On the application of one M.G.Krein's result to the spectral analysis of Sturm-Liouville operators.

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Abstract

Discovered by M.G.Krein analogy between polynomials orthogonal on the unit circle and generalized eigenfunctions of certain differential systems is used to obtain some new results in spectral analysis of Sturm-Liouville operators.

Section A.

In this section we remind some results obtained by M.G.Krein in his famous article [1]. In this paper author develops the "theory of polinomials, orthogonal on the positive half-line". And this polinoms are constructed from exponents rather than from the powers of independent variable. It's well known that there are many ways to construct the system of orthogonal polinomials on the unit circle. One of them is to start from the moments matrix. This way was chosen by M.G.Krein to obtain his results for positive half-line.

Let's assume that $H(t)=\overline{H(-t)}$ - function summable on each segment $(-r,r)$.

Proposition. *If for any continuous $\varphi(t)$ the following inequaliy holds*

$$\int_0^r |\varphi(s)|^2\,ds+\int_0^r\int_0^r H(t-s)\varphi(t)\overline{\varphi(s)}dtds\geq 0 \tag{1}$$

for each $r>0$, so, and in this case only, there exists the non-decreasing function $\sigma(\lambda)$ ($\lambda\in R,\sigma(0)=0,\sigma(\lambda-0)=\sigma(\lambda)$), such that

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{d\sigma(\lambda)}{1+\lambda^2} &< \infty\\ \int_0^t (t-s)H(s)ds &= \int_{-\infty}^{\infty}(1+\frac{i\lambda t}{1+\lambda^2}-e^{i\lambda t})\frac{d\sigma(\lambda)}{\lambda^2}+(i\gamma-\frac{sign(t)}{2})t\end{aligned}$$

where γ is real constant.

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If in addition we presume that the equality in (1) is possible for $\varphi = 0$ only then the Hermit kernel $H(t-s)$, $(0 \leq t, s \leq r)$ has Hermit resolvent $\Gamma_r(s,t) = \overline{\Gamma_r(s,t)}$ that satisfies the relation

$$\Gamma_r(t,s) + \int_0^r H(t-u)\Gamma_r(u,s)du = H(t-s) \ (0 \leq s, t \leq r)$$

The continuous analogues of polinomials orthogonal on the unit circle are defined by the formula

$$\begin{aligned} P(r,\lambda) &= e^{i\lambda r}(1 - \int_0^r \Gamma_r(s,0)e^{-i\lambda s}ds), \\ P_*(r,\lambda) &= 1 - \int_0^r \Gamma_r(0,s)e^{i\lambda s}ds, \ r \geq 0. \end{aligned}$$

Using the well known properties of resolvents we obtain the following system

$$\begin{aligned} \frac{dP(r,\lambda)}{dr} &= i\lambda P(r,\lambda) - \overline{A(r)}P_*(r,\lambda), \\ \frac{dP_*(r,\lambda)}{dr} &= -A(r)P(r,\lambda), \end{aligned} \tag{2}$$

where $A(r) = \Gamma_r(0,r)$.

Proposition. *For each finite $f(x) \in L^2(R^+)$ we have the following equality*

$$\|f\|_2^2 = \int_{-\infty}^{\infty} |F_P(\lambda)|^2 \, d\sigma(\lambda), \text{where } F_P(\lambda) = \int_0^{\infty} f(r)P(r,\lambda)dr.$$

Consequently we have the isometric mapping U_P from $L^2(R^+)$ into $L^2(\sigma, R)$.

Theorem. *The mapping U_P is unitary if and only if the following integral diverges (equals to $-\infty$)*

$$\int_{-\infty}^{\infty} \frac{\ln \sigma'(\lambda)}{1+\lambda^2} d\lambda. \tag{3}$$

Theorem. *The following statements are equivalent*

(1) The integral (3) *is finite.*

(2) At least for some λ, $\Im\lambda > 0$ the integral

$$\int_0^{\infty} |P(r,\lambda)|^2 \, dr \tag{4}$$

converges.

(3) At least for some λ ($\Im\lambda > 0$) the function $P_*(r,\lambda)$ is bounded.

(4) On any compact set in the open upper half-plane integral (4) *converges uniformly. That is equivalent to the existence of uniform limit $\Pi(\lambda) = \lim_{r\to\infty} P_*(r,\lambda)$.*

It's easy to verify that in cases $A(r) \in L^1(R^+)$, $A(r) \in L^2(R^+)$ the conditions (1)-(4) are satisfied. What is more, in the first case measure σ is continuously differentiable with certain estimates for its derivative.

