Ideal Quantum Gases in D-dimensional Space and Power-Law Potentials

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Abstract

We investigate ideal quantum gases in D-dimensional space and confined in a generic external potential by using the semiclassical approximation. In particular, we derive density of states, density profiles and critical temperatures for Fermions and Bosons trapped in isotropic power-law potentials. ¿From such results, one can easily obtain those of quantum gases in a rigid box and in a harmonic trap. Finally, we show that the Bose-Einstein condensation can set up in a confining power-law potential if and only if $D/2 + D/n > 1$, where D is the space dimension and n is the power-law exponent.

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1 Introduction

For dilute alkali-metal atoms in magnetic or magneto-optical traps at very low temperatures, the Bose-Einstein condensation has been achieved[1] in 1995 and the Fermi quantum degeneracy[2] in 1999. These results have renewed the theoretical investigation on Bose and Fermi gases.

In the experiments with Bosons, the system is weakly-interacting and the thermodynamical properties depend on the s-wave scattering length (for a review see Ref. 3). Nevertheless, by using Feshback resonances, it is now possible to modify and also switch-off the atom-atom interaction.[4] In the case of Fermions, the s-wave scattering between atoms in the same hyperfine state is inhibited due the Pauli principle. It follows that at low temperature the dilute Fermi gas, in a fixed hyperfine state, is practically ideal.[2]

In previous papers we analyzed ground-state and vortex properties of Bose condensates in different external potentials: harmonic potential,[5−10], toroidal potential[11] and double-well potential.[12] Recently, we have also afforded the study of the thermodynamics of interacting Bose gases in harmonic potential.[13,14]

In this paper, we investigate the thermal properties of both Bose and Fermi ideal gases in a generic confining external potential. All the calculations are performed by assuming a D-dimensional space. Such an assumption is motivated by esthetic criteria but also by recent experiments with degenerate gases in systems with reduced or fractal dimension.[3] We analyze in detail the isotropic power-law potential, from which one easily deduces the results of a rigid box and a harmonic trap.

2 Confined Ideal Fermi and Bose Gases

Let us consider a confined quantum gas of non-interacting identical Fermions (or Bosons) in D-dimensional space. In the grand canonical ensemble of equilibrium statistical mechanics,[15] the average number N_α of particles in the single-particle state $|\alpha\rangle$ with energy ϵ_α is given by

$$N_\alpha = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} \pm 1} \; , \qquad (1)$$

where the sign $+$ $(-)$ is for Fermions (Bosons), μ is the chemical potential and $\beta = 1/(kT)$ with k the Boltzmann constant and T the absolute temperature. In general, given the single-particle function N_α, the average total number N of particles of the system reads

$$N = \sum_\alpha N_\alpha \; . \qquad (2)$$

This condition fixes the chemical potential μ. Thus, μ is a function of β and N. In the case of Fermions, μ has no limitations and at zero temperature μ is called Fermi energy E_F. ¿From the Fermi energy E_F one immediately obtains the Fermi temperature $T_F = E_F/k$. Below the Fermi temperature, the Fermions begin to fill the lowest available single-particle states in accordance with the Pauli exclusion Principle: one has the Fermi quantum degeneracy.[15] In the case of Bosons, μ cannot be higher than the lowest single-particle energy level ϵ_0, i.e., it must be $\mu < \epsilon_0$. When $\mu \to \epsilon_0$ the function N_0 diverges and consequently also N diverges. The physical meaning is that the lowest single-particle state becomes macroscopically occupied and one has the so-called Bose-Einstein condensation (BEC).[15] It is a standard procedure to calculate the condensed fraction N_0/N and also the BEC transition temperature T_B by studying the non divergent quantity $N - N_0$ at $\mu = \epsilon_0$ as a function of the temperature.[3,15]

