Preprint-KUL-TF-2000/02

Goldstone Boson Normal Coordinates

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January 20, 2000

Abstract

The phenomenon of spontaneous symmetry breaking is well known. It is known to be accompanied with the appearance of the 'Goldstone boson'. In this paper we construct the canonical coordinates of the Goldstone boson, for quantum spin systems with short range as well as long range interactions.

Keywords

Spontaneous symmetry breaking, Goldstone theorem, plasmon frequency, long wavelength limits, canonical coordinates.

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Contents

1 Introduction

As is well known, spontaneous symmetry breakdown (SSB) is one of the basic phenomena accompanying collective phenomena, such as phase transitions in statistical mechanics, or ground state excitations in field theory. SSB is a representative tool for the analysis of many phenomena in modern physics. The study of SSB goes back to the Goldstone Theorem [1], which was the subject of much analysis. This theorem refers usually to the ground state property that for short range interacting systems, SSB implies the absence of an energy gap in the excitation spectrum [2, 3].

In this paper we concentrate on the non-relativistic Goldstone Theorem, and we mean by this spontaneous symmetry breaking of a continuous symmetry group in condensed matter homogeneous many particles systems, with short range as well as long range interactions.

There are many different situations to consider. For short range interactions, it is typical that SSB yields a dynamics which remains symmetric in the thermodynamic limit. At temperature $T = 0$, one has as main characteristics the absence of an energy gap. However for equilibrium states ($T > 0$), SSB is better characterized by bad clustering properties [4, 5].

For long range interactions, it is typical that SSB breaks also the symmetry of the dynamics. This situation has been studied extensively in the literature. In physics the phenomenon is known as the occurence of oscillations with energy spectrum taking a finite value $\epsilon(k \to 0) \neq 0$. Different approximation methods, typical here is the random phase approximation, yield the computation of these frequencies. For mean field models, such as the BCS-model [6], the Overhauser model [7], a spin density wave model [8], the anharmonic crystal model [9], and for the jellium model [10], one is able to give the rigorous mathematical status of these frequencies as elements of the spectrum of typical fluctuation operators [11, 12]. The typical operators entering in the discussion are the generator of the broken symmetry and the order parameter operator. In a physical language they are the charge density and current density operators. It is proved that their fluctuation operators form a quantum canonical pair, which decouples from the other degrees of freedom of the system. As fluctuation operators are collective operators, they describe the collective mode accompanying the SSB phenomenon. Hence for long range interacting systems, we realised mathematically rigorously in these models, the so-called Anderson theorem [13, 14] of 'restoration of symmetry', stating that there exists a spectrum of collective modes $\epsilon(k \to 0) \neq 0$ and that the mode in the limit $k \to 0$ is the operator which connects the set of degenerate temperature states, i.e. 'rotates' one ergodic state into an other. We conjecture that our results of [6–10] can be proved for general long range two-body interacting systems as a universal theorem.

However Anderson did formulate his theorem in the context of the Goldstone theorem for short range interacting systems, i.e. in the case $\epsilon(k \to 0) = 0$ of absence of an energy gap in the ground state. Of course one knows that there is no one-to-one relation between long range interactions and the presence of an energy gap for symmetry breaking systems (see e.g. [9]). The imperfect Bose gas and the weakly interacting Bose gas are examples of long range interacting systems showing SSB, but without energy gap. In [15] we realise for these boson models the above described programme of construction of the collective modes operators of condensate density and condensate current, as normal modes dynamically independent from the other degrees of freedom of the system. We consider the whole temperature range, the ground state included. In particular the ground state situation is interesting, because it yields a non-trivial quantum mechanical canonical pair of conjugate operators, giving an explicit representation of the field variables of the socalled Goldstone boson.

In this paper we are able to present the analogous proof for general interacting quantum lattice systems, and hence give *a model independent construction*. We construct the fluctuation operators of the generator of a broken symmetry and of the order parameter and prove that they form a canonical pair. We prove that this pair is dynamically indenpendent from the other degrees of freedom of the system.

In the case of long range interactions, we prove that the appearance of a plasmon fequency is a natural phenomenon corresponding to the spectrum of the above mentioned canonical pair. Moreover these fluctuation operators are normal. Our main contribution here is *the construction of a canonical order parameter*. Usually there are many order parameter operators. Therefore the identification of the right one for the purpose is important.

For short range interactions in the ground state, we find again the phenomenon of squeezing of the fluctuation operator of the generator of the broken symmetry. In the literature this is sometimes referred to the statement that in case of SSB, the broken symmetry behaves like an approximate symmetry. The amount of squeezing is inversely related to the anormality of the fluctuation operator of the order parameter, which itself is directly related to the degree of off-diagonal long range order. Using an appropriate volume scaling, which is determined by the long wavelength behaviour of the spectrum, we arrive at the construction of the Goldstone boson normal coordinates. We consider this result as a formal step forward, beyond the known analysis of the Goldstone phenomenon. We repeat that our construction is solely determined by the long wavelength behaviour of the microscopic energy spectrum of the system.

Finally, we want to throw the attention of the reader to the direct open questions which should keep our attention. There is first of all the problem of SSB of more dimensional symmetries. One should expect a more dimensional Goldstone boson. There is also the problem of SSB of non-commutative symmetry groups. An insight in this situation would certainly contribute to information on the situation of SSB in gauge theories in relativistic field theory.

We finish the introduction by formulating the systems and the technical settings. We consider quantum spin systems on $\mathbb{Z}^\nu$. They are described as follows [16]: to each bounded region $\Lambda \subset \mathbb{Z}^\nu$ one assigns a C^*-algebra of observables $\mathcal{A}_\Lambda$ such that

$$\Lambda_1 \cap \Lambda_2 = \varnothing \Rightarrow [\mathcal{A}_{\Lambda_1}, \mathcal{A}_{\Lambda_2}] = 0 \qquad \text{(locality)}$$
$$\Lambda_1 \subseteq \Lambda_2 \Rightarrow \mathcal{A}_{\Lambda_1} \subseteq \mathcal{A}_{\Lambda_2} \qquad \text{(isotony)}$$

Denote by $\mathcal{A}_L$ the algebra of all local observables:

$$\mathcal{A}_L = \bigcup_{\Lambda \subset \mathbb{Z}^\nu} \mathcal{A}_\Lambda.$$

The norm closure $\mathcal{A}$ of $\mathcal{A}_L$ is again a C^*-algebra and considered as the algebra of quasi-local observables of the system.

The group $\mathbb{Z}^\nu$ of space translations of the lattice acts as a group of *-automorphisms on $\mathcal{A}$ by

$$\tau_x : A \in \mathcal{A}_\Lambda \to \tau_x(A) \in \mathcal{A}_{\Lambda+x}$$

We assume there exists a time evolution, i.e. a one parameter group of automorphisms, α_t, $t \in \mathbb{R}$, on $\mathcal{A}$. This dynamics is supposed to be translation invariant:

$$[\alpha_t, \tau_x] = 0 \tag{1}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{Z}^\nu$. Let ω be an equilibrium state of α_t, i.e. ω satisfies the KMS condition at inverse temperature β:

$$\omega(A\alpha_{i\beta}B) = \omega(BA)$$

for $A, B \in \mathcal{A}$. We also assume ω to be extremal translation invariant.

Through the GNS construction, ω defines the Gelfand triple $(\mathcal{H}, \pi, \Omega)$, where $\mathcal{H}$ is a Hilbert space, π a *-representation of $\mathcal{A}$ as bounded operators on $\mathcal{H}$ and Ω a cyclic vector of $\mathcal{H}$ such that

$$\omega(A) = (\Omega, \pi(A)\Omega).$$

Since ω is translation and time invariant, there exist unique groups of unitary operators $\{U_x | x \in \mathbb{Z}^\nu\}$ and $\{V_t | t \in \mathbb{R}\}$ in $B(\mathcal{H})$ with

$$\pi(\tau_x(A)) = U_x \pi(A) U_x^{-1}$$
$$\pi(\alpha_t(A)) = V_t \pi(A) V_t^{-1}.$$

In fact U_x and V_t are defined by

$$U_x \pi(A)\Omega = \pi(\tau_x(A))\Omega$$
$$V_t \pi(A)\Omega = \pi(\alpha_t(A))\Omega.$$

These are well defined unitaries on $\mathcal{H}$, and we have $U_x^{-1} = U_{-x}$ and $V_t^{-1} = V_{-t}$. By Stone's theorem we may write

$$V_t = e^{itH}$$

and H is called the physical Hamiltonian.

Since ω is extremal translation invariant the spectral resolution of U_x is given by

$$U_x = |\Omega)(\Omega| + \int e^{ik.x} dP(k),$$

where k is in the set $[-\pi, \pi]^\nu$. This notation indicates that the spectrum of U_x has one discrete point at $k = 0$, the corresponding eigenspace being one-dimensional; the rest of the spectrum ($k \neq 0$) is continuous.

We now consider the phenomenon of spontaneous symmetry breaking. For a more elaborate discussion of this phenomenon in quantum statistical mechanics, see [17]. Suppose $\{\gamma_s | s \in \mathbb{R}\}$ is a one-parameter symmetry group of our system. This is a one-parameter group of automorphisms on the algebra $\mathcal{A}$ wich leaves the dynamics and the equilibrium state invariant:

$$\begin{aligned} \gamma_s \circ \alpha_t &= \alpha_t \circ \gamma_s \\ \omega(\gamma_s(A)) &= \omega(A). \end{aligned}$$

In this case γ_s can be implemented by a group of unitaries in the representation space:

$$\pi\left(\gamma_s(A)\right) = U_s^\gamma \pi(A) (U_s^\gamma)^{-1}$$

and as above for translations and time evolution, $U_s^\gamma \pi(A)\Omega = \pi\left(\gamma_s(A)\right)\Omega$ and $(U_s^\gamma)^{-1} = U_{-s}^\gamma$. By Stone's theorem we may again write

$$U_s^\gamma = e^{isQ}$$

and Q is the infinitesimal generator of the symmetry which satisfies

$$\begin{aligned} Q\Omega &= 0 \\ \omega([Q, A]) &= (\Omega, [Q, A]\Omega) = 0, \end{aligned}$$

where possible domain problems are usually bypassed by suitable assumptions.

At low temperatures a phase transition may occur such that there are at least two distinct extremal states. The symmetry γ then breaks down spontaneously if one of these states is no longer invariant under γ, i.e. there is some $A \in \mathcal{A}$ such that

$$\omega(\gamma_s(A)) \neq \omega(A).$$

Then γ can no longer be implemented by a group of unitaries in the representation space, in other words, the global infinitesimal generator Q no longer exists. However in the case of a continuous symmetry, we are usually given in each representation a *formal* infinitesimal generator expressed as the sum over a certain domain of a generator density

$$Q_\Lambda = \sum_{x \in \Lambda} q_x.$$

To give the global generator Q a rigorous meaning, one has to approximate it by a certain localized sequence

$$Q\pi(A)\Omega \equiv \lim_{|\Lambda| \to \infty} [Q_\Lambda, \pi(A)]\Omega,$$

for $A \in \mathcal{A}_L$. We say spontaneous symmetry breaking (SSB) occurs if there exists an $A \in \mathcal{A}_L$ such that

$$\lim_{|\Lambda| \to \infty} \omega([Q_\Lambda, A]) \neq 0.$$

Such an operator A is then called an order parameter operator.