Consider $E(r,\lambda) = e^{-i\lambda r}P(2r,\lambda) = \Phi(r,\lambda) + i\Psi(r,\lambda)$. Let $E(-r,\lambda) = \overline{E(r,\lambda)} = \Phi(r,\lambda) - i\Psi(r,\lambda)$. From (2) we infer that

$$\begin{aligned} \frac{d\Phi}{dr} &= -\lambda\Psi - a(r)\Phi + b(r)\Psi, \ \Phi(0,\lambda) = 1; \\ \frac{d\Psi}{dr} &= \lambda\Phi + b(r)\Phi + a(r)\Psi, \ \Psi(0,\lambda) = 0. \end{aligned}$$

where $a(r) = 2\Re A(2r), \ b(r) = 2\Im A(2r)$.

Proposition. *The mapping $U_E : f(r) \to F_E(\lambda) = \int\limits_{-\infty}^{\infty} f(r)E(r,\lambda)dr$, defined on the finite functions $f(r) \in L^2(R)$, generates the unitary operator from $L^2(R)$ onto $L^2(\sigma, R)$.*

The trivial case $a = b = 0$ yields $E(r,\lambda) = e^{i\lambda r}, \ \Psi(r,\lambda) = \sin(r\lambda), \ \Phi(r,\lambda) = \cos(r\lambda)$.

In case when $H(t)$ is real, the function $\sigma(\lambda)$ is odd. Consequently $b(r) = 0$. Assuming that $H(t)$ is absolutely continuous, we have that Φ and Ψ are solutions of the equations

$$\begin{array}{lll} \Psi'' - q\Psi + \lambda^2\Psi = 0, & \Psi(0) = 0, & \Psi'(0) = \lambda; \\ \Phi'' - q_1\Phi + \lambda^2\Phi = 0, & \Phi(0) = 1, & \Phi'(0) + a(0)\Phi(0) = 0, \end{array} \tag{5}$$

where $q_1(x) = a^2(x) - a'(x)$ and $q(x) = a^2(x) + a'(x)$.

Section B.

Consider the Sturm-Liouville operator on the half-line with Dirichlet boundary condition at zero

$$l(u) = -u'' + qu, \ u(0) = 0. \tag{6}$$

Let's assume that real-valued $q(x)$ admits the following representation $q(x) = a^2(x) + a'(x)$ where $a(x)$ is absolutely continuous function on the half-line. That means that $a(x)$ is the solution of the Ricatti equation.

Consider also the corresponding differential Dirac-type system (see [2] or [3])

$$\left\{ \begin{array}{ll} \Phi'(x,\lambda) = & -\lambda\Psi(x,\lambda) - a(x)\Phi(x,\lambda) \\ \Psi'(x,\lambda) = & \lambda\Phi(x,\lambda) + a(x)\Psi(x,\lambda) \end{array} \right. , \tag{7}$$

where $\Phi(0,\lambda) = 1, \Psi(0,\lambda) = 0$.

From the result stated in Section A it follows that the spectral measure $\rho(\lambda)$ of problem (6) is connected with the spectral measure $\hat{\sigma}(\lambda)$ of system (7) [1] by the following relation

$$\rho(t) = 2\int\limits_0^{\sqrt{t}} \alpha^2 d\hat{\sigma}(\alpha). \tag{8}$$

From this and results stated in Section A we can infer one very simple but significant corollary

Corollary. *If $q(x)$ is real-valued function such that*

[1] See the definition of spectral measure for the differential system in [3]. Here the function $\hat{\sigma}(\lambda)$ is connected with $\sigma(\lambda)$ from the section A by the relation: $\hat{\sigma}(\lambda) = 2\sigma(\lambda)$.

$$\sup_{x\in R}\int_x^{x+1}|q(s)|^2\,ds<\infty, \tag{9}$$

the improper integral $W(x)=\int_x^\infty q(s)ds$ *exists and satisfies the condition* $W(x)\in L^2(R^+)$, *then the absolutely continuous part of spectrum of operator* H_h, *generated by differential expression* $l(u)=-u''+qu$ *and boundary condition* $u(0)=hu'(0)$, $(h\in R\cup\infty\ )$ *fills the whole positive half-line.*