In the semiclassical limit, the D-dimensional system is described by a continuum of states[15−17] and, instead of ϵ_α, one uses the classical single-particle phase-space energy $\epsilon(\mathbf{r},\mathbf{p})$, where $\mathbf{r} = (r_1, ..., r_D)$ is the position vector and $\mathbf{p} = (p_1, ..., p_D)$ is the linear momentum vector. In this way one obtains from Eq. (1) the single-particle phase-space distribution

$$n(\mathbf{r},\mathbf{p}) = \frac{1}{e^{\beta(\epsilon(\mathbf{r},\mathbf{p})-\mu)} \pm 1} \; . \tag{3}$$

Note that the accuracy of the semiclassical approximation is expected to be good if the number of particles is large and the energy level spacing is smaller then kT.[15−17] Because of the Heisenberg principle, the quantum elementary volume of the single-particle 2D-dimensional phase-space is given by $(2\pi\hbar)^D$, where $\hbar$ is the Planck constant.[16] It follows that the average number N of particles in the D-dimensional space can be written as

$$N = \int \frac{d^D\mathbf{r}\; d^D\mathbf{p}}{(2\pi\hbar)^D}\; n(\mathbf{r},\mathbf{p}) = \int d^D\mathbf{r}\; n(\mathbf{r}) = \int d^D\mathbf{p}\; n(\mathbf{p}) \; , \tag{4}$$

where

$$n(\mathbf{r}) = \int \frac{d^D\mathbf{p}}{(2\pi\hbar)^D} n(\mathbf{r},\mathbf{p}) \tag{5}$$

is the spatial distribution, and

$$n(\mathbf{p}) = \int \frac{d^D\mathbf{r}}{(2\pi\hbar)^D} n(\mathbf{r},\mathbf{p}) \tag{6}$$

is the momentum distribution. It is important to observe that the total number N of particle can also be written as

$$N = \int_0^\infty d\epsilon \; \rho(\epsilon) \; \frac{1}{e^{\beta(\epsilon-\mu)} \pm 1} \; , \tag{7}$$

where $\rho(\epsilon)$ is the density of states. It can be obtained from the semiclassical formula

$$\rho(\epsilon) = \int \frac{d^D\mathbf{r} \; d^D\mathbf{p}}{(2\pi\hbar)^D} \delta(\epsilon - \epsilon(\mathbf{p}, \mathbf{r})) \; , \tag{8}$$

where $\delta(x)$ is the Dirac delta function.

In the case of Fermions, at zero temperature, i.e., in the limit $\beta \to \infty$ where $\mu \to E_F$ (the Fermi energy), the phase-space distribution (3) becomes

$$n(\mathbf{r}, \mathbf{p}) = \Theta\left(E_F - \epsilon(\mathbf{r}, \mathbf{p})\right) \; , \tag{9}$$

where $\Theta(x)$ is the Heaviside step function.[15,16]

In the case of Bosons, below the BEC transition temperature T_B, the equation (3) describes only the non-condensed thermal cloud. Thus, the semi-classical quantization renormalizes the exact Bose distribution (1) that is divergent. Because there is not an unique way to translate wave-functions into a phase-space distribution,[16] one cannot introduce an exact single-particle phase-space distribution for the Bose condensate. Nevertheless, the exact spatial distribution of the Bose condensate is $n_0(\mathbf{r}) = |\Psi(\mathbf{r})|^2$, where $\Psi(\mathbf{r})$ is called order parameter or macroscopic wave-function of the condensate, normalized to the number N_0 of condensed Bosons. For an ideal Bose gas, the function $\Psi(\mathbf{r})$ is simply the eigenfunction of the lowest single-particle state of the system. In this paper we do not study the density profiles of the Bose condensed fraction because, for a non-interacting gas, their shape is not temperature dependent: only their normalization is a function of temperature.

Actually, to calculate the BEC transition temperature T_B and the condensed fraction N_0/N it is sufficient to study the non-condensed fraction (thermal cloud). (For a recent discussion of the properties of the Bose condensate, see Ref. 3 and also Ref. 5-14).

