Perturbations of One-Dimensional Schrödinger Operators Preserving the Absolutely Continuous Spectrum

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Rowan Killip

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To my grandfathers,
who taught me how to put things back together
—I figured out how to pull them apart myself.

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Abstract

We study the stability of the absolutely continuous spectrum of one-dimensional Schrödinger operators

$$[Hu](x) = -u''(x) + q(x)u(x)$$

with periodic potentials $q(x)$. Specifically, it is proved that any perturbation of the potential, $V \in L^2$, preserves the essential support (and multiplicity) of the absolutely continuous spectrum. This is optimal in terms of L^p spaces and, for $q \equiv 0$, it answers in the affirmative a conjecture of Kiselev, Last and Simon.

By adding constraints on the Fourier transform of V, it is possible to relax the decay assumptions on V. It is proved that if $V \in L^3$ and $\hat{V}$ is uniformly locally square integrable, then preservation of the a.c. spectrum still holds. If we assume that $q \equiv 0$, still stronger results follow: if $V \in L^3$ and $\hat{V}(k)$ is square integrable on an interval $[k_0, k_1]$, then the interval $[k_0^2/4, k_1^2/4]$ is contained in the essential support of the absolutely continuous spectrum of the perturbed operator.

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Chapter 1 Introduction

We wish to consider perturbations of one-dimensional Schrödinger operators with bounded periodic potentials. So given a bounded periodic function $q(x)$ on $\mathbf{R}$ we define

$$[H_0 u](x) = -u''(x) + q(x)u(x)$$

as an operator acting in $L^2(\mathbf{R})$. (More strictly, the domain of operator H_0 is the Sobolev space H^2, which is dense in L^2.) The assumption that q is bounded is made purely to avoid distracting complications. More generally, one may consider q which are locally integrable. We shall also have cause to discuss the discrete version of this operator, namely

$$[h_0 u](n) = u(n+1) + u(n-1) + q(n)u(n)$$

which acts in $\ell^2(\mathbf{Z})$. For the most part though, we shall discuss the traditional (non-discretized) Schrödinger operator.

The spectral theory of one-dimensional Schrödinger operators with periodic potentials is very well understood. Ultimately this stems from the following fact: if L denotes the period of q and ψ is a solution of

$$-\psi'' + q\psi = z\psi$$

for some $z \in \mathbf{C}$, then $\phi(x) = \psi(x + L)$ is also a solution of this differential equation. By iterating this fact, we see that solutions are either steadily growing (as $x \to \infty$ or $-\infty$) or are uniformly bounded. The set of z for which there are bounded solutions is a closed subset of $\mathbf{R}$. Indeed, it is the spectrum of H_0, $\sigma(H_0)$. It consists of closed intervals (known as the spectral bands) and is bounded from below but unbounded above. For any z in the interior of the spectrum, all solutions are bounded. This does not hold at the band edges [17].

One of the core results of subordinacy theory is that bounded solutions correspond to absolutely continuous spectrum [32]. Employing this theorem in the current instance tells us that the spectrum $\sigma(H_0)$ of H_0 is purely absolutely continuous with multiplicity two. The nature of the spectrum has been known for a very long time, in particular from before the advent of subordinacy theory. However, we choose to present it in this way as it foreshadows some of the discussion to come.

The question at the centre of this thesis is the following: what perturbations V may be added to the potential without destroying the absolutely continuous spectrum? In particular, what type of decay at infinity is permissible. At the moment this question is rather imprecise; in particular, we have yet to explain in what sense we wish the absolutely continuous spectrum to be preserved. For the moment though, we introduce the following notation for the perturbed operator

$$[Hu](x) = -u''(x) + q(x)u(x) + V(x)u(x).$$

Undoubtedly the oldest result on spectral stability is due to Weyl [22, Theorem XIII.14]: any relatively compact perturbation preserves the essential spectrum. For the operators we consider, this means that if V decays in some reasonable fashion, for example,

$$\int_{n-1}^{n} V^2 dx \to 0 \text{ as } n \to \pm\infty,$$

then $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0) = \sigma(H_0)$. This tells us that we need only focus on what happens to the spectral type on $\sigma(H_0)$. Any spectrum of H, lying outside $\sigma(H_0)$, must consist of discrete eigenvalues. In the discrete case, one merely needs $V(n) \to 0$ as $n \to \pm\infty$ to apply the Weyl Theorem.

The next general result we wish to consider is the Kato-Rosenblum Theorem (more accurately, a version of it due to Birman and Kuroda). As well as being interesting in its own right, it suggests what type of spectral stability result we should try to prove. If $V \in L^1$ then V is a relatively trace class perturbation of H_0. That is to say, for each $z \in \mathbf{C} \setminus \mathbf{R}$, $R - R_0 = (H-z)^{-1} - (H_0 - z)^{-1}$ is trace class. The Kato-Rosenblum Theorem then tells us that wave operators exist and are complete and hence that the absolutely continuous parts of H and H_0 are unitarily equivalent.

Here we have a very clear statement of what it means for the absolutely continuous spectrum to be preserved—namely, if the absolutely continuous parts of the perturbed and unperturbed operators are unitarily equivalent. This is much stronger than the statement $\sigma_{\mathrm{ac}}(H) = \sigma_{\mathrm{ac}}(H_0)$. To better explain this, we introduce the following:

Definition 1. *Given a self-adjoint operator A on a Hilbert space H, an essential support of the a.c. spectrum is any set Σ_{ac} such that*

i) *for every $\phi \in \mathsf{H}$ the corresponding spectral measure μ obeys $\mu_{\mathrm{ac}}(S \cap \Sigma_{\mathrm{ac}}) = \mu_{\mathrm{ac}}(S)$ for any Borel set S, and*

ii) *$\exists \phi \in \mathsf{H}$ such that $d\mu \gg \chi_{\Sigma_{\mathrm{ac}}}(E)dE$.*

This determines Σ_{ac} uniquely up to sets of zero Lebesgue measure.

Remark In order to simplify the presentation, equations involving Σ_{ac} should be interpreted as holding for *some* choice of essential support.

The statement 'the absolutely continuous parts of H and H_0 are unitarily equivalent' is logically equivalent to '$\Sigma_{\mathrm{ac}}(H) = \Sigma_{\mathrm{ac}}(H_0)$ and the spectral multiplicities are equal on this set'. It should be clear that $\Sigma_{\mathrm{ac}} \subseteq \sigma_{\mathrm{ac}}$, indeed σ_{ac} is the smallest closed set which contains (a realization of) Σ_{ac}. However, in general $\Sigma_{\mathrm{ac}} \neq \sigma_{\mathrm{ac}}$; they may differ by any set with empty interior and positive Lebesgue measure. In the case of H_0 or h_0, $\Sigma_{\mathrm{ac}} = \sigma_{\mathrm{ac}}$.

As with most recent work on this subject, we shall prefer to talk about equality of the essential support of the absolutely continuous spectrum rather than unitary equivalence of absolutely continuous parts.

For potentials with Coulombic decay, $V(x) \sim 1/x$, the Kato-Rosenblum Theorem is inapplicable. The wave operators simply do not exist. If we wish to consider potentials with slower than L^1 decay, we must introduce modified wave operators. The best known conditions for existence and completeness of wave operators [10] require decay restrictions on V, V' and V''. Alternatively, this monograph contains a proof (Theorem 30.2.10) that the absolutely continuous spectrum is preserved under weaker assumptions on just V, V' by directly obtaining estimates on the resolvent. In the one-dimensional case we are considering, however, these are all subsumed by the following result of Weidmann [35]. If V is the sum of two terms, one integrable and the other tending to zero at infinity and of bounded variation, then $\Sigma_{\text{ac}}(H) = \Sigma_{\text{ac}}(H_0)$ with equal multiplicity. Actually this theorem is due to Weidmann only in the case that $q \equiv 0$. For periodic background potentials, it is due to Stolz [34].

Perhaps the earliest results in the opposite direction arose from Pearson studies of sparse potentials [21]. Suppose $W(x)$ is continuous and of compact support and that $g_n \notin \ell^2$. Pearson showed that if $x_n \to \infty$ sufficiently quickly, then the operator H with $q \equiv 0$ and

$$V(x) = \sum g_n W(|x| - x_n)$$

has purely singular continuous spectrum. This tells us, for example, that if $p > 2$ then there exists a $V \in L^p$ so that $\Sigma_{\text{ac}}(H) = \emptyset$. More recently, Kiselev, Last and Simon [14] treated sparse potentials for x_n obeying $x_n/x_{n+1} \to 0$ by showing that if $g_n \in \ell^2$ then H has purely absolutely continuous spectrum on $(0, \infty)$, whilst if $g_n \notin \ell^2$ then H has purely singular continuous spectrum on this interval. Remling [24] also gave related results.

Whilst these results give a very strong indication of what decay properties are insufficient to prove the preservation of Σ_{ac}, at least in an L^p sense, there is a still more instructive result due to Kotani and Ushiroya [16]. Once again we consider $q \equiv 0$. Given any bounded positive decreasing function $a(x) \notin L^2(\mathbf{R}^+)$, they construct a probability space of choices for V, each obeying $|V(x)| \le a(|x|)$ such that $H = H_0 + V$ has purely singular spectrum with probability one. It is also shown that if $|V(x)|(1+|x|)^\epsilon \in L^2$ then the spectrum on $(0, \infty)$ is purely absolutely continuous with probability one. Later we shall see that L^2 perturbations preserve the a.c. spectrum. The Kotani-Ushiroya result implies that L^2 is the border-line in a very strong sense. For example, if Φ is a non-negative convex function with $x^{-2}\Phi(x) \to 0$ as $x \to 0$ then there is a V with $\int \Phi\big(|V(x)|\big)dx < \infty$ that gives rise to purely singular spectrum.

In the discrete case, random decaying potentials were studied by Delyon, Simon and Souillard [7, 8, 29]. Specifically, they studied potentials of the form $V(n) = a_n X_n$ where a_n decays as $|n| \to \infty$ and X_n are independent identically distributed random variables for each $n \in \mathbf{Z}$. Suppose, for simplicity, that the distribution of X_n is compactly supported. If $a_n = (1 + |n|)^{-\alpha}$, then the following are known to hold with probability one:

i) If $\alpha < 1/2$, the spectrum is pure point with rapidly decaying eigenfunctions.
ii) If $\alpha = 1/2$, then the spectrum is purely singular (continuous at low energies, pure point at higher energies) and
iii) if $\alpha > 1/2$, then Kotani has proved that the spectrum is purely absolutely

continuous.