Proof. Let's consider (7) with $a(x)=-W(x)$. The spectral measure of system (7) with chosen $a(x)$ has the needed property. Consequently the Sturm-Liouville operator with potential $q^*(x)=a'+a^2$ and Dirichlet boundary condition also has the a.c. component which fills the whole positive half-line. But initial potential $q(x)$ differs from $q^*(x)$ by $L^1(R^+)$ term only. Consequently, Kuroda's theorem [4] guarantees that operator H_h has the needed property for $h=0$. But it means that this statement is true for any h since the essential support of a.c. component doesn't depend on h. That follows, for example, from the subordinate solutions theory.$\square$

In the next theorems of this section we will show how Krein's results will help to establish asymptotics for generalized eigenfunctions and analyze the spectrum of some Sturm-Liouville operators.

Theorem 1. *If* $q(x)$ *is real-valued function such that*

$$\sup_{x\in R}\int_x^{x+1}\min\{0,q(s)\}ds>-\infty, \tag{10}$$

the improper integral $W(x)=\int_x^\infty q(s)ds$ *exists and satisfies the condition* $|W(x)|\le\frac{\gamma}{x+1}$, $(0<x,0<\gamma<1/4)$, *then operator* H *generated by* (6) *is non-negative and*

$$\left|\int_0^\infty\frac{\ln\rho'(\lambda)}{\sqrt{\lambda}(1+\lambda)}d\lambda\right|<\infty. \tag{11}$$

What is more for a.e. positive spectral parameter the generalized eigenfunctions of differential expression $l(u)=-u''+qu$ *has the following asymptotic* $u(x,\lambda,\alpha)=C(\lambda,\alpha)\sin(x\lambda+\varphi(\lambda,\alpha))+\overline{o}(1)$, $u(0,\lambda,\alpha)=\cos(\alpha)$, $u'(0,\lambda,\alpha)=\sin(\alpha)$. *The spectrum of operator* H, *generated by (6), is purely absolutely continuous on the positive half-line.*

Proof.

We will separate the proof on two parts.

1. Reducing to the system.

Let's consider Ricatti equation $q(x)=a^2(x)+a'(x)$. We will find solution which is absolutely continuous, tends to zero at the infinity and belongs to the class $L^2(R^+)$. Integrating we will get the following nonlinear integral equation $\int_x^\infty a^2(s)ds-a(x)=W(x)$. Our goal is to study operator B: $Bf(x)=\int_x^\infty f^2(s)ds-W(x)$ that acts in complete metric space Ω of measurable functions which admit the estimate $|g(x)|\le\frac{æ}{x+1}$, $(æ=\frac{1-\sqrt{1-4\gamma}}{2}\ )$.

The metric is introduced by the formula $\rho(g_1, g_2) = ess\sup_{x\geq 0}\{(x+1)|g_1(x) - g_2(x)|\}$. To use the contraction operators principle one should verify that the following conditions hold

1. B is acting from Ω to Ω,
2. B is contraction operator.

It is not difficult to show that the both conditions are satisfied. Really

$$|Bf| \leq \frac{\gamma}{x+1} + \int\limits_x^\infty \frac{æ^2}{(s+1)^2} ds = \frac{æ^2 + \gamma}{x+1} = \frac{æ}{x+1},$$

$$|Bg_1 - Bg_2| \leq \int\limits_x^\infty |g_1 - g_2|\,|g_1 + g_2|\,ds \leq \rho(g_1, g_2)\int\limits_x^\infty \frac{2æ}{(s+1)^2} ds = \frac{2æ}{x+1}\rho(g_1, g_2),$$

that means $\rho(Bg_1, Bg_2) \leq 2æ\rho(g_1, g_2)$ which implies the contraction property since $2æ < 1$. Thus we have the single fixed point $a(x) \in \Omega$, so that $Ba = a$. Certainly this function satisfies the Ricatti equation as well. So it suffices to use Proposition and (8) to obtain (11).

