Convergence of spectra of mesoscopic systems collapsing onto a graph.

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Abstract

Let M be a finite graph in the plane and M^{ε} be a domain that looks like the ε-fattened graph M (exact conditions on the domain are given). It is shown that the spectrum of the Neumann Laplacian on M^{ε} converges when $\varepsilon \to 0$ to the spectrum of an ODE problem on M. Presence of an electromagnetic field is also allowed. Considerations of this kind arise naturally in mesoscopic physics and other areas of physics and chemistry. The results of the paper extend the ones previously obtained by J. Rubinstein and M. Schatzman.

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1 Introduction

In recent years one has witnessed growing interest in spectral theory of differential (versus difference) operators on graphs. Although probably one of the first such studies was done in physical chemistry [47], the main thrust in this direction came from the mesoscopic physics [29]. Recent progress in nanotechnology and microelectronics enables one to manufacture circuits of thin semiconductor strips called "quantum wires" or thin superconducting structures. This triggered mathematical investigation of spectra of differential operators on thin domain shrinking to a graph (see the survey [15] and references therein and papers [43]-[46] and [49]). Quasi-one-dimensional models that reduce to differential problems on graphs are known to describe quantum wires ([50], [51], [15]), atomic [30] and molecular wires [2], high temperature granular superconductors ([1], [11], [43]-[46]), and free-electron theory of conjugated molecules ([27], [28], [47]). Similar, although not identical problems arise in study of thin high contrast photonic crystals (see the surveys [5, 34] and papers [18]-[22], [35, 36]). Consideration of such structures leads to spectral problems for differential and pseudo-differential operators on graphs. In 3D analogous analysis produces similar problems on surface structures. It is interesting to mention that at the same time often similar but seemingly unrelated graph models arouse in scattering theory, spectral theory, tomography, theory of dynamical systems, probability, and other areas of physics and mathematics ([4], [6], [7]-[14], [23]-[25], [31], [32], [39]-[41]). In most of these cases the domain was assumed to shrink to a graph, and a natural asymptotic consideration was applied. The resulting eigenvalue problems on graphs usually look as follows: along each edge of the graph one has the spectral problem

$$-\frac{d^2u}{ds^2} = \lambda^2 u$$

or some its generalization, with "appropriate" boundary conditions imposed at each vertex. Finding the boundary conditions at the vertices and proving convergence of the spectrum when the domain shrinks to a graph are still problematic in many cases. The main difficulty comes from the behavior around the vertices of the graph. The case when the domain shrinks to a smooth curve or surface is much simpler and was successfully treated for the Dirichlet and Neumann Laplace operators (see correspondingly [15] and references therein and [49]). The most interesting case of spectra in narrow domains surrounding nonsmooth graphs was considered in [46] for the Neumann Laplacian and in [5, 19, 20, 21, 22, 34, 35, 36] for the problems of the photonic crystals theory. The recent preprint [48] provides some type of resolvent convergence for the Neumann Laplacian. Convergence of solutions of the heat equation generated by the Neumann Laplacian on a "fattened graph" domain was obtained in [23, 24]. However, [23, 24, 48] do not discuss convergence of the spectrum.

This paper addresses the case of the Neumann Laplace operator on a domain shrinking to a graph. In comparison with [46], the proofs are streamlined and simplified, smoothness condition on the geometry of the domain in vicinity of vertices is removed, the domain is allowed to have variable thickness along the edges of the graph (which changes the limit of the spectrum), and presence of an electric potential is allowed. The results are easily transferable to graphs in higher dimensions, to the case of slightly thicker vertex junctions, and to periodic systems.

The next section contains the definitions of the main objects, the formulation of the problem, and the statement of the main results. In order to simplify the exposition of the proofs, we chose to start with a simpler particular case. This is done in the Section 3, where the proof is provided of the convergence of the spectrum for the Neumann Laplace operator in absence of electromagnetic field. This result is extended to the case of presence of electric and magnetic potentials in the Section 4. Final remarks are presented in the Section 5, where in particular some extensions and open questions are mentioned. The last section contains acknowledgments.

2 Definitions of main objects and formulation of the main results

Let M be a finite graph smoothly embedded into $\mathbb{R}^2$ in such a way that each edge is a finite segment of a C^2-curve. The (finite) set of vertices is denoted by $V = \{v_l\}$ and edges are denoted by $\{M_j\}$. Each edge M_j is equipped with the arc length coordinate x_j, whose range spans an interval $[a_j, b_j]$. It is assumed that the arc length parametrization represents the edge M_j as the image of this interval under a C^2-mapping $\varphi_j : [a_j, b_j] \to \mathbb{R}^2$ with non-vanishing derivative. The edges are assumed to intersect transversely and only at the vertices (loops are allowed). The notation $J(v)$ will be used for the set of all indices j such that the edge M_j contains the vertex v. We will be interested in "fattened graph" domains M^ε in the plane that can be roughly described as ε-neighborhoods of the graph M. Let us make this notion more precise. We will assume that for any $\varepsilon > 0$ the domain M^ε is a neighborhood of M that consists of narrow "tubes" M_j^ε going along the edges and of suitable neighborhoods $V_l^{\,\varepsilon}$ of vertices:

$$M^{\,\varepsilon} = \left(\underset{j}{\cup} M_j^{\varepsilon}\right) \cup \left(\underset{l}{\cup} V_l^{\,\varepsilon}\right).$$

We will now impose the precise conditions on these parts of $M^{\,\varepsilon}$. Although the conditions might look complicated, they actually are not, as the figure below will indicate.

1. For each edge M_j and $\varepsilon > 0$ positive numbers γ_ε^2 and γ_ε^1 are defined such that $\gamma_\varepsilon^i \sim \gamma^i \varepsilon$ when $\varepsilon \to 0$, and $\gamma^2 > \gamma^1 > 0$. If the edge M_j is represented as $\varphi_j\left([a_j, b_j]\right)$ (see the notations above), we define $a_{j,\varepsilon}^i = a_j + \gamma_\varepsilon^i$ and $b_{j,\varepsilon}^i = b_j - \gamma_\varepsilon^i$, where $i = 1, 2$. Then for small ε we have $a_{j,\varepsilon}^2 > a_{j,\varepsilon}^1 > a_j$, $b_{j,\varepsilon}^2 < b_{j,\varepsilon}^1 < b_j$, and the expressions $\left(a_{j,\varepsilon}^2 - a_{j,\varepsilon}^1\right)$, $\left(a_{j,\varepsilon}^1 - a_j\right)$, $\left(b_{j,\varepsilon}^2 - b_{j,\varepsilon}^1\right)$, and $\left(b_{j,\varepsilon}^1 - b_j\right)$ are of order ε when $\varepsilon \to 0$. The meaning of these numbers will be clarified in further conditions and the figure below.

2. For each edge M_j two strictly positive functions g_j, $h_j \in C^1[a_j, b_j]$ are defined. The "tube" M_j^{ε} is defined as follows:

$$M_j^{\varepsilon} = \left\{(x_j, y_j)|\, x_j \in [a_{j,\varepsilon}^1, b_{j,\varepsilon}^1], -\varepsilon g_j(x_j) < y_j < \varepsilon h_j(x_j)\right\},$$

   where y_j is the coordinate normal to the edge and x_j is the arc length coordinate along the edge as before. So, the tube does not reach the ends of the edge M_j, but rather ends at the distance γ_ε^1 of order ε away from them. For sufficiently small ε the tubes M_j^{ε} are assumed to be disjoint.

3. The neighborhoods $V_l^{\,\varepsilon}$ of vertices are disjoint with the tubes M_j^{ε}. The narrow sides $\left\{(x_j, y_j) \in M_j^{\varepsilon}|\, x_j = a_{j,\varepsilon}^1 \text{ or } x_j = b_{j,\varepsilon}^1\right\}$ are the only joint boundaries of M_j^{ε} and the adjacent neighborhoods $V_l^{\,\varepsilon}$.

4. There exist positive numbers r and R such that for any ε and l the neighborhood $V_l^{\,\varepsilon}$ contains the open ball B_l^{ε} of radius $r\varepsilon$ centered at v_l and is contained in the similar ball of radius $R\varepsilon$.

5. Let an edge M_j have the vertices v_l (where $x_j = a_j$) and v_m (where $x_j = b_j$). We denote by $S_{j,l}^{\varepsilon}$ and $S_{j,m}^{\varepsilon}$ respectively the end portions, or "sleeves" of the tube M_j^{ε} defined as follows:

$$S_{j,l}^{\varepsilon} = \left\{(x_j, y_j) \in M_j^{\,\varepsilon} \mid x_j \in [a_{j,\varepsilon}^1, a_{j,\varepsilon}^2]\right\},$$

$$S_{j,m}^{\varepsilon} = \left\{ (x_j, y_j) \in M_j^{\varepsilon} \mid x_j \in [b_{j,\varepsilon}^2, b_{j,\varepsilon}^1] \right\}.$$

We assume that the enlarged neighborhood

$$U_l^{\varepsilon} = V_l^{\varepsilon} \cup \left(\bigcup_{j \in J(v_l)} S_{j,l}^{\varepsilon} \right)$$

is starshaped with respect to the ball B_l^{ε}. This means that for any point in this neighborhood and any point in the ball the segment connecting these points belongs to the neighborhood.

Fig. 1.

The structure of the domain M^{ε}. Here U_l^{ε} is V_l^{ε} with attached "sleeves" $S_{j,l}^{\varepsilon}$.

The Neumann Laplace operator $-\Delta_{\varepsilon}$ in $L^2(M^{\varepsilon})$ is defined in the standard way by means of its quadratic form

$$E^{\varepsilon}(u, u) = \int_{M^{\varepsilon}} |\nabla u|^2 \, dA$$

with the domain $H^1(M^{\varepsilon})$ (see for instance [12]). Here dA denotes the area element. The goal is to understand the behavior of the spectrum $\sigma(-\Delta_{\varepsilon})$ of this operator when $\varepsilon \to 0$. Our main result states that this spectrum converges to the spectrum of a second order differential operator on the graph M. This operator, which depends on the shape of the tubes M_j^{ε}, is defined below.