Using this result as an input to a general method, Simon [31] has shown that if $p > 2$ then for a dense G_δ of $V \in \ell^p$ the operator $u(n+1) + u(n-1) + V(n)u(n)$ has purely singular continuous spectrum on $[-2, 2]$. This set is the essential spectum of the operator. For traditional Schrödinger operators it is shown that for a dense G_δ of $V \in L^p(\mathbf{R}), p > 2$, the operator $-u'' + Vu$ has purely singular continuous spectrum on $(0, \infty)$.

The dichotomy between L^2 perturbations preserving a.c. spectrum and non-L^2 perturbations destroying it, at least for sparse and random perturbations, led to the conjecture by Kiselev, Last and Simon [14] that if V is square integrable then $\Sigma_{\text{ac}}(H) = \Sigma_{\text{ac}}(H_0)$, at least when $q \equiv 0$.

Recent positive results began with Kiselev's proof [11] that if $|V(x)| \leq C(1 + |x|)^{-\epsilon - 3/4}$ then $\Sigma_{\text{ac}}(H) = \Sigma_{\text{ac}}(H_0)$. This paper actually proves a much stronger result: for almost every $E \in \Sigma_{\text{ac}}(H_0)$ all solutions of $-\psi'' + q\psi + V\psi = E\psi$ are bounded and, if $q \equiv 0$, have WKB-type asymptotics. That this implies $\Sigma_{\text{ac}}(H) = \Sigma_{\text{ac}}(H_0)$ follows from the subordinacy theory result mentioned earlier. By stating that the solutions have WKB asymptotics we mean that there are solutions

$$\psi_\pm = \exp\Big\{ \pm ikx \mp \tfrac{i}{2k} \int_0^x V(s)ds \Big\} + o(1)$$

with $k^2 = E$. Of course, any other solution of $-\psi'' + V\psi = E\psi$ can be written as a linear combination of ψ_+ and ψ_-.

Later Kiselev [13] and Molchanov improved this to $|V(x)| \leq C(1 + |x|)^{-\epsilon - 2/3}$. Finally, Christ-Kiselev [1] and Remling [25] obtained the optimal result in this direction, namely if $|V(x)| \leq C(1 + |x|)^{-\epsilon - 1/2}$ then for almost every energy $E \in (0, \infty)$ all solutions are bounded and, if $q \equiv 0$, obey WKB-type asymptotics. Consequently, $\Sigma_{\text{ac}}(H) = \Sigma_{\text{ac}}(H_0)$. (That this is optimal in terms of power-law decay follows from the work of Kotani-Ushiroya mentioned earlier.) By quoting these power-law theorems we have not quite presented their results in full generality. For example they are able to admit singularities: $(1 + |x|)^\epsilon V(x) \in L^p$ for any $1 \leq p < 2$. However, they were not able to find a purely L^p-type condition and offered this as a direction for further study [3]. All these works also treat power decaying potentials for the discrete Schrödinger operator.

Recently Christ and Kiselev extended their method to potentials $V \in L^p$ or $V' \in L^p$ provided $p < 2$. See [2].

For vanishing background potentials, $q \equiv 0$, Deift and Killip [5] were able to prove if $V \in L^2$ then $\Sigma_{\text{ac}}(H) = \Sigma_{\text{ac}}(H_0)$ and so confirm the conjecture of Kiselev, Last and Simon. Later we shall present a proof of this result.

The main ingredient of the Deift-Killip proof is one of the Buslaev-Faddeev-Zakharov trace formulae which are perhaps best known for their role in the study of the KdV equation [20]. Suppose V is of compact support; then we may consider solutions to $-u''(x) + V(x)u(x) = k^2 u(x)$ which agree with e^{ikx} to the right of the support of the potential. To the left of the support of the potential we can write $u(x) = a_p(k)e^{ikx} + b_p(k)e^{-ikx}$ where a_p, b_p are analytic functions of the momentum

$k \neq 0$. The subscript p is to indicate that a, b are parameterized as functions of momentum k; later we shall define a function a which is parameterized by energy $z = k^2$.

In the time-dependent picture, $e^{\pm ikx}$ represent waves travelling to the right/left. Thus it is natural to term $t(k) = 1/a_p(k)$ the transmission coefficient and $r(k) = b_p(k)/a_p(k)$ the reflection coefficient. The second of the BFK trace formulae is

$$\int_{\mathbf{R}} \log |a_p(k)| k^2 dk = \tfrac{\pi}{8} \int V(x)^2 dx - \tfrac{2\pi}{3} \sum E_n^{3/2}, \tag{1.1}$$

where $-E_n$ are the negative eigenvalues of the Schrödinger operator with potential $V(x)$. We will derive this formula in Chapter 4.

All the main results of this thesis rest on the fact that preservation of absolutely continuous spectrum follows from appropriate a priori bounds on $|a_p|$, the reciprocal of the transmission coefficient. This is made precise in Proposition 3.4. Because the eigenvalue term, $-\frac{2\pi}{3}\sum E_n^{3/2}$, is negative, (1.1) gives an a priori bound on $|a_p(k)|$. This is then sufficient to prove that square summable perturbations preserve the absolutely continuous spectrum when $q \equiv 0$.

Using higher-order trace formulae it is possible to prove similar results with estimates on derivatives. For example: if $V \in L^4$ and $V'' \in L^2$ then $\Sigma_{\mathrm{ac}}(H) = \Sigma_{\mathrm{ac}}(H_0) = (0, \infty)$. We will also prove this in Chapter 4. Though known to the authors at the time, this theorem does not appear in [5]. The spectral consequences of the full family of trace formulae have recently been studied systematically by Molchanov, Novitskii and Vainberg [18].

The trace formula method is effective only in the case $q \equiv 0$. The natural analogue of (1.1) in the presence of a periodic background potential suffers from the following problem: the contribution from the eigenvalues lying in the spectral gaps is no longer assuredly negative. The greater part of this thesis is devoted to the presentation of a different, though intimately related, method which is able to treat general periodic q. Specifically we shall interpret a as a Fredholm determinant and then use regularized determinants to study the integral of $\log |a|$ over bounded intervals in energy $E = k^2$; the trace formula is for the integral over all energies.

Although our main goal is to prove results for the whole-line operator H, this will be effected by first proving results for half-line operators. Define

$$[H_+ u](x) = -u''(x) + q(x)u(x) + V(x)u(x) \text{ in } L^2(\mathbf{R}^+) \text{ with } u(0) = 0$$

that is, a Schrödinger operator on the positive half-axis with a Dirichlet boundary condition at the origin. The half-line version of the Deift-Killip results described above is

Theorem 1. *If $q \equiv 0$ and either*
 i) *$V \in L^2$ or*
 ii) *$V \in L^4$ and $V'' \in L^2$,*
then $\Sigma_{\mathrm{ac}}(H_+) = (0, \infty)$.

In the next section, under the heading *Proof that Theorems $\Rightarrow$ Corollaries*, we will explain how one obtains whole-line results from theorems about the half-line operators. For example

Corollary 1. *Suppose $q \equiv 0$ and either*
i) $V \in L^2(\mathbf{R})$*, or*
ii) $V \in L^4(\mathbf{R})$ *and* $V'' \in L^2(\mathbf{R})$.
Then $\Sigma_{\mathrm{ac}}(H) = \Sigma_{\mathrm{ac}}(H_0)$, that is, $(0,\infty)$ and with equal spectral multiplicity two.

In the presence of a periodic background potential q we prove

Theorem 2. *If q is periodic and $V \in L^2(\mathbf{R}^+)$ then $\Sigma_{\mathrm{ac}}(H_+) = \Sigma_{\mathrm{ac}}(H_0)$.*

Corollary 2. *Suppose q is periodic and $V \in L^2(\mathbf{R})$. Then $\Sigma_{\mathrm{ac}}(H) = \Sigma_{\mathrm{ac}}(H_0)$ with spectral multiplicity two.*

Note that $\Sigma_{\mathrm{ac}}(H_0)$ is also the essential support of absolutely continuous spectrum for the unperturbed ($V \equiv 0$) half-line operator. Hence its appearance in the theorem. That this is the optimal L^p space follows from the work of Kotani-Ushiroya mentioned earlier. By adding a mild constraint, it is possible to obtain a result for $V \in L^3$.

Theorem 3. *Suppose that q is periodic, $V \in L^3(\mathbf{R}^+)$ and its Fourier transform $\hat{V} \in \ell^\infty(L^2)$ (that is the distribution $\hat{V}$ is uniformly locally L^2). Then $\Sigma_{\mathrm{ac}}(H_+) = \Sigma_{\mathrm{ac}}(H_0)$.*

Corollary 3. *Suppose that q is periodic, $V \in L^3(\mathbf{R})$ and $\hat{V} \in \ell^\infty(L^2)$, then $\Sigma_{\mathrm{ac}}(H) = \Sigma_{\mathrm{ac}}(H_0)$ with spectral multiplicity two.*

For means of later convenience we adopt the following norm on $\ell^\infty(L^2)$

$$\|f\|^2_{\ell^\infty(L^2)} = \sup_n \int_{-2\pi/L}^{2\pi/L} \left| f\left(x + \tfrac{2\pi n}{L}\right)\right|^2 dx, \tag{1.2}$$

where L denotes the period of q. In the case of zero background, $q \equiv 0$, we are able to obtain information which is local energy:

Theorem 4. *Let I be an interval in $(0,\infty)$ and define $\tilde{I} = \{k \in \mathbf{R} : k^2/4 \in I\}$. Suppose $q \equiv 0$, $V \in L^3$ and the distribution $\hat{V}$ restricts to an L^2 function on the set $\tilde{I}$. Then $I \subseteq \Sigma_{\mathrm{ac}}(H_+)$.*

Corollary 4. *Under the assumptions of the previous theorem, $I \subseteq \Sigma_{\mathrm{ac}}(H)$ with multiplicity two.*

Remarks 1. In all the above theorems/corollaries, the theory of relatively trace class perturbations states that one may add a further L^1 perturbation to the potential without changing $\Sigma_{\mathrm{ac}}(H)$.
2. Our methods do permit us to obtain a version of Theorem 3 which is local in energy; however, the criterion on V seems difficult to check in general. Specifically, one needs to bound (5.14) in $L^1(I, dE)$ to show that $I \subset \Sigma_{\mathrm{ac}}(H)$ as this would give a replacement for (5.12).