3 Gases in External Potential

Let us consider the ideal Fermi (Bose) gas in a confining external potential $U(\mathbf{r})$ and in a D-dimensional space. The classical single-particle energy is defined as

$$\epsilon(\mathbf{r},\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + U(\mathbf{r}) \; , \tag{10}$$

where $\mathbf{p}^2/(2m)$ is the kinetic energy and m is the mass of the particle. The D-dimensional vectors $\mathbf{r} = (x_1, ..., x_D)$ and $\mathbf{p} = (p_1, ..., p_D)$ are respectively the position and momentum of the particle.

First, we note that, by using Eq. (8) and Eq. (10), the semiclassical density of states can be written as

$$\rho(\epsilon) = \left(\frac{m}{2\pi\hbar^2}\right)^{\frac{D}{2}} \frac{1}{\Gamma(\frac{D}{2})} \int d^D\mathbf{r} \; \left(\epsilon - U(\mathbf{r})\right)^{\frac{(D-2)}{2}} \; . \tag{11}$$

where $\Gamma(n)$ is the factorial function. Then, we introduce the Fermi and Bose functions.[18,19]

DEFINITION 1. *The Fermi function is given by*

$$f_n(z) = \frac{1}{\Gamma(n)} \int_0^{\infty} dy \frac{z e^{-y} y^{n-1}}{1 + z e^{-y}} \; ,$$

and the Bose function is

$$g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty dy \frac{z e^{-y} y^{n-1}}{1 - z e^{-y}} \ ,$$

where $\Gamma(n)$ *is the factorial function.*

REMARK 1. *The Fermi and Bose functions are connected by the relation* $f_n(z) = -g_n(-z)$. *For* $|z| < 1$ *the two functions can be written as*

$$g_n(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^n} \ ,$$

$$f_n(z) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{z^i}{i^n} \ .$$

One finds also that $g_n(1) = \zeta(n)$, *where* $\zeta(n)$ *is the Riemann* ζ*-function.*[18,19]

Now we state two theorems about ideal Fermi and Bose gases in external potential. Remind that we work in the semiclassical limit.[3,15–16]

THEOREM 1.1 *For an ideal Fermi gas in an external potential* $U(\mathbf{r})$ *and D-dimensional space, the finite temperature spatial distribution is given by*

$$n(\mathbf{r}) = \frac{1}{\lambda^D} f_{\frac{D}{2}} \left(e^{\beta(\mu - U(\mathbf{r}))} \right) \ ,$$

where $\lambda = (2\pi\hbar^2\beta/m)^{1/2}$ *is the thermal length and* μ *is the chemical potential. The zero temperature spatial distribution is*

$$n(\mathbf{r}) = \left(\frac{m}{2\pi\hbar^2} \right)^{D/2} \frac{1}{\Gamma(\frac{D}{2}+1)} \left(E_F - U(\mathbf{r}) \right)^{\frac{D}{2}} \Theta\left(E_F - U(\mathbf{r}) \right) \ ,$$

where E_F *is the Fermi energy.*

Proof. One finds the spatial distribution from the Eq. (5) by integrating over momenta the Eq. (3) (with the sign +). In particular, one has

$$n(\mathbf{r}) = \int \frac{d^D\mathbf{p}}{(2\pi\hbar)^D} \frac{1}{e^{\beta(\frac{\mathbf{p}^2}{2m}+U(\mathbf{r})-\mu)}+1} =$$

$$= \frac{1}{(2\pi\hbar^2)^D} \frac{D\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2}+1)} \int_0^\infty dp \frac{p^{D-1} e^{\beta(\mu-U(\mathbf{r}))} e^{-\beta(\frac{p^2}{2m})}}{1+e^{\beta(\mu-U(\mathbf{r}))} e^{-\beta(\frac{p^2}{2m})}} ,$$

where $D\pi^{\frac{D}{2}}/\Gamma(\frac{D}{2}+1)$ is the volume of the D-dimensional unit sphere. Then, with the position $y^2 = \beta\frac{p^2}{2m}$ and using the Fermi function $f_{\frac{D}{2}}(z)$ with $z = e^{\beta(\mu-U(\mathbf{r}))}$, one gets the finite temperature spatial distribution. Finally, one obtains the zero-temperature result by observing that