Let us denote by $p(x)$ the piecewise-smooth (with possible discontinuity at the vertices) function on M that coincides on each edge M_j with the sum $h_j(x) + g_j(x)$. Now our ambient space is the weighted L^2-space $H = L^2(M, p(x)dx)$. Here dx denotes the arc length measure along edges. The space $H^1(M)$ consists of functions f on M whose restriction f_j to each edge M_j belongs to $H^1(M_j)$ and such that f is continuous on M (i.e., the boundary values of f at any vertex are the same for any edge adjacent to this vertex). We now define a self-adjoint operator A_p in H by its quadratic form

$$E(u, u) = \sum_j \int_{M_j} |u'(x_j)|^2 \, p(x_j) dx_j.$$

with the domain $H^1(M)$. One can check that this is a closed form and hence it defines a self-adjoint operator A_p:

$$A_p f(x_j) = -\frac{1}{p}\frac{d}{dx_j}\left(p\frac{df}{dx_j}\right),\ f \in D(A_p),\ x_j \in M_j.$$

It is also not hard to check that any function f from the domain $D(A_p)$ of the operator A_p satisfies the following boundary conditions at each vertex v:

1. f is continuous through the vertex (since $f \in H^1(M)$).

2. $$\sum_{j \in J(v)} p_j(v)\frac{df_j}{dx_j}(v) = 0,$$

   where $J(v)$ is the set of all indices j such that the j-th edge M_j is adjacent to v, x_j is the arc length coordinate on M_j measured starting from v, and f_j and p_j denote the restrictions of the functions f and p to the edge M_j (so $p_j(v)$ must be understood as the limit value along M_j of p_j at the vertex v).

It is easy to show that the spectra of both positive operators $-\Delta_\varepsilon$ and A_p are discrete (due to compactness of embedding of the corresponding H^1-spaces into L^2). We denote the eigenvalues counted in increasing order with their multiplicity by $\lambda_n(-\Delta_\varepsilon)$ and $\lambda_n(A_p)$ respectively.

The main results of this paper are the following theorems:

Theorem 1 *Under the conditions imposed above on the graph M and the domains M^ε, the following equality holds for each $n = 1, 2, \ldots$:*

$$\lim_{\varepsilon \to 0} \lambda_n(-\Delta_\varepsilon) = \lambda_n(A_p).$$

This theorem can be generalized for the case of presence of an electromagnetic field. Let a Lipschitz real valued function q (electric potential) and an $\mathbb{R}^2$-valued vector function $\mathbf{A}$ of class C^1 (magnetic potential) be defined in a fixed (i.e., independent of ε) neighborhood of the graph M. The definitions of the operators $-\Delta_\varepsilon$ and A_p can be modified to include these fields.

Namely, the domains of the corresponding quadratic forms remain the same, while the expressions for the quadratic forms now are modified as follows:

$$\int\limits_{M^{\varepsilon}} \left(|(\nabla + i\mathbf{A})u|^2 + q\,|u|^2 \right) dx \tag{1}$$

and

$$\sum_j \int\limits_{M_j} \left(\left| \frac{df}{dx_j} + iA_j^{\tau} f \right|^2 + q\,|f|^2 \right) p\, dx_j \tag{2}$$

correspondingly. Here A_j^{τ} denotes the tangential component of the field $\mathbf{A}$ along the edge M_j. We will denote these operators by $H_{\varepsilon}(\mathbf{A}, q)$ and $H_M(A, q)$ correspondingly. As a differential expression, $H_M(A, q)$ acts as

$$H_M(A, q)f = -\frac{1}{p}\left(\frac{d}{dx_j} + iA_j^{\tau}\right) p \left(\frac{d}{dx_j} + iA_j^{\tau}\right) f + qf$$

along the edge M_j, with the domain consisting of all functions $f \in H^1(M)$ such that $f \in H^2(M_j)$ for each j and such that at each vertex v

$$\sum_{j, v \in M_j} p_j(v) \left(\frac{df}{dx_j}(v) + iA_j^{\tau}(v) f(v) \right) = 0.$$

Here the derivatives are taken in the direction away from the vertex v along the edges M_j containing v, and p_j is the restriction of p to the edge M_j. Notice that the values $p_j(v)$ do not have to be the same for different M_j containing v.

Theorem 2 *The statement of the Theorem 1 holds true for the operators $H_{\varepsilon}(\mathbf{A}, q)$ and $H_M(A, q)$ with the quadratic forms (1) and (2) correspondingly.*

3 Proof of the Theorem 2.1

The main idea of the proof is clear and has been used many times before, see for instance [46, 49, 48, 22]. One represents the eigenvalues $\lambda_n(-\Delta_{\varepsilon})$ and $\lambda_n(A_p)$ using the minimax principle and then one needs to be able to transplant H^1-functions from M^{ε} to M and back in such a way that the

Rayleigh ratio does not increase much. Let us introduce now the "extension" $Q^{\varepsilon} : H^1(M) \to H^1(M^{\varepsilon})$ and "averaging" $P^{\varepsilon} : H^1(M^{\varepsilon}) \to H^1(M)$ mappings that do this job.

The idea of the definition of the mapping Q^{ε} is natural. It was used in [46]. A function $f \in H^1(M)$ is shrunk on each edge to a little bit shorter segment, making at the same time its value around each vertex constant. Then the function is extended to M^{ε} in such a way that it is constant around vertices and independent on the normal variables around the shortened edges. Let us now make this more precise. The same local coordinates (x_j, y_j) as before (arc length and normal) will be used. We will denote by $\psi_j : [a^1_{j,\varepsilon}, b^1_{j,\varepsilon}] \to [a_j, b_j]$ the 1-to-1 linear mapping between these two segments.

Definition 3 *For $u \in H^1(M)$ we define the function $Q^{\varepsilon}u$ on M^{ε} as follows:*

$$Q^{\varepsilon}u = \begin{cases} u(v_l) & \text{in } V_l^{\varepsilon} \\ u \circ \psi_j \text{ extended as a constant with respect to } y_j & \text{in } M_j^{\varepsilon} \end{cases} \tag{3}$$

Lemma 4 *1. For any sufficiently small $\varepsilon > 0$ the operator Q^{ε} maps $H^1(M)$ into $H^1(M^{\varepsilon})$.*

2. For any sufficiently small $\varepsilon > 0$ and for any $u \in H^1(M)$, the following inequalities hold:

$$||Q^{\varepsilon}u||^2_{L^2(M^{\varepsilon})} \geq \varepsilon\,(1 + O(\varepsilon))\,||u||^2_{L^2(M)}, \tag{4}$$

$$E^{\varepsilon}(Q^{\varepsilon}u, Q^{\varepsilon}u) \leq \varepsilon\,(1 + O(\varepsilon))\,E(u, u), \tag{5}$$

where $O(\varepsilon)$ does not depend on u.

Proof: The first statement of the lemma is elementary. Indeed, if $u \in H^1(M)$, then from the construction it is obvious that $Q^{\varepsilon}u \in H^1\left(M_j^{\varepsilon}\right)$ for any j and $Q^{\varepsilon}u \in H^1(V_l^{\varepsilon})$ for any l. Since the construction guarantees that $Q^{\varepsilon}u$ is continuous through the joint straight boundaries of M_j^{ε} and V_l^{ε}, this implies that $u \in H^1(M^{\varepsilon})$.

Let us now prove the second statement. We estimate the L^2-norm first. Neglecting the sets V_l^{ε} we get

$$||Q^{\varepsilon}u||^2_{L^2(M^{\varepsilon})} \geq \sum_j \int_{M_j^{\varepsilon}} |Q^{\varepsilon}u|^2\,dA.$$

In each tube M_j^ε we can change coordinates in the integral to the arc length and normal coordinates (x_j, y_j), which via a direct calculation leads to

$$dA = (1 + O(\varepsilon))dx_j dy_j. \tag{6}$$

Since the function $Q^\varepsilon u$ does not depend on y_j in this set, we get

$$\sum_j \int_{M_j^\varepsilon} |Q^\varepsilon u|^2 \, dA = (1 + O(\varepsilon)) \sum_j \int\limits_{a_{j,\varepsilon}^1}^{b_{j,\varepsilon}^1} |u_j \circ \psi_j|^2 \int\limits_{-g_j(x_j)\varepsilon}^{h_j(x_j)\varepsilon} dy_j dx_j$$

$$= \varepsilon(1 + O(\varepsilon)) \sum_j \int\limits_{a_{j,\varepsilon}^1}^{b_{j,\varepsilon}^1} |u_j \circ \psi_j|^2 \, p(x_j) \, dx_j$$

$$= \varepsilon(1 + O(\varepsilon))[\sum_j \int\limits_{a_{j,\varepsilon}^1}^{b_{j,\varepsilon}^1} |u_j \circ \psi_j|^2 \, p \circ \psi_j \, dx_j$$

$$+ \sum_j \int\limits_{a_{j,\varepsilon}^1}^{b_{j,\varepsilon}^1} |u_j \circ \psi_j|^2 \, \frac{(p - p \circ \psi_j)}{p \circ \psi_j} p \circ \psi_j \, dx_j]. \tag{7}$$

Due to smoothness and positivity of p_j and since $\psi_j(x_j) - x_j = O(\varepsilon)$, we obtain that

$$\frac{(p - p \circ \psi_j)}{p \circ \psi_j} = O(\varepsilon) \tag{8}$$

and hence the sum in (7) is equal

$$\varepsilon(1 + O(\varepsilon)) \sum_j \int\limits_{a_{j,\varepsilon}^1}^{b_{j,\varepsilon}^1} |u_j \circ \psi_j|^2 \, p \circ \psi_j \, dx_j$$

Changing variables $\psi_j(x) \to x$ and using the obvious estimate

$$\psi_j' = (1 + O(\varepsilon)) \tag{9}$$

we obtain

$$\sum_j \int_{M_j^\varepsilon} |Q^\varepsilon u|^2 \, dA = \varepsilon \left(1 + O(\varepsilon)\right) ||u||^2_{L^2(M)} \,. \tag{10}$$

which implies the required inequality

$$||Q^\varepsilon u||^2_{L^2(M^\varepsilon)} \geq \varepsilon \left(1 + O(\varepsilon)\right) ||u||^2_{L^2(M)} \,.$$

Let us move now to estimating the quadratic form $E^\varepsilon \left(Q^\varepsilon u, Q^\varepsilon u\right)$. According to the definition of $Q^\varepsilon u$, we get

$$E^\varepsilon \left(Q^\varepsilon u, Q^\varepsilon u\right) = \sum_j \int_{M_j^\varepsilon} |\nabla Q^\varepsilon u|^2 \, dA,$$

which up to a factor $(1 + O(\varepsilon))$ is

$$\sum_j \int\limits_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} \int\limits_{-g_j(x_j)\varepsilon}^{h_j(x_j)\varepsilon} \left|(u \circ \psi_j)'_{x_j}\right|^2 dy_j dx_j = \sum_j \int\limits_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} \left|(u \circ \psi_j)'_{x_j}\right|^2 \left(\int\limits_{-g_j(x_j)\varepsilon}^{h_j(x_j)\varepsilon} dy_j \right) dx_j$$

$$= \varepsilon \sum_j \int\limits_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} \left|(u \circ \psi_j)'_{x_j}\right|^2 p(x_j) \, dx_j .$$

Now replacing p_j by $p_j \circ \psi_j$ with the estimate (8) like before, we arrive at the expression

$$\varepsilon(1 + O(\varepsilon)) \sum_j \int\limits_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} |(u \circ \psi_j)'|^2 \, p \circ \psi_j \, dx_j,$$

and consequently by changing variables and using (9) at

$$\varepsilon(1 + O(\varepsilon)) \sum_j \int\limits_{a_j}^{b_j} | \, u'|^2 \, p \, dx_j = \varepsilon(1 + O(\varepsilon)) E\left(u, u\right).$$

The lemma is proven.