3. Let $\dot{\mathsf{H}}^s$ denote the Sobolev space of functions with square integrable derivative of order s. For non-integer s, $\dot{\mathsf{H}}^s = \{f : |k|^s \hat{f}(k) \in L^2(dk)\}$. Then from Corollary 4: if $V \in L^3 \cap \dot{\mathsf{H}}^s$ for some $s \in \mathbf{R}$ then $\Sigma_{\text{ac}}(H) = (0, \infty)$.
4. The method presented here extends readily to perturbations of the discrete Schrödinger operator h_0. One obtains the following analogues of the corollaries given above. If $q(n)$ is periodic and $V \in \ell^2(\mathbf{Z})$ then $\Sigma_{\text{ac}}(h) = \Sigma_{\text{ac}}(h_0)$ with equal multiplicity (namely two). Suppose $q \equiv 0$, $V \in \ell^3$. If $\hat{V}(\theta) = \sum V(n)e^{in\theta}$ is square integrable on an interval I then the interval $\tilde{I} = \{2\cos(\theta/2) : \theta \in I\}$ is contained in $\Sigma_{\text{ac}}(h)$. Moreover, the spectral multiplicity is equal to two on $\tilde{I}$. Notice that because $\hat{V}$ is uniformly locally L^2 iff V is ℓ^2, the analogues of Corollaries 2 and 3 coincide.

Let us briefly discuss the occurrence of $k^2/4$ in Theorem 4. By the Weyl Theorem, we know that if $V \in L^3$ then $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0)$. Consequently, the only way to obtain absence of absolutely continuous spectrum is to generate enough singular spectrum to take its place. Wigner and von Neumann discovered a potential with asymptotics $V(x) \sim \sin(kx)/x$ which has a positive eigenvalue at $E = k^2/4$. The physical explanation for this is that the separation of the bumps in the potential is half the wavelength of a particle with energy $k^2/4$. In this way, they set up a coherent reflection which is sufficient to generate an eigenstate. This gives rise to the heuristic that singularities in $\hat{V}$ at frequency k give rise to singular spectum at energy $k^2/4$. If $\hat{V}$ is square integrable on an interval I then, in some sense, it is regular at almost every frequency in that interval.

Naboko [19] and Simon [33] have extended the Wigner-von Neumann idea to construct potentials with $xV(x) \to \infty$ arbitrarily slowly which produce eigenvalues at a dense set of $E \in (0, \infty)$. By the theorems presented earlier, these eigenvalues are embedded in $\Sigma_{\text{ac}}(H) = (0, \infty)$. In this context we should also mention that Remling [26] has given bounds on the Hausdorff dimension of any possible embedded singular continuous spectrum.

We would like to present some sample applications of these theorems to sparse potentials. Given a C^∞ function of compact support f, we wish to consider potentials of the form

$$V(x) = \sum_n a_n f\big(\lambda_n[x - n!]\big). \tag{1.3}$$

From the work of Kiselev-Last and Simon mentioned above, we know that if $\lambda_n \equiv 1$ then the Schrödinger operator with potential V has purely absolutely continuous or purely singular continuous spectrum on $(0, \infty)$ depending on whether $a_n \in \ell^2$ or $a_n \notin \ell^2$ respectively. As mentioned earlier, this shows that Theorem 2 is optimal. Notice also that it also has implications for part ii) of Theorem 1: the condition that $V'' \in L^2$ cannot be replaced by an $L^p, p > 2$ condition. Indeed for any $s \in \mathbf{R}$, $V \in \dot{\mathsf{H}}^s$ iff $a_n \in \ell^2$.

Let us now consider $a_n = n^{-\alpha}$, $\lambda_n = n^\beta$. By direct calculation, it is easy to see that $V(x)$ will be L^3 if $\sum a_n^3 \lambda_n^{-1} < \infty$, that is, if $3\alpha + \beta > 1$. Because the potential is so sparse, it is easy to estimate the $\dot{\mathsf{H}}^s$ norm of V. Specifically, given distinct summands S_n, S_m from (1.3), the $\dot{\mathsf{H}}^s$ inner product of S_n and S_m is an extremely rapidly decaying function of $|n-m|$. In fact, for $s \in \mathbf{Z}$, S_n and S_m are orthogonal. As a result, $V \in \dot{\mathsf{H}}^s$ iff $2\alpha + (1-2s)\beta > 1$.

If $1/2 > \alpha > (1-\beta)/3$ and $\beta > 0$ then by choosing s sufficiently negative, $V \in L^3 \cap \dot{\mathsf{H}}^s$ and so, by Theorem 4, the corresponding Schrödinger operator has absolutely continuous spectrum on $(0,\infty)$. While if $1/2 > \alpha > (1-\beta)/3$ and $\beta < 0$ then by choosing $s > 0$ sufficiently large, $V \in L^3 \cap \dot{\mathsf{H}}^s$ and the preservation of a.c. spectrum again follows from Theorem 4.

To produce an example of the applicability of Theorem 3, we would like to consider a slightly different potential

$$V(x) = \sum_n a_n f(x-n)\cos(nx).$$

This is in L^3 iff $a_n \in \ell^3$. Taking the Fourier transform gives

$$\hat{V}(\xi) = \tfrac{1}{2}\sum_n a_n[\hat{f}(\xi-n) + e^{2i\xi n}\hat{f}(\xi+n)],$$

which is then uniformly locally square integrable. This means that one may apply Theorem 3 to see that for any periodic background potential $q(x)$, the addition of this perturbation preserves the absolutely continuous spectrum.

The text is arranged as follows: In the next chapter we present some background material and show how the corollaries follow from their respective theorems. Chapter 3 introduces the main object for study, $a(z)$, the reciprocal of the transmission coefficient, as the perturbation determinant and derives some of its basic properties. In particular, Proposition 3.4 shows how one may obtain spectral information from estimates on $a(z)$. In Chapter 4 we discuss the trace formulae and use them to prove Theorem 1. To prove the other theorems, we use regularized determinants. These are introduced in Chapter 5 and some estimates are derived. The final chapter combines this material with some facts about harmonic functions to prove Theorems 2–4.

Chapter 2 Background

In this chapter we introduce notation and some of the background material that we shall be using. Of course, proofs and more extensive discussions are left to the references. The reader may find [4] and [9] of most use. As a natural part of these discussions we prove that each of the corollaries follows from its corresponding theorem.

Let us begin with the unperturbed operator

$$[H_0 u](x) = -u''(x) + q(x)u(x) \text{ acting in } L^2(\mathbf{R}),$$

where q is periodic and bounded. The spectral theory of such operators is well understood and is usually studied through the Bloch (or Floquet) solutions. For each $z \in \overline{\mathbf{C}^+} = \{z : \operatorname{Im} z \geq 0\}$, these are the non-zero solutions of

$$-\psi''(x) + q(x)\psi(x) = z\psi(x) \tag{2.1}$$

for which there exists $\beta \in \mathbf{C} \setminus \{0\}$ such that

$$\psi(x + L) = \beta\psi(x).$$

Here L denotes the period of q. When q is identically constant the period is ill-defined. This case is excluded and will be treated in the next paragraph. For each $z \in \mathbf{C}^+ = \{z : \operatorname{Im} z > 0\}$ there are only two possibilities for β: $\beta^+(z)$ with $|\beta^+| < 1$ and $\beta^-(z) = 1/\beta^+(z)$. It is possible to extend $\beta^\pm(z)$ continuously to $\mathbf{R}$; in fact, $\sigma(H_0) = \{E \in \mathbf{R} : |\beta^+(E)| = 1\}$. Of course, $\beta^+ = 1/\beta^-$ still holds on $\mathbf{R}$. However, at the discrete set of points where $\beta^+ = \pm 1$, they are no longer distinct. For this reason, and others related to it, we wish to discard these points from consideration. So we define

$$\Sigma_0 = \{E \in \sigma(H_0) : \beta^+(E) \neq \pm 1\}. \tag{2.2}$$

Because the set of points excluded is discrete, this set forms an essential support for the absolutely continuous spectrum of H_0. Moreover, by fixing the normalisation $\psi_0^\pm(0; z) = 1$, there are solutions $\psi_0^\pm(x; z)$ of (2.1) which obey

$$\psi_0^\pm(x + L; z) = \beta^\pm(z)\psi_0^\pm(x; z)$$

respectively and which depend analytically on $z \in \mathbf{C}^+ \cup \Sigma_0$. By the definitions of $\beta^\pm$, if $z \in \mathbf{C}^+$ then $\psi_0^\pm$ decay exponentially as $x \to \pm\infty$ respectively. In particular, they are square integrable near $\pm\infty$ which will be important for defining the Weyl m-functions. Notice however that for $E \in \Sigma_0$, the solutions do not decay at all; in fact, they are almost periodic.

The case where q is identically a constant reduces easily to $q \equiv 0$. In this case, $\Sigma_0 = (0, \infty)$ and $\psi_0^{\pm}(x; z) = \exp(\pm ix\sqrt{z})$ where the branch of the square root is chosen to make $\psi_0^{\pm}$ decay in the appropriate directions.

Before moving on, we wish to make three further remarks about $\psi_0^{\pm}$. Firstly,

$$\psi_0^+(x, E) = \overline{\psi_0^-(x, E)} \quad \text{for all } E \in \Sigma_0. \tag{2.3}$$

Secondly, it is possible to give an explicit formula for the Green function in terms of $\psi_0^{\pm}$. That is, the resolvent $R_0(z) = (H_0 - z)^{-1}$ is an integral operator with kernel

$$G(x, x'; z) = \frac{\psi_0^+(x^>; z)\psi_0^-(x^<; z)}{W[\psi_0^+, \psi_0^-]}, \tag{2.4}$$

where $x^> = \max\{x, x'\}$ and $x^<$ is the minimum. In the denominator we have the Wronskian of the two solutions, $W[f, g](x) = f(x)g'(x) - f'(x)g(x)$, which does not depend on x. Lastly, in what follows the z-dependence will often be left implicit: $\psi_0^+(x) = \psi_0^+(x; z)$ and $\psi_0^-(x) = \psi_0^-(x; z)$.

As described in the introduction, our study of the perturbed operator

$$[Hu](x) = -u''(x) + q(x)u(x) + V(x)u(x) \text{ acting in } L^2(\mathbf{R})$$

will employ its relation to half-line operators, in particular, to

$$[H_{\pm}u](x) = -u''(x) + q(x)u(x) + V(x)u(x) \text{ in } L^2(\mathbf{R}^{\pm}) \text{ with } u(0) = 0.$$

These are the half-line Schrödinger operators with Dirichlet boundary conditions at the origin which have the same potential as H.

Proof that Theorems $\Rightarrow$ Corollaries. This is a well-known application of the Kato-Rosenblum theorem on relatively trace class perturbations [23]: The introduction of a Dirichlet boundary condition at the origin reduces H to $H_+ \oplus H_-$. As it is a relatively trace class perturbation, the absolutely continuous parts of H and $H_+ \oplus H_-$ are unitarily equivalent. In our preferred nomenclature: $\Sigma_{\text{ac}}(H) = \Sigma_{\text{ac}}(H_+) \cup \Sigma_{\text{ac}}(H_-)$ with multiplicity two on the set $\Sigma_{\text{ac}}(H_+) \cap \Sigma_{\text{ac}}(H_-)$. Whilst the theorems do not mention H_-, it is naturally equivalent to H_+ with the potential $V(-x)$. In this way each of the corollaries follows directly from its theorem. □

The method of Dirichlet decoupling used in this proof is very well known; for example, it is used in the standard proof of Weyl's law on the eigenvalue asymptotics for vibrating membranes. For a discussion of its use in scattering theory (in arbitrarily many dimensions) see [6].