2 Canonical coordinates

2.1 Introduction

In [11, 12] a dynamical system of macroscopic quantum fluctuations is constructed for sufficiently clustering states. We repeat the main results in order to fix the notation and refer to the original papers for more details and proofs. The main issue of this section is the construction of creation and annihilation operators for this system of macroscopic fluctuation observables.

With each $x \in \mathbb{Z}^\nu$ we associate the algebra $\mathcal{A}_x$, a copy of the matrix algebra M_N of $N \times N$ matrices. For each $\Lambda \subset \mathbb{Z}^\nu$, consider the tensor product $\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{A}_x$. The algebra of all local observables is

$$\mathcal{A}_L = \bigcup_{\Lambda\subset\mathbb{Z}^\nu} \mathcal{A}_\Lambda.$$

The norm closure $\mathcal{A}$ of $\mathcal{A}_L$ is again a C^*-algebra

$$\mathcal{A} = \bar{\mathcal{A}}_L = \overline{\bigcup_{\Lambda\subset\mathbb{Z}^\nu} \mathcal{A}_\Lambda},$$

and is considered the algebra of quasi-local observables of our system.

The group $\mathbb{Z}^\nu$ of space translations of the lattice acts as a group of *-automorphisms on $\mathcal{A}$ by:

$$\tau_x : A \in \mathcal{A}_\Lambda \to \tau_x(A) \in \mathcal{A}_{\Lambda+x}\ ,\quad x \in \mathbb{Z}^\nu.$$

The dynamics of our system is determined in the usual way by the local Hamiltonians

$$H_\Lambda = \sum_{X\subseteq\Lambda} \Phi(X),\quad \Lambda \subset \mathbb{Z}^\nu$$

with self adjoint $\Phi(X) \in \mathcal{A}_X$ for all $X \subset \mathbb{Z}^\nu$. The interaction Φ is supposed to be translation invariant:

$$\tau_x\Phi(X) = \Phi(X + x).$$

For each $\Lambda \subset \mathbb{Z}^\nu$, the local dynamics α_t^Λ is given by

$$\begin{aligned} \alpha_t^\Lambda &: \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda \\ \alpha_t^\Lambda(A) &= e^{itH_\Lambda} A e^{-itH_\Lambda},\quad A \in \mathcal{A}_\Lambda. \end{aligned}$$

If there exists $\lambda > 0$ such that

$$\|\Phi\|_\lambda \equiv \sum_{0\in X} |X| N^{2|X|} e^{\lambda d(X)} \|\Phi(X)\| < \infty, \tag{2}$$

with $d(X) = \sup_{x,y\in X} |x-y|$ the diameter of the set X and $|X|$ the number of elements in X, then the global dynamics α_t is well defined as the norm limit of the local dynamics α_t^Λ [16].

The state ω is an (α_t, β)-KMS state which is supposed to have good spatial clustering expressed by

$$\sum_{x\in\mathbb{Z}^\nu} \alpha_\omega(|x|) < \infty, \tag{3}$$

with α_ω the following clustering function:

$$\alpha_\omega(d) = \sup_{\Lambda,\Lambda'} \sup_{A\in\mathcal{A}_\Lambda, B\in\mathcal{A}_{\Lambda'}} \left\{ \frac{1}{\|A\|\|B\|} |\omega(AB) - \omega(A)\omega(B)| \;\Big|\; d \le d(\Lambda, \Lambda') \right\}. \tag{4}$$

2.2 Normal fluctuations

Denote by Λ_n the cube centered around the origin with edges of length $2n+1$. For any $A \in \mathcal{A}$, the local fluctuation $F_n(A)$ of A in the state ω is given by

$$F_n(A) = \frac{1}{|\Lambda_n|^{1/2}} \sum_{x\in\Lambda_n} \big(\tau_x A - \omega(A)\big).$$

Definition 1 (Normal fluctuations [12]). *Let $\mathcal{A}_0$ be any *-subspace of $\mathcal{A}$, the system $(\mathcal{A}, \mathcal{A}_0, \omega)$ is said to have normal fluctuations for $\mathcal{A}_0$ if*

(i) $\forall A, B \in \mathcal{A}_0$

$$\sum_{x\in\mathbb{Z}^\nu} \big|\omega(A\tau_x B) - \omega(A)\omega(B)\big| < \infty \tag{5}$$

(ii) $\forall A = A^* \in \mathcal{A}_0$, $t \in \mathbb{R}$

$$\lim_{n\to\infty} \omega\big(e^{itF_n(A)}\big) = e^{-\frac{t^2}{2} s_\omega(A,A)} < \infty \tag{6}$$

where $s_\omega(A,B) = \lim_{n\to\infty} \mathrm{Re}\, \omega\big(F_n(A)^* F_n(B)\big)$ *for all* $A, B \in \mathcal{A}_0$.

Remark that by (5)

$$s_\omega(A,B) = \mathrm{Re} \sum_{x\in\mathbb{Z}^\nu} \big(\omega(A^*\tau_x B) - \omega(A^*)\omega(B)\big).$$

Theorem 1 (Central Limit Theorem [12]). *If the state ω of $\mathcal{A}$ satisfies the L^1-space-factorization condition (3) then for all $A, B \in \mathcal{A}_{L,sa}$ (self-adjoint elements of $\mathcal{A}_L$)*

$$\begin{aligned}\lim_{n\to\infty} \omega\big(e^{iF_n(A)}e^{iF_n(B)}\big) &= \lim_{n\to\infty} \omega\big(e^{iF_n(A+B)}\big)e^{-\frac{1}{2}\omega([F_n(A),F_n(B)])}\\ &= \exp\Big\{-\frac{1}{2}s_\omega\left(A+B,A+B\right)-\frac{i}{2}\sigma_\omega(A,B)\Big\}\end{aligned}$$

where $\sigma_\omega(A,B) = \lim_{n\to\infty} 2 \text{ Im } \omega\big(F_n(A)^*F_n(B)\big)$. In particular the system has normal fluctuations for $\mathcal{A}_0 \subseteq \mathcal{A}_L$.

Now we are able to introduce the algebra of normal fluctuations of the system $(\mathcal{A}, \mathcal{A}_L, \omega)$. Consider the symplectic space $(\mathcal{A}_{L,sa}, \sigma_\omega)$. Denote by $W(\mathcal{A}_{L,sa}, \sigma_\omega)$ the CCR-algebra generated by the Weyl operators $\{W(A)|A \in \mathcal{A}_{L,sa}\}$, satisfying the product rule

$$W(A)W(B) = W(A+B)e^{-\frac{i}{2}\sigma_\omega(A,B)}.$$

The central limit theorem fixes a representation of this CCR-algebra. This is proved in the following theorem.

Theorem 2 (Reconstruction Theorem [12]). *For each $A \in \mathcal{A}_{L,sa}$ the limits*

$$\lim_{n\to\infty} \omega\big(e^{iF_n(A)}\big)$$

define a quasi-free state $\tilde{\omega}$ of the CCR-algebra $W(\mathcal{A}_{L,sa}, \sigma_\omega)$ by

$$\tilde{\omega}\big(W(A)\big) = e^{-\frac{1}{2}s_\omega(A,A)}.$$

*Moreover if γ is a *-automorphism of $\mathcal{A}$ leaving $\mathcal{A}_L$ invariant, commuting with the space translations and leaving the state ω invariant, then $\tilde{\gamma}$ given by*

$$\tilde{\gamma}(W(A)) = W(\gamma(A))$$

*defines a quasi-free *-automorphism of $W(\mathcal{A}_{L,sa}, \sigma_\omega)$.*

The quasi-free state $\tilde{\omega}$ induces a GNS-triplet $(\tilde{\mathcal{H}}, \tilde{\pi}, \tilde{\Omega})$ and yields a von Neumann algebra

$$\tilde{\mathcal{M}} = \tilde{\pi}\big(W(\mathcal{A}_{L,sa}, \sigma_\omega)\big)''.$$

This algebra will be called the *algebra of normal (macroscopic) fluctuations.*

By the above theorems and the fact that the representation $\tilde{\pi}$ is regular, we are able to identify the operator $F_0(A)$,

$$\lim_{n\to\infty} \omega\big(e^{iF_n(A)}\big) = \tilde{\omega}\big(e^{iF_0(A)}\big),$$

where $F_0(A)$ is a boson field satisfying the boson commutation rules

$$[F_0(A), F_0(B)] = i\sigma_\omega(A, B).$$

We identify the macroscopic fluctuations of the system $(\mathcal{A}, \omega)$ with the boson field $F_0(\cdot)$, i.e.

$$\lim_{n\to\infty} F_n(A) = F_0(A).$$

The limit $n \to \infty$ is in the sense of the central limit theorem and the reconstruction theorem.

Let $(\mathcal{H}, \pi, \Omega)$ be the GNS-triplet induced by the state ω and consider the sesquilinear form $\langle\cdot,\cdot\rangle_0$ on $\mathcal{H}$ with domain $\pi(\mathcal{A}_L)\Omega$ which we simply denote by $\mathcal{A}_L$:

$$\begin{aligned} \langle A, B\rangle_0 &= s_\omega(A, B) + \frac{i}{2}\sigma_\omega(A, B) \\ &= \sum_{x\in\mathbb{Z}^\nu} \big(\omega(A^*\tau_x B) - \omega(A^*)\omega(B)\big). \end{aligned} \tag{7}$$

We call A and B in $\mathcal{A}_L$ equivalent, denoted $A \equiv B$ if

$$\langle A - B, A - B\rangle_0 = 0.$$

Theorem 3 (Coarse graining [12]). *Let $A, B \in \mathcal{A}_{L,sa}$, then the following are equivalent:*

(i) $A \equiv B$

(ii) $\tilde{\pi}\left(W(A)\right) = \tilde{\pi}\left(W(B)\right).$

Proof. Suppose first that (ii) is satisfied, then

$$\big[\tilde{\pi}(W(A)), \tilde{\pi}(W(B))\big] = 0$$

and hence $\sigma_\omega(A, B) = 0$. Further

$$1 = \tilde{\omega}\big(W(A)W(B)^*\big) = \tilde{\omega}\big(W(A - B)\big) = e^{-\frac{1}{2}s_\omega(A-B,A-B)}$$

and from (7), $\langle A - B, A - B\rangle_0 = 0$, proving (i).

Conversely, suppose $A \equiv B$ then

$$\frac{1}{4}\big|\sigma_\omega(A - B, X)\big|^2 \le \langle A - B, A - B\rangle_0 \langle X, X\rangle_0$$

implies $\sigma_\omega(A - B, X) = 0$ for all $X \in \mathcal{A}_{L,sa}$, i.e. $\tilde{\pi}(W(A - B)) \in \tilde{\mathcal{M}}'$ (commutant of $\mathcal{M}$).