Corollary 5 *Under the conditions imposed above on the graph M and the domains M^{ε}, the following inequality holds for each $n = 1, 2, ...$:*

$$\limsup_{\varepsilon \to 0} \lambda_n(-\Delta_{\varepsilon}) \leq \lambda_n(A_p). \tag{11}$$

Proof. We use the standard min-max formula for the eigenvalues that works for both operators $A = -\Delta_{\varepsilon}$ and $A = A_p$:

$$\lambda_n = \inf_{\dim W = n} \sup_{\substack{u \in W \\ u \neq 0}} R(u).$$

Here

$$R(u) = \frac{(Au, u)}{(u, u)}$$

is the Rayleigh quotient, λ_n is the n-th eigenvalue (counted in non-decreasing order) of the self-adjoint operator A (as $-\Delta_{\varepsilon}$ or A_p), and W is an n-dimensional subspace of the domain of the quadratic form of the operator. Let $W \subset H^1(M)$ be the n-dimensional subspace spanned by the eigenfunctions $u_1, u_2, ... u_n$ of A_p corresponding to the eigenvalues $\lambda_1, \lambda_2, ... \lambda_n$. For this choice of W we have

$$\lambda_n = \sup_{\substack{u \in W \\ u \neq 0}} R(u).$$

Consider the subspace $Q^{\varepsilon} W \subset H^1(M^{\varepsilon})$. Notice that by the construction the linear mapping Q^{ε} has zero kernel, so $\dim Q^{\varepsilon} W = \dim W = n$. Then

$$\lambda_n(-\Delta_{\varepsilon}) \leq \sup_{u \in Q^{\varepsilon} W} \frac{(\nabla u, \nabla u)}{(u, u)}$$

$$= \sup_{u \in W} \frac{(\nabla Q^{\varepsilon} u, \nabla Q^{\varepsilon} u)}{(Q^{\varepsilon} u, Q^{\varepsilon} u)} = \sup_{u \in W} \frac{E^{\varepsilon}(Q^{\varepsilon} u, Q^{\varepsilon} u)}{(Q^{\varepsilon} u, Q^{\varepsilon} u)}.$$

The lemma above implies the following estimate for the last expression:

$$\frac{E^{\varepsilon}(Q^{\varepsilon} u, Q^{\varepsilon} u)}{(Q^{\varepsilon} u, Q^{\varepsilon} u)} \leq (1 + O(\varepsilon)) \frac{E(u, u)}{(u, u)},$$

which leads to the inequality

$$\lambda_n(-\Delta_{\varepsilon}) \leq (1 + O(\varepsilon)) \lambda_n(A_p)$$

and proves the corollary.

Remark 1 *This proof does not require the starshapedness condition on the neighborhoods of vertices.*

We now move to the averaging operator P^{ε} and to proving the opposite inequality for the eigenvalues $\lambda_n(-\Delta_{\varepsilon})$ and $\lambda_n(A_p)$.

We define the normal average (that we will denote $N_j u$) of a function u in M_j^{ε} in the natural way:

$$N_j u\left(x_j\right) = \frac{1}{\varepsilon p\left(x_j\right)} \int\limits_{-g_j(x_j)\varepsilon}^{h_j(x_j)\varepsilon} u\left(x_j, y_j\right) dy_j.$$

Let B be the unit disk in $\mathbb{R}^2$ centered at the origin. Consider a function $\omega \in C_0^{\infty}(B)$ with unit average. One can translate and homothetically shrink ω to produce functions $\omega_l \in C_0^{\infty}(B_l^{\varepsilon})$ of unit average, where $B_l^{\varepsilon} \subset V_l^{\varepsilon}$ are the disks described in the condition 4 imposed on the domain M^{ε}. Let us also introduce a cut-off function

$$\rho(x) \in C_0^{\infty}(\mathbb{R}),\ \rho(0) = 1,\ \rho(x) = 0 \text{ when } |x| > 0.5 \min_j \left|b_{j,\varepsilon}^1 - a_{j,\varepsilon}^1\right|.$$

We are ready now to define the operator P^{ε}. Its definition differs from the one suggested in [46].

Given $u \in H^1\left(M^{\varepsilon}\right)$, we introduce a function $P^{\varepsilon}u$ on the graph M as follows.

Definition 6 *Let M_j be an edge connecting the vertices v_l (where $x_j = a_j$) and v_m (where $x_j = b_j$). Then we define:*

$$\left(P^{\varepsilon}u\right)\left(x_j\right) = c\left(v_l\right) = \frac{1}{vol(B_l^{\varepsilon})} \int\limits_{B_l^{\varepsilon}} \omega_l\left(y\right) u\left(y\right) dy \tag{12}$$

if $x_j \in \left[a_j, a_{j,\varepsilon}^1\right]$,

$$\left(P^{\varepsilon}u\right)\left(x_j\right) = c\left(v_m\right) = \frac{1}{vol(B_m^{\varepsilon})} \int\limits_{B_m^{\varepsilon}} \omega_m\left(y\right) u\left(y\right) dy \tag{13}$$

if $x_j \in \left[b_{j,\varepsilon}^1, b_{j,\varepsilon}\right]$, and finally

$$\left(P^{\varepsilon}u\right)\left(x_j\right) = N_j u\left(x_j\right) + \left[c\left(v_l\right) - N_j u\left(a_{j,\varepsilon}^1\right)\right] \rho(x_j - a_{j,\varepsilon}^1) \tag{14}$$

$$+\left[c\left(v_m\right)-N_j u\left(b_{j,\varepsilon}^1\right)\right]\rho(x_j-b_{j,\varepsilon}^1)$$

if $x_j\in\left[a_{j,\varepsilon}^1,b_{j,\varepsilon}^1\right]$.

The meaning of these formulas is quite simple. Namely, the formulas (12) and (13) mean that around any vertex v_l the value of $P^{\varepsilon}u$ is defined as the weighted (with the weight ω_l) average of u over the ball B_l^{ε}. The main contribution into $P^{\varepsilon}u$ along the edge (formula (14)) is just the normal average $N_j u$ of u. However, this average would not fit continuously with the constant values (12) and (13) around the vertices. In order to correct this, the two additional terms are added in (14) in order to adjust the values of $N_j u$ close to the vertices.

Our last major step in proving Theorem 1 is to establish the following estimates:

Lemma 7 *1. For any sufficiently small $\varepsilon>0$ the operator P^{ε} maps $H^1\left(M^{\varepsilon}\right)$ into $H^1\left(M\right)$.*

2. For any sufficiently small $\varepsilon>0$ and for all $u\in H^1\left(M^{\varepsilon}\right)$

$$E\left(P^{\varepsilon}u,P^{\varepsilon}u\right)\leq\varepsilon^{-1}(1+O(\sqrt{\varepsilon}))E^{\varepsilon}\left(u,u\right)\tag{15}$$

and

$$||P^{\varepsilon}u||_{L^2(M)}^2\geq\varepsilon^{-1}\left[\left(1+O(\sqrt{\varepsilon})\right)||u||_{L^2(M^{\varepsilon})}^2-O(\sqrt{\varepsilon})E^{\varepsilon}\left(u,u\right)\right].\tag{16}$$

Here $O(\sqrt{\varepsilon})$ is uniform with respect to u.

Before attempting the proof of this lemma, we obtain its corollary that is of interest to us.

Corollary 8 *For every $n=1,2,...$*

$$\liminf_{\varepsilon\to0}\lambda_n\left(-\Delta_{\varepsilon}\right)\geq\lambda_n\left(A_p\right).$$

The Corollaries 5 and 8 obviously imply the Theorem 1.

Let us show that the Corollary 8 follows from the Lemma 7. The proof is mostly similar to the proof of the Corollary 5, except that the mapping P^{ε} might (and actually does) have a non-zero kernel and that the inequality (16) has an additional term $O(\sqrt{\varepsilon})E^{\varepsilon}\left(u,u\right)$. Let us fix an integer $N>0$ and look only at the eigenvalues $\lambda_n(-\Delta_\varepsilon)$ with $n < N$. Then the estimate (11) implies that

$$|\lambda_n(-\Delta_\varepsilon)| < C$$

when $n < N$ and $\varepsilon \to 0$. Consider now the subspace $W_\varepsilon \subset H^1(M^\varepsilon)$ spanned by the eigenfunctions corresponding to these eigenvalues $\lambda_n(-\Delta_\varepsilon)$. The proof in fact only requires that the operator P^ε considered on the subspace W_ε has zero kernel. We can now justify that this is true if ε is small enough. Indeed, since $|\lambda_n(-\Delta_\varepsilon)| < C$, we have the estimate

$$E^\varepsilon(u,u) \leq C\,||u||_{L^2(M^\varepsilon)} \tag{17}$$

on W_ε. If now $u \in W_\varepsilon$, we get from (16) the estimate

$$||P^\varepsilon u||^2_{L^2(M)} \geq \varepsilon^{-1}\left(1 - C\sqrt{\varepsilon}\right)||u||^2_{L^2(M^\varepsilon)},$$

which obviously implies that for ε small enough $P^\varepsilon u = 0$ if and only if $u = 0$. This finishes the proof of the Corollary 8.

Our last task is to prove the Lemma 7. The whole proof hinges on a string of simple Poincare type inequalities.