The reason for performing this reduction from the whole line to the half-line is because the spectral properties of $H_{\pm}$ are directly accessible from the Weyl m-functions. Although we might now dispense with discussing the whole-line operator H and concentrate just on H_+, some of the special properties of the whole-line operator will make our life easier.

The Weyl m-functions are the analytic functions m^+ and m^- defined by

$$m^\pm(z) = \pm\frac{\psi^{\pm\prime}(0)}{\psi^\pm(0)} \text{ for each } z \in \mathbf{C}^+. \tag{2.5}$$

Here $\psi^\pm$ are the unique (up to scalar multiples) solutions of

$$-\psi''(x) + q(x)\psi(x) + V(x)\psi(x) = z\psi(x)$$

which are square integrable at $\pm\infty$ respectively. The functions $\psi^\pm$ are typically called the Weyl solutions.

Any such m-function admits a unique representation as

$$m(z) = A + \tfrac{1}{\pi}\int_{\mathbf{R}} \frac{1+Ez}{E-z}\,d\mu(E) \tag{2.6}$$

with $A \in \mathbf{R}$ and $d\mu$ a finite positive measure. In many references this representation is written with the measure $d\nu = (1+E^2)d\mu$ and the corresponding change to the kernel. The significance of this representation is as follows: if $d\mu^\pm$ are the measures used to represent $m^\pm$ in (2.6) then $H_\pm$ are unitarily equivalent to the operator $f(E) \mapsto Ef(E)$ acting in $L^2(\mathbf{R}, d\mu^\pm)$ respectively. Moreover, from the representation (2.6), one may infer that

$$(1+E^2)d\mu(E) = \operatorname*{weak-lim}_{\epsilon\to 0} \operatorname{Im}[m(E+i\epsilon)]dE. \tag{2.7}$$

For the unperturbed problem, $V = 0$, the Weyl solutions are the Bloch solutions and so extend analytically from $\mathbf{C}^+$ to Σ_0. Since we chose the normalisation $\psi_0^\pm(0) = 1$, the m-functions for the unperturbed problem, m_0^+, m_0^- also extend analytically to Σ_0. By (2.3) we have the additional property that $m_0^-(E) = -\overline{m_0^+(E)}$ for all $E \in \Sigma_0$.

For a general $V \in L^2$ (or L^3) the Weyl solutions need not extend analytically, or even continuously, to Σ_0. As in Deift-Killip, we shall circumvent this problem by treating the true perturbation V as the limit of a sequence of potentials V_n which are C^∞ and of compact support. A second simplification is that we may assume that V is supported on $[0,\infty)$. This is possible because the theorems refer to H_+ which only depends on $V(x)$ for $x \geq 0$. For such potentials the Weyl solutions behave very nicely:

Lemma 2.1. *Suppose $V \in C^\infty$ is supported on a compact subset of $[0,\infty)$. For each $z \in \mathbf{C} \cup \Sigma_0$ there exist solutions $\psi^\pm$ of*

$$-\psi''(x) + q(x)\psi(x) + V(x)\psi(x) = z\psi(x) \tag{2.8}$$

such that $\psi^\pm(\pm x) = \psi_0^\pm(\pm x)$ for sufficiently large $x > 0$. In fact, $\psi^-(x) = \psi_0^-(x)$ for all $x \leq 0$. Moreover, $\psi^\pm$ are analytic functions of $z \in \mathbf{C}^+ \cup \Sigma_0$.

Proof. This lemma amounts to little more than the existence theorem for linear ordinary differential equations, though it is quite instructive to first apply variation of parameters with respect to ψ_0^+, ψ_0^-. □

This lemma implies that $m^-(z) = m_0^-(z)$ and so is analytic on $\mathbf{C}^+ \cup \Sigma_0$. It also shows that $m^+(z)$ is meromorphic on $\mathbf{C}^+ \cup \Sigma_0$. In fact, if $m^+(z)$ were to have a pole inside Σ_0 this would force H^+ to have an eigenvalue. This is impossible because as $x \to \infty$ any non-trivial solution of the perturbed ODE is a non-trivial linear combination of the Bloch solutions and so not L^2. Therefore, $m^\pm(z)$ are analytic on Σ_0 and, by (2.6), on $\mathbf{C}^+$ also. As a result we may take the weak limit (2.7) explicitly to find

$$(1 + E^2) d\mu^+(E) = \operatorname{Im}\big[m^+(E)\big]\, dE \qquad \text{on } \Sigma_0. \tag{2.9}$$

This suggests that we need to bound $\operatorname{Im} m^+(z)$ away from zero almost everywhere on the spectrum of the unperturbed operator. Actually, since this equation is only for $V \in C^\infty$ of compact support, it suggests that we should bound Im $m(z)$ uniformly for a sequence of potentials converging to the true perturbation. This intuition will be made precise in Proposition 3.4.

Chapter 3 The Perturbation Determinant

The last chapter closed by suggesting that we shall be able to prove the theorems by bounding $\operatorname{Im} m(z)$ from below. The main result of this chapter, Proposition 3.4, shows how this may be achieved by first obtaining a priori bounds on the function $a(z)$. We introduce the function $a(z)$ in terms of a Fredholm determinant:

Proposition 3.1. *Suppose $V \in C^\infty$ is supported on a compact subset of $[0,\infty)$. For each $z \in \mathbf{C}^+$, the operator $R_0(z)V$ is of trace class, so we may define*

$$a(z) = \det\big(1 + R_0(z)V\big), \tag{3.1}$$

which may be written directly in terms of the Bloch and Jost solutions

$$a(z) = \frac{W[\psi^+, \psi^-]}{W[\psi_0^+, \psi_0^-]}. \tag{3.2}$$

This shows that $a(z)$ may be extended analytically to Σ_0.

Proof. Recall that R_0 is the resolvent of H_0, that is, $R_0(z) = (H_0 - z)^{-1}$. For $q \equiv 0$, that $R_0(z)V$ is trace class follows immediately from a theorem of Birman-Solomjak [27, Theorem 4.5]. This may be extended to $q \not\equiv 0$ by employing the resolvent formula. Section 3 of the above reference proves that the determinant exists and Proposition 5.7 proves (3.2) in the case that $q \equiv 0$. This proof can easily be adapted to the present cause.

Naively (3.2) only shows that $a(z)$ is meromorphic on Σ_0. However, the theory of periodic Schrödinger operators shows that $W[\psi_0^+, \psi_0^-]$ is non-zero throughout $\mathbf{C}^+ \cup \Sigma_0$. □

When $q \equiv 0$ we described another function a_p in the introduction as the reciprocal of the transmission coefficient. An explanation of their equivalence, up to parameterization, will be posponed until the next chapter where the trace formula 1.1 is proved. In order to relate a and the m-functions we will need a further

Lemma 3.2. *Suppose V is of compact support. Then for each point $E \in \Sigma_0$,*

$$\frac{\operatorname{Im} m^+(E)}{\operatorname{Im} m_0^+(E)} = \left| \frac{\psi_0^+(0)}{\psi^+(0)} \right|^2 \tag{3.3}$$

and similarly for m^-, m_0^-. Moreover for such E, all four m-functions have positive imaginary part.

Proof. Because E is real, both $\psi^+(x)$ and $\overline{\psi^+}(x)$ (its complex conjugate) are solutions to the perturbed ODE (2.8). From the invariance of Wronskians, it then follows that

$$\begin{aligned} -2i\mathrm{Im}(m^+)|\psi^+(0)|^2 &= {\psi^+}'(0)\overline{\psi^+(0)} - \psi^+(0)\overline{{\psi^+}'(0)} \\ &= W\left[\psi^+, \overline{\psi^+}\right](0) \\ &= W\left[\psi^+, \overline{\psi^+}\right](x) \quad \forall x. \end{aligned}$$

Similarly, for all $x \in \mathbf{R}$, $-2i\mathrm{Im}(m_0)|\psi_0^+(0)|^2 = W\left[\psi_0^+, \overline{\psi_0^+}\right](x)$ which is non-zero for $E \in \Sigma_0$ because $\overline{\psi_0^+} = \psi_0^-$ for such E and the Bloch solutions are linearly independent. Now for x sufficiently large $\psi^+(x)$ and $\psi_0^+(x)$ agree. This means that the Wronskians must be equal, which proves (3.3). It also shows that $\mathrm{Im}\, m^\pm(E)$ and $\mathrm{Im}\, m_0^\pm(E)$ are non-zero. That they are not negative is a consequence of the representation (2.6). $\square$

Evaluating the Wronskians of (3.2) at the point $x = 0$, we have

$$\begin{aligned} a(z) &= \frac{\psi^+(0){\psi^-}'(0) - \psi^+(0){\psi^-}'(0)}{\psi_0^+(0){\psi_0^-}'(0) - \psi_0^+(0){\psi_0^-}'(0)} \\ &= \frac{\psi^+(0)\psi^-(0)}{\psi_0^+(0)\psi_0^-(0)} \frac{m^+(z) + m^-(z)}{m_0^+(z) + m_0^-(z)}. \end{aligned}$$

However, because we only wish to prove theorems about H_+ we continue to assume V is supported in $[0, \infty)$. This means $\psi^-(0) = \psi_0^+(0)$ and $m^-(z) = m_0^-(z)$ (refer to Lemma 2.1). Employing these identities,

$$a(z) = \frac{\psi^+(0)}{\psi_0^+(0)} \frac{m^+(z) + m_0^-(z)}{m_0^+(z) + m_0^-(z)}$$

for each $z \in \mathbf{C}^+ \cup \Sigma_0$. If we consider just $E \in \Sigma_0$, Lemma 3.2 permits us to write

$$\begin{aligned} |a(E)|^2 &= \frac{\mathrm{Im}\, m_0^+(E)}{\mathrm{Im}\, m^+(E)} \left| \frac{m^+(E) + m_0^-(E)}{m_0^+(E) + m_0^-(E)} \right|^2 \\ &= \frac{|m^+(E) + m_0^-(E)|^2}{4\, \mathrm{Im}\, m^+(E)\, \mathrm{Im}\, m_0^-(E)}. \end{aligned} \tag{3.4}$$

To obtain the second equality, we have used the fact that (2.3) implies $\overline{m_0^+(E)} = -m_0^-(E)$ for each $E \in \Sigma_0$.