Also

$$\left\|\left(\tilde{\pi}(W(A-B))-1\right)\tilde{\Omega}\right\|^2 = \tilde{\omega}\Big(\left[W(A-B)-1\right]^*\left[W(A-B)-1\right]\Big)$$
$$= 2-\tilde{\omega}\big(W(A-B)\big)-\tilde{\omega}\big(W(B-A)\big)$$
$$= 0.$$

As $\tilde{\Omega}$ is cyclic for $\tilde{\mathcal{M}}$, it is separating for $\tilde{\mathcal{M}}'$. Hence

$$\tilde{\pi}(W(A-B)) = 1$$

or

$$\tilde{\pi}(W(A)) = \tilde{\pi}(W(B)),$$

proving *(ii)*. □

This is the property of coarse graining: different micro observables yield the same macroscopic fluctuation operator.

Denote by $[\mathcal{A}_L]$ the equivalence classes of $\mathcal{A}_L$ for the equivalence relation $\equiv$. The form $\langle\cdot,\cdot\rangle_0$ is a scalar product on $[\mathcal{A}_L]$. Denote by $\mathcal{K}_\omega$ the Hilbert space obtained as the completion of $[\mathcal{A}_L]$. Clearly s_ω and σ_ω extend continuously to $\mathcal{K}_\omega$. Denote by $\mathcal{K}_\omega^{Re}$ the real subspace of $\mathcal{K}_\omega$ generated by $[\mathcal{A}_{L,sa}]$. Now one considers the CCR-algebra $W\left(\mathcal{K}_\omega^{Re},\sigma_\omega\right)$ in the same representation induced by the state $\tilde{\omega}$.

Corollary 4 ([12]). *With the notations of above one has the following equality:*

$$\tilde{\mathcal{M}} = \tilde{\pi}\big(W(\mathcal{K}_\omega^{Re},\sigma_\omega)\big)''.$$

2.3 Reversible dynamics of fluctuations

Theorem 2 is not directly applicable with $\gamma = \alpha_t$, because with this choice it is not clear, and generally not true that $\alpha_t\mathcal{A}_L \subseteq \mathcal{A}_L$. Nevertheless, since $\alpha_t F_n(A) = F_n(\alpha_t A)$ one is tempted to define the dynamics $\tilde{\alpha}_t$ of the fluctuations by the formula

$$\tilde{\alpha}_t F_0(A) = F_0(\alpha_t A).$$

The non-trivial point in this formula is that it is unclear whether the central limit of the non-local observable $\alpha_t A$ exists or not. Furthermore if $F_0(\alpha_t A)$ exists it remains to prove that $(\tilde{\alpha}_t)_t$ defines a weakly continuous group of *-automorphisms on the fluctuation algebra $\tilde{\mathcal{M}}$.

Theorem 5 ([12]). *Let the interaction Φ be of short range, i.e. let Φ satisfy condition (2). For all local observables $A \in [\mathcal{A}_L]$, one has that for all $t \in \mathbb{R}$, $\alpha_t A \in \mathcal{K}_\omega$ and if $A \in [\mathcal{A}_{L,sa}]$ then $\alpha_t A \in \mathcal{K}_\omega^{Re}$. $W(\alpha_t A)$ is a well defined element of $\tilde{\mathcal{M}}$ and as*

$$W(\alpha_t A) = e^{iF_0(\alpha_t A)},\ A \in [\mathcal{A}_{L,sa}]$$

the fluctuation $F_0(\alpha_t A)$ exists for all $t \in \mathbb{R}$.

The map $U_t : [\mathcal{A}_L] \to \mathcal{K}_\omega$, $U_t A = \alpha_t A$ is a well defined linear operator on the Hilbert space $(\mathcal{K}_\omega, \langle\cdot,\cdot\rangle_0)$ extending to a unitary operator for all $t \in \mathbb{R}$. The map $t \to U_t$ is a strongly continuous one-parameter group.

*For all elements $A \in \mathcal{K}_\omega^{Re}$ define $\tilde{\alpha}_t W(A) = W(U_t A)$. Then $\tilde{\alpha}_t$ extends to a weakly continuous one-parameter group of *-automorphisms of $\tilde{\mathcal{M}}$.*

The following theorem proves that if the microsystem is in an equilibrium state, then also the macro system of fluctuations is in an equilibrium state for the dynamics constructed in the previous theorem, i.e. the notion of equilibrium is preserved under the operation of coarse graining induced by the central limit.

Theorem 6 ([12]). *If ω is an α_t-KMS state of $\mathcal{A}$ at $\beta > 0$, then $\tilde{\omega}$ is an $\tilde{\alpha}_t$-KMS state of the von Neumann algebra $\tilde{\mathcal{M}}$ at the same temperature.*

2.4 Canonical coordinates

Now we proceed to the explicit construction of creation and annihilation operators of fluctuations in the algebra $\tilde{\mathcal{M}}$. For product states this construction can be found in [18]. Here we work out the construction for the most general system.

From the definition of $\mathcal{K}_\omega^{Re}$ and $\mathcal{K}_\omega$ we can write

$$\mathcal{K}_\omega = \mathcal{K}_\omega^{Re} + i\mathcal{K}_\omega^{Re}.$$

Let * be the operation on $\mathcal{K}_\omega$ defined by

$$A^* = (A_1 + iA_2)^* = A_1 - iA_2,\ A_1, A_2 \in \mathcal{K}_\omega^{Re}.$$

Clearly for $A \in \mathcal{A}_L$ one has $[A]^* = [A^*]$ and it follows from Theorem 5 that

$$(U_t A)^* = U_t A^*$$

for all $A \in \mathcal{K}_\omega$.

Let $\mathcal{D}$ denote the set of infinitely differentiable functions on $\mathbb{R}$ with compact support. $\mathcal{D}$ is dense in $\mathcal{C}_0(\mathbb{R})$, the continuous functions vanishing at ∞, for the supremum norm. If $\hat{f} \in \mathcal{D}$ then the inverse Fourier transform

$$f(z) = \int_{-\infty}^{+\infty} d\lambda \hat{f}(\lambda) e^{i\lambda z}$$

is an entire analytic function. If supp $\hat{f} \in [-R, R]$ then it follows from the theorem of Paley-Wiener [16] that for all $n \in \mathbb{N}$ there exists a constant C_n such that

$$|f(z)| \leq C_n(1+|z|)^{-n} e^{R|\mathrm{Im} z|}.$$

Let $U_t = e^{it\tilde{h}} = \int e^{-it\lambda} d\tilde{E}_\lambda$ be the spectral resolution of the unitary group U_t and for $A \in \mathcal{K}_\omega$, $f \in L^1(\mathbb{R})$ denote

$$\begin{aligned} A(f) &= \int_{-\infty}^{+\infty} dt f(t) U_t A \\ &= \int_{-\infty}^{+\infty} \hat{f}(\lambda) d\tilde{E}_\lambda A \\ &= \hat{f}(-\tilde{h})A. \end{aligned}$$

Clearly one has $A(f)^* = A^*(\bar{f})$.
Let W be an open set in $\mathbb{R}$ and let $\tilde{E}_W = \int_W d\tilde{E}_\lambda$ be the spectral projection onto the spectral subspace $\mathcal{K}_W$. It follows from the spectral theory [16,19] that $\mathcal{K}_W$ is generated by the set

$$\{A(f) | A \in \mathcal{K}_\omega,\ f \in \mathcal{D},\ \text{supp}\ \hat{f} \subset W\}.$$

Finally for $A \in \mathcal{K}_\omega$ denote the associated spectral measure by

$$d\mu_A(\lambda) = \langle A, d\tilde{E}_\lambda A \rangle_0$$

and its spectral support Δ_A

$$\Delta_A = \{\lambda \in \mathbb{R} \mid \mu_A([\lambda-\epsilon, \lambda+\epsilon]) > 0\ \forall \epsilon > 0\}.$$

It is easy to see that Δ_A is also given by

$$\Delta_A = \{\lambda \in \mathbb{R} \mid \hat{f}(\lambda) = 0,\ \forall \hat{f} \in \mathcal{D} \text{ such that } A(f) = 0\}.$$

From this expression and $\hat{\bar{f}}(\lambda) = \bar{\hat{f}}(-\lambda)$ it follows that

$$\Delta_{A^*} = -\Delta_A$$

and from the same argument one also has

$$\tilde{E}_+ A^* = (\tilde{E}_- A)^* \tag{8}$$

where $\tilde{E}_+ = \tilde{E}_{(-\infty,0)}$ and $\tilde{E}_- = \tilde{E}_{(0,+\infty)}$ are the projections onto positive, respectively negative energy.

Lemma 7. *Let ω be an (α_t, β)-KMS state on the algebra $\mathcal{A}$. For all $A \in \mathcal{K}_\omega$, $\hat{f} \in \mathcal{D}$*

$$\int \hat{f}(\lambda) d\mu_A(\lambda) = \int \hat{f}(\lambda) e^{-\beta\lambda} d\mu_{A^*}(-\lambda).$$

Proof. Follows from the KMS-properties of $\tilde{\omega}$. $\square$

Let $\mathcal{K}^{Re}_{\omega,0} = \tilde{E}_0 \mathcal{K}^{Re}_\omega$ and $\mathcal{K}^{Re}_{\omega,1} = (\tilde{E}_+ + \tilde{E}_-)\mathcal{K}^{Re}_\omega$.
Define the operator J on $\mathcal{K}^{Re}_{\omega,1}$ by

$$J = i(\tilde{E}_+ - \tilde{E}_-). \tag{9}$$

From (8) one has for all $A \in \mathcal{K}^{Re}_{\omega,1}$

$$(JA)^* = JA^*$$

and thus $J\mathcal{K}^{Re}_{\omega,1} \subseteq \mathcal{K}^{Re}_{\omega,1}$.

Proposition 8. *The operator J defined above is a complex structure on the symplectic space $(\mathcal{K}^{Re}_{\omega,1}, \sigma_\omega)$:*

(i) $J^2 = -1$

(ii) $\sigma_\omega(A, JB) = -\sigma_\omega(JA, B), \quad A, B \in \mathcal{K}^{Re}_{\omega,1}$

(iii) $\sigma_\omega(A, JA) > 0, \quad 0 \neq A \in \mathcal{K}^{Re}_{\omega,1}$

Proof. From the definition of J and $\sigma_\omega = 2 \ \mathrm{Im}\ \langle\cdot,\cdot\rangle_0$, (i) and (ii) are trivially satisfied. Now we prove (iii). Let $\mathcal{E}$ be the set of real functions f such that $\hat{f} \in \mathcal{D}$ and $0 \notin \mathrm{supp}\ \hat{f}$. By the spectral theory, the set generated by

$$\{A(f) | A \in \mathcal{K}^{Re}_{\omega,1},\ f \in \mathcal{E}\}$$

is dense in $\mathcal{K}^{Re}_{\omega,1}$.
Take such an element $A(f)$. Using the previous lemma one computes

$$\begin{aligned}
\langle \tilde{E}_- A(f), \tilde{E}_- A(f)\rangle_0 &= \int |\hat{f}(\lambda)|^2 \chi_{(0,\infty)}(\lambda) d\mu_A(\lambda) \\
&= \int |\hat{f}(-\lambda)|^2 e^{\beta\lambda} \chi_{(-\infty,0)}(\lambda) d\mu_A(\lambda) \\
&= \langle \tilde{E}_+ A(f), e^{-\beta\tilde{h}} \tilde{E}_+ A(f)\rangle_0.
\end{aligned}$$

Because $\tilde{E}_+, \tilde{E}_-$ are projections and $e^{-\beta\tilde{h}} = \int e^{\beta\lambda} d\tilde{E}_\lambda$ is bounded on $\tilde{E}_+\mathcal{K}^{Re}_{\omega,1}$, this relation holds for all $B \in \mathcal{K}^{Re}_{\omega,1}$.