2. Absence of singular component.

Consider the system (7). From Section A we know that function

$$P(x,\lambda) = \exp(i\lambda\frac{x}{2})\left(\Phi(x/2,\lambda) + i\Psi(x/2,\lambda)\right) \tag{12}$$

satisfies the following system

$$\begin{cases} \frac{dP}{dx} &= i\lambda P - AP_*, \\ \frac{dP_*}{dx} &= -AP. \end{cases}$$

Where $P(0,\lambda) = P_*(0,\lambda) = 1$, and $A(x) = \frac{1}{2}a(\frac{x}{2})$. Let's introduce the following function $Q(x) = e^{-i\lambda x}P(x)$. So we will have

$$\begin{cases} \frac{dQ}{dx} = & -Ae^{-i\lambda x}P_*, \\ \frac{dP_*}{dx} = & -Ae^{i\lambda x}Q. \end{cases}$$

$P_*(0,\lambda) = Q(0,\lambda) = 1$. It's easy to see that $Q = \overline{P_*}$. Consequently $Q(x) = 1 - \int\limits_0^x A(s)e^{-i\lambda s}\overline{Q(s)}ds$; $|Q(x)| \leq 1 + \int\limits_0^x |A(s)|\,|Q(s)|\,ds$. Gronuol Lemma yields the estimate $|Q(x)| \leq \exp\left(\int\limits_0^x |A(s)|\,ds\right) \leq \left(\frac{x+2}{2}\right)^æ$. From (5) it follows that $u(x,\lambda) = \frac{\Psi(x,\lambda)}{\lambda}$ $(\lambda \neq 0)$ satisfies the conditions $-u'' + qu = \lambda^2 u$, $u(0,\lambda) = 0$, $u'(0,\lambda) = 1$. From (12) we have $|u(x,\lambda)| \leq \frac{(x+1)^æ}{2^æ\lambda}$, ($æ < 1/2$). The similar estimate can be proved for linear independent solution $v(x,\lambda)$ such that $v(0,\lambda) = 1$, $v'(0,\lambda) = 0$. Indeed it suffices to consider second equation of (5) letting $q_1 = q$, solve equation $q = b^2 - b'$ and repeat the same arguments to obtain the desired inequality for linear independent solution $w(x,\lambda)$ which satisfies the condition $w'(0) + b(0)w(0) = 0$. Since $v(x,\lambda)$ is linear combination of $u(x,\lambda)$ and $w(x,\lambda)$ we have the needed estimate. In the same way derivatives u', v' can be estimated. It follows from the inequality $|Q^{'}| = |A||Q| \le \frac{æ}{2^{æ}(2+x)^{1-æ}}$ and (12). Consequently from the constancy of Wronskian $W(u,v)$ and equivalence of $\int\limits_x^{x+1} u^2(s,\lambda)ds$ and $\int\limits_x^{x+1} u^{'2}(s,\lambda)ds$ [2] we get the inequalities

$$
\begin{aligned}
C_2(\lambda)x^{1-2æ} &\le \int\limits_0^x u^2(s,\lambda)ds \le C_1(\lambda)x^{2æ+1}, \\
C_2(\lambda)x^{1-2æ} &\le \int\limits_0^x v^2(s,\lambda)ds \le C_1(\lambda)x^{2æ+1},
\end{aligned}
$$

where the constants $C_1,\ C_2$ are positive. So we can find $\zeta > 0$ so that $\left(\int\limits_0^x u_1^2(s,\lambda)ds\right)\left(\int\limits_0^x u_2^2(s,\lambda)ds\right)^{-\zeta} \to \infty$ for any two linearly independent solutions u_1, u_2 of equation from (6).

The refined subordinacy theory [5], [6] yields that there is $\eta(\lambda) > 0$ so that $(D_\eta \rho)(\lambda^2) = \overline{\lim}_{\varepsilon\to 0} \frac{\rho(\lambda^2-\varepsilon,\lambda^2+\varepsilon)}{(2\varepsilon)^\eta} = 0$. Consequently ρ gives zero weight to every $\Omega \subset R^+$ with $\dim \Omega = 0$. On the other hand we will prove that u_1, u_2 might be unbounded at the infinity only on the λ set with the zero Hausdorff dimension. Consider $Q(x,\lambda):\ Q^{'} = -Ae^{-i\lambda x}\overline{Q(x)},\ Q(0) = 1.$