From the representation (2.6) it follows that all m-functions have non-negative imaginary parts. From this and some simple manipulations of (3.4) it follows that $|a(E)|^2 \geq 1$ for all $E \in \Sigma_0$. In fact, $a(z)$ does not have any zeros in $\mathbf{C}^+$ either. If it did, (3.2) would imply that ψ^+, ψ^- were linearly dependent and hence that the perturbed ODE (2.8) has an L^2 solution for this z. This is impossible because H is self-adjoint and so cannot have a non-real eigenvalue. Let us repeat our recent deductions for future reference.

Lemma 3.3. *Suppose $V \in C^\infty$ is supported on a compact subset of $[0, \infty)$. The func-*

tion $a(z)$ is analytic and non-zero on $\mathbf{C}^+ \cup \Sigma_0$. Consequently, $\log\left|a(z)\right|$ *is harmonic there. Moreover,*

$$\left|a(E)\right|^2 = \frac{|m^+(E) + m_0^-(E)|^2}{4\,\mathrm{Im}\, m^+(E)\,\mathrm{Im}\, m_0^-(E)} \geq 1 \tag{3.5}$$

for all $E \in \Sigma_0$.

We have bounded $\log|a(E)|$ from below. But really, as the following proposition suggests, we should bound it from above. Since $a(z)$ is non-zero, $\log|a(z)|$ is harmonic. This will prove instrumental in obtaining the desired inequality.

Proposition 3.4. *Let $V \in L^p(\mathbf{R}^+)$ (for any $1 \leq p < \infty$) and extend it to a function on $\mathbf{R}$ by $V(x) = 0$ for $x < 0$. Further, let V_n be a sequence of C^∞ functions on $\mathbf{R}$, each supported on a compact subset of $[0,\infty)$, which converge to the true perturbation V in L^p sense. To each such approximate perturbation, associate its own function $a_n(z) = \det[1 + R_0(z)V_n]$. Suppose $I \subset \Sigma_0$ is a compact interval and that there is a bounded continuous function $w : I \to [0,\infty)$ with support I such that*

$$\int_I \log\left|a_n(z)\right| w(E)dE \leq C \tag{3.6}$$

holds uniformly in n. Then I is contained in the essential support of the absolutely continuous spectrum of H_+, in symbols $I \subset \Sigma_{\mathrm{ac}}(H_+)$.

Proof. Let $m_n^\pm$ denote the Weyl m-functions for $H_n = H_0 + V_n$ and $m^\pm$ those for the full perturbed problem $H = H_0 + V$. For consistency of notation we fix $V_0 \equiv 0$. Because $V_n \to V$ in L^p, $m_n^\pm(z) \to m^\pm(z)$ uniformly on compact subsets of $\mathbf{C}^+$. From the representation (2.6) it follows that $d\mu_n^\pm$, the measures occurring in the representation of $m_n^\pm$, converge weakly to $d\mu^\pm$, the measures for $m^\pm$. From the fact that $\mathrm{Im}\, m_n^\pm(i) \to \mathrm{Im}\, m^\pm(i)$ one may also infer that the total mass of each $d\mu_n^\pm$ is uniformly bounded.

From Lemma 3.2 we know that $\mathrm{Im}\, m_n^\pm(E)$ are positive for all $n \geq 0$. Consequently, $|m_n^+(E) + m_0^-(E)| > \mathrm{Im}\, m_0^-(E)$ and it follows, from (3.4), that

$$\left|a(E)\right|^2 \geq \frac{\mathrm{Im}\, m_0^-(E)}{4\,\mathrm{Im}\, m_n^+(E)} \text{ for all } E \in \Sigma_0.$$

As we just remarked, $\mathrm{Im}\, m_0^-(E)$ is positive on Σ_0 and hence it is bounded below on the compact set I. As a result, there is a constant $c \geq 0$ so that

$$2\log\left|a(E)\right| \geq -\log\left[\mathrm{Im}\, m_n^+(E)\right] - c.$$

Integrating and applying (3.6) it follows that there exists a constant C' such that

$$-\int_I \log\left[\mathrm{Im}\, m_n^+(E)\right] w(E)\, dE \leq C'. \tag{3.7}$$

This suggests that $\mathrm{Im}\, m_n^+(E)$ cannot be too small. But first we must exclude the possibility that cancellations are occurring. Let $\log^+(x) = \max\{0, \log(x)\}$ which obeys $\log^+(x) \le x$. Then

$$\int_I \log^+ \big[\operatorname{Im} m_n^+(E)\big] w(E)\, dE \le \int_I \operatorname{Im} m_n^+(E) w(E)\, dE =$$
$$= \int_I w(E) d\mu_n^+(E) \to \int w(E) d\mu^+$$

by using (2.9) and the weak convergence of $d\mu_n^\pm$ discussed at the beginning of this proof. This means that the integral of $\log^+[\operatorname{Im} m_n^+(E)]$ is uniformly bounded and (3.7) reduces to

$$\int_I \log^- \big[\operatorname{Im} m_n^+(E)\big] w(E)\, dE \le C''$$

uniformly in n. Here we have used the notation $\log^-(x) = \max\{0, -\log(x)\}$. Since $\log^-$ is convex, we may apply Jensen's inequality and then (2.9) to infer

$$C'' \ge w(K)\ \log^- \Big\{ \tfrac{1}{w(K)} \int_K \operatorname{Im} m_n^+(E) w(E)\, dE \Big\}$$
$$\Rightarrow \qquad \int_K w(E)\, d\mu_n^+(E) \ge w(K)\ \exp\big\{-C''/w(K)\big\}$$

for any compact $K \subset I$. Here $w(K) = \int_K w(E)\, dE$.

Observe the following fact: if the measure $d\nu$ (on $\mathbf{R}$) is the weak limit of $d\nu_n$ and K is a compact set then $\nu(K) \ge \overline{\lim}\, \nu_n(K)$. This is a simple consequence of the fact that Borel measures on $\mathbf{R}$ are regular (c.f. Lemma 2 of [5]). Applying this fact with $d\nu_n = w(E)\, d\mu_n^+(E)$,

$$\int_K w(E)\, d\mu^+(E) \ge w(K) \exp\big\{-C'/w(K)\big\}.$$

Since w is positive Lebesgue almost everywhere on I, and so on K, this proves that the Lebesgue measure on I is absolutely continuous with respect to $d\mu^+$. That is, I is a subset of the essential support of the absolutely continuous part of $d\mu^+$. This completes the proof for, as we mentioned after introducing the representation (2.6), H_+ is unitarily equivalent to multiplication by E in $L^2\big(d\mu^+(E)\big)$. $\square$

Chapter 4 Trace Formulae

This chapter is dedicated to the case $q \equiv 0$ and presents joint work with P. Deift [5]. This permits some considerable simplifications, especially if we replace the former spectral parameter z by k^2 with $k \in \mathbf{C}$. For example, for each $z \in \mathbf{C}^+$, the Weyl solutions are given by $\psi_0^\pm(x; z) = \exp(ikx)$ where k is the square root of z which lies in the upper half-plane. With this parameterization, we obtain Weyl solutions which are entire functions of k; as functions of z they have a branch cut. In analogy with Lemma 2.1 we have

Lemma 4.1. *If V is supported on a compact subset of $[0, \infty)$ then $\psi^\pm(x)$ are entire functions of k and, for $k \neq 0$, we may write*

$$\psi^+(x) = a_p(k)e^{ikx} + b_p(k)e^{-ikx} \quad \forall x < 0 \tag{4.1}$$

with a_p and b_p analytic functions of $k \in \mathbf{C} \setminus \{0\}$.

Now we are ready to demonstrate the relation between $a_p(k)$ and the function $a(z)$ of the previous chapter. Evaluating the Wronskians of (3.2) at any point $x < 0$ we have

$$\begin{aligned} a(z) &= \frac{W[a_p e^{ikx} + b_p(k)e^{-ikx}, e^{-ikx}]}{W[e^{ikx}, e^{-ikx}]} \\ &= a_p(k) \end{aligned}$$

and so $a(z) = a_p(k)$ provided k is chosen to be the square root of z which lies in the upper half-plane.

In order to apply Proposition 3.4 we wish to obtain a bound on a weighted integral of $\log |a(E)|$ over $[0, \infty)$ for potentials $V \in C_c^\infty$. The second of the BFZ trace formulae (1.1) arises from the evaluation of

$$\int_0^\infty \log \left|a(E)\right| \sqrt{E}\, dE$$

and is valid for potentials V with $(1 + |x|)V(x) \in L^1$. As we do not need it in this generality, we shall derive it for $V \in C_c^\infty$.

By taking complex conjugates in (4.1), it follows that $a_p(-k) = \overline{a_p(k)}$. As a result we can rewrite the above integral as

$$\int_{\mathbf{R}} \log \left[a_p(k)\right] k^2 dk.$$

We would like to evaluate this integral by closing the contour in the upper half-plane. While $a_p(k)$ is analytic, $\log \left[a_p(k)\right]$ fails to be at the zeros of $a_p(k)$. By (3.5) we know that a_p does not have zeros for $k \in \mathbf{R}$. Moreover, as discussed just before Lemma 3.3, complex zeros can only occur if $z = k^2$ is an eigenvalue of (2.8). Therefore, the zeros $a_p(k)$ which lie in the upper half-plane occur at finitely many purely imaginary points $i\beta_1, ..., i\beta_m$ corresponding to the negative eigenvalues $-E_1 = -\beta_1^2, ..., -E_m = -\beta_m^2$ of (2.8). To remove the problem of its zeros, we divide $a_p(k)$ by the Blaschke product

$$B(k) = \prod_{j=1}^{m} \frac{k - i\beta_j}{k + i\beta_j}.$$

Then the Cauchy Theorem says

$$\int_{-R}^{R} \log\left[a_p(k)/B(k)\right] k^2 dk = -\int_0^{\pi} \log\left[a(Re^{i\theta})/B(Re^{i\theta})\right] iR^3 e^{3i\theta} d\theta.$$

Because for $k \in \mathbf{R}$ both $B(-k) = \overline{B(k)}$ and $|B(k)| = 1$, the left-hand integral is unchanged by the introduction of $B(k)$. So taking the limit $R \to \infty$

$$\begin{aligned}\int_0^{\infty} \log\left|a(E)\right| \sqrt{E}\, dE &= -\lim_{R\to\infty} \int_0^{\pi} \log\left[a_p(Re^{i\theta})/B(Re^{i\theta})\right] iR^3 e^{3i\theta} d\theta \\ &= -\tfrac{2\pi}{3} \sum_{j=1}^{m} E_j^{3/2} - \lim_{R\to\infty} \int_0^{\pi} \log\left[a_p(Re^{i\theta})\right] iR^3 e^{3i\theta} d\theta. \qquad (4.2)\end{aligned}$$