Lemma 9 *Let $f \in H^1[a,b]$, then*

$$\int_a^b \left| f(y) - \frac{1}{b-a}\int_a^b f(t)dt \right|^2 dy \leq \left(\frac{b-a}{\pi}\right)^2 \int_a^b \left|\frac{df}{dy}\right|^2 dy. \tag{18}$$

Proof. Expressions in both sides of (18) after dividing by $(b-a)$ are invariant with respect to shifts and dilations, so it is sufficient to prove (18) for one interval $[a,b]$, for instance $[0,\pi]$. Extending a function $f \in H^1[0,\pi]$ as an even function to $[-\pi,\pi]$ and expanding the extension into Fourier series $\sum f_n e^{inx}$, one reduces (18) to the obvious inequality

$$\sum_{n\neq 0} |f_n|^2 \leq \sum_{n\neq 0} n^2 |f_n|^2.$$

Another proof of (18) follows from the equivalence of (18) to the inequality

$$\frac{\int_a^b |f'|^2\,dy}{\int_a^b |f|^2\,dy} \geq \mu_1$$

on functions orthogonal to constants, where $\mu_1 = (\pi/(b-a))^2$ is the first non-zero eigenvalue of the operator $-\frac{d^2}{dy^2}$ with the Neumann boundary conditions.

Lemma 10 *Let $[a,b] \subset [a^1_{j,\varepsilon}, b^1_{j,\varepsilon}]$ and $\Omega = \left\{(x_j, y_j) \in M^\varepsilon_j \,|\, x_j \in [a,b]\right\}$ be the corresponding piece of the tube M^ε_j. Then for any $u \in H^1(\Omega)$ we have*

$$|N_j u(a)| = \left| \frac{1}{\varepsilon p(a)} \int\limits_{-\varepsilon g_j(a)}^{\varepsilon h_j(a)} u\,(a,y)\,dy \right| \le \frac{C}{\sqrt{\varepsilon\,|b-a|}}\,||u||_{H^1(\Omega)}\,, \tag{19}$$

where the constant C depends on the graph M and on the functions g_j and h_j only. In particular, for any $u \in H^1(M^\varepsilon)$

$$\left|N_j u(a^1_{j,\varepsilon})\right| \le \frac{C}{\sqrt{\varepsilon}}\,||u||_{H^1(M^\varepsilon_j)}\,. \tag{20}$$

Proof. Changing the variables $x = (x_j - a)/(b-a)$, $y = y_j/\varepsilon$, one reduces the domain to the tube

$$\Omega^0 = \{(x,y)|\, x \in [0,1],\, y \in [-g_j(x), h_j(x)]\}\,.$$

Then the standard embedding theorems [12] provide the inequality

$$\left| \frac{1}{p(a)} \int\limits_{-g_j(a)}^{h_j(a)} U\,(0,y)\,dy \right| \le C\,||U||_{H^1(\Omega^0)}\,, \tag{21}$$

where

$$U\,(x,y) = u(a + (b-a)x, \varepsilon y). \tag{22}$$

Replacing $||U||_{H^1(\Omega^0)}$ by the equivalent expression $||\nabla U||_{L^2(\Omega^0)} + ||U||_{L^2(\Omega^0)}$ and changing the variables back in the right hand side of (21), one gets

$$(\varepsilon(b-a))^{-1/2} \left(\int\limits_{\Omega} \left(\left|\frac{\partial u}{\partial x_j}\right|^2 (b-a)^2 + \left|\frac{\partial u}{\partial y_j}\right|^2 \varepsilon^2 \right) dx_j dy_j \right)^{1/2}$$

$$+ (\varepsilon(b-a))^{-1/2} \left(\int\limits_{\Omega} |u|^2\, dx_j dy_j \right)^{1/2} \le \frac{C}{\sqrt{\varepsilon\,|b-a|}}\,||u||_{H^1(\Omega)}\,.$$

This proves (19). The inequality (20) is a particular case of (19) when $[a,b] = [a^1_{j,\varepsilon}, b^1_{j,\varepsilon}]$.

Let us recall now the following Sobolev's representation theorem:

Lemma 11 *(Theorem 1 on page 20 of [37]) Let $\Omega \subset \mathbb{R}^m$ be a bounded domain of diameter d starshaped with respect to a ball $B_\delta \subset \Omega$ of radius δ and let $u \in H^1(\Omega)$. Let also $\omega \in C_0^\infty(B_\delta)$ be such that the average of ω over B_δ is equal to 1. Then for almost all $x \in \Omega$*

$$u(x) = \frac{1}{vol(B_\delta)} \int\limits_{B_\delta} \omega(y) u(y)\, dy + \int\limits_{\Omega} \left(\frac{f_1(x;r,\theta)}{r} \frac{\partial u}{\partial y_1} + \frac{f_2(x;r,\theta)}{r} \frac{\partial u}{\partial y_2} \right) dy_1 dy_2,$$

where $r = |y-x|$, $\theta = (y-x) r^{-1}$, and f_i are infinitely differentiable functions such that

$$|f_i| \le C(d/\delta).$$

Here the constant C is independent of Ω and u.

We will use further for each vertex v_l a function $\omega_l \in C_0^\infty(B_l^\varepsilon)$ described before. The function ω_l is chosen as the $r\varepsilon$-dilation of a fixed function ω, and we drop ε in its notation.

Lemma 12 *Let Ω be a bounded domain in $\mathbb{R}^2$ of diameter d starshaped with respect to a ball B_δ of radius δ. Let also $u \in H^1(\Omega)$. Then there exists a constant u_0 such that*

$$||u - u_0||_{L^2(\Omega)} \le Cd^2/\delta \, ||\nabla u||_{L^2(\Omega)},$$

where C is independent of Ω and u.

Proof: Let ω be the function from the Lemma 11. If we define

$$u_0 = \frac{1}{vol(B_\delta)} \int\limits_{B_\delta} \omega(x,y) u(x,y)\, dx dy,$$

then the statement follows immediately from the Lemma 11 and the following known result (see for instance Theorem 7.3.1 in [38]):

Lemma 13 *Let Ω be a bounded domain of diameter d in $\mathbb{R}^m$, A be a continuous function defined on the closure of $\Omega \times \Omega$, and T be the weakly singular integral operator of order $\beta \in [0,m]$*

$$Tf(x) = \int_\Omega \frac{A(x,y)}{|x-y|^\beta} f(y)\, d\Omega.$$

Then T is defined and bounded on $L^2(\Omega)$ and

$$||Tf||_{L^2(\Omega)} \le Cd^{m-\beta} ||A||_{L^\infty} ||f||_{L^2(\Omega)},$$

where C depends only on m and β.

The Lemma 12 has the following immediate consequence:

Corollary 14 *There exists a constant C such that for sufficiently small values of ε and for any $u \in H^1(U_l^\varepsilon)$ the following inequalities hold:*

$$\|u - c(v_l)\|_{L^2(U_l^\varepsilon)} \le C\varepsilon \|\nabla u\|_{L^2(U_l^\varepsilon)} \tag{23}$$

and hence

$$\|u - c(v_l)\|_{H^1(U_l^\varepsilon)} \le C \|\nabla u\|_{L^2(U_l^\varepsilon)}. \tag{24}$$

Here the average $c(v_l)$ is defined in (12).

Our last auxiliary lemma provides estimates for the deviation between the cross-sectional and volume averages $N_j u\left(a_{j,\varepsilon}^1\right)$ and $c(v_l)$ of a function $u \in H^1(M^\varepsilon)$.

Lemma 15 *There exists a constant C independent on ε such that for all $u \in H^1(U_l^\varepsilon)$*

$$\left|c(v_l) - N_j u\left(a_{j,\varepsilon}^1\right)\right| \le C \|\nabla u\|_{L^2(U_l^\varepsilon)}. \tag{25}$$

Proof: Let us homothetically enlarge the domain U_l^ε with the factor ε^{-1}. and call the new domain W^ε. Both sides of the inequality (25) do not change under homothety, so it is sufficient to prove it for the domains W^ε only. All the domains W^ε are uniformly bounded and starshaped with respect to the same disk of radius r. In this case due to standard embedding theorems applied to the enlarged sleeve $S_{j,l}^\varepsilon$ we get

$$\left|c(v_l) - N_j u\left(a_{j,\varepsilon}^1\right)\right| \le C \|u - c(v_l)\|_{H^1(S_{j,l}^\varepsilon)} \le C \|u - c(v_l)\|_{H^1(W_l^\varepsilon)}.$$

Now the Lemma 12 applies, which gives the uniform with respect to ε estimate

$$\|u - c(v_l)\|_{H^1(W_l^\varepsilon)} \le C ||\nabla u||_{L^2(W_l^\varepsilon)}.$$

Shrinking the domain back we get (25).

Corollary 16 *For any $u \in H^1(M^\varepsilon)$ the following inequality holds*

$$||u||_{L^2(U_l^\varepsilon)} \le C\sqrt{\varepsilon}\, ||u||_{H^1(M^\varepsilon)} \tag{26}$$

with a constant C independent of u and ε.

Proof. We estimate $||u||_{L^2(U_l^\varepsilon)}$ as follows:

$$||u||_{L^2(U_l^\varepsilon)} \le ||u - c(v_l)||_{L^2(U_l^\varepsilon)} + \left|\left|c(v_l) - N_j u(a_{j,\varepsilon}^1)\right|\right|_{L^2(U_l^\varepsilon)} + \left|\left|N_j u(a_{j,\varepsilon}^1)\right|\right|_{L^2(U_l^\varepsilon)}.$$

According to (23), the first term is estimated from above by $C\varepsilon\, ||u||_{H^1(M^\varepsilon)}$. The second and third terms together can be estimated by

$$\left(\left|c(v_l) - N_j u(a_{j,\varepsilon}^1)\right| + \left|N_j u(a_{j,\varepsilon}^1)\right|\right) (vol(U_l^\varepsilon))^{1/2} \le C\varepsilon \left(\left|c(v_l) - N_j u(a_{j,\varepsilon}^1)\right| + \left|N_j u(a_{j,\varepsilon}^1)\right|\right).$$

Since (20) and (25) show that

$$\left|c(v_l) - N_j u(a_{j,\varepsilon}^1)\right| + \left|N_j u(a_{j,\varepsilon}^1)\right| \le C\varepsilon^{-1/2}\, ||u||_{H^1(M^\varepsilon)},$$

this proves (26).