So

$$
\begin{aligned}
Q(x) &= 1 - \int\limits_0^x A(s)e^{-i\lambda s}\overline{Q(s)}ds = 1 + \int\limits_0^x \left(\int\limits_s^\infty A(\tau)e^{-i\lambda\tau}d\tau\right)^{'} \overline{Q(s)}ds = \\
& 1 + \left(\int\limits_s^\infty A(\tau)e^{-i\lambda\tau}d\tau\right)\overline{Q(s)}\Bigg|_{s=0}^{s=x} - \int\limits_0^x \left(\int\limits_s^\infty A(\tau)e^{-i\lambda\tau}d\tau\right)\overline{Q^{'}(s)}ds, \qquad (13)
\end{aligned}
$$

where λ is such that integral $\int\limits_0^\infty A(\tau)e^{-i\lambda\tau}d\tau$ converges. The last term in (13) can be rewritten as $\int\limits_0^x \left(\int\limits_s^\infty A(\tau)e^{-i\lambda\tau}d\tau\right) A(s)e^{i\lambda s}Q(s)ds$. Finally we get

$$
Q(x) = J(x) - \int\limits_0^x \left(\int\limits_s^\infty A(\tau)e^{-i\lambda\tau}d\tau\right) A(s)e^{i\lambda s}Q(s)ds,
$$

where $J(x) = C(\lambda) + o(1)\overline{Q(x)}$. From argument used in [7] (Theorem 1.3, 1.4) it follows that the set Ξ of λ for which $\left(\int\limits_s^\infty A(\tau)e^{i\lambda\tau}d\tau\right) A(s)e^{-i\lambda x} \notin L^1(R^+)$ has zero Hausdorff dimension. But one can easily verify that this fact leads to the boundedness of $Q(x)$ at the infinity for $\lambda \notin \Xi$. From formula (12) it follows that generalized eigenfunctions $u(x,\lambda)$ which correspond to the Dirichlet boundary condition at zero are bounded at the infinity for $\lambda \notin \Xi$. At the same way we can prove that the linear independent solution $v(x,\lambda)$ is bounded at the infinity if $\lambda \notin \Upsilon$ ($\dim \Upsilon = 0$). Consequently

[2] It's the only place in the whole proof where we use the condition (10).

results obtained by Stolz [8] guarantee that the support of singular measure has the zero Hausdorff dimension. Meanwhile as we have shown above spectral measure gives the zero weight to any set of zero Hausdorff measure. So the singular spectrum is absent.□

Remark 1. We could have got rid of the condition (10) and prove the absence of positive eigenvalues by making use of Hardy inequality for the equation $Q(x) = \int\limits_x^\infty A(s)e^{-i\lambda s}\overline{Q(s)}ds$ which follows from $Q' = -Ae^{-i\lambda x}\overline{Q}$ and $Q(\infty) = 0$. Really, it would mean the absence of nonzero eigenvalues for differential system (7). By (8) we would have the absence of positive eigenvalues for (6).

Remark 2. Conditions of Theorem 1 are often fulfilled for potentials that oscillate at the infinity (see [9], [10] and bibliography there). We used method different from common ones such as modified Prüfer transform, $I + Q$ asymptotic integration and so on.

Remark 3. If we consider potentials which satisfy the estimate $|q_\varepsilon(x)| \le \frac{\varepsilon}{x+1}$, then the point spectrum may occur on $[0, \frac{4\varepsilon^2}{\pi^2}]$ (see [11], [7]) for any $\varepsilon > 0$. The situation for potentials considered in Theorem 1 is different. Really, the von Neumann -Wigner example $q_{vNW} = 8\frac{\sin 2x}{x} + O(\frac{1}{x^2})$ shows that the condition

$$\left|\int\limits_x^\infty q_\gamma(s)ds\right| \le \frac{\gamma}{x+1} \tag{14}$$

doesn't guarantee the absence of the positive eigenvalues for γ large enough. Meanwhile, for small γ $(0 < \gamma < 1/4)$ the singular spectrum disappears on the whole positive half-line. [3]

Remark 4. From the proof of the Theorem 1 we can infer that the asymptotics for generalized eigenfunctions may not be true on the set of zero Hausdorff dimension only. And this statement holds even without the constraint (10). In [12] the following simple statement is proved

If $q(x)$ is continuous function which admits the representation $q = \frac{\partial W}{\partial x}$, $W \in L^1(1, \infty)$ then equation $-\varphi'' + q\varphi = k^2\varphi$ doesn't have nontrivial square integrable solutions for $k \ne 0$. What is more, if $|W(x)| \le C|x|^{-1-\varepsilon}$ then for any $k \ne 0$ there exists the solution $\varphi(x, k)$ of the same equation which has the following asymptotic $\left|\varphi(x, k) - e^{-ixk}\right| \le C(k)\,|x|^{-\varepsilon}$, $x \ge 1$, where $|C(k)| < C(k_0)$ for $|k| \ge k_0 > 0$.

We improved this result in power scale in some extent using method from [7].