$$n(\mathbf{r}) = \int \frac{d^D\mathbf{p}}{(2\pi\hbar)^D} \Theta\left(E_F - \frac{\mathbf{p}^2}{2m} - U(\mathbf{r})\right) =$$

$$= \frac{1}{(2\pi\hbar^2)^D} \frac{D\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2}+1)} \Theta\left(E_F - U(\mathbf{r})\right) \int_0^{\sqrt{2m(E_F-U(\mathbf{r}))}} dp\, p^{D-1} =$$

$$= \frac{1}{(2\pi\hbar^2)^D} \frac{D\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2}+1)} \Theta\left(E_F - U(\mathbf{r})\right) \frac{1}{D} \left(\sqrt{2m(E_F - U(\mathbf{r})}\right)^D ,$$

where the spatial distribution is taken from the Eq. (8). □

In the same way, but using the sign − in Eq. (3) and the Bose function $g_{\frac{D}{2}}(z)$ with $z = e^{\beta(\mu-U(\mathbf{r}))}$, one can easily prove also the following theorem.

THEOREM 1.2 *For an ideal Bose gas in an external potential* $U(\mathbf{r})$*, the finite temperature non-condensed spatial distribution is given by*

$$n(\mathbf{r}) = \frac{1}{\lambda^D} g_{\frac{D}{2}}\left(e^{\beta(\mu-U(\mathbf{r}))}\right) ,$$

where $\lambda = (2\pi\hbar^2\beta/m)^{1/2}$ *is the thermal length and* μ *is the chemical potential.* $\square$

These two theorems are the generalization, of the formulas for ideal homogenous Fermi and Bose gases in a box of volume V (for $D = 3$ see Ref. 15). They show that, in the semiclassical limit, the non-homogenous results are obtained with the substitution $\mu \to \mu - U(\mathbf{r})$, also called *local density approximation*. In particular, with $U(\mathbf{r}) = 0$, from the previous theorems, one obtains the Fermi temperature T_F and the Bose temperature T_B for quantum gases in a rigid box, by imposing the normalization condition (4). The results are

$$E_F = kT_F = \left(\frac{2\pi\hbar^2}{m}\right)\left[\Gamma\left(\frac{D}{2}+1\right)n\right]^{2/D}$$

and

$$kT_B = \left(\frac{2\pi\hbar^2}{m}\right)\left(\frac{n}{\zeta(\frac{D}{2})}\right)^{2/D} ,$$

where $n = N/V$ is the homogenous density of particles (again, for $D = 3$ see Ref. 15).

In general, to find the momentum distribution, the Fermi temperature and the Bose temperature, it is necessary to specify the external potential. In many experiments with alkali-metal atoms, the external trap can be accurately modelled by a harmonic potential.[3] More generally, one can consider power-law potentials, which are important for studying the effects of adiabatic changes in the trap. The density of states of a quantum gas in the power-law potential $U(\mathbf{r}) = A\, r^n$ can be calculated from Eq. (11) and reads

$$\rho(\epsilon) = \left(\frac{m}{2\hbar^2}\right)^{\frac{D}{2}}\left(\frac{1}{A}\right)^{\frac{D}{\alpha}}\frac{\Gamma(\frac{D}{\alpha}+1)}{\Gamma(\frac{D}{2}+1)\Gamma(\frac{D}{2}+\frac{D}{\alpha})}\epsilon^{\frac{D}{2}+\frac{D}{\alpha}-1} . \tag{12}$$

We can now state two theorems about ideal Fermi and Bose gases in isotropic power-law potentials.