To calculate the remaining integral, we need to determine the asymptotics of $\log[a(k)]$ as k goes to infinity in the upper half-plane. If we write $\psi^+(x) = p(x)e^{ikx}$, then $p(x)$ obeys the integral equation

$$p(x) = 1 + \tfrac{i}{2k} \int_x^{\infty} \left[1 - e^{2ik(y-x)}\right] V(y) p(y) dy. \qquad (4.3)$$

This equation is of Volterra type and so may be solved by iteration. If $\operatorname{Im} k \geq 0$ the kernel is $O(1/|k|)$, so for $x < 0$ and $\operatorname{Im} k \geq 0$ we have

$$\begin{aligned}p(x) = 1 &+ \tfrac{i}{2k} \int_0^{\infty} \left[1 - e^{2ik(y-x)}\right] V(y) dy \\ &- \tfrac{1}{4k^2} \int_0^{\infty}\int_y^{\infty} \left[1 - e^{2ik(y-x)}\right]\left[1 - e^{2ik(z-y)}\right] V(y)V(z) dz dy \\ &- \tfrac{i}{8k^3} \int_0^{\infty}\int_y^{\infty}\int_z^{\infty} \left[1 - e^{2ik(y-x)}\right]\left[1 - e^{2ik(z-y)}\right]\left[1 - e^{2ik(u-z)}\right] V(y)V(z)V(u) du dz dy \\ &+ O\big(|k|^{-4}\big).\end{aligned}$$

Because $x < 0$, $\psi^+(x) = a_p(k)e^{ikx} + b_p(k)e^{-ikx}$ and so $p(x) = a_p(k) + b_p(k)e^{-2ikx}$. Therefore, to find a_p, we just need to collect those terms in the above expansion without a leading factor of e^{-2ikx}:

$$\begin{aligned}a_p(k) =&1 + \tfrac{i}{2k}\int V(y)dy - \tfrac{1}{8k^2}\iint \left[1 - e^{2ik|y-z|}\right] V(y)V(z)dydz\\&+ \int_0^\infty\int_y^\infty\int_z^\infty \left[1 - e^{2ik(z-y)}\right]\left[1 - e^{2ik(u-z)}\right] V(y)V(z)V(u)dudzdy\\&+ O(|k|^{-4}).\end{aligned}$$

The change in the term with two integrals arises from symmetrizing the integrand then taking half of the integral over all space.

Now we proceed to expand $\log[a_p(k)]$ using $\log(1+\epsilon) = \epsilon - \epsilon^2/2 + \epsilon^3/3 + O(\epsilon^4)$; again this is for $\mathrm{Im}k \geq 0$:

$$\begin{aligned}\log[a_p(k)] =&\tfrac{i}{2k}\int V(y)dy + \tfrac{1}{8k^2}\iint e^{2ik|y-z|}V(y)V(z)dydz\\&- \tfrac{i}{16k^3}\int V(x)dx \iint e^{2ik|y-z|}V(y)V(z)dydz\\&- \tfrac{i}{8k^3}\int_0^\infty\int_y^\infty\int_z^\infty \left[e^{2ik(u-y)} - e^{2ik(z-y)} - e^{2ik(u-z)}\right] V(y)V(z)V(z)dydzdu\\&+ O(|k|^{-4}).\end{aligned}$$

Notice that something remarkable has happened; the three terms of the form $\frac{C}{k^3}\left[\int V\right]^3$ which occur, cancel. To see this we have used the fact that

$$\int_0^\infty\int_y^\infty\int_z^\infty V(y)V(z)V(u)\, dydzdu = \tfrac{1}{6}\iiint V(y)V(z)V(u)\, dydzdu.$$

We are now ready to perform the contour integral

$$\lim_{R\to\infty}\int_0^\pi \log\left[a_p(Re^{i\theta})\right] iR^3e^{3i\theta}d\theta.$$

Naive estimation shows that terms $O(|k|^{-4})$ make no contribution as $R \to \infty$. The term of order $|k|^{-1}$, $\frac{i}{2k}\int V$, gives zero identically for all R. The two terms $O(|k|^{-3})$ also give zero contribution in the limit $R \to \infty$ because both integrands are $O\left(\frac{1}{1+R\sin(\theta)}\right)$. (Notice that this would not be the case had the terms of the form $\frac{C}{k^3}\left[\int V\right]^3$ not cancelled.) With these observations, (4.2) reduces to

$$\int_0^\infty \log\left|a(E)\right|\sqrt{E}\,dE = -\tfrac{2\pi}{3}\sum_{j=1}^m E_j^{3/2} - \lim_{R\to\infty}\tfrac{i}{8}\int_0^\pi\iint e^{-2iR|y-z|e^{i\theta}}V(y)V(z)Re^{i\theta}dydzd\theta.$$

To perform this last integral, it is better to use the Cauchy Theorem once more—this time to replace the semi-circular contour $Re^{i\theta}$, $\theta \in [0,\pi]$ with a straight line:

$$\int_0^\infty \log\big|a(E)\big|\sqrt{E}\,dE = -\tfrac{2\pi}{3}\sum_{j=1}^m E_j^{3/2} + \lim_{R\to\infty}\tfrac{1}{8}\iiint_{-R}^{R} e^{2ik|y-z|}V(y)V(z)\,dkdydz.$$

Performing the k integral first gives a factor $\sin[2R(y-z)]/(y-z)$. As $R\to\infty$ this factor converges distributionally to $\pi\delta(y-z)$. Since we have assumed that V is C^∞ and compactly supported, this leads to

$$\int_0^\infty \log\big|a(E)\big|\sqrt{E}\,dE = -\tfrac{2\pi}{3}\sum_{j=1}^m E_j^{3/2} + \tfrac{\pi}{8}\int V^2 \tag{4.4}$$

which is the trace formula we set out to prove.

Proceeding more systematically it is possible to derive the full family of trace formulae. See for example [20] for a proof of

Proposition 4.2. *If V is C^∞ and of compact support then*

$$\int_0^\infty \log|a(E)|E^{n-1/2}dE = \frac{(-1)^n\pi}{2^{2n+1}}\int \xi_{2n+1}(x)dx + \frac{(-1)^n 2\pi}{2n+1}\sum E_m^{n+1/2}, \tag{4.5}$$

where $\xi_0(x)=0, \xi_1(x)=V(x)$ and $\xi_{n+1}+\xi_n' + \sum_{\ell=1}^{n}\xi_\ell\xi_{n-\ell}=0$.

Proof of Theorem 1. We begin by proving part i). Let V_n be a sequence of C^∞ functions of compact support which converge to the true perturbation V in L^2. Then it follows from (4.4) that

$$\int_0^\infty \log\big|a_n(E)\big|\sqrt{E}\,dE \le \tfrac{\pi}{8}\int V_n^2 \le C$$

because the eigenvalue term is always negative, or zero. But this provides the conditions of Proposition 3.4 and so proves that $\Sigma_{\rm ac}=[0,\infty)$.

The proof of part ii) is almost identical, but employs (4.5) with $n=3$; (4.4) corresponds to $n=1$. More explicitly we have

$$\begin{aligned}\int_0^\infty \log|a(E)|E^{5/2}dE &= -\tfrac{2\pi}{7}\sum E_m^{7/2} + \tfrac{\pi}{2^7}\int (V'')^2 - 5V^2V'' + 5V^4\,dx\\ &\le \tfrac{\pi}{2^7}\int (V'')^2 - 5V^2V'' + 5V^4\,dx\\ &\le \tfrac{7\pi}{2^8}\big\|V''\big\|_{L^2}^2 + \tfrac{15\pi}{2^8}\big\|V\big\|_{L^4}^4,\end{aligned}$$

where the last inequality follows from Hölder and $2XY \le X^2+Y^2$. In this way, if we choose a sequence of C^∞ functions V_n each of compact support so that $V_n \to V$ in both L^4 and $\dot{H}^2$, then we have a uniform bound on

$$\int_0^\infty \log|a_n(E)|E^{5/2}dE.$$

This means that we may once again employ Proposition 3.4 to complete the proof.

Chapter 5 Regularized Determinants

This chapter introduces regularized determinants and provides some estimates. As in the previous chapter, we are interested in a priori bounds and shall assume that V is C^∞ and of compact support.

If one regards determinants as the product of the eigenvalues, regularized determinants are the analogue of the convergence factors used in the Weierstrass Factorisation Theorem. A thorough treatment of these functions may be found in [27]. We shall need only the first two regularized determinants:

$$\det_2(1+A) = \det(1+A)\exp\{-\mathrm{tr}(A)\}, \tag{5.1}$$
$$\det_3(1+A) = \det(1+A)\exp\{-\mathrm{tr}(A) + \tfrac{1}{2}\mathrm{tr}(A^2)\}. \tag{5.2}$$

Whilst at first these are only defined for elements of trace class, $A \in \mathfrak{I}_1$, they may be extended to the Hilbert-Schmidt class, $\mathfrak{I}_2$, and $\mathfrak{I}_3 = \{A : \mathrm{tr}(|A|^3) < \infty\}$ respectively. The possibility of this extension is underlined by the estimates

$$\big|\det_2(1+A)\big| \le \exp\{C_2\|A\|_2^2\}, \tag{5.3}$$
$$\big|\det_3(1+A)\big| \le \exp\{C_3\|A\|_3^3\}, \tag{5.4}$$

where C_2, C_3 are universal constants and $\|A\|_n^n = \mathrm{tr}(|A|^n)$ denotes the nth power of the norm on $\mathfrak{I}_n$. These estimates are proved in Section 9 of [27]. Our application of these estimates is

Lemma 5.1. *There is a constant C depending only on q, so that the estimates*

$$\big\|R_0(E+i\epsilon)V\big\|_2 \le C\epsilon^{-1/2}(1+\epsilon^{-1/2})\|V\|_{L^2},$$
$$\big\|R_0(E+i\epsilon)V\big\|_3 \le C\epsilon^{-1/2}(1+\epsilon^{-1/2})\|V\|_{L^3}$$

hold for all $E \in \mathbf{R}$ and all $\epsilon > 0$. And consequently, for a new constant C,

$$\log\big|\det_2\big(1+R_0(E+i\epsilon)V\big)\big| \le C\epsilon^{-1}(1+\epsilon^{-1})\|V\|_{L^2}^2,$$
$$\log\big|\det_3\big(1+R_0(E+i\epsilon)V\big)\big| \le C\epsilon^{-3/2}(1+\epsilon^{-3/2})\|V\|_{L^3}^3$$

hold for this same set of E, ϵ.