Theorem 2. *If $q(x)$ is real-valued function such that the improper integral $W(x) = \int\limits_x^\infty q(s)ds$ exists and satisfies the condition $W(x) \in L^p(R^+) \cap L^2(R^+)$, $(1 < p < 2)$, then for a.e. positive spectral parameter the generalized eigenfunctions has the following asymptotic $u(x, \lambda, \alpha) = C(\lambda, \alpha)\sin(x\lambda + \varphi(\lambda, \alpha)) + \overline{o}(1)$, where $u(0, \lambda, \alpha) = \cos(\alpha)$, $u'(0, \lambda, \alpha) = \sin(\alpha)$.*

Proof.

Let's consider the system (7) with $a(x) = -W(x)$. Introducing $P(x, \lambda)$ by (12) and $Q(x, \lambda) = e^{-i\lambda x}P(x)$ we have the following equation for $Q(x, \lambda)$:

$$Q' = -Ae^{-i\lambda x}\overline{Q}, \tag{15}$$

where $Q(0, \lambda) = 1, A(x) = \frac{1}{2}a(\frac{x}{2})$. So we can see that $Q(x, \lambda) = 1 - \int\limits_0^x A(s)\overline{P(s, \lambda)}ds = 1 - \overline{\int\limits_0^x A(s)P(s, \lambda)ds}$. Since $P(x, \lambda)$ play the role of orthogonal polynomials, we can expect that the

[3] It's interesing to find out is the constant 1/4 optimal or not.

analogue of Menchoff theorem for orthonormal systems [13] will guarantee the convergence of the last integral almost everywhere.

But for our purpose it's more convenient not to derive the generalization of Menchoff results for $P(x,\lambda)$ system but to give the reference to the very recent results of M.Christ, A.Kiselev [14] Really, for the solution of equation (15) we have the following formal series:

$$
\begin{aligned}
Q(x,\lambda) &= Q_\infty(\lambda)(1+\int_x^\infty A(s_1)e^{-i\lambda s_1}ds_1+\dots \\
&+\int_x^\infty A(s_1)e^{-i\lambda s_1}\int_{s_1}^\infty A(s_2)e^{i\lambda s_2}\dots\int_{s_{j-1}}^\infty A(s_j)e^{(-1)^j\lambda s_j}ds_j\dots ds_1+\dots).
\end{aligned}
$$

The convergence of this kind of series for a.e. λ w.r.t. Lebesgue measure was established in [14] for $A(s)\in L^p(R^+)$, $1\le p<2$. By the methods of this paper we can show that $Q(x,\lambda)$ satisfies the equation (15) for a.e. λ. Thus we have that a.e. $Q(x,\lambda)$ is bounded at the infinity. Let's consider now the Sturm-Liouville operator on the half-line with potential $q^*(x)=a'+a^2$ with Dirichlet boundary condition at zero. From (5) and (12) we can infer that the generalized eigenfunctions of this equation which satisfy the Dirichlet condition at zero are bounded at the infinity for a.e. positive spectral parameter.

It's obvious that $q^*(x)$ can be represented in the following form $q^*(x)=-T'+T^2$, where $T=W$. Repeating the same argument for Dirac-type system (7) with $a(x)=T(x)$ we see, that generalized eigenfunctions, which satisfy the conditions $\Phi(0)=1$, $\Phi'(0,\lambda)+T(0)\Phi(0,\lambda)=0$ are bounded at the infinity as well. At the same way we can show that the whole transfer matrix of Sturm-Liouville equation with potential $q^*(x)$ is bounded at the infinity for a.e. positive spectral parameter. But $q^*(x)=W^2-W'=W^2+q$, so it differs from the initial potential q by the $W^2\in L^1(R^+)$ term only. It's easy to show, that if the transfer matrix of S.L. operator is bounded for some λ, so the transfer matrix of operator obtained by the $L^1(R^+)$ perturbation is bounded also. Consequently the transfer matrix of the initial S.L. operator is bounded for a.e. positive spectral parameter. □

Remark 1. The fulfillment of the conditions of Theorem 2 doesn't mean that the singular component of spectrum is absent. Moreover the von Neumann- Wigner potential illustrates that at least the positive eigenvalue may occur.

Now we will show that conditions of Theorem 2 are satisfied under some assumption imposed on the cos -transform of potential. From then on we suppose that $q(x)$ is such that

(A) its cos -transform $\widehat{q(\omega)}=\lim_{N\to\infty}\int_0^N q(x)\cos(\omega x)dx$ exists in $L^1_{loc}(R^+)$ sence.