THEOREM 2.1 *Let us consider an ideal Fermi gas in a power-law isotropic potential* $U(\mathbf{r}) = A\ r^n$ *with* $r = |\mathbf{r}| = (\sum_{i=1}^{D} x_i^2)^{1/2}$. *The finite temperature momentum distribution is given by*

$$n(\mathbf{p}) = \frac{1}{(2\hbar\sqrt{\pi})^D} \frac{\Gamma(\frac{D}{n}+1)}{\Gamma(\frac{D}{2}+1)} \left(\frac{1}{\beta A}\right)^{\frac{D}{n}} f_{\frac{D}{n}}\left(e^{\beta\left(\mu - \frac{\mathbf{p}^2}{2m}\right)}\right) .$$

The zero temperature momentum distribution is

$$n(\mathbf{p}) = \frac{1}{(2\hbar\sqrt{\pi})^D} \left(\frac{1}{A}\right)^{\frac{D}{n}} \left(E_F - \frac{\mathbf{p}^2}{2m}\right)^{\frac{D}{n}} \Theta\left(E_F - \frac{\mathbf{p}^2}{2m}\right) .$$

The Fermi energy E_F *and the Fermi temperature* T_F *are given by*

$$E_F = kT_F = \left[\left(\frac{2\hbar^2}{m}\right)^{\frac{D}{2}} A^{\frac{D}{n}} \frac{\Gamma(\frac{D}{2}+1)}{\Gamma(\frac{D}{n}+1)} \Gamma\left(\frac{D}{2}+\frac{D}{n}+1\right) N\right]^{\frac{1}{\frac{D}{2}+\frac{D}{n}}} ,$$

where N *is the number of Fermions in the gas.*

Proof. One finds the finite temperature momentum distribution from the Eq. (6) and by integrating over space coordinates the Eq. (3) (with the sign +). In particular, one has

$$n(\mathbf{p}) = \int \frac{d^D\mathbf{r}}{(2\pi\hbar)^D} \frac{1}{e^{\beta(\frac{\mathbf{p}^2}{2m}+Ar^n-\mu)}+1} =$$

$$= \frac{1}{(2\pi\hbar^2)^D} \frac{D\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2}+1)} \int_0^\infty dr \frac{r^{D-1} e^{\beta(\mu-\frac{p^2}{2m})} e^{-\beta Ar^n}}{1+e^{\beta(\mu-\frac{p^2}{2m})} e^{-\beta Ar^n}} ,$$

where again $D\pi^{\frac{D}{2}}/\Gamma(\frac{D}{2}+1)$ is the volume of the D-dimensional unit sphere. Setting $y^2 = \beta A r^n$ and using the definition of Fermi function $f_{\frac{D}{2}}(z)$ with $z =$

$e^{\beta(\mu-\frac{p^2}{2m})}$, one finds the first formula of the theorem. The zero-temperature results are obtained by observing that

$$n(\mathbf{p}) = \int \frac{d^D\mathbf{r}}{(2\pi\hbar)^D}\Theta\left(E_F - \frac{\mathbf{p}^2}{2m} - U(\mathbf{r})\right) =$$

$$= \frac{1}{(2\pi\hbar^2)^D}\frac{D\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2}+1)}\Theta\left(E_F - \frac{p^2}{2m}\right)\int_0^{A^{-\frac{1}{n}}(E_F-\frac{\mathbf{p}^2}{2m})^{1/n}} dr\ r^{D-1} =$$

$$= \frac{1}{(2\pi\hbar^2)^D}\frac{D\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2}+1)}\Theta\left(E_F - \frac{p^2}{2m}\right)\frac{1}{D}\left(A^{-\frac{1}{n}}(E_F - \frac{\mathbf{p}^2}{2m})^{1/n}\right)^D ,$$

where the momentum distribution is taken from the Eq. (9). The Fermi energy E_F and the Fermi temperature T_F are found from the normalization condition of the zero-temperature momentum distribution. Namely, one finds

$$N = \int d^D\mathbf{p}\frac{1}{(2\hbar\sqrt{\pi})^D}\left(\frac{1}{A}\right)^{\frac{D}{n}}\left(E_F - \frac{\mathbf{p}^2}{2m}\right)^{\frac{D}{n}}\Theta\left(E_F - \frac{\mathbf{p}^2}{2m}\right) =$$

$$= \frac{1}{(2\hbar\sqrt{\pi})^D}\left(\frac{1}{A}\right)^{\frac{D}{n}}\int_0^{\sqrt{2mE_F}} dp\ p^{D-1}\left(E_F - \frac{p^2}{2m}\right)^{\frac{D}{n}} .$$