Proof. The first two estimates follow directly from Theorem B.9.3 of [28]. More explicitly, since $|E+i\epsilon|^{-1}$ is bounded by a constant multiple of $\epsilon^{-1/2}(1+\epsilon^{-1/2})(1+|E|)^{-1/2}$, this theorem shows that there exists C_p with

$$\big\|R_0(E+i\epsilon)V\big\|_p \le C_p\epsilon^{-1/2}(1+\epsilon^{-1/2})\big\|V\big\|_{L^p}$$

for any $p \ge 2$. Employing these bounds in (5.3) and (5.4) gives the second pair of estimates. ∎

Given the specificity of the situation we are treating, these estimates are quite crude. However, they will prove ample for our purposes.

For $E + i\epsilon \in \mathbf{C}^+$ we can also obtain lower bounds on $\log|\det(1 + R_0V)|$.

Proposition 5.2. *Suppose V is C^∞ and of compact support. Then there is a constant C so that*

$$\log\left|\det{}_2\bigl(1 + R_0(E + i\epsilon)V\bigr)\right| \geq -C\epsilon^{-1}(1 + \epsilon^{-1})\|V\|_{L^2}^2, \tag{5.5}$$

$$\log\left|\det{}_3\bigl(1 + R_0(E + i\epsilon)V\bigr)\right| \geq -C\epsilon^{-3/2}(1 + \epsilon^{-3/2})\|V\|_{L^3}^3 \tag{5.6}$$

hold for all $E \in \mathbf{R}$ and $\epsilon > 0$.

Proof. Let us begin by calculating some derivatives. Note that all resolvents appearing in this proof are evaluated at the point $E + i\epsilon \in \mathbf{C}^+$.

$$\begin{aligned}\frac{d}{d\epsilon}\log\left|\det(1 + R_0V)\right| &= \operatorname{Re}\,\operatorname{tr}\bigl(i(1 + R_0V)^{-1}R_0R_0V\bigr)\\ &= \operatorname{Re}\,\operatorname{tr}\bigl(iRR_0V\bigr)\end{aligned} \tag{5.7}$$

by the resolvent formula. Also

$$\frac{d}{d\epsilon}\operatorname{Re}\,\operatorname{tr}(R_0V) = \operatorname{Re}\,\operatorname{tr}\bigl(iR_0R_0V\bigr) \tag{5.8}$$

$$\frac{d}{d\epsilon}\tfrac{1}{2}\operatorname{Re}\,\operatorname{tr}(R_0VR_0V) = \operatorname{Re}\,\operatorname{tr}\bigl(iR_0VR_0R_0V\bigr) \tag{5.9}$$

By combining (5.7) and (5.8) we have

$$\begin{aligned}\frac{d}{d\epsilon}\log\left|\det{}_2(1 + R_0V)\right| &= \operatorname{Re}\,\operatorname{tr}\bigl(i(R - R_0)R_0V\bigr)\\ &= -\operatorname{Re}\,\operatorname{tr}\bigl(iR_0VRR_0V\bigr).\end{aligned} \tag{5.10}$$

By employing the trace ideal version of Hölder's inequality, this implies

$$\left|\frac{d}{d\epsilon}\log\left|\det{}_2(1 + R_0V)\right|\right| \leq \|R_0V\|_2^2\,\|R\|$$

Here $\|\cdot\|$ denotes the operator norm. Of course, $\|R\| \leq \epsilon^{-1}$. This and Lemma 5.1 permit us to conclude

$$\left|\frac{d}{d\epsilon}\log\left|\det{}_2(1 + R_0V)\right|\right| \leq C'\epsilon^{-2}(1 + \epsilon^{-1})\|V\|_{L^2}^2.$$

Continuing from (5.10), one can show

$$
\begin{aligned}
\frac{d}{d\epsilon}\log\left|\det_3(1+R_0V)\right| &= -\mathrm{Re}\,\mathrm{tr}\big(iR_0VRR_0V\big) + \mathrm{Re}\,\mathrm{tr}\big(iR_0VR_0R_0V\big)\\
&= \mathrm{Re}\,\mathrm{tr}\big(iR_0V(R_0-R)R_0V\big)\\
&= \mathrm{Re}\,\mathrm{tr}\big(iR_0VR_0VRR_0V\big)\\
\left|\frac{d}{d\epsilon}\log\left|\det_3(1+R_0V)\right|\right| &\le \|R_0V\|_3^3\|R\| \le C''\epsilon^{-5/2}(1+\epsilon^{-3/2})\|V\|_{L^3}^3
\end{aligned}
$$

by similar means.

Since V is of compact support, it is easy to show that

$$\lim_{\epsilon\to\infty}\det_n\big[1+R_0(E+i\epsilon)V\big]=1 \quad \forall E\in\mathbf{R}.$$

Hence the proposition follows from our derivative bounds and the Fundamental Theorem of Calculus. □

Our goal is to obtain (3.6) so that we may apply Proposition 3.4. This requires us to understand the behaviour of $a(E)$ for $E\in\Sigma_0$ whilst thus far we have been estimating the regularized determinants for $z\in\mathbf{C}^+$. The significance of these estimates at non-real z will become clearer in the next chapter. For the moment, however, we wish to relate $a(E)$ and the regularized determinants for $E\in\Sigma_0$. We begin by considering $\det_2$.

Proposition 5.3. *Suppose V is C^∞ and of compact support. Then the harmonic function*

$$f(z)=\log\left|\det_2\big(1+R_0(z)V\big)\right|$$

extends continuously from $\mathbf{C}^+$ to Σ_0. Moreover, $f(E)=\log\big|a(E)\big|$ for $E\in\Sigma_0$.

Proof. To see why $f(z)$ is harmonic, just recall Lemma 3.3 and

$$f(z)=\log|a(z)|-\mathrm{Re}\,\mathrm{tr}\big(R_0(z)V\big)$$

from the definition of $\det_2$. Because V is integrable, we can write

$$\mathrm{Re}\,\mathrm{tr}\big(R_0(z)V\big)=\mathrm{Re}\int G(x,x;z)V(x)dx \tag{5.11}$$

and so extend the left-hand side continuously to Σ_0. By Lemma 3.3 this means that $f(z)$ extends continuously to Σ_0. To complete the proof we need to show that (5.11) vanishes for $z=E\in\Sigma_0$. For such E, we can use (2.3) and (2.4) to write

$$\mathrm{Re}\int G(x,x;E)V(x)dx=\mathrm{Re}\int\frac{|\psi^+(x)|^2}{W\big[\psi^+,\overline{\psi^+}\big]}V(x)dx$$

which is zero because $W\big[\psi^+,\overline{\psi^+}\big]$ is purely imaginary. □

The fact that the real part of the Green function vanishes inside the spectrum is what permitted us to conclude that $f(E) = \log|a(E)|$. This holds not only for periodic q but on the absolutely continuous spectrum of any ergodic Schrödinger operator [15, 30]. It is possible to extend our results to prove that any interval contained within the a.c. spectrum of an operator with ergodic potential remains in the a.c. spectrum under an L^2 perturbation. For these potentials, the results that we obtained quite easily from the properties of the Bloch solutions must be obtained more circuitously. For example, the fact that the Weyl solutions extend analytically from $\mathbf{C}^+$ to Σ_0 must be obtained by first observing that the Green function may be extended analytically by an application of the Schwarz reflection principle.

The equivalent of Proposition 5.3 for $\det_3$ is more involved, especially if $q \not\equiv 0$.

Proposition 5.4. *Suppose $V \in C^\infty$ and is of compact support. Then the harmonic function*

$$g(z) = \log\left|\det_3\bigl(1 + R_0(z)V\bigr)\right|$$

extends continuously from $\mathbf{C}^+$ to Σ_0. Suppose $q \equiv 0$ and $I \subset \Sigma_0 = (0,\infty)$ is a compact interval. If we define $\tilde{I} = \{k \in \mathbf{R} : k^2/4 \in I\}$ then

$$\int_I \Bigl|g(E) - \log|a(E)|\Bigr| dE \le C_I \bigl\|\hat{V}\bigr\|^2_{L^2(\tilde{I};dk)}. \tag{5.12}$$

The version for $q \not\equiv 0$ is slightly weaker: for any compact interval $I \subset \Sigma_0$

$$\int_I \Bigl|g(E) - \log|a(E)|\Bigr| dE \le C_I \bigl\|\hat{V}\bigr\|^2_{\ell^\infty(L^2)}. \tag{5.13}$$

The definition of $\ell^\infty(L^2)$ was given in (1.2) *of Chapter 2.*

Proof. The result on continuous extension is essentially the same as the previous proposition because

$$\begin{aligned} g(z) &= f(z) + \tfrac{1}{2}\mathrm{Re}\ \mathrm{tr}\bigl[R_0(z)VR_0(z)V\bigr] \\ &= f(z) + \tfrac{1}{2}\mathrm{Re} \iint G(x,y;z)^2 V(x)V(y)\,dx\,dy. \end{aligned}$$

As $f(E) = \log|a(E)|$ for all $E \in \Sigma_0$, we need only attend to the additional term. Given $E \in \Sigma_0$, we may employ (2.3) and (2.4) to write

$$g(E) - f(E) = 2\mathrm{Re} \iint_{x>y} \frac{\psi_0^+(x)^2\overline{\psi_0^+}(y)^2}{W\bigl[\psi_0^+, \overline{\psi_0^+}\bigr]^2} V(x)V(y)\,dx\,dy.$$

However, the Wronskian is purely imaginary and taking the complex conjugate of

$\psi_0^+(x)^2\overline{\psi_0^+}(y)^2$ is equivalent to interchanging x, y. So

$$g(E) - f(E) = \frac{1}{W\left[\psi_0^+, \overline{\psi_0^+}\right]^2} \iint \psi_0^+(x)^2\overline{\psi_0^+}(y)^2 V(x)V(y)dx\,dy$$
$$= \frac{1}{W\left[\psi_0^+, \overline{\psi_0^+}\right]^2} \left| \int \psi_0^+(x)^2 V(x)dx \right|^2. \tag{5.14}$$

Controlling this when $q \equiv 0$ is the easiest because $\psi_0^+(x; E) = \exp(ikx)$ where $k^2 = E$ and $k > 0$. In this case, therefore,

$$g(E) - f(E) = \frac{-1}{4k^2}\left|\hat{V}(2k)\right|^2$$

and for each compact $I \subset (0, \infty)$ we have

$$\int \left|g(E) - f(E)\right| dE \leq C_I \left\|\hat{V}\right\|^2_{L^2(\tilde{I})},$$

which proves (5.12) because $f(E) = \log|a(E)|$.