We also will use the following elementary inequalities:

Lemma 17 *For any real numbers a, b, c, and $\delta \in (0,1)$ the following inequalities hold:*

$$\begin{aligned} (1-2\delta)\, a^2 - \frac{b^2}{\delta} - \frac{c^2}{\delta} &\le (a+b+c)^2 \\ &\le (1+2\delta)\, a^2 + \left(1 + \frac{2}{\delta}\right) b^2 + \left(1 + \frac{2}{\delta}\right) c^2. \end{aligned} \tag{27}$$

Let us go back to the proof of the Lemma 7, which is our last task in the proof of the Theorem 1.

Proof of the Lemma 7. The proof of the first statement of the lemma is straightforward and we omit it. Let us prove the second statement. Since all the expressions in (15) and (16) are continuous in $H^1(M^\varepsilon)$, it is sufficient to prove the lemma for infinitely smooth functions u only. So we will assume that $u \in C^\infty(M^\varepsilon)$. We will start with proving the inequality (15) first. The expression in the left hand side is

$$E(P^\varepsilon u, P^\varepsilon u) = \sum_j \int_{a_j}^{b_j} \left|\frac{dP^\varepsilon u}{dx_j}\right|^2 p(x_j) dx_j.$$

Since the function $P^{\varepsilon}u$ is constant in a vicinity of the vertex, the integrand is non-zero on $[a^1_{j,\varepsilon}, b^1_{j,\varepsilon}]$ only. The definition (14) of $P^{\varepsilon}u$ contains three terms. We need to estimate the (weighted) integrals of squares of derivatives of each of them. Due to (6), the measures dA and $dx_j dy_j$ on M_j^{ε} differ by a factor $(1+O(\varepsilon))$ only, so in order to shorten the expressions we will neglect their difference. Let us start with $N_j u$. It is beneficial to change the variables in the tube M_j^{ε} as follows:

$$x = x_j,\; y = \frac{(y_j + \varepsilon g_j(x_j))}{\varepsilon p(x_j)}.$$

This turns M_j^{ε} it into a flat constant width strip $[a^1_{j,\varepsilon}, b^1_{j,\varepsilon}] \times [0,1]$. Then we can rewrite

$$N_j u(x_j) = \frac{1}{\varepsilon p(x_j)} \int\limits_{-g_j(x_j)\varepsilon}^{h_j(x_j)\varepsilon} u(x_j, y_j)\, dy_j = \int\limits_0^1 u(x_j, \varepsilon(p(x_j)\, y - g_j(x_j)))\, dy.$$

This leads to

$$\frac{dN_j u}{dx_j} = \int_0^1 \left(\frac{\partial u}{\partial x_j} + \frac{\partial u}{\partial y_j}\left[\varepsilon\left(yp' - g_j'\right)\right]\right) dy$$

$$= \frac{1}{\varepsilon p(x_j)} \int\limits_{-\varepsilon g_j(x_j)}^{\varepsilon h_j(x_j)} \left[\frac{\partial u}{\partial x_j} + \varepsilon \frac{\partial u}{\partial y_j}\left(\frac{y_j}{\varepsilon}\frac{p'(x_j)}{p(x_j)} + \frac{g_j(x_j)\, p'(x_j)}{p(x_j)} - g_j'(x_j)\right)\right] dy_j. \tag{28}$$

Hence,

$$\left|\frac{dN_j u}{dx_j}\right| \le \frac{1}{\varepsilon p(x_j)} \int\limits_{-\varepsilon g_j(x_j)}^{\varepsilon h_j(x_j)} \left(\left|\frac{\partial u}{\partial x_j}\right| + C\varepsilon \left|\frac{\partial u}{\partial y_j}\right| \right) dy_j,$$

which due to the Hölder inequality gives

$$\left|\frac{dN_j u}{dx_j}\right|^2 \le \frac{1}{\varepsilon p(x_j)} \int\limits_{-\varepsilon g_j(x_j)}^{\varepsilon h_j(x_j)} \left(\left|\frac{\partial u}{\partial x_j}\right| + C\varepsilon \left|\frac{\partial u}{\partial y_j}\right| \right)^2 dy_j \le \frac{1+O(\varepsilon)}{\varepsilon p(x_j)} \int\limits_{-\varepsilon g_j(x_j)}^{\varepsilon h_j(x_j)} |\nabla u|^2\, dy_j.$$

So we get

$$\int_{a_{j,\varepsilon}^{1}}^{b_{j,\varepsilon}^{1}} p\left(x_{j}\right)\left|\frac{dN_{j}u}{dx_{j}}\right|^{2} dx_{j} \leq \frac{1+O(\varepsilon)}{\varepsilon}\int_{M_{j}^{\varepsilon}} \left|\nabla u\right|^{2} dA. \tag{29}$$

The remaining two terms in $P^{\varepsilon}u$ are similar, so we will estimate only one of them:

$$\int_{a_{j,\varepsilon}^{1}}^{b_{j,\varepsilon}^{1}} p\left(x_{j}\right)\left|\frac{d}{dx_{j}}\left(\left[c\left(v_{l}\right)-N_{j}u\left(a_{j,\varepsilon}^{1}\right)\right]\rho(x_{j}-a_{j,\varepsilon}^{1})\right)\right|^{2} dx_{j}$$

$$=\left|c\left(v_{l}\right)-N_{j}u\left(a_{j,\varepsilon}^{1}\right)\right|^{2}\int_{a_{j,\varepsilon}^{1}}^{b_{j,\varepsilon}^{1}} p\left(x_{j}\right)\left|\rho'(x_{j}-a_{j,\varepsilon}^{1})\right|^{2} dx_{j}$$

$$\leq C\left|c\left(v_{l}\right)-N_{j}u\left(a_{j,\varepsilon}^{1}\right)\right|^{2},$$

which after using (25) can be estimated from above by $C\left|\left|\nabla u\right|\right|_{L^{2}(M^{\varepsilon})}^{2}$. Bringing all these estimates together and using (27), we can now complete the proof of the bound (15). Namely, choosing in (27) $\delta=\sqrt{\varepsilon}$, we get

$$\int_{a_{j,\varepsilon}^{1}}^{b_{j,\varepsilon}^{1}} p\left(x_{j}\right)\left|\frac{dP^{\varepsilon}u}{dx_{j}}\right|^{2} dx_{j} \leq \left(\left(1+2\sqrt{\varepsilon}\right)\frac{1+O(\varepsilon)}{\varepsilon}+C\left(1+\frac{2}{\sqrt{\varepsilon}}\right)\right)\left|\left|\nabla u\right|\right|_{L^{2}(M^{\varepsilon})}^{2}$$

$$=\frac{1+O(\sqrt{\varepsilon})}{\varepsilon}\left|\left|\nabla u\right|\right|_{L^{2}(M^{\varepsilon})}^{2},$$

This gives (15)

Let us now move to the proof of (16). We will still assume that $u\in C^{\infty}(M^{\varepsilon})$. We need to compare the L_{2} norms of the function u and of its averaged version $P^{\varepsilon}u$. The inequality (18) gives

$$\int_{-\varepsilon g_{j}(x_{j})}^{\varepsilon h_{j}(x_{j})}\left|u(x_{j},y_{j})-\frac{1}{\varepsilon p\left(x_{0}\right)}\int_{-\varepsilon g_{j}(x_{j})}^{\varepsilon h_{j}(x_{j})} u(x_{j},t)dt\right|^{2} dy_{j} \leq \varepsilon^{2}p^{2}\left(x_{j}\right)\int_{-\varepsilon g_{j}(x_{j})}^{\varepsilon h_{j}(x_{j})}\left|\frac{\partial u}{\partial y_{j}}\right|^{2} dy_{j}$$

or

$$\int_{-\varepsilon g_j(x_j)}^{\varepsilon h_j(x_j)} |u - N_j u|^2 \, dy_j \le \varepsilon^2 p^2(x_j) \int_{-\varepsilon g_j(x_j)}^{\varepsilon h_j(x_j)} \left| \frac{\partial u}{\partial y_j} \right|^2 dy_j$$

Integrating with respect to x_j and using (6), we get

$$\int_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} \int_{-\varepsilon g_j(x_j)}^{\varepsilon h_j(x_j)} |u - N_j u|^2 \, dy_j dx_j \le C\varepsilon^2 \int_{M_j^\varepsilon} |\nabla u|^2 \, dA. \tag{30}$$

Let us estimate from below the norm

$$||N_j u||^2_{L^2(M_j, pdx)} \ge ||N_j u||^2_{L^2([a^1_{j,\varepsilon}, b^1_{j,\varepsilon}], pdx)} = \int_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} |N_j u|^2 \, p dx_j = \varepsilon^{-1} \int_{M_j^\varepsilon} |N_j u|^2 \, dy_j dx_j.$$

Representing $N_j u = u - (u - N_j u)$ and using the inequality (27) with $a = u$, $b = u - N_j u$, $c = 0$, and $\delta = \frac{\sqrt{\varepsilon}}{2}$, we get

$$\int_{M_j^\varepsilon} |N_j u|^2 \, dy_j dx_j \ge \left(1 - \sqrt{\varepsilon}\right) \int_{M_j^\varepsilon} |u|^2 \, dy_j dx_j - \frac{2}{\sqrt{\varepsilon}} \int_{M_j^\varepsilon} |u - N_j u|^2 \, dy_j dx_j.$$

Using now (30), we obtain

$$||N_j u||^2_{L^2(M_j, p_j dx)} \ge \varepsilon^{-1} \left(1 + O(\sqrt{\varepsilon})\right) \left(||u||^2_{L^2(M_j^\varepsilon)} - C\varepsilon^{3/2} \, ||\nabla u||^2_{L^2(M_j^\varepsilon)} \right). \tag{31}$$