Setting $x = \frac{p^2}{2m}$ and observing[18,19] that

$$\int_0^{E_F} dx\ x^{\frac{D}{2}-1}(E_F - x)^{\frac{D}{n}} = \frac{\Gamma(\frac{D}{2})\Gamma(\frac{D}{n}+1)}{\Gamma\left(\frac{D}{2}+\frac{D}{n}+1\right)}E_F^{\frac{D}{2}+\frac{D}{n}} ,$$

one obtains

$$N = \left(\frac{m}{2\hbar^2}\right)^{\frac{D}{2}}\left(\frac{1}{A}\right)^{\frac{D}{n}}\frac{\Gamma(\frac{D}{2})\Gamma(\frac{D}{n}+1)}{\Gamma\left(\frac{D}{2}+\frac{D}{n}+1\right)}E_F^{\frac{D}{2}+\frac{D}{n}} .$$

Finally, by inverting this formula one gets the Fermi energy E_F. □

THEOREM 2.2 *Let us consider an ideal Bose gas in a power-law isotropic potential* $U(\mathbf{r}) = A\ r^n$ *with* $r = |\mathbf{r}| = (\sum_{i=1}^D x_i^2)^{1/2}$. *The finite temperature* *non-condensed momentum distribution is given by*

$$n(\mathbf{p}) = \frac{1}{(2\hbar\sqrt{\pi})^D}\frac{\Gamma(\frac{D}{n}+1)}{\Gamma(\frac{D}{2}+1)}\left(\frac{1}{\beta A}\right)^{\frac{D}{n}} g_{\frac{D}{n}}\left(e^{\beta\left(\mu-\frac{\mathbf{p}^2}{2m}\right)}\right) .$$

The Bose transition temperature T_B reads

$$kT_B = \left[\left(\frac{2\hbar^2}{m}\right)^{\frac{D}{2}} A^{\frac{D}{n}}\frac{\Gamma(\frac{D}{2}+1)}{\Gamma(\frac{D}{n}+1)}\frac{1}{\zeta(\frac{D}{2}+\frac{D}{n})}N\right]^{\frac{1}{\frac{D}{2}+\frac{D}{n}}}$$

and the condensed fraction is

$$\frac{N_0}{N} = 1-\left(\frac{T}{T_B}\right)^{\frac{D}{2}+\frac{D}{n}} ,$$

where N is the number of Bosons in the gas.

Proof. The finite temperature momentum distribution can be found by following the procedure used in the proof of the previous theorem: from the Eq. (6) and by integrating over space coordinates the Eq. (3) (but with the sign $-$). It follows that one must use the Bose function $g_{\frac{D}{2}}(z)$ with $z = e^{\beta(\mu-\frac{p^2}{2m})}$. At the BEC transition temperature T_B, the chemical potential μ is zero and at $\mu = 0$ the number N of particles can be analytically determined. One has

$$N = \int d^D\mathbf{p}\frac{1}{(2\hbar\sqrt{\pi})^D}\frac{\Gamma(\frac{D}{n}+1)}{\Gamma(\frac{D}{2}+1)}\left(\frac{1}{\beta A}\right)^{\frac{D}{n}} g_{\frac{D}{n}}\left(e^{-\beta\frac{\mathbf{p}^2}{2m}}\right) =$$

$$= \frac{1}{(2\hbar\sqrt{\pi})^D}\frac{\Gamma(\frac{D}{n}+1)}{\Gamma(\frac{D}{2}+1)}\frac{D\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2}+1)}\left(\frac{1}{\beta A}\right)^{\frac{D}{n}}\sum_{i=1}^{\infty}\frac{1}{i^{\frac{D}{2}}}\int_0^{\infty} dpp^{D-1}e^{-i\beta\frac{p^2}{2m}} ,$$