(B) $\frac{2}{\pi}\lim_{N\to\infty}\int_0^N\widehat{q(\omega)}\cos(\omega x)d\omega=q(x)$ in $L^1_{loc}(R^+)$.

Theorem 3. *If $q(x)$ is such that (A) and (B) are satisfied and $\widehat{q(\omega)}=\widehat{q(0)}+\widehat{\varphi(\omega)}$ where $\frac{\widehat{\varphi(|\omega|)}}{\omega}\in L^{\varepsilon,2}(R)$ for some positive ε, then the asymptotics from the Theorem 2 is true.*

Proof. Let's consider even infinitely smooth function $\widehat{\chi(\omega)}$ such that

$$
\widehat{\chi(\omega)}=\begin{cases}1, & |\omega|\le 1/2;\\ 0, & |\omega|\ge 1.\end{cases}
$$

Then $\widehat{q(\omega)} = \widehat{q(0)}\widehat{\chi(\omega)} + \widehat{\psi(\omega)}$ where $\widehat{\psi(\omega)} = \widehat{q(0)}[1 - \widehat{\chi(\omega)}] + \widehat{\varphi(\omega)}$. It's easy to see that $\frac{\widehat{\psi(|\omega|)}}{\omega} \in L^{\varepsilon,2}(R)$. And it suffices to prove that S.L. operator with potential $\psi(x) = \frac{2}{\pi}\int\limits_0^\infty \widehat{\psi(\omega)}\cos(\omega x)d\omega$ has the transfer matrix bounded at the infinity for a.e. positive spectral parameter. Really, the initial potential $q(x) = \psi(x) + \chi(x)\widehat{q(0)}$ where $\chi(x) = \frac{2}{\pi}\int\limits_0^\infty \widehat{\chi(\omega)}\cos(\omega x)d\omega = \frac{1}{\pi}\int\limits_{-\infty}^\infty \widehat{\chi(\omega)}\cos(\omega x)d\omega \in L^1(R^+)$. So the transfer matrix of initial operator would be bounded at the infinity for a.e. positive spectral parameter.

Consider $\widehat{L(\omega)} = \frac{\widehat{\psi(|\omega|)}}{\omega}$.

$$\begin{aligned}
W(x) &= \lim_{N\to\infty}\int\limits_x^N \psi(s)ds = \frac{2}{\pi}\lim_{N\to\infty}\int\limits_x^N\left\{\lim_{T\to\infty}\int\limits_0^T \widehat{\psi(\omega)}\cos(\omega s)d\omega\right\}ds = \\
&= \frac{2}{\pi}\lim_{N\to\infty}\lim_{T\to\infty}\int\limits_0^T \widehat{\psi(\omega)}\left\{\int\limits_x^N \cos(\omega s)ds\right\}d\omega = \frac{1}{\pi}\lim_{N\to\infty}\lim_{T\to\infty}\int\limits_{-T}^T \widehat{L(\omega)}(\sin(N\omega) - \sin(x\omega))d\omega = \\
&= -\sqrt{\frac{2}{\pi}}\Im\widehat{L(x)},
\end{aligned}$$

where $L(x)$ is from $L^p(R^+)$ (for some $p < 2$) as the Fourier transform of $L^{\varepsilon,2}(R)$ function and so the arguments from the Theorem 2 are applicable. □

We see that roughly speaking these conditions are fulfilled if $\widehat{q(\omega)}$ is relatively smooth near the zero and admits some bounds at the infinity.

Remark 1. Local condition in zero of Theorem 3 is satisfied if $\widehat{q(\omega)}$ is from $W^{1+\varepsilon,2}(0,\delta)$ for some positive $\delta > 0$.

Remark 2. In paper [15] it was considered the dependence of absence of singular component on certain interval on the local smoothness of Fourier transform.

Remark 3. From the corollary and method used in theorem 3 it follows that

if $q(x)$ is such that (A) and (B) are satisfied, $\widehat{q(\omega)} = \widehat{q(0)} + \widehat{\varphi(\omega)}$ where $\left|\widehat{\varphi(|\omega|)}\right| \le C|\omega|^{1/2+\varepsilon}$ in the vicinity of zero for some positive ε and $\frac{\widehat{\varphi(|\omega|)}}{|\omega|+1} \in L^2(R)$, then the essential support of spectral measure of operator is R^+.