Using this property one has

$$\begin{aligned}\sigma_\omega(A, JA) &= -2i \text{ Im } \langle A, JA\rangle_0 \\ &= 2\left(\langle \tilde{E}_+ A, \tilde{E}_+ A\rangle_0 - \langle \tilde{E}_- A, \tilde{E}_- A\rangle_0\right) \\ &= 2\int_{-\infty}^{0} (1 - e^{\beta\lambda})\langle A, d\tilde{E}_\lambda A\rangle_0 \geq 0.\end{aligned}$$

The strict inequality holds because the spectral measure $d\mu_A(\lambda)$ is regular and $\tilde{E}_0 A = 0$. $\square$

The existence of a complex structure J yields the existence of creation and annihilation operators by

$$a_0^\pm(A) = \frac{F_0(A) \mp iF_0(JA)}{\sqrt{2}}$$

for all $A \in \mathcal{K}^{Re}_{\omega,1}$. They satisfy the property

$$a_0^\pm(JA) = \pm i a_0^\pm(A).$$

It is easily seen that

$$\begin{aligned}\tilde{\omega}\left(a_0^-(A)a_0^+(B)\right) &= 2\langle \tilde{E}_+ A, \tilde{E}_+ B\rangle_0 \\ \tilde{\omega}\left(a_0^+(A)a_0^-(B)\right) &= 2\langle \tilde{E}_- A, \tilde{E}_- B\rangle_0.\end{aligned}$$

2.5 Spectral representation

The GNS-triplet induced by ω is denoted $(\mathcal{H}, \pi, \Omega)$. The translations are implemented by unitaries in the representation space

$$\begin{aligned}\pi(\tau_x(A)) &= U_x A U_{-x} \\ U_x &= |\Omega)(\Omega| + \int e^{ik.x} dP(k).\end{aligned}$$

Remark that we omitted the notation π.

Let H be the Hamiltonian which generates the microscopic time evolution α_t, i.e.

$$\alpha_t(A) = e^{itH} A e^{-itH}.$$

The spectral decomposition of H is denoted

$$H = \int_{-\infty}^{\infty} \lambda dE_\lambda.$$

Denote $L(A) = [H, A]$, the spectral decomposition of L is given by

$$L(A) = \iint (\lambda - \lambda') dE_\lambda A dE_{\lambda'}.$$

Each observable $A \in \mathcal{A}_L$ can be separated in a 'positive energy' part and a 'negative energy' part, i.e. $A = A_+ + A_-$ where

$$A_+ = \iint \chi_{(-\infty,\lambda)}(\lambda') dE_\lambda A dE_{\lambda'}, \tag{10}$$

with $\chi_\Delta(\lambda) = 1$ if $\lambda \in \Delta$ and 0 otherwise, and

$$A_- = (A_+)^* = \iint \chi_{(\lambda,\infty)}(\lambda') dE_\lambda A dE_{\lambda'} = \iint \chi_{(-\infty,\lambda)}(\lambda') dE_{\lambda'} A dE_\lambda.$$

Also denote $\tilde{L} = \frac{d}{idt}\tilde{\alpha}_t\big|_{t=0}$, i.e.

$$\tilde{L}F_0(A) = F_0(\tilde{h}A).$$

Lemma 9. *For $f \in \mathcal{D}$, $A, B \in \mathcal{A}_{L,sa}$,*

$$\lim_{n\to\infty} \omega\Big(F_n(A)\hat{f}(H)F_n(B)\Big) = \tilde{\omega}\Big(F_0(A)\hat{f}(\tilde{L})F_0(B)\Big), \tag{11}$$

in other words, under the central limit, the measure $\omega\big(F_n(A)dE_\lambda F_n(B)\big)$ converges to the measure $\tilde{\omega}\big(F_0(A)d\tilde{E}_{-\lambda}F_0(B)\big)$.

Proof. For $f \in \mathcal{D}$ we have

$$\lim_{n\to\infty} \int dt f(t)\omega\left(F_n(A)\alpha_t F_n(B)\right) = \int dt f(t)\tilde{\omega}\left(F_0(A)\tilde{\alpha}_t F_0(B)\right),$$

proving the lemma. □

Note that on the r.h.s. of (11), we should have written $[A]$ instead of A, a notation we omit for simplicity.

Lemma 10. *For $A, B \in \mathcal{A}_{L,sa}$,*

$$\lim_{n\to\infty} \omega\big(F_n(A_-)F_n(B_+)\big) = \frac{1}{2}\tilde{\omega}\big(a_0^-(A)a_0^+(B)\big)$$
$$\lim_{n\to\infty} \omega\big(F_n(A_+)F_n(B_-)\big) = \frac{1}{2}\tilde{\omega}\big(a_0^+(A)a_0^-(B)\big)$$

Proof. By Lemma 9,

$$\begin{aligned}\lim_{n\to\infty}\omega\big(F_n(A_-)F_n(B_+)\big) &= \lim_{n\to\infty}\int_0^\infty \omega\big(F_n(A)dE_\lambda F_n(B)\big)\\ &= \int_0^\infty \tilde{\omega}\big(F_0(A)d\tilde{E}_{-\lambda}F_0(B)\big)\\ &= \langle \tilde{E}_+A, \tilde{E}_+B\rangle_0\\ &= \frac{1}{2}\tilde{\omega}\left(a_0^-(A)a_0^+(B)\right),\end{aligned}$$

and similarly for $\lim_{n\to\infty}\omega\big(F_n(A_+)F_n(B_-)\big)$. □

Let $\{dP(k)|k \in [0,2\pi)^\nu\}$ be the spectral measure of the translations in the GNS-representation of ω. Denote $\mathcal{H}_+ = \overline{\{A_+\Omega|A\in\mathcal{A}_L\}}$. The notation A_+ is introduced in equation (10).

Assumption 1. *Assume that the family of spectral projections $\mathcal{S} = \{P(\Delta)|\Delta\subset[0,2\pi)^\nu\}$ is maximally abelian on $\mathcal{H}_+$. Then there exists an energy spectrum ϵ_k^β such that*

$$HA_+\Omega = \int \epsilon_k^\beta dP(k)A_+\Omega, \tag{12}$$

with $\epsilon_0^\beta = 0$ and $\epsilon_k^\beta > 0$ for $k\neq 0$.

This can be seen as follows. If $\mathcal{S}$ is maximally abelian (i.e. $\mathcal{S}' = \mathcal{S}''$) on $\mathcal{H}_+$, then every operator on $\mathcal{H}_+$ belonging to $\mathcal{S}'$ can be written as a function of the spectral measure P, i.e.

$$[X,U_x] = 0 \ \ \forall x \Rightarrow XA_+\Omega = \int f_X(k)dP(k)A_+\Omega.$$

Loosely speaking this means that every translation invariant operator is a convolution operator in configuration space. In particular when the Hilbert space under consideration is a $L^p(\mathbb{R}^\nu)$ space, this result is always true, as is proved in [20]. Since we have assumed translation invariant interactions, i.e. $[H,U_x]=0 \ \ \forall x$, there must indeed exist a spectrum ϵ_k^β such that for each observable A,

$$HA_+\Omega = \int \epsilon_k^\beta dP(k)A_+\Omega,$$

with $\epsilon_0^\beta = 0$ and $\epsilon_k^\beta > 0$ for $k\neq 0$.

Physically the assumption means that the spectrum of H is non-degenerate in the sense that with each wave vector k there corresponds a unique energy ϵ_k^β. Of course for temperature states one first has to exclude the trivial degeneracy $\lambda\in \text{sp } H \Leftrightarrow -\lambda \in \text{sp } H$ for this statement to make sense, which corresponds to restricting $\mathcal{H}$ to $\mathcal{H}_+$.

Let

$$\epsilon^\beta = \lim_{k\to 0} \epsilon_k^\beta.$$

If $\epsilon^\beta = 0$, we say that the Hamiltonian H is *gapless*, otherwise, if $\epsilon^\beta > 0$, we say that the Hamiltonian H has a *gap* ϵ^β.

Lemma 11. *For $f \in \mathcal{D}$, $A, B \in \mathcal{A}_{L,sa}$:*

$$\lim_{n\to\infty} \omega\big(F_n(A)\hat{f}(H)F_n(B)\big) = \frac{1}{2}\hat{f}(\epsilon^\beta)\tilde{\omega}\left(a_0^-(A)a_0^+(B)\right) + \frac{1}{2}\hat{f}(-\epsilon^\beta)e^{-\beta\epsilon^\beta}\tilde{\omega}\left(a_0^-(B)a_0^+(A)\right).$$

Proof. Let $\omega(A) = \omega(B) = 0$,

$$a_{A,B}(z) = \omega(A\tau_z B)$$

and

$$\hat{a}_{A,B}(k) = \sum_z a_{A,B}(z)e^{-ik.z}.$$

Since $\sum_z |a_{A,B}(z)| < \infty$,

$$\lim_{n\to\infty} \omega\big(F_n(A)F_n(B)\big) = \hat{a}_{A,B}(0) = \lim_{k\to 0} \hat{a}_{A,B}(k).$$

For $h \in \mathcal{D}_{\mathbb{Z}^\nu}$, the set of sequences on $\mathbb{Z}^\nu$ with compact support,

$$\sum_z \overline{h(z)}a_{A,B}(z) = \sum_z \overline{h(z)} \int e^{ik.z}\omega\left(AdP(k)B\right) = \int \overline{\hat{h}(k)}\omega\left(AdP(k)B\right).$$

By the Parsival equality

$$\sum_z \overline{h(z)}a_{A,B}(z) = \int \frac{dk}{(2\pi^\nu)}\overline{\hat{h}(k)}\hat{a}_{A,B}(k).$$

Hence for $h \in \mathcal{D}_{\mathbb{Z}^\nu}$

$$\int \hat{h}(k)\omega\left(AdP(k)B\right) = \int \frac{dk}{(2\pi^\nu)}\hat{h}(k)\hat{a}_{A,B}(k).$$

Then for $A, B \in \mathcal{A}_{L,sa}$, $\omega(A) = \omega(B) = 0$:

$$
\begin{aligned}
&\omega(A\tau_{y-x}\hat{f}(H)B) \\
&= \int_{-\infty}^{0} \hat{f}(\lambda)\omega(A\tau_{y-x}dE_\lambda B) + \int_0^\infty \hat{f}(\lambda)\omega(A\tau_{y-x}dE_\lambda B) \\
&= \int_0^\infty \hat{f}(-\lambda)e^{-\beta\lambda}\omega(B\tau_{x-y}dE_\lambda A) + \int_0^\infty \hat{f}(\lambda)\omega(A\tau_{y-x}dE_\lambda B) \\
&= \int_0^\infty \hat{f}(-\lambda)e^{-\beta\lambda}\omega(B_-\tau_{x-y}dE_\lambda A_+) + \int_0^\infty \hat{f}(\lambda)\omega(A_-\tau_{y-x}dE_\lambda B_+) \\
&= \omega\big(B_-\tau_{x-y}\hat{f}(-H)e^{-\beta H}A_+\big) + \omega\big(A_-\tau_{y-x}\hat{f}(H)B_+\big) \\
&= \int \hat{f}(-\epsilon_k^\beta)e^{-\beta\epsilon_k^\beta}e^{ik.(x-y)}\omega\left(B_-dP(k)A_+\right) + \int \hat{f}(\epsilon_k^\beta)e^{-ik.(x-y)}\omega\left(A_-dP(k)B_+\right) \\
&= \int \frac{dk}{(2\pi)^\nu}\hat{f}(-\epsilon_k^\beta)e^{-\beta\epsilon_k^\beta}e^{ik.(x-y)}\hat{a}_{B_-,A_+}(k) + \int \frac{dk}{(2\pi)^\nu}\hat{f}(\epsilon_k^\beta)e^{-ik.(x-y)}\hat{a}_{A_-,B_+}(k).
\end{aligned}
$$