The quantity that we need to estimate from below in (16) is $||P^\varepsilon u||^2$. The definition of P^ε, (27), and (31) result in

$$\int_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} |P^\varepsilon u(x_j)|^2 \, p(x_j) \, dx_j \ge (1 - 2\sqrt{\varepsilon}) \int_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} |N_j u(x_j)|^2 \, p(x_j) \, dx_j$$

$$- \frac{\left| c(v_l) - N_j u\left(a^1_{j,\varepsilon}\right) \right|^2}{\sqrt{\varepsilon}} \int_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} \left| \rho(x_j - a^1_{j,\varepsilon}) \right|^2 p(x_j) \, dx_j$$

$$-\frac{\left|c\left(v_m\right)-N_j u\left(b^1_{j,\varepsilon}\right)\right|^2}{\sqrt{\varepsilon}}\int\limits_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}}\left|\rho(x_j-b^1_{j,\varepsilon})\right|^2 p\left(x_j\right)dx_j$$

$$\geq\frac{(1-C\sqrt{\varepsilon})}{\varepsilon}\left[\int_{M_j^\varepsilon}|u|^2\,dx_jdy_j-C\varepsilon^{3/2}\int_{M_j^\varepsilon}|\nabla u|^2\,dx_jdy_j\right]$$

$$-\frac{C}{\sqrt{\varepsilon}}\left|c\left(v_l\right)-N_j u\left(a^1_{j,\varepsilon}\right)\right|^2-\frac{C}{\sqrt{\varepsilon}}\left|c\left(v_m\right)-N_j u\left(b^1_{j,\varepsilon}\right)\right|^2.$$

According to the Lemma 15, we can estimate the last expression from below to get

$$\int\limits_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}}\left|P^\varepsilon u\left(x_j\right)\right|^2 p\left(x_j\right)dx_j$$

$$\geq\frac{(1-C\sqrt{\varepsilon})}{\varepsilon}\left[\int_{M_j^\varepsilon}|u|^2\,dx_jdy_j-C\varepsilon^{3/2}\int_{M_j^\varepsilon}|\nabla u|^2\,dx_jdy_j\right]-\frac{C}{\sqrt{\varepsilon}}\left\|\nabla u\right\|^2_{L^2\left(U_l^\varepsilon\right)}$$

$$\geq\frac{(1-C\sqrt{\varepsilon})}{\varepsilon}\left[\int_{M_j^\varepsilon}|u|^2\,dx_jdy_j-C\varepsilon^{3/2}\int_{M_j^\varepsilon}|\nabla u|^2\,dx_jdy_j-C\sqrt{\varepsilon}\left\|\nabla u\right\|^2_{L^2\left(U_l^\varepsilon\right)}\right].$$

Adding these inequalities for all values of j and using then (26), we obtain

$$\sum_j\int\limits_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}}\left|P^\varepsilon u\left(x_j\right)\right|^2 p\left(x_j\right)dx_j$$

$$\geq\frac{(1-C\sqrt{\varepsilon})}{\varepsilon}\left[\|u\|^2_{L^2(M^\varepsilon)}-C\sqrt{\varepsilon}\left\|\nabla u\right\|^2_{L^2(M^\varepsilon)}-\sum_l\|u\|^2_{L^2\left(U_l^\varepsilon\right)}\right]$$

$$\geq\frac{(1-C\sqrt{\varepsilon})}{\varepsilon}\left[\|u\|^2_{L^2(M^\varepsilon)}-C\sqrt{\varepsilon}\left\|\nabla u\right\|^2_{L^2(M^\varepsilon)}\right].$$

This finishes the proof of the Lemma 7 and hence of the Theorem 1.

4 Operators with electromagnetic fields

Let a Lipschitz real valued function q (electric potential) and an $\mathbb{R}^2$-valued vector function $\mathbf{A}$ of class C^1 (magnetic potential) be defined in a neighborhood of the graph M. The definitions of the operators $-\Delta_\varepsilon$ and A_p can be modified to include these fields. Namely, the ambient spaces and the domains of the corresponding quadratic forms remain the same, while the expressions for the quadratic forms now are modified as follows:

$$E^\varepsilon(u, u; \mathbf{A}, q) = \int\limits_{M^\varepsilon} \left(|(\nabla + i\mathbf{A})u|^2 + q\,|u|^2\right) dx \tag{32}$$

and

$$E(f, f; A, q) = \sum_j \int\limits_{M_j} \left(\left|\frac{df}{dx_j} + iA_j^\tau f\right|^2 + q\,|f|^2\right) p dx_j \tag{33}$$

correspondingly. Here A_j^τ denotes the tangential component of the field $\mathbf{A}$ along the edge M_j. Namely, if $\phi_j : [a_j, b_j] \to \mathbb{R}^2$ is the C^2-mapping representing the edge M_j, then $A_j^\tau = A \cdot \frac{d\varphi_j}{dx_j}$. We will denote these operators by $H_\varepsilon(\mathbf{A}, q)$ and $H_M(A, q)$ correspondingly. As a differential expression, $H_M(A, q)$ acts as

$$H_M(A, q)f = -\frac{1}{p}\left(\frac{d}{dx_j} + iA_j^\tau\right) p \left(\frac{d}{dx_j} + iA_j^\tau\right) f + qf$$

along the edge M_j, with the domain consisting of all functions $f \in H^1(M)$ such that $f \in H^2(M_j)$ for each j and that at each vertex v

$$\sum_{j \in J(v)} p_j(v) \left(\frac{df}{dx_j}(v) + iA_j^\tau(v) f(v)\right) = 0.$$

Here the derivatives df/dx_j are taken in the direction away from the vertex v along the edges M_j containing v, and p_j is the restriction of p to the edge M_j. The values $p_j(v)$ do not have to be (and usually are not) the same for different indices j such that $v \in M_j$. Our goal now is prove the statement of the Theorem 2 for these operators:

Theorem 2. *Under the imposed conditions,*

$$\lim_{\varepsilon \to 0} \lambda_n(H_\varepsilon(\mathbf{A}, q)) = \lambda_n(H_M(A, q)) \tag{34}$$

for any $n = 1, 2, \ldots$.

We notice first of all that shifting if necessary by a constant, we can assume without loss of generality that $q > c > 0$. The proof of the theorem follows the same lines as (and actually uses several parts of) the proof of Theorem 1. The convergence (34) will be established if we introduce appropriate extension and averaging operators $Q^{\varepsilon} : H^1(M) \to H^1(M^{\varepsilon})$ and $P^{\varepsilon} : H^1(M^{\varepsilon}) \to H^1(M)$ such that they do not increase significantly the corresponding Rayleigh ratios. The definition (12) - (14) of P^{ε} is the same as when the electromagnetic field is absent, while the definition (3) of Q^{ε} needs to be modified a little to allow for the presence of the magnetic potential.

Let us define the tubular domain B_j^{ε} (which is just a little bit shortened M_j^{ε}) as follows:

$$B_j^{\varepsilon} = \left\{(x_j, y_j) | \, x_j \in [a_{j,\varepsilon}^2, b_{j,\varepsilon}^2], -\varepsilon g_j(x_j) < y_j < \varepsilon h_j(x_j)\right\}.$$

The following objects will be needed: $\nu = \nu(x_j, y_j)$ is the unit tangent vector at the point $(x_j, y_j) \in B_j^{\varepsilon}$ to the coordinate line $x_j = const$ in the direction of increase of y_j and $\tau = \tau(x_j, y_j)$ is the unit normal vector to this line. Analogously, $\rho = \rho(x_j, y_j)$ is the unit tangent vector to the coordinate curve $y_j = const$ in the direction of increase of x_j. Notice that $\tau - \rho = O(\varepsilon)$. The vector fields (i.e., first order differential operators) corresponding to ν, τ, and ρ respectively, are denoted by N, T, and R. So, in particular $R = \partial/\partial x_j$, $N = \partial/\partial y_j$, and $T = \partial/\partial x_j + L(\varepsilon)$, where $L(\varepsilon)$ is a first order differential operator with continuous $O(\varepsilon)$ coefficients. We also introduce in M_j^{ε} the normal and tangential components $A_j^{\nu} = A \cdot \nu$ and $A_j^{\tau} = A \cdot \tau$ of the magnetic potential A.

Let us move on now to the definition of the extension operator Q^{ε}. While in the absence of magnetic potential we extended functions from M_j as constants in y_j, now we have to extend them as appropriate exponents. Namely,

Definition 18 *For* $u \in H^1(M)$, *we define the function* $Q^{\varepsilon}u$ *on* M^{ε} *as follows:*

$$Q^{\varepsilon}u = \begin{cases} u(v) & in\ V_l^{\varepsilon}(v) \\ \frac{1}{a_{j,\varepsilon}^2 - a_{j,\varepsilon}^1}\left[a_{j,\varepsilon}^2 - x_j + \left(x_j - a_{j,\varepsilon}^1\right) e^{r_j}\right] u(v) & in\ S_{j,l}^{\varepsilon} \\ e^{r_j}\left(u \circ \psi_j\right)(x_j, 0) & in\ B_j^{\varepsilon} \end{cases}. \quad (35)$$

Here, analogously to (3), ψ_j *is the linear mapping that stretches the segment*

$\left[a_{j,\varepsilon}^2, b_{j,\varepsilon}^2\right]$ *to* $[a_j, b_j]$ *and*

$$r_j(x_j, y_j) = -i \int_0^{y_j} A_j^\nu (x_j, y)\, dy.$$

We will denote later the function $(u \circ \psi_j)(x_j, 0)$ involved in (35) by $\widetilde{u}$, so in B_j^ε we have $Q^\varepsilon u = e^{r_j} \widetilde{u}$.

There are two differences between this definition and the one in the absence of magnetic potential. First of all, the exponential factor $\exp r_j$ arises in the tubular part. Secondly, since in the vicinity of the vertex the extended function is constant, while at the edge of the tube B_j^ε it is not, in order to preserve continuity of the extended function one needs a (linear) homotopy between them in the sleeve $S_{j,l}^\varepsilon$.

Our goal now is to prove Lemma 19 and Lemma 21 below that are analogs of Lemmas 4 and 7. This would immediately lead to the statement of the theorem.

Lemma 19 *1. For any sufficiently small $\varepsilon > 0$ the operator Q^ε maps $H^1(M)$ into $H^1(M^\varepsilon)$.*

2. For any sufficiently small $\varepsilon > 0$ and for any $u \in H^1(M)$, the following inequalities hold:

$$||Q^\varepsilon u||_{L^2(M^\varepsilon)}^2 \geq \varepsilon \left(1 + O(\sqrt{\varepsilon})\right) ||u||_{L^2(M)}^2, \tag{36}$$

$$E^\varepsilon (Q^\varepsilon u, Q^\varepsilon u; \mathbf{A}, q) \leq \varepsilon \left(1 + O(\sqrt{\varepsilon})\right) \left(E(u, u; A, q) + O(\varepsilon) ||u||_{L^2(M)}^2\right), \tag{37}$$

where $O(\varepsilon)$ and $O(\sqrt{\varepsilon})$ depend on ε and the potentials only (and do not depend on u).