where the Bose function has been written as a power series (see Remark 1). Setting $x = i\beta\frac{p^2}{2m}$ and observing[18,19] that

$$\int_0^{\infty} dx\, x^{\frac{D}{2}}e^{-x} = \Gamma\left(\frac{D}{2}+1\right) ,$$

and also

$$\sum_{i=0}^{\infty} \frac{1}{i^{\frac{D}{2}+\frac{D}{n}}} = \zeta\left(\frac{D}{2}+\frac{D}{n}\right) \; ,$$

one obtains

$$N = (kT)^{\frac{D}{2}+\frac{D}{n}} \left(\frac{m}{2\hbar^2}\right)^{\frac{D}{2}} \left(\frac{1}{A}\right)^{\frac{D}{n}} \frac{\Gamma(\frac{D}{n}+1)\zeta(\frac{D}{2}+\frac{D}{n})}{\Gamma(\frac{D}{2}+1)} \; .$$

By inverting the function $N = N(T)$ one finds the transition temperature T_B. Below T_B, a macroscopic number N_0 of particle occupies the single-particle ground-state of the system. It follows that the previous equation gives the number $N - N_0$ of non-condensed particles and the condensed fraction is $N_0/N = 1 - (T/T_B)^{D/2+D/n}$. □

This last theorem generalizes the BEC results obtained with $D = 3$ by Bagnato, Pritchard and Kleppner.[17]

It is important to observe that from the two previous theorems one easily derives the thermodynamic properties of quantum gases in harmonic traps and in a rigid box. In fact, by setting $n = 2$ and $A = m\omega^2 r^2/2$ one gets the formulas for the Bose and Fermi gases in a harmonic trap (in the case of a anisotropic harmonic potential, ω is the geometric average of the frequencies of the trap). The results for a rigid box are instead obtained by letting $\frac{D}{n} \to 0$, where the density of particles per unit length is given by N/Ω_D and $\Omega_D = D\pi^{\frac{D}{2}}/\Gamma(\frac{D}{2}+1)$ is the volume of the D-dimensional unit sphere.

Finally, one notes that in the formula of the BEC transition temperature T_B it appears the function $\zeta(\frac{D}{2}+\frac{D}{n})$. Because $\zeta(x) < \infty$ for $x > 1$ but $\zeta(1) = \infty$,[17,18] one easily deduces the following corollary.

COROLLARY 1. *Let us consider an ideal Bose gas in a power-law isotropic potential* $U(\mathbf{r}) = A\ r^n$ *with* $r = |\mathbf{r}| = (\sum_{i=1}^{D} x_i^2)^{1/2}$. *BEC is possible if and only if the following condition is satisfied*

$$\frac{D}{2} + \frac{D}{n} > 1 \ ,$$

where D is the space dimension and n is the exponent of the confining power-law potential. $\square$

This is a remarkable inequality. For example, for $D = 2$ one finds the familiar result that there is no BEC in a homogenous gas ($\frac{D}{n} \to 0$) but BEC is possible in a harmonic trap ($n = 2$). Moreover, one obtains that for $D = 1$ BEC is possible with $1 < n < 2$.

4 Conclusions

By using the grand canonical ensemble of statistical mechanics and the semiclassical approximation, we have derived some thermodynamic properties of ideal quantum gases in a generic isotropic power-law confining external potential. We have calculated the density of states, spatial and momentum distributions and obtained analytical formulas for the Fermi energy, the BEC transition temperature and the Bose condensed fraction. Note that nowadays the spatial and momentum density profiles are quantities easily experimentally measured. We have also shown that BEC in an isotropic power-law potential is possible if and only if $D/2 + D/n$, where D is the space dimension and n is the exponent of the confining power-law potential.

The present investigation is the starting point for future analyses of interacting quantum gases in D-dimensional space and generic trapping potential.

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