The proof for $q \not\equiv 0$ is more involved. What we present now is actually just a minor alteration of an argument from the "Proof of Theorem 1.8" in [12]. Recall from Chapter 2 that there is a unimodular complex number $\beta^+(E)$ associated to each $E \in \Sigma_0$ so that

$$\psi_0^+(x + L; E) = \beta^+(E)\psi_0^+(x; E), \tag{5.15}$$

where L is the period of q. Moreover, $\beta^+(E) \neq \pm 1$ so we may define $\gamma(E) = \arg(\beta^+(E)) \in (-\pi, \pi)$. It is well known that $\gamma(E)$ is strictly monotonic on each connected subset of Σ_0. Indeed, there exists $\epsilon_I > 0$ so that $|\gamma'(E)| > \epsilon_I$ for all E in the compact set I. From (5.15) we see that for each $E \in \Sigma_0$, $\sigma(x; E) = \exp[-i\gamma(E)x/L]\psi_0^+(x; E)$ is L-periodic (in x) and hence so is $\sigma(x; E)^2$. Because I is compact the family $\{\sigma(x; E)^2 : E \in I\}$ is uniformly continuously differentiable in x. That is to say,

$$\sup_{E \in I} \|\sigma(\cdot, E)\|_{C^1} \leq C_I < \infty.$$

As a result, when we develop σ^2 as a Fourier series

$$\sigma(x; E)^2 = \sum_n c_n(E)e^{2\pi inx/L}$$

the sequence $\tilde{c}_n = \sup_E |c_n(E)|$ is bounded in ℓ^1.

Let us now apply what we have learned to the estimation of (5.14):

$$\begin{aligned}\int \psi_0^+(x)^2 V(x)dx &= \int \sigma(x;E)^2 e^{2i\gamma(E)x/L} V(x)dx \\ &= \int \sum_n c_n(E) \exp\big\{i\big[2\pi n + 2\gamma(E)\big]x/L\big\} V(x)dx \\ &= \sum_n c_n(E)\hat{V}\big(\tfrac{2\pi n+2\gamma(E)}{L}\big).\end{aligned}$$

By using the triangle inequality, then Hölder's inequality, this implies

$$\left\| \int \psi_0^+(x)^2 V(x)dx \right\|_{L^2(I;dE)} \leq \sum_n \tilde{c}_n \big\| \hat{V}\big(\tfrac{2\pi n+2\gamma(E)}{L}\big)\big\|_{L^2(I;dE)}.$$

Remember that $|\gamma'(E)| > \epsilon_I$ for $E \in I$. So, if $\gamma(I) = [\gamma_0, \gamma_1]$, then by a change of variables,

$$\int_I \big|V\big(\tfrac{2\pi n+2\gamma(E)}{L}\big)\big|^2 dE \leq \int_{2\gamma_0/L}^{2\gamma_1/L} \big|V\big(\tfrac{2\pi n}{L} + y\big)\big|^2 \epsilon_I^{-1}\, dy.$$

But $[\gamma_0, \gamma_1] \subset (-\pi, \pi)$ and so, by our choice of $\ell^\infty(L^2)$ norm, it is immediate that

$$\left\| \int \psi_0^+(x)^2 V(x)dx \right\|_{L^2(I;dE)} \leq \epsilon_I^{-1} \|\tilde{c}_n\|_{\ell^1} \|V\|_{\ell^\infty(L^2)}.$$

Because $W \neq 0$ this provides sufficient control on (5.14) to prove (5.13). $\square$

Chapter 6 Proofs of Theorems 2–4

In Chapter 2 we showed that if V is of compact support, then the function

$$\log\big|a(z)\big| = \mathrm{Re}\,\log\big\{\det\big(1 + R_0(z)V\big)\big\}$$

is harmonic on the set $\mathbf{C}^+ \cup \Sigma_0$. Recall that Σ_0 is an essential support for the absolutely continuous spectrum of H_0. Indeed, Σ_0 is just $\sigma(H_0)$ with a discrete set of points removed.

In Chapter 5 we introduced regularized determinants and defined

$$f(z) = \log\big|\det{}_2\big(1 + R_0(z)V\big)\big|, \tag{6.1}$$

$$g(z) = \log\big|\det{}_3\big(1 + R_0(z)V\big)\big|. \tag{6.2}$$

We further showed that these functions are also harmonic on $\mathbf{C}^+ \cup \Sigma_0$, obtained estimates for $z \in \mathbf{C}^+$ and bounded $f(E) - a(E)$ and $g(E) - a(E)$ for $E \in \Sigma_0$.

The current chapter is devoted to obtaining the estimate (3.6) and so proving Theorems 2–4. We will use the fact that f, g are harmonic to convert our bounds for non-real z to bounds for real $z \in \Sigma_0$. For this reason we begin with some results about harmonic functions.

Given a closed interval $I \subset \Sigma_0$, we define a triangle over I as follows: If $I = [a, b]$, define $c = (a + b)/2 + i(b - a)/2$. Then the points a, b, c form a right-angled isosceles triangle with hypotenuse I. The remaining two sides we denote by Λ. Let $\triangle = \{x + iy : a + y \le x \le b - y, y \ge 0\}$, that is, the solid triangle with boundary $\Lambda \cup I$, and let $z_0 \in \triangle$ be the centroid (centre of mass) of $\triangle$, $z_0 = (a+b)/2 + i(b-a)/6$. The choice of the centroid for z_0 is essentially arbitrary. This construction is depicted more clearly in Figure 1.

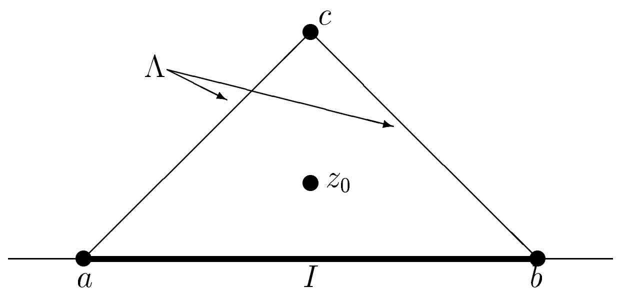

Figure 1: The definition of Λ, z_0.

Let $\mathcal{P}_{z_0}(z)$ denote the Poisson kernel for our triangle evaluated at the point z_0. That is to say, for any function F, continuous and harmonic throughout the solid triangle $\triangle$,

$$F(z_0) = \int_{\Lambda\cup I} \mathcal{P}_{z_0}(z) F(z)|dz|.$$

Moreover, we have the following estimate on $\mathcal{P}_{z_0}(z)$.

Lemma 6.1. *There is a constant C_{z_0} such that*

$$\mathcal{P}_{z_0}(z) \leq C_{z_0} \mathrm{Im}(z)^3 \quad \forall z \in \Lambda. \tag{6.3}$$

Proof. By scaling and translation, it suffices to consider just one triangle and its centroid, say one with $a = 0$. By conformal mapping it is easy to calculate the Poisson kernel for the infinite sector $\{z : 0 \leq \arg(z) \leq \pi/4\}$ with the chosen z_0. Doing so, one obtains a bound of the form (6.3) for this kernel. The chosen triangle is a subset of this sector and harmonic measure is monotone with respect to the domain. This means that (6.3) holds for all z from the line $\overline{ac}$. For z on the other side, $\overline{bc}$, the same estimate holds by symmetry.

The fact that we obtain a third power in (6.3) is because the angles $\angle cab$ and $\angle cba$ are $\pi/4$. With angles α one would obtain the power $(\pi/\alpha) - 1$. $\square$

We have now accumulated enough lemmas and propositions to prove Theorems 2–4. We begin with the

Proof of Theorem 2. Let $V \in L^2(\mathbf{R}^+)$ and extend it to the whole of $\mathbf{R}$ by $V(x) = 0$ for $x < 0$. Next we choose a sequence of C^∞ functions, V_n, each supported on a compact subset of $[0, \infty)$ which converge to V in L^2 sense. To each V_n we associate functions

$$a_n(z) = \det\bigl(1 + R_0(z)V_n\bigr)$$

as in Proposition 3.4, and

$$f_n(z) = \log\bigl|\det_2\bigl(1 + R_0(z)V_n\bigr)\bigr|$$

as in Proposition 5.3.

We shall prove the theorem by showing that any compact interval $I \subset \Sigma_0$ is also contained in $\Sigma_{\mathrm{ac}}(H)$. For in this way $\Sigma_0 \subset \Sigma_{\mathrm{ac}}(H_+)$. By Weyl's Relative Compactness Theorem, this implies that $\Sigma_0 = \Sigma_{\mathrm{ac}}(H_+)$, as discussed in the introduction. So let us fix I and define Λ, z_0 as was done earlier (c.f. Fig. 1).

From the fact that the functions $f_n(z)$ are harmonic (Proposition 5.3) and the definition of the Poisson kernel, we know that

$$f_n(z_0) = \int_{\Lambda\cup I} \mathcal{P}_{z_0}(z) f_n(z)\,|dz|. \tag{6.4}$$

As the sequence is V_n uniformly bounded in L^2, Lemma 5.1 tells us that $f_n(z_0)$ is bounded uniformly from above. For the same reason, Propositions 5.2 and 6.1 combine to show that the integral over Λ is bounded uniformly from below. Combining these two deductions, it follows that

$$\int_I \mathcal{P}_{z_0}(E) f_n(E)\, dE \le C$$

uniformly in n. But $f_n(E) = \log|a(E)|$ by Proposition 5.3. Therefore, we may apply Proposition 3.4 with $w(E) = \mathcal{P}_{z_0}(E)$ and so conclude that $I \subset \Sigma_{\mathrm{ac}}(H_+)$. As I was chosen arbitrarily, this completes the proof. □

The proofs of Theorems 3 and 4 follow along the same lines. However, for completeness we include a brief sketch of each.

Proof of Theorem 3. In this case we choose V_n so that $V_n \to V$ in L^3 sense and such that $\hat{V}_n$ are uniformly bounded in $\ell^\infty(L^2)$. Then, repeating the deductions of the previous proof, but with Proposition 5.4 replacing Proposition 5.3, we find

$$\int_I \mathcal{P}_{z_0}(E) g_n(E)\, dE \le C \tag{6.5}$$

uniformly in n. But now we may employ (5.13) to infer that Proposition 3.4 holds for any compact $I \subset \Sigma_0$. This completes the proof just as it did for Theorem 1. □

Proof of Theorem 4. It makes no harm to assume that the interval I in the theorem is compact, for any interval is a countable union of compact intervals. Next we choose $V_n \to V$ in L^3 such that $\int_{\tilde{I}} |\hat{V}|^2 dk$ is bounded. Lemma 5.1 may again be used to estimate $g_n(z_0)$ from above while Proposition 6.1 and (5.6) of Proposition 5.2 combine to bound the integral over Λ from below. This gives (6.5) and then (3.6) follows from (5.12) of Proposition 5.4. With this input, we can apply Proposition 3.4 and so complete the proof. □

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