Proof: The first statement of the lemma is obvious, since the extended function $Q^\varepsilon u$ by the construction belongs to H^1 in each of the parts $V_l^\varepsilon(v)$, $S_{j,l}^\varepsilon$, and B_j^ε, and is continuous through their boundaries.

Let us shift to proving (36) of the second claim of the lemma. Suppose that $u \in H^1(M)$. Neglecting the sets U_l^ε, we get

$$||Q^\varepsilon u||_{L^2(M^\varepsilon)}^2 \geq \sum_j \int_{M_j^\varepsilon} |Q^\varepsilon u|^2\, dxdy.$$

Since $|\exp(r_j)| = 1$, starting from here the estimates of the corresponding part of the proof of Lemma 4 are applicable, which yields (36).

The last task is to prove (37). Let us first of all notice a straightforward inequality that holds for any $u \in H^1(M)$:

$$||u||^2_{H^1(M)} \leq C \left(E(u, u; A, q) + ||u||^2_{L^2(M,pdx)} \right). \tag{38}$$

The integration over M^ε in the expression for $E^\varepsilon(Q^\varepsilon u, Q^\varepsilon u; \mathbf{A}, q)$ can be split into two: over $U_l^\varepsilon(v)$ (which is the union of $V_l^\varepsilon(v)$ with the sleeve $S^\varepsilon_{j,l}$) and over B_j^ε. In $U_l^\varepsilon(v)$, according to (35), $Q^\varepsilon u = \zeta_\varepsilon u(v)$, where ζ_ε is a function which is bounded with its derivative uniformly with respect to ε. Hence, we can estimate

$$\int\limits_{U_l^\varepsilon(v)} \left(|(\nabla + i\mathbf{A})Q^\varepsilon u|^2 + q\,|Q^\varepsilon u|^2 \right) dA = |u(v)|^2 \int\limits_{U_l^\varepsilon(v)} \left(|(\nabla + i\mathbf{A})\zeta_\varepsilon|^2 + q\,|\zeta_\varepsilon|^2 \right) dA$$

$$\leq C\varepsilon^2\,|u(v)|^2 \leq C\varepsilon^2\,||u||^2_{H^1(M)} \leq C\varepsilon^2 \left(E(u, u; A, q) + ||u||^2_{L^2(M,pdx)} \right), \tag{39}$$

where we used (38). Consider now the tubular part B_j^ε. Then we have the following relations:

$$\nabla = \nu N + \tau T, \mathbf{A} = A_j^\nu \nu + A_j^\tau \tau, N r_j = -iA_j^\nu, \text{ and } Re^{r_j} = O(\varepsilon). \tag{40}$$

The integrand of the integral over the tube B_j^ε can be transformed as follows:

$$|(\nabla + i\mathbf{A})Q^\varepsilon u|^2 + q\,|Q^\varepsilon u|^2 = |(\nabla + i\mathbf{A})e^{r_j}\widetilde{u}|^2 + q\,|\widetilde{u}|^2$$

$$= |(\nabla + \nabla(r_j) + i\mathbf{A})\widetilde{u}|^2 + q(x_j, 0)\,|\widetilde{u}|^2 + (q - q(x_j, 0))\,|\widetilde{u}|^2. \tag{41}$$

Using (40) and $T = \frac{\partial}{\partial x_j} + L(\varepsilon)$, we transform the first term into

$$|(\nabla + \nabla(r_j) + i\mathbf{A})\widetilde{u}|^2 = \left| \left(\nu N + \tau T + (\nu N + \tau T)(r_j) + i\left(A_j^\nu \nu + A_j^\tau \tau \right) \right) \widetilde{u} \right|^2$$

$$= \left| \tau\left(\frac{\partial}{\partial x_j} + iA_j^\tau\right)\widetilde{u} + L_1(\varepsilon)\widetilde{u} \right|^2,$$

where $L_1(\varepsilon)$ is a first order differential operator with continuous $O(\varepsilon)$ coefficients. Using this and the Lipschitz property of q, one can estimate (41) from above by

$$\left| \tau\left(\frac{\partial}{\partial x_j} + iA_j^\tau\right)\widetilde{u} + L_1(\varepsilon)\widetilde{u} \right|^2 + q(x_j, 0)\,|\widetilde{u}|^2 + O(\varepsilon)\,|\widetilde{u}|^2.$$

Now integrating over B_j^ε, eliminating the stretching function ψ_j as this was done in (7) - (10), and using (38) again, we get the estimate

$$\int\limits_{B_j^\varepsilon} \left(|(\nabla + i\mathbf{A})Q^\varepsilon u|^2 + q\,|Q^\varepsilon u|^2\right) dA \leq \varepsilon\left(1 + O(\sqrt{\varepsilon})\right)\left(E\left(u, u; A, q\right) + O\left(\varepsilon\right)||u||^2_{L^2(M)}\right).$$

This inequality together with (39) proves (37) and finishes the proof of the lemma.

Application of the minimax description of the eigenvalues leads to the

Corollary 20 *For any $n = 1, 2, ...$*

$$\limsup_{\varepsilon\to 0} \lambda_n(H_\varepsilon(\mathbf{A}, q)) \leq \lambda_n(H_M(A, q)).$$

In order to get an estimate from below for $\lambda_n(H_\varepsilon(\mathbf{A}, q))$, we use the same averaging operator $P^\varepsilon : H^1(M^\varepsilon) \to H^1(M)$ as in the preceding section.(see Definition 6).

Lemma 21 *For any sufficiently small $\varepsilon > 0$ and for any $u \in H^1\left(M^\varepsilon\right)$*

$$||P^\varepsilon u||^2_{L^2(M)} \geq \varepsilon^{-1}\left[\left(1 + O(\sqrt{\varepsilon})\right)||u||^2_{L^2(M^\varepsilon)} - O(\sqrt{\varepsilon})E^\varepsilon\left(u, u; \mathbf{A}, q\right)\right] \quad (42)$$

and

$$E\left(P^\varepsilon u, P^\varepsilon u; A, q\right) \leq \varepsilon^{-1}(1+O(\sqrt{\varepsilon}))(E^\varepsilon\left(u, u; \mathbf{A}, q\right)+O(\varepsilon)\,||u||^2_{L^2(M^\varepsilon)}). \quad (43)$$

Here $O(\sqrt{\varepsilon})$ is uniform with respect to u.

Proof: Let us mention first the following analog of (38): for $u \in H^1(M^\varepsilon)$

$$||u||^2_{H^1(M^\varepsilon)} \leq C\left(E^\varepsilon\left(u, u; \mathbf{A}, q\right) + ||u||^2_{L^2(M^\varepsilon)}\right). \quad (44)$$

Inequality (42) follows from (16) and (44). So, let us prove (43). We prove it in two steps. Since

$$E\left(P^\varepsilon u, P^\varepsilon u; A, q\right) = E\left(P^\varepsilon u, P^\varepsilon u; A, 0\right) + \left(qP^\varepsilon u, P^\varepsilon u\right), \quad (45)$$

we will estimate each of the two terms separately. Consider $E\left(P^{\varepsilon}u, P^{\varepsilon}u; A, 0\right)$ first. The definition (14) of P^{ε} in M_j^{ε} consists of three terms. Let us tackle the first (and the main) of them N_j. Simple calculation (see (28)) shows that

$$\left|\frac{dN_j u}{dx_j} + iA_j^{\tau} N_j u\right|^2$$

$$= \left|\frac{1}{\varepsilon p\left(x_j\right)} \int_{-\varepsilon g_j(x_j)}^{\varepsilon h_j(x_j)} \left(\frac{\partial u}{\partial x_j} + \varepsilon \frac{\partial u}{\partial y_j} w_j + iA_j^{\tau}(x_j, 0)u\right) dy_j\right|^2$$

$$\leq \frac{1}{\varepsilon p\left(x_j\right)} \int_{-\varepsilon g_j(x_j)}^{\varepsilon h_j(x_j)} \left|\frac{\partial u}{\partial x_j} + \varepsilon \frac{\partial u}{\partial y_j} w_j + iA_j^{\tau}(x_j, 0)u\right|^2 dy_j \tag{46}$$

with the uniformly bounded function

$$w_j = \frac{y_j}{\varepsilon} \frac{p'\left(x_j\right)}{p\left(x_j\right)} + \frac{g_j\left(x_j\right) p'\left(x_j\right)}{p\left(x_j\right)} - g_j'\left(x_j\right).$$

The integrand $\left|\frac{\partial u}{\partial x_j} + \varepsilon \frac{\partial u}{\partial y_j} w_j + iA_j^{\tau}(x_j, 0)u\right|^2$ can be transformed into

$$\left|\left(\frac{\partial u}{\partial x_j} + iA_j^{\tau}(x_j, y_j)u\right) + i\left(A_j^{\tau}(x_j, 0) - A_j^{\tau}(x_j, y_j)\right)u + \varepsilon w_j \frac{\partial u}{\partial y_j}\right|^2.$$

Since the magnetic potential is in C^1 and w is bounded, Lemma 17 gives an estimate of (46) from above by

$$\leq \left(1 + O(\sqrt{\varepsilon})\right)\left|\frac{\partial u}{\partial x_j} + iA_j^{\tau}(x_j, y_j)u\right|^2 + O\left(\varepsilon^{3/2}\right)\left(|u|^2 + \left|\frac{\partial u}{\partial y_j}\right|^2\right)$$

$$\leq \left(1 + O(\sqrt{\varepsilon})\right)|\nabla u + i\mathbf{A}u|^2 + O\left(\varepsilon^{3/2}\right)\left(|u|^2 + |\nabla u|^2\right). \tag{47}$$

Thus,

$$\int_{a_{j,\varepsilon}^1}^{b_{j,\varepsilon}^1} \left|\frac{dN_j u}{dx_j} + iA_j^{\tau} N_j u\right|^2 p(x_j) dx_j$$

$$\leq \frac{1 + O\left(\sqrt{\varepsilon}\right)}{\varepsilon} \int_{M_j^{\varepsilon}} |\nabla u + i\mathbf{A}u|^2 \, dA + O\left(\varepsilon^{1/2}\right) \int_{M_j^{\varepsilon}} \left(|u|^2 + |\nabla u|^2\right) dA \tag{48}$$