Let

$$
\delta_n(k) = \frac{1}{|\Lambda_n|}\sum_{x,y\in\Lambda_n} e^{ik.(x-y)};
$$

δ_n is a δ-convergent series, i.e. for all $h \in \mathcal{D}_{\mathbb{Z}^\nu}$

$$
\lim_{n\to\infty}\int \frac{dk}{(2\pi)^\nu}\hat{h}(k)\delta_n(k) = \lim_{k\to 0}\hat{h}(k).
$$

We find

$$
\begin{aligned}
&\lim_{n\to\infty}\omega\left(F_n(A)\hat{f}(H)F_n(B)\right) \\
&= \lim_{n\to\infty}\left\{\int \frac{dk}{(2\pi)^\nu}\hat{f}(-\epsilon_k^\beta)e^{-\beta\epsilon_k^\beta}\hat{a}_{B_-,A_+}(k)\delta_n(k) + \int \frac{dk}{(2\pi)^\nu}\hat{f}(\epsilon_k^\beta)\hat{a}_{A_-,B_+}(k)\delta_n(k)\right\} \\
&= \lim_{k\to 0}\left\{f(-\epsilon_k^\beta)e^{-\beta\epsilon_k^\beta}\hat{a}_{B_-,A_+}(k) + \hat{f}(\epsilon_k^\beta)\hat{a}_{A_-,B_+}(k)\right\} \\
&= \frac{1}{2}\hat{f}(-\epsilon^\beta)e^{-\beta\epsilon^\beta}\tilde{\omega}\left(a_0^-(B)a_0^+(A)\right) + \frac{1}{2}\hat{f}(\epsilon^\beta)\tilde{\omega}\left(a_0^-(A)a_0^+(B)\right)
\end{aligned}
$$

□

Theorem 12. *Let the system $(\mathcal{A}, \alpha_t, \omega)$ satisfy the central limit theorem (Theorem 1), i.e. ω satisfies the clustering condition (3) and the interaction Φ is of short range (2). Then:*

(i) If $\epsilon^\beta = 0$: the time evolution $\tilde{\alpha}_t$ on $\tilde{\mathcal{M}}$ is trivial, i.e.

$$
\tilde{\alpha}_t F_0(A) = F_0(A)
$$

for all $A \in \mathcal{K}_\omega^{Re}$. In particular this means that $\tilde{\mathcal{M}}$ is a classical algebra, i.e.

$$[F_0(A), F_0(B)] = i\sigma_\omega(A, B) = 0$$

for all $A, B \in \mathcal{K}_\omega^{Re}$.

(ii) If $\epsilon^\beta > 0$: If $\mathcal{K}_{\omega,1}^{Re}$ is not empty, then for all $A \in \mathcal{K}_{\omega,1}^{Re}$,

$$\tilde{\alpha}_t F_0(A) = F_0(e^{\epsilon^\beta t J} A),$$

i.e.

$$U_t = e^{it\tilde{h}} = e^{\epsilon^\beta t J}.$$

Moreover with each observable $A \in \mathcal{K}_{\omega,1}^{Re}$ one can associate a quantum harmonic oscillator pair $\big(Q_0(A), P_0(A)\big)$ oscillating with a frequency ϵ^β, in the following canonical way:

$$Q_0(A) = F_0(A) \qquad P_0(A) = \frac{1}{\epsilon^\beta} F_0(JA).$$

The commutator

$$\big[Q_0(A), P_0(A)\big] = ic_\omega(A), \;\; c_\omega(A) > 0$$

where

$$c_\omega(A) = \lim_{n\to\infty} \beta\big(F_n(A), F_n(A)\big)_\sim,$$

where $(\cdot,\cdot)_\sim$ is the Duhamel two-point function [21],

$$(X, Y)_\sim = \frac{1}{\beta}\int_0^\beta du\, \omega(X\alpha_{iu}Y). \tag{13}$$

The equations of motion for the pair $\big(Q_0(A), P_0(A)\big)$ are

$$\begin{aligned}\tilde{\alpha}_t Q_0(A) &= Q_0(A)\cos\epsilon^\beta t + \epsilon^\beta P_0(A)\sin\epsilon^\beta t\\ \tilde{\alpha}_t P_0(A) &= -\frac{1}{\epsilon^\beta} Q_0(A)\sin\epsilon^\beta t + P_0(A)\cos\epsilon^\beta t.\end{aligned}$$

The creation and annihilation operators are now redefined to be

$$a_0^\pm(A) = \frac{Q_0(A) \mp i\epsilon^\beta P_0(A)}{\sqrt{2\epsilon^\beta}} = \frac{F_0(A) \mp iF_0(JA)}{\sqrt{2\epsilon^\beta}},$$

satsfying

$$\begin{aligned}\big[a_0^-(A), a_0^+(A)\big] &= c_\omega(A)\\ \tilde{\alpha}_t a_0^\pm(A) &= e^{\pm it\epsilon^\beta} a_0^\pm(A).\end{aligned}$$

The $(\tilde{\alpha}_t, \beta)$-KMS property of ω is expressed by

$$\tilde{\omega}\left(a_0^+(A)a_0^-(A)\right) = \frac{c_\omega(A)}{e^{\beta\epsilon^\beta}-1}$$
$$\tilde{\omega}\left(a_0^+(A)a_0^+(A)\right) = 0 \ ,$$

or equivalently

$$\tilde{\omega}\left(Q_0(A)^2\right) = (\epsilon^\beta)^2\tilde{\omega}\left(P_0(A)^2\right) = \frac{c_\omega(A)\epsilon^\beta}{2}\coth\frac{\beta\epsilon^\beta}{2}.$$

Proof. First let $\epsilon^\beta = 0$. By Lemma 11, for all $A, B \in \mathcal{K}^{Re}_{\omega,1}$

$$\tilde{\omega}\big(F_0(A)F_0(B)\big) = \frac{1}{2}\tilde{\omega}\left(a_0^-(A)a_0^+(B)\right) + \frac{1}{2}\tilde{\omega}\left(a_0^-(B)a_0^+(A)\right) = \tilde{\omega}\big(F_0(B)F_0(A)\big).$$

Hence $\sigma_\omega(A,B) = 0$ for $A, B \in \mathcal{K}^{Re}_{\omega,1}$. But if $0 \neq A \in \mathcal{K}^{Re}_{\omega,1}$, then $\sigma_\omega(A, JA) > 0$. Hence there does not exist $0 \neq A \in \mathcal{K}^{Re}_{\omega,1}$. This shows that $\mathcal{K}^{Re}_\omega = \mathcal{K}^{Re}_{\omega,0}$. But by definition $\mathcal{K}^{Re}_{\omega,0} = \tilde{E}_0\mathcal{K}^{Re}_\omega$, i.e. $\mathcal{K}^{Re}_{\omega,0}$ is the set of U_t invariant vectors in $\mathcal{K}^{Re}_\omega$. Hence for all $A \in \mathcal{K}^{Re}_\omega = \mathcal{K}^{Re}_{\omega,0}$

$$\tilde{\alpha}_t F_0(A) = F_0(A),$$

and the dynamics $\tilde{\alpha}_t$ is trivial on $\tilde{\mathcal{M}}$.

The KMS-condition then yields for $A, B \in \mathcal{K}^{Re}_\omega$

$$\tilde{\omega}\big(F_0(A)F_0(B)\big) = \tilde{\omega}\big(F_0(B)\tilde{\alpha}_{i\beta}F_0(A)\big) = \tilde{\omega}\big(F_0(B)F_0(A)\big).$$

Hence $\sigma_\omega(A,B) = 0$ for $A, B \in \mathcal{K}^{Re}_\omega$ and $\tilde{\mathcal{M}}$ is a classical algebra.

Next let $\epsilon^\beta > 0$ and assume there exists $0 \neq A \in \mathcal{K}^{Re}_{\omega,1}$. By Lemma 11,

$$\begin{aligned}
&\tilde{\omega}\big(F_0(i\tilde{h}A)^2\big) = \lim_{n\to\infty} -\omega\big(F_n(L(A))^2\big) = \lim_{n\to\infty}\omega\big(F_n(A)H^2F_n(A)\big)\\
&= (\epsilon^\beta)^2e^{-\beta\epsilon^\beta}\tilde{\omega}\left(a_0^-(A)a_0^+(A)\right) + (\epsilon^\beta)^2\tilde{\omega}\left(a_0^-(A)a_0^+(A)\right)\\
&= (\epsilon^\beta)^2\tilde{\omega}\left(F_0(A)^2\right) = (\epsilon^\beta)^2\tilde{\omega}\left(F_0(JA)^2\right),
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\omega}\big(F_0(JA)F_0(i\tilde{h}A)\big) = \lim_{n\to\infty} i\omega\big(F_n(JA)HF_n(A)\big)\\
&= -i\epsilon^\beta e^{-\beta\epsilon^\beta}\tilde{\omega}\left(a_0^-(A)a_0^+(JA)\right) + i\epsilon^\beta\tilde{\omega}\left(a_0^-(JA)a_0^+(A)\right)\\
&= \epsilon^\beta e^{-\beta\epsilon^\beta}\tilde{\omega}\left(a_0^-(A)a_0^+(A)\right) + \epsilon^\beta\tilde{\omega}\left(a_0^-(A)a_0^+(A)\right) = \epsilon^\beta\tilde{\omega}\left(F_0(A)^2\right).
\end{aligned}$$

Hence

$$\langle\epsilon^\beta JA - i\tilde{h}A, \epsilon^\beta JA - i\tilde{h}A\rangle_0 = 0,$$

and by the equivalence relation (Theorem 3)

$$F_0(i\tilde{h}A) = \epsilon^\beta F_0(JA).$$

By integration

$$\tilde{\alpha}_t F_0(A) = F_0(e^{\epsilon^\beta tJ}A)$$

or

$$U_t = e^{\epsilon^\beta tJ}. \tag{14}$$

Let $Q_0(A) = F_0(A)$, $P_0(A) = (\epsilon^\beta)^{-1}F_0(JA)$,

$$\big[Q_0(A), P_0(A)\big] = \frac{i}{\epsilon^\beta}\sigma_\omega(A, JA)$$

and hence

$$c_\omega(A) = \frac{1}{\epsilon^\beta}\sigma_\omega(A, JA).$$

On the other hand by Lemma 9 and Lemma 11

$$\begin{aligned}
&\lim_{n\to\infty}\big(F_n(A), F_n(A)\big)_\sim = \lim_{n\to\infty}\omega\big(F_n(A)[1-e^{-\beta L}]L^{-1}F_n(A)\big) \\
&= \frac{1-e^{\beta\epsilon^\beta}}{-2\epsilon^\beta}e^{-\beta\epsilon^\beta}\tilde{\omega}\left(a_0^-(A)a_0^+(A)\right) + \frac{1-e^{-\beta\epsilon^\beta}}{2\epsilon^\beta}\tilde{\omega}\left(a_0^-(A)a_0^+(A)\right) \\
&= \frac{1-e^{-\beta\epsilon^\beta}}{\epsilon^\beta}\tilde{\omega}\left(a_0^-(A)a_0^+(A)\right).
\end{aligned}$$

With

$$\sigma_\omega(A, JA) = \tilde{\omega}\left(a_0^-(A)a_0^+(A)\right) - \tilde{\omega}\left(a_0^+(A)a_0^-(A)\right) = (1-e^{-\beta\epsilon^\beta})\tilde{\omega}\left(a_0^-(A)a_0^+(A)\right)$$

we find indeed

$$c_\omega(A) = \lim_{n\to\infty}\big(F_n(A), F_n(A)\big)_\sim.$$

The equations of motion follow straightforwardly from (14) and $a_0^\pm(JA) = \pm i a_0^\pm(A)$.

