac{|V_N(\mathbf{y})|}{|\mathbf{y}-\mathbf{x}|}\,d\mathbf{y}\cdot\|(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T\psi_\rho\|_1. \tag{4.10}$$

Note that

$$\sup_{\mathbf{x}\in\Omega}\int_\Omega\frac{|V_N(\mathbf{y})|}{|\mathbf{y}-\mathbf{x}|}\,d\mathbf{y}\leq\sup_{\mathbf{x}\in\Omega}\int_{|\mathbf{y}-\mathbf{x}|\leq r}\frac{|V(\mathbf{y})|}{|\mathbf{y}-\mathbf{x}|}\,d\mathbf{y}+\frac{1}{r}\int_\Omega|V_N(\mathbf{y})|\,d\mathbf{y}.$$

It follows that

$$\lim_{N\to\infty}\sup_{\mathbf{x}\in\Omega}\int_\Omega\frac{|V_N(\mathbf{y})|}{|\mathbf{y}-\mathbf{x}|}\,d\mathbf{y}\leq\lim_{r\to 0}\sup_{\mathbf{x}\in\Omega}\int_{|\mathbf{y}-\mathbf{x}|\leq r}\frac{|V(\mathbf{y})|}{|\mathbf{y}-\mathbf{x}|}\,d\mathbf{y}=0.$$

This implies that, if N is sufficiently large,

$$\|V_N\psi_\rho\|_1\leq\frac{1}{2}\|(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T\psi_\rho\|_1. \tag{4.11}$$

In view of (4.9) and (4.11), we obtain

$$\|(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T\psi_\rho\|_1\leq 2(|E|+N)\|\psi_\rho\|_1.$$

This, together with Theorem 4.6, gives

$$\frac{C\rho^{1/2}}{\ln(\rho+1)}\|\psi_\rho\|_1\leq 2(|E|+N)\|\psi_\rho\|_1$$

or

$$\frac{C\rho^{1/2}}{\ln(\rho+1)}\leq 2(|E|+N)$$

for any $\rho\geq 1$. This is impossible if we let $\rho\to\infty$.

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