Section C.

In this section we will discuss the dependence of spectral measure $\sigma(\lambda)$ on the coefficient $A(x)$ of system (2).

In fact function $A(x)$ plays the role of sequence a_n for polynomials orthogonal on the unit circle (see [16]).

We will see that for the Dirac-type systems the situation is not so simple. The basic reason is the possible oscillation of $A(x)$. The following Lemma is true

Lemma. If measurable bounded function $A(x)$ is such that

$$\int\limits_x^\infty e^{-s}A(s)ds = \overline{o}(e^{-x}),\ A(x)e^x\int\limits_x^\infty A(s)e^{-s}ds \in L_1(R^+)$$

then conditions (1)-(4) from section A are satisfied.

Proof. Consider system (2) with $\lambda = i$. If $P = e^{-x}Q$ then we have

$$\begin{aligned} Q' &= -Ae^{x}P_* \\ P_*' &= -Ae^{-x}Q \end{aligned} \qquad (16)$$

Consequently

$$P_*(x,i) = 1 - \int_0^x A(s)e^{-s}ds + \int_0^x A(s)e^{s}P_*(s,i)\int_s^\infty A(\xi)e^{-\xi}d\xi ds - \int_x^\infty A(\xi)e^{-\xi}d\xi \int_0^x A(s)e^{s}P_*(s,i)ds$$

And now it suffices to use the standard argument. Let $M_n = \max_{x\in[0,n]}|P_*(x,i)| = |P_*(x_n,i)|$

So $M_n \le 1 + C + M_n \int_0^\infty |A(s)|e^s \left|\int_s^\infty A(\xi)e^{-\xi}d\xi\right| ds + \bar{o}(1)M_n$

If the whole integral in the last formula is less then 1 then M_n is bounded. Otherwise we should start to solve the equations (16) not from zero but from some other point x_0 for which this condition is satisfied.

Example. $A(x) = (x^2+1)^{-\alpha}\sin(x^\beta)$ where $\alpha, \beta > 0$.

One can easily verify that conditions of Lemma are satisfied if $2\alpha + \beta/2 > 1$. Meanwhile $A(x) \in L_2(R^+)$ if and only if $\alpha > 1/4$. Nevertheless for nonpositive $A(x)$ with bounded derivative the condition $A(x) \in L^2(R^+)$ is necessary for (1)-(4) from Section A to be true.

Proposition. If one of the conditions (1)-(4) is true, $A(x) \le 0$ and $A'(x)$ is bounded then $A(x)$ is from $L^2(R^+)$.

Proof.

Really, since $A(x) \le 0$ both P and Q are not less then 1. Consequently if one of (1)-(4) holds then $P_*(x,i)$ is bounded and as it follows from (16) $\int_0^x |A(s)|e^s \int_s^x |A(\xi)|e^{-\xi}d\xi ds$ is bounded as well. But we have the inequality

$$\int_0^x |A(s)|e^s \int_s^x |A(\xi)|e^{-\xi}d\xi ds \ge e^{-1}\int_0^{x-1}|A(s)|\int_s^{s+1}|A(\xi)|d\xi ds \ge C\int_0^{x-1}|A(s)|^2 ds.$$

which concludes the proof of proposition. The latter inequality follows from the boundedness of $A'(x)$. □

Function $A(x)$ from the example above with $\alpha = 1/4$, $1 < \beta \le 3/2$ satisfies the conditions of Lemma (consequently (1)-(4) holds), has bounded derivative but is not from $L^2(R^+)$. The explanation is that this function is not nonpositive.

We would like to conclude the paper with two open problems the first of which is much more difficult then the second one.

Open problems.

1. Prove that Theorem 2 holds for $W \in L^2(R^+)$.

2. Prove that the presence of a.c. component on the half-line pertains to those potentials which Fourier transform is from L^2 near the zero. Specifically, if q admits the Fourier transform $\widehat{q}$ such that $\widehat{q} \in L^2_{loc}(R)$ and $\frac{\widehat{q}}{|\omega|+1} \in L^2(R)$, then the a.c. part of the spectrum fills the whole positive half-line. This conjecture seems reasonable at least with some additional constraints since we can

represent $\widehat{q} = \widehat{q}_1 + \widehat{q}_2$. Where the $\widehat{q}_1$ is localized near the zero and is from L^2 so the methods of paper [15] works. The other function $\widehat{q}_2$ is such that Theorem 3 can be applied.

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