From now on we work with the redefined creation and annihilation operators. The equilibrium expectation values for the two-point function follow from the correlation inequalities [22, 23] for the state $\tilde{\omega}$: for all $X \in \tilde{\mathcal{M}}$

$$\beta\tilde{\omega}\big(X^*\tilde{L}(X)\big) \geq \tilde{\omega}\left(X^*X\right)\ln\frac{\tilde{\omega}\left(X^*X\right)}{\tilde{\omega}\left(XX^*\right)}.$$

Hence for all X

$$\frac{-\beta\tilde{\omega}\big(X\tilde{L}(X^*)\big)}{\tilde{\omega}\,(XX^*)} \leq \ln\frac{\tilde{\omega}\,(X^*X)}{\tilde{\omega}\,(XX^*)} \leq \frac{\beta\tilde{\omega}\big(X^*\tilde{L}(X)\big)}{\tilde{\omega}\,(X^*X)}.$$

Let now $X = a_0^-(A)$. Since $\tilde{L}(a_0^\pm(A)) = \pm\epsilon^\beta a_0^\pm(A)$, we get

$$-\beta\epsilon^\beta \leq \ln\frac{\tilde{\omega}\left(a_0^+(A)a_0^-(A)\right)}{\tilde{\omega}\left(a_0^+(A)a_0^-(A)\right)+c_\omega(A)} \leq -\beta\epsilon^\beta,$$

or

$$\tilde{\omega}\left(a_0^+(A)a_0^-(A)\right) = \frac{c_\omega(A)}{e^{\beta\epsilon^\beta}-1}.$$

Since $\tilde{\omega}\,(F(A)^2) = \tilde{\omega}\,(F(JA)^2)$ we get

$$\tilde{\omega}\left(Q_0(A)^2\right) = (\epsilon^\beta)^2\tilde{\omega}\left(P_0(A)^2\right).$$

Together with time invariance, this implies

$$\begin{aligned}
&\tilde{\omega}\left(a_0^+(A)a_0^+(A)\right)\\
&= \frac{1}{2\epsilon^\beta}\Big\{\tilde{\omega}\big(Q_0(A)^2\big) - (\epsilon^\beta)^2\tilde{\omega}\big(P_0(A)^2\big) - i\epsilon^\beta\tilde{\omega}\big(Q_0(A)P_0(A)+P_0(A)Q_0(A)\big)\Big\}\\
&= \frac{i}{2}\tilde{\omega}\left(\tilde{L}(P_0(A))P_0(A)+P_0(A)\tilde{L}(P_0(A))\right) = \frac{i}{2}\tilde{\omega}\left(\tilde{L}\big(P_0(A)^2\big)\right)\\
&= 0.
\end{aligned}$$

Finally

$$\begin{aligned}
\tilde{\omega}\left(Q_0(A)^2\right) &= \frac{\epsilon^\beta}{2}\tilde{\omega}\left(\left(a_0^+(A)+a_0^-(A)\right)^2\right)\\
&= \frac{\epsilon^\beta}{2}\tilde{\omega}\left(a_0^+(A)a_0^-(A)+a_0^-(A)a_0^+(A)\right)\\
&= \frac{c_\omega(A)\epsilon^\beta}{2}\coth\frac{\beta\epsilon^\beta}{2}.
\end{aligned}$$

□

This theorem shows that the system of macroscopic fluctuations determined by a given microdynamic system is completely integrable or solvable in terms of its microscopic spectrum ϵ_k^β (see Assumption 1). This result has already its own intrinsic value, universally applicable to diverse concrete models and situations.

The content of this paper is to apply it to the situation of spontaneous symmetry breaking. It will become clear later that this theorem gets a basic content when there is a broken symmetry in the system, since the fluctuation of the symmetry generator is then not conserved. If we let $\beta \to \infty$ in order to obtain the ground state of the system, for short range interactions, the Goldstone theorem guarantees a gapless spectrum and part (i) of Theorem 12 tells us that ω can not satisfy the clustering condition (3), i.e. there is *off diagonal long range order* in the system. One goes around this problem by looking at the k-mode fluctuations, $k \neq 0$,

$$F_{n,k}(A) = \sum_{x \in \Lambda_n} \big(\tau_x A - \omega(A)\big) \cos k.x.$$

Since for the ground state $\epsilon_k > 0$ for $k \neq 0$, these operators will essentially behave as in part (ii) of Theorem 12, i.e. oscillating harmonically with a frequency ϵ_k. This oscillation is related to the Goldstone mode. This is worked out in full detail in section 4.

Systems with long range interactions might have a gap in the energy spectrum, so in this case the fluctuation operator of the symmetry generator provides a non-trivial operator satisfying part (ii) of Theorem 12. Since we have assumed short range interactions throughout this section, these results can not simply be transported to the case of long range interacting systems. Therefore we give in section 3 a prototype example of a long range interacting system for which the previous results can be calculated explicitly.

3 Long range interactions

3.1 Introduction

In this section we study symmetry breaking systems whose Hamiltonian has a gap in the ground state. These systems typically have long range interactions, but since there is no general criterium whether a long range interacting system has a spectral gap or not, and since an infinite volume time evolution in general may not exist for these systems (see condition (2)), we restrict ourself to mean field systems which are long range interacting systems with a well defined time evolution in the thermodynamic limit and with a spectral gap. Similar results as the ones presented here have already been obtained for different other mean field models [7, 8], and for the jellium model [10], albeit by different methods. Moreover our main contribution in this section is *the construction of a canonical order parameter*.

Mean field models are prototypes of quantum systems with normal fluctuations outside the critical point [18]. Mathematically they have the following structure. Denote by M the algebra of $m \times m$ complex matrices and by $\mathcal{A}$ the C^*-algebra generated by the sequence $\otimes_{i=1}^{n} M_i$ where M_i is a copy of M. For each $A \in M$ denote by A_i the embedding of A in $\mathcal{A}$:

$$A_i = 1 \otimes \cdots \otimes A \otimes 1 \otimes \cdots$$

where A is in the i^{th} site.

Let ρ be a state of M, then we denote by ω_ρ the product state of $\mathcal{A}$ defined by

$$\omega_\rho(A \otimes B \otimes \cdots \otimes C) = \rho(A)\rho(B)\cdots\rho(C)$$

for $A, B, \cdots, C \in M$. Remark also that a state ρ of M is determined by a density matrix which we denote by the same symbol ρ, i.e. $\rho(A) = \mathrm{tr}\, \rho A$.

A mean field model is given by the local Hamiltonians

$$H_N = \sum_{i=1}^{N} A_i + \frac{1}{2N} \sum_{i \neq j, i,j=1}^{N} B_{ij}$$

where A_i are copies of $A^* = A \in M$ and B_{ij} are copies of $B^* = B \in M \otimes M$ such that B_{ij} is invariant under the symmetry $B_{ij} = B_{ji}$.

The extremal equilibrium states of these systems are given by states ω_ρ such that the density matrix ρ satisfies the gap equation [24]

$$\rho = \frac{e^{-\beta H_\rho}}{\mathrm{tr}\, e^{-\beta H_\rho}}$$

where

$$H_\rho = A + B_\rho\,, \qquad B_\rho = \mathrm{tr}_2\, (1 \otimes \rho) B$$

and tr_2 is the partial trace over the second space.

The effective time evolution on M in the equilibrium state is now given by

$$\alpha_t(A) = e^{itH_\rho} A e^{-itH_\rho}, \quad A \in M.$$

The dynamics on the lattice algebra $\mathcal{A}$ is the product evolution

$$\alpha_t(A \otimes B \otimes \cdots \otimes C) = \alpha_t(A) \otimes \alpha_t(B) \otimes \cdots \otimes \alpha_t(C)$$

for all $A, B, \cdots, C \in M$. The state ρ is an (α_t, β)-KMS state on M and ω_ρ is a β-KMS state for the product evolution $\alpha_t \equiv \otimes \alpha_t$.

This section serves as an illustration of the general structure presented in Theorem 12. Therefore let us consider a specific model, namely the strong coupling BCS-model for superconductivity. It is described by the Hamiltonian [25, 26]

$$H_N = \epsilon \sum_{i=-N}^{N} \sigma_i^z - \frac{1}{2N+1} \sum_{i,j=-N}^{N} \sigma_i^+ \sigma_j^- \ , \ \epsilon < \frac{1}{2}$$

where $\sigma^z, \sigma^\pm$ are the usual Pauli matrices (i.e. $m = 2$). H_N acts on the Hilbert space $\otimes_{i=-N}^{N} \mathbb{C}_i^2$.

The solutions of the KMS equation are given by the product states $\omega_\lambda = \omega_{\rho_\lambda}$ on the infinite tensor product algebra $\mathcal{A} = \otimes_{i=-\infty}^{\infty} (M_2)_i$ of the system; ρ_λ is a 2×2 density matrix, given by the solutions of the gap equation

$$\rho_\lambda = \frac{e^{-\beta h_\lambda}}{\mathrm{tr}\, e^{-\beta h_\lambda}} \ , \quad \lambda = \mathrm{tr}\, \rho_\lambda \sigma^- = \omega_\lambda(\sigma^-) \ , \quad h_\lambda = \epsilon \sigma^z - \lambda \sigma^+ - \bar{\lambda} \sigma^-.$$

This is easily turned into the equation for λ:

$$\lambda \left(1 - \frac{\tanh \beta\mu}{2\mu} \right) = 0 \tag{15}$$

with $\mu = (\epsilon^2 + |\lambda|^2)^{1/2}$. Clearly, this equation has always the solution $\lambda = 0$, describing the so-called normal phase. We are interested in the solutions $\lambda \neq 0$ which exist in the case $\beta > \beta_c$ where β_c is determined by the equation

$$\tanh \beta_c \epsilon = 2\epsilon.$$

These solutions $\lambda \neq 0$ are understood to describe the superconducting phase. Remark that if $\lambda \neq 0$ is a solution of (15), then for all $\phi \in [0, 2\pi)$, $\lambda e^{i\phi}$ is a solution as well. There is an infinite degeneracy of the states for the superconducting phase. The degeneracy is due to the breaking of the gauge symmetry. As

$$\sigma^z = \sigma^+ \sigma^- - \sigma^- \sigma^+$$

it is clear that the Hamiltonian H_N is invariant under the continuous gauge transformations automorphism group $\mathcal{G} = \{\gamma_\phi | \phi \in [0, 2\pi)\}$ of $\mathcal{A}$

$$\gamma_\phi : \sigma_i^+ \to \gamma_\phi(\sigma_i^+) = e^{-i\phi}\sigma_i^+.$$

However the solutions ω_λ are not invariant for this symmetry transformation, because:

$$\omega_\lambda(\gamma_\phi(\sigma_i^+)) = e^{-i\phi}\omega_\lambda(\sigma_i^+) \neq \omega_\lambda(\sigma_i^+). \tag{16}$$

The gauge symmetry of the system is spontaneously broken. Remark also that h_λ is no longer invariant under the symmetry transformation, this is a typical feature of long range interacting systems.