Let us now move to estimating the energy of the remaining two terms in $P^{\varepsilon}u$ (see (14)). They are defined in the same way, so it is sufficient to estimate the first of them. This leads to the expression

$$\left|c(v_l) - N_j u\left(a^1_{j,\varepsilon}\right)\right|^2 \int\limits_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} p(x_j)\left|\rho'(x_j - a^1_{j,\varepsilon}) + iA^{\tau}_j \rho(x_j - a^1_{j,\varepsilon})\right|^2 dx_j$$
$$\leq C\left|c(v_l) - N_j u\left(a^1_{j,\varepsilon}\right)\right|^2, \tag{49}$$

which according to Lemma 15 can be estimated from above by $C\|\nabla u\|^2_{L^2(M^{\varepsilon})}$. Now using lemma 17 again and putting (48) and (49) together, we obtain

$$\int\limits_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} p(x_j)\left|\frac{dP^{\varepsilon}u}{dx_j} + iA^{\tau}_j P^{\varepsilon}u\right|^2 dx_j$$
$$\leq \frac{1+O(\sqrt{\varepsilon})}{\varepsilon}\left[\int\int\limits_{M^{\varepsilon}} |\nabla u + i\mathbf{A}u|^2 dA + O(\varepsilon)\,||u||^2_{H^1(M^{\varepsilon})}\right]. \tag{50}$$

We also need to estimate the part of the energy functional containing integration over the neighborhood U^{ε}_l of a vertex v_l. In this neighborhood $P^{\varepsilon}u$ equals the constant $c(v_l)$ (see (12)). Hence, the only non-zero part of the energy functional (remember that we do not take yet the electric potential into account) is

$$\int\limits_{a_j}^{a^1_{j,\varepsilon}} \left|iA^{\tau}_j c(v_l)\right|^2 dx_j \leq C\varepsilon\,|c(v_l)|^2 \leq C\,\|u\|^2_{H^1(M^{\varepsilon})}. \tag{51}$$

The last inequality above follows from (20) and (25). Now (50), (51), and (44) imply

$$E\left(P^{\varepsilon}u, P^{\varepsilon}u; A, 0\right) \leq \frac{(1+O(\sqrt{\varepsilon}))}{\varepsilon}\left[E^{\varepsilon}(u,u;\mathbf{A},0) + O(\varepsilon)\,\|u\|^2_{L^2(M^{\varepsilon})}\right]. \tag{52}$$

What remains is to estimate the expression $(qP^{\varepsilon}u, P^{\varepsilon}u)$. It consists of two types of terms:

$$\int\limits_{a_j}^{a^1_{j,\varepsilon}} q(x_j, 0)\,|c(v_l)|^2\, p(x_j)\, dx_j \tag{53}$$

and

$$\int_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} q\left(x_j,0\right)\left|P^{\varepsilon}u\right|^2 p\left(x_j\right)dx_j. \tag{54}$$

According to the last inequality in (51) and boundedness of q and p, the expression in (53) does not exceed $C\left|\left|u\right|\right|^2_{H^1(M^{\varepsilon})}$. The integral (54) can be estimated from above using Lemma 17 by

$$\left(1+O(\sqrt{\varepsilon})\right)\int_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} q\left(x_j,0\right)\left|N_j u\right|^2 p\left(x_j\right)dx_j$$

$$+\left(1+O\left(\frac{1}{\sqrt{\varepsilon}}\right)\right)\int_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} q_j\left(x,0\right)\left[c\left(v_l\right)-N_j u(a^1_{j,\varepsilon})\right]^2\rho^2\left(x_j-a^1_{j,\varepsilon}\right)p\left(x_j\right)dx_j$$

$$+\left(1+O\left(\frac{1}{\sqrt{\varepsilon}}\right)\right)\int_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} q_j\left(x,0\right)\left[c\left(v_m\right)-N_j u(b^1_{j,\varepsilon})\right]^2\rho^2\left(x_j-b^1_{j,\varepsilon}\right)p\left(x_j\right)dx_j. \tag{55}$$

After adding expressions (55) for all edges M_j, we can bounded the first tem as follows:

$$\sum_j\int_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} q\left(x_j,0\right)\left|N_j u\right|^2 p\left(x_j\right)dx_j \le \frac{1}{\varepsilon}\sum_j\int_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}} q\left(x_j,0\right)\left(\int_{-\varepsilon g_j(x_j)}^{\varepsilon h_j(x_j)}\left|u(x_j,y_j)\right|^2 dy_j\right)dx_j$$

$$\le \frac{1}{\varepsilon}\sum_j\int_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}}\int_{-\varepsilon g_j(x_j)}^{\varepsilon h_j(x_j)}\left|q\left(x_j,0\right)-q\left(x_j,y_j\right)\right|\left|u(x_j,y_j)\right|^2 dy_j dx_j$$

$$+\frac{1}{\varepsilon}\sum_j\int_{a^1_{j,\varepsilon}}^{b^1_{j,\varepsilon}}\int_{-\varepsilon g_j(x_j)}^{\varepsilon h_j(x_j)} q\left(x_j,y_j\right)\left|u(x_j,y_j)\right|^2 dy_j dx_j$$

$$\leq (1+O(\varepsilon))\left(C\,||u||^2_{L^2(M^{\varepsilon})}+\frac{1}{\varepsilon}\int_{M^{\varepsilon}} q\,|u|^2\,dA\right). \tag{56}$$

The last step in proving the lemma is to estimate the second (and similarly the third) term in (55). Here we use (25) to get

$$\int_{a_j}^{b_j} q(x_j,0)\left[c(v_l)-N_j u(a^1_{j,\varepsilon})\right]^2 \rho^2(x-a_j)\,p(x_j)\,dx_j$$

$$\leq C\left[c(v_l)-N_j u(a^1_{j,\varepsilon})\right]^2 \leq C\,\|\nabla u\|^2_{L^2\left(U_l^{\varepsilon}\right)} \tag{57}$$

Inequalities (52), (44), (56), and (57) together lead to (43). This finishes the proof of the lemma.

As in the previous section, the last lemma implies

Corollary 22 *For any* $n=1,2,...$

$$\liminf_{\varepsilon\to 0}\lambda_n(H_{\varepsilon}(\mathbf{A},q))\geq \lambda_n(H_M(A,q)).$$

The proof of Theorem 2 consists in combining the Corollaries 20 and 22.

5 Final remarks

1. We were inspired by the ground breaking paper [46]. Let us provide here a little bit more detailed comparison between its results and proofs and those of the current work.

   Additional features that we allowed are variable thickness of the pipes around the graph edges and presence of an electric potential. As our results show, variable thickness leads to a different limit operator on graph.

   One of the main difficulties is to handle neighborhoods of vertices of the graph. As we have realized, the details of the geometry of the domain M^{ε} near a graph vertex (for instance, smoothness of the boundary) are irrelevant. What is important, though, is the validity of suitable Poincare inequalities uniformly with respect to ε (see Corollary 14 and Lemma 15). It is achieved in [46] by imposing significant smoothness condition on the boundary of the junction, which involves some uniform with respect to ε estimates ((4.6) and (4.7) in [46]) on up to the second derivative of the function parametrizing the boundary. However, when one manufactures quantum wires circuits, it is probably hard to achieve smoothness of the junction. We instead impose a much simpler star-shapedness condition that does not require any smoothness. Although we believe that this condition is not restrictive in practical situations, it is certainly also not necessary. We believe that an additional effort would yield a weaker (some kind of a "uniform C") sufficient condition on the domain (see related discussion on the conditions of validity of the Poincare inequality in [37]). The construction of the averaging operator P^{ε} is different from the one used in [46], which lead to significant simplification of the proof and to elimination of most of what was called in [46] "plumber shop".

2. One of the conditions on the domain was that the diameters of the junction regions V_l^{ε} decay as ε. One can see that the proof still works when these diameters decay faster than $\sqrt{\varepsilon}$ (but not faster than ε). However, as the results of [23] suggest, if the diameters decay slower than $\sqrt{\varepsilon}$ and hence the vertex neighborhoods look like large protrusions in narrow tubes, the asymptotics of the spectrum should look differently. We plan to address this issue elsewhere.

3. It is often interesting to consider periodic meshes of quantum wires (see for instance [3], [17], [16]). Then the graph M instead of being finite is assumed to be periodic with respect to a lattice Γ on the plane. The domain M^{ε} and the magnetic and electric potentials $\mathbf{A}$ and q are assumed to be periodic with respect to Γ as well. In this situation the operators $H_{\varepsilon}(\mathbf{A}, q)$ on M^{ε} and $H_M(A, q)$ on M have continuous spectra. An analog of the Theorem 2 holds in the following form:

   Theorem 23 *For any $C > 0$ the following convergence takes place:*

   $$\lim_{\varepsilon \to 0} \sigma(H_{\varepsilon}(\mathbf{A}, q)) \cap [-C, C] = \sigma(H_M(A, q)) \cap [-C, C].$$

   Here the limit is understood in the meaning of Hausdorff distance between sets.

   The proof is based on applying the Floquet theory first (see [42], [33]), which reduces both operators to the direct integrals of the corresponding Bloch Hamiltonians on the torus depending on the quasimomentum. Treating these operators on the torus is done exactly as for the finite graphs on the plane, while quasimomenta are included into the magnetic potential and hence treated as in the proof of Theorem 2. So, the periodic situation boils down to the one for the finite graphs on the torus in presence of electromagnetic fields, which is covered by our considerations above.

4. The statements and the proofs of the main theorems 1 and 2 also hold in multidimensional case, i.e. for "fattened graph" domains in $\mathbb{R}^m$. One only needs to assume that the tubes M_j^ε have sufficiently smooth boundaries and denote by $\varepsilon p(x)$ the $(m-1)$-dimensional area of the normal section of M_j^ε at a point x of the edge M_j.

5. The case of the Dirichlet Laplacian is more interesting from the physics point of view, but at the same time much more complicated and less studied than that of the Neumann Laplacian. One complication is obvious: when the domain shrinks, the lower bound of the spectrum of the Dirichlet Laplacian diverges, so in order to have any finite limit the spectrum must be rescaled, for instance by shifting the lowest eigenvalue to zero. Another is that it is not clear what vertex boundary conditions the limit operator should satisfy. In the smooth case (i.e., when vertices are absent), this procedure works well and leads to a limit operator on the curve M [15]. Another interesting difference with the Neumann case is that the limit operator involves the curvature of M, which does not occur for the Neumann conditions. There are apparently no results of the kind presented in this paper for the Dirichlet Laplacian when the vertices are present. One can find a survey of this topic and additional references in [15].

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