From (16) it follows also that

$$\omega_\lambda \circ \gamma_\phi = \omega_{\lambda e^{i\phi}},$$

i.e. one solution ω_λ is transformed into another solution $\omega_{\lambda e^{i\phi}}$ by the gauge transformation γ_ϕ.

The gauge group $\mathcal{G}$ is not implemented by unitaries in any of the representations induced by the solutions ω_λ. Locally however, the gauge transformation γ_ϕ is implemented by unitaries: take any finite set Λ of indices, then

$$\gamma_\phi\Big(\prod_{i\in\Lambda}\sigma_i^-\Big) = (U_\phi^\Lambda)^*\Big(\prod_{i\in\Lambda}\sigma_i^-\Big)U_\phi^\Lambda$$

where

$$U_\phi^\Lambda = e^{\frac{i}{2}\phi Q_\Lambda}\ , \quad Q_\Lambda = \sum_j \sigma_j^z.$$

The operator Q_Λ is called the local charge or symmetry generator and σ^z the charge density or symmetry generator density.

3.2 Canonical coordinates of the Goldstone mode

In this section we introduce the algebra of fluctuations and show how the Goldstone mode operators are to be defined in a canonical way. The relation between symmetry breaking and quantum fluctuations in the strong coupling BCS model has been studied before in [6]. This analysis is here extended.

Per lattice site $j \in \mathbb{Z}$ one has the local algeba of observables, the real 2×2 matrices, M_2, generated by the Pauli matrices in the representation:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

As state we consider a particular equilibrium state ω_λ with $\beta > \beta_c$ which reduces per lattice point to the trace state

$$\omega_\lambda(A) = \mathrm{tr}\, \rho_\lambda A\ , \quad A \in M_2.$$

Because of the product character of the algebra, the state and the time evolution, it is sufficient to consider fluctuations of one-point observables. Locally the fluctuation of A in the state ω_λ is:

$$F_N(A) = \frac{1}{(2N+1)^{1/2}} \sum_{i=-N}^{N} \big(A_i - \rho_\lambda(A)\big)\ , \quad A \in M_2.$$

The commutator of two fluctuations is a mean, indeed:

$$\big[F_N(A), F_N(B)\big] = \frac{1}{2N+1} \sum_{i=-N}^{N} \big([A,B]\big)_i.$$

For $A, B \in M_2$ define

$$\begin{aligned} s_\lambda(A,B) &= \mathrm{Re}\, \rho_\lambda\Big(\big(A - \rho_\lambda(A)\big)\big(B - \rho_\lambda(B)\big)\Big) \\ \sigma_\lambda(A,B) &= \mathrm{Im}\, \rho_\lambda\Big(\big[A - \rho_\lambda(A)\big]\big[B - \rho_\lambda(B)\big]\Big) = -i\rho_\lambda\big([A,B]\big). \end{aligned}$$

Clearly $(M_{2,sa}, \sigma_\lambda)$ is a symplectic space and s_λ is a symmetric positive bilinear form on $M_{2,sa}$.

Because ρ_λ is time invariant, $\rho_\lambda \circ \alpha_t = \rho_\lambda$ and because the evolution α_t is local, $\alpha_t : M_{2,sa} \to M_{2,sa}$, one has that α_t is a symplectic operator on $(M_{2,sa}, \sigma_\lambda)$: for all $t \in \mathbb{R}$

$$\sigma_\lambda(\alpha_t A, \alpha_t B) = \sigma_\lambda(A, B).$$

The structure $(M_{2,sa}, \sigma_\lambda, s_\lambda, \alpha_t)$ defines in a canonical way the CCR-dynamical system $\Big(\overline{W(M_{2,sa}, \sigma_\lambda)}, \tilde{\omega}_\lambda, \tilde{\alpha}_t\Big)$; $\tilde{\omega}_\lambda$ is a quasi-free state on the CCR-algebra $\overline{W(M_{2,sa}, \sigma_\lambda)}$:

$$\tilde{\omega}_\lambda\big(W(A)\big) = e^{-\frac{1}{2}s_\lambda(A,A)}$$

and

$$\tilde{\alpha}_t\big(W(A)\big) = W\big(\alpha_t(A)\big)$$

for all $A \in M_{2,sa}$.

Theorem 13 ([6]). *For all $A, B \in M_{2,sa}$ one has*

$$\begin{aligned} \lim_{N\to\infty} \omega_\lambda\big(e^{iF_N(A)} \alpha_t e^{iF_N(B)}\big) &= \tilde{\omega}_\lambda\big(W(A) \tilde{\alpha}_t W(B)\big) \\ &= e^{-\frac{1}{2}s_\lambda(A+\alpha_t B, A+\alpha_t B)} e^{-\frac{i}{2}\sigma_\lambda(A, \alpha_t B)}. \end{aligned}$$

The state $\tilde{\omega}_\lambda$ is an $(\tilde{\alpha}_t, \beta)$-KMS state on $\overline{W(M_{2,sa}, \sigma_\lambda)}$, the CCR-algebra of fluctuations.

This theorem is nothing but an application for the strong coupling BCS-model of the results in section 2.

Let $(\tilde{\mathcal{H}}_\lambda, \tilde{\pi}_\lambda, \tilde{\Omega}_\lambda)$ be the GNS triplet of $\tilde{\omega}_\lambda$. As the state $\tilde{\omega}_\lambda$ is regular, there exists a real linear map, called the bose field

$$F_\lambda : M_{2,sa} \to \mathcal{L}(\tilde{\mathcal{H}}_\lambda)$$

such that $\tilde{\pi}_\lambda\big(W(A)\big) = e^{iF_\lambda(A)}$ and the commutation relations $\big[F_\lambda(A), F_\lambda(B)\big] = i\sigma_\lambda(A,B)$. As in section 2.2 the construction suggests the identification

$$\lim_{N\to\infty} F_N(A) = F_\lambda(A).$$

$\tilde{\omega}_\lambda$ is completely characterized by the two-point function on the algebra of fluctuations

$$\begin{aligned} \tilde{\omega}_\lambda\big(F_\lambda(A)F_\lambda(B)\big) &= \lim_{N\to\infty} \tilde{\omega}_\lambda\big(F_N(A)F_N(B)\big) \\ &= \omega_\lambda\left(\big[A - \rho_\lambda(A)\big]\big[B - \rho_\lambda(B)\big]\right) \\ &= s_\lambda(A,B) + \frac{i}{2}\sigma_\lambda(A,B). \end{aligned}$$

Now we proceed to the construction of the complex structure J (see section 2.4). First note that the Hamiltonian h_λ,

$$h_\lambda = \begin{pmatrix} \epsilon & -\lambda \\ -\bar{\lambda} & -\epsilon \end{pmatrix}$$

can be diagonalised

$$h_\lambda = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}$$

with $\mu = (\epsilon^2 + |\lambda|^2)^{1/2}$. The eigenvectors are

$$\varphi_1 = \left(\frac{\mu-\epsilon}{2\mu}\right)^{1/2} \begin{pmatrix} -\frac{\lambda}{\mu-\epsilon} \\ \\ 1 \end{pmatrix} \quad , \quad \varphi_2 = \left(\frac{\mu+\epsilon}{2\mu}\right)^{1/2} \begin{pmatrix} \frac{\lambda}{\mu+\epsilon} \\ \\ 1 \end{pmatrix}.$$

Let $P_- = |\varphi_2)(\varphi_2|$ and $P_+ = |\varphi_1)(\varphi_1|$ be the corresponding spectral projections. The spectral resolution of h_λ is then given by

$$h_\lambda = -\mu P_- + \mu P_+.$$

In order to construct J we need to know the spectral resolution of $[h_\lambda, \cdot]$ considered as operator on M_2. The spectrum of $[h_\lambda, \cdot]$ is given by $\{-2\mu, 0, 2\mu\}$, the corresponding spectral projections are respectively:

$$\begin{aligned} E_- &= E(-2\mu) = P_- \cdot P_+ \\ E_0 &= P_- \cdot P_- + P_+ \cdot P_+ \\ E_+ &= E(2\mu) = P_+ \cdot P_-, \end{aligned}$$

and $[h_\lambda, A] = -2\mu E_-(A) + 2\mu E_+(A)$.

On $M^1_{2,sa} \equiv (E_+ + E_-)M_{2,sa}$ define J as in section 2 (equation (9)) by

$$J(E_+ + E_-)(A) = i(E_+ - E_-)(A).$$

The operator J defined above is a complex structure on the symplectic space $(M^1_{2,sa}, \sigma_\lambda)$, satisfying the properties of Proposition 8: $J^2 = -1$, $\sigma_\lambda(A, JB) = -\sigma_\lambda(JA, B)$, $\quad A, B \in M^1_{2,sa}$ and $\sigma_\lambda(A, JA) > 0$, $\;$ if $0 \neq A \in M^1_{2,sa}$. Remark that on $M^1_{2,sa}$, $[h_\lambda, \cdot] = -2i\mu J(\cdot)$ (Cfr. Theorem 12).

If there is no spontaneous breaking of the gauge symmetry generated by σ^z (i.e. if $\lambda = 0$), then $h_0 = \epsilon\sigma^z$ and obviously $[h_0, \sigma^z] = 0$, $\sigma^z = E_0(\sigma^z)$ and we can write

$$h_0 = \frac{\mu_0^2}{\epsilon} E_0(\sigma^z).$$

Now let $\lambda \neq 0$, then also $[h_\lambda, \sigma^z] \neq 0$. However $[h_\lambda, E_0(\sigma^z)] = 0$,

$$E_0(\sigma^z) = -\frac{\epsilon}{\mu}|\varphi_2)(\varphi_2| + \frac{\epsilon}{\mu}|\varphi_1)(\varphi_1|$$

and

$$h_\lambda = \frac{\mu^2}{\epsilon} E_0(\sigma^z).$$

Hence the state ω_λ and the corresponding time evolution α_t are still invariant under the symmetry generated by $E_0(\sigma^z)$:

$$\lim_{N\to\infty} \omega_\lambda\Big(\Big[\sum_{i=-N}^{N} E_0(\sigma^z)_i, A\Big]\Big) = 0$$

for all local A. Symmetry breaking is only concerned with the operator

$$\hat{\sigma}^z \equiv \sigma^z - E_0(\sigma^z) = (E_+ + E_-)(\sigma^z);$$

$\hat{\sigma}^z \in M^1_{2,sa}$ and we are interested in in the fluctuations of the operators $\hat{\sigma}^z$ together with its adjoint $J\hat{\sigma}^z$. Calculate:

$$E_\pm(\sigma^z) = \frac{|\lambda|^2}{2\mu^2}\begin{pmatrix} 1 & \pm\frac{\lambda}{\mu\mp\epsilon} \\ \mp\frac{\bar{\lambda}}{\mu\pm\epsilon} & -1 \end{pmatrix}.$$

Then

$$\hat{\sigma}^z = \frac{|\lambda|^2}{\mu^2}\sigma^z + \frac{\epsilon}{\mu^2}(\lambda\sigma^+ + \bar{\lambda}\sigma^-)$$

$$J\hat{\sigma}^z = \frac{i}{\mu}(\lambda\sigma^+ - \bar{\lambda}\sigma^-).$$

Note that $J\hat{\sigma}^z$ is the usual order parameter operator for the BCS model, but now constructed by means of σ^z and the spectrum of the Hamiltonian. We have also $\omega_\lambda\big(J\hat{\sigma}^z\big) = 0$ and $0 = \omega_\lambda\big([h_\lambda, J\hat{\sigma}^z]\big) = -2i\mu\omega_\lambda\big(J(J\hat{\sigma}^z)\big) = 2i\mu\omega_\lambda(\hat{\sigma}^z)$.

The fluctuations are given by

$$F_\lambda(E_0(\sigma^z)) = \lim_{N\to\infty} \frac{1}{(2N+1)^{1/2}} \sum_i \big[E_0(\sigma^z)_i - \rho_\lambda\big(E_0(\sigma^z)\big)\big]$$
$$F_\lambda(\hat{\sigma}^z) = \lim_{N\to\infty} \frac{1}{(2N+1)^{1/2}} \sum_i \hat{\sigma}_i^z$$
$$F_\lambda(J\hat{\sigma}^z) = \lim_{N\to\infty} \frac{1}{(2N+1)^{1/2}} \sum_i J\hat{\sigma}_i^z,$$

and of course $F_\lambda(\sigma^z) = F_\lambda(E_0(\sigma^z)) + F_\lambda(\hat{\sigma}^z)$.

The variances are easily calculated since

$$(E_0\sigma^z)^2 = \frac{\epsilon^2}{\mu^2} \qquad (\hat{\sigma}^z)^2 = (J\hat{\sigma}^z)^2 = \frac{|\lambda|^2}{\mu^2}.$$

Note $1 = (\sigma^z)^2 = E_0(\sigma^z)^2 + (\hat{\sigma}^z)^2$. Also

$$\rho_\lambda(\sigma^z) = \rho_\lambda(E_0\sigma^z) = -\frac{\epsilon}{\mu}\tanh\beta\mu = -2\epsilon.$$

Hence

$$\tilde{\omega}_\lambda\big(F_\lambda(E_0\sigma^z)^2\big) = s_\lambda(E_0\sigma^z, E_0\sigma^z) = \rho_\lambda\big((E_0\sigma^z)^2\big) - \rho_\lambda(E_0\sigma^z)^2 = \frac{\epsilon^2}{\mu^2} - 4\epsilon^2$$
$$\tilde{\omega}_\lambda\big(F_\lambda(\hat{\sigma}^z)^2\big) = s_\lambda(\hat{\sigma}^z, \hat{\sigma}^z) = \rho_\lambda\big((\hat{\sigma}^z)^2\big) = \frac{|\lambda|^2}{\mu^2}$$
$$\tilde{\omega}_\lambda\big(F_\lambda(J\hat{\sigma}^z)^2\big) = s_\lambda(J\hat{\sigma}^z, J\hat{\sigma}^z) = \rho_\lambda\big((J\hat{\sigma}^z)^2\big) = \frac{|\lambda|^2}{\mu^2}.$$

The only non-trivial commutator is

$$\big[F_\lambda(\hat{\sigma}^z), F_\lambda(J\hat{\sigma}^z)\big] = i\sigma_\lambda(\hat{\sigma}^z, J\hat{\sigma}^z) = \omega_\lambda\big([\hat{\sigma}^z, J\hat{\sigma}^z]\big) = \omega_\lambda\big([\sigma^z, J\hat{\sigma}^z]\big) = i\frac{4|\lambda|^2}{\mu},$$

expressing the bosonic character of the fluctuations. Remark on the other hand that the microscopic observables $\hat{\sigma}^z$ and $J\hat{\sigma}^z$ do not satisfy canonical commutation relations, their fluctuations do. Remark also that indeed $\sigma_\lambda(\hat{\sigma}^z, J\hat{\sigma}^z) > 0$.

The flucuation operator $F_\lambda(E_0\sigma^z)$ is invariant under the dynamics $\tilde{\alpha}_t$, but the operators $F_\lambda(\hat{\sigma}^z)$ and $F_\lambda(J\hat{\sigma}^z)$ satisfy the equations of motion

$$\frac{d}{idt}\tilde{\alpha}_t\big(F_\lambda(\hat{\sigma}^z)\big) = F_\lambda\big([h_\lambda, \alpha_t(\hat{\sigma}^z)]\big) = -2i\mu F_\lambda\big(\alpha_t(J\hat{\sigma}^z)\big) = -2i\mu\tilde{\alpha}_t F_\lambda(J\hat{\sigma}^z) \qquad (17)$$
$$\frac{d}{idt}\tilde{\alpha}_t\big(F_\lambda(J\hat{\sigma}^z)\big) = F_\lambda\big([h_\lambda, \alpha_t(J\hat{\sigma}^z)]\big) = 2i\mu F_\lambda\big(\alpha_t(\hat{\sigma}^z)\big) = 2i\mu\tilde{\alpha}_t F_\lambda\big(\hat{\sigma}^z\big). \qquad (18)$$

In integrated form one gets:

$$\tilde{\alpha}_t F_\lambda(\hat{\sigma}^z) = F_\lambda(\hat{\sigma}^z)\cos 2\mu t + F_\lambda(J\hat{\sigma}^z)\sin 2\mu t$$
$$\tilde{\alpha}_t F_\lambda(J\hat{\sigma}^z) = -F_\lambda(\hat{\sigma}^z)\sin 2\mu t + F_\lambda(J\hat{\sigma}^z)\cos 2\mu t$$

Denote the non-conserved part of the fluctuation operator of the generator of the broken symmetry by

$$Q_\lambda \equiv F_\lambda(\hat{\sigma}^z) \quad \text{and} \quad P_\lambda \equiv \frac{F_\lambda(J\hat{\sigma}^z)}{2\mu}.$$

We define the pair (Q_λ, P_λ) as *the canonical pair of the Goldstone bosons*. This pair shares indeed all physical properties for Goldstone bosons.

First of all they satisfy the correct commutation rule for the position and momentum variable of a boson particle:

$$[Q_\lambda, P_\lambda] = \frac{1}{2\mu}\left[F_\lambda(\hat{\sigma}^z), F_\lambda(J\hat{\sigma}^z)\right] = 2i\frac{|\lambda|^2}{\mu^2} = ic_\lambda,$$

where $c_\lambda \equiv 2\frac{|\lambda|^2}{\mu^2} = \left(\sigma^z - \rho_\lambda(\sigma^z), \sigma^z - \rho_\lambda(\sigma^z)\right)_\sim$, mostly called the order parameter. Here it enters as a quantization parameter or Planck's constant.

One expects the Goldstone bosons to behave like free bosons, independent of all other degrees of freedom of the system. This follows immediately from the equations of motion (17) and (18), in terms of Q_λ and P_λ

$$\tilde{\alpha}_t Q_\lambda = Q_\lambda \cos 2\mu t + 2\mu P_\lambda \sin 2\mu t$$
$$\tilde{\alpha}_t P_\lambda = -\frac{Q_\lambda}{2\mu}\sin 2\mu t + P_\lambda \cos 2\mu t,$$

which describe the evolution of a quantum harmonic oscillator pair (Q_λ, P_λ) oscillating with a frequency 2μ. This is the phenomenon of the doubling of the frequency for the inherent plasmon frequency.

The state $\tilde{\omega}_\lambda$ is an equilibrium state for this dynamics, and a straightforward computation yields

$$\tilde{\omega}_\lambda\left(Q_\lambda^2\right) = (2\mu)^2\tilde{\omega}\left(P_\lambda^2\right) = \frac{|\lambda|^2}{\mu^2} = \frac{c_\lambda(2\mu)}{2}\coth\frac{\beta(2\mu)}{2},$$

a quantum mechanical expression of a virial theorem. Remark that in the normal phase $(\lambda \to 0)$, $Q_{\lambda=0} = P_{\lambda=0} = 0$, i.e. the Goldstone boson disappears.

The creation and annihilation operators of the Goldstone bosons are as usual

$$a_\lambda^\pm = \frac{Q_\lambda \mp i2\mu P_\lambda}{\sqrt{4\mu}}.$$

The state $\tilde{\omega}_\lambda$ is gauge-invariant and quasi-free with respect to the gauge transformations of these creation and annihilation operators,

$$\tilde{\omega}_\lambda \left(a_\lambda^+ a_\lambda^+ \right) = 0 = \tilde{\omega}_\lambda \left(a_\lambda^+ \right).$$

The two-point function is given by

$$\tilde{\omega}_\lambda \left(a_\lambda^+ a_\lambda^- \right) = \frac{1}{e^{2\beta\mu} - 1}.$$

4 Short range interactions

4.1 Goldstone theorem and canonical order parameter

Let ω be an extremal translation invariant (α_t, β)-KMS state, α_t a dynamics generated by a translation invariant Hamiltonian H and let γ_s be a strongly continuous one-parameter symmetry group which is locally generated by a generator

$$Q_n = \sum_{x\in\Lambda_n} q_x,$$

where $\Lambda_n = [-n, n]^\nu \cap \mathbb{Z}^\nu$ and q_x is the symmetry generator density. Denote $q = q_{x=0}$.

For systems with short range interactions, assuming spontaneous symmetry breaking amounts to:

Assumption 2. *Assume that there exists an $(\alpha_t,\ \beta)$-KMS or ground state ω such that ω is not invariant under the symmetry transformation γ, while the dynamics α_t remains invariant under γ, i.e. $\exists A \in \mathcal{A}_L$ such that*

$$\omega\big(\gamma_s(A)\big) \neq \omega(A), \tag{19}$$

while

$$\alpha_t \circ \gamma_s = \gamma_s \circ \alpha_t. \tag{20}$$

The invariance of the dynamics (20) is crucial in this context (see [27] and Proposition 15 and 20 below).

An operator A satisfying (19) is again called an order parameter operator. Equation (19) is equivalent to

$$\frac{d}{ds}\omega\big(\gamma_s(A)\big)\Big|_{s=0} = \lim_{n\to\infty} \omega\big([Q_n, A]\big) \neq 0.$$

The local Hamiltonians are determined by an interaction Φ

$$H_n = \sum_{X\subseteq\Lambda_n} \Phi(X)$$

and the infinite volume Hamiltonian H is defined such that for $A \in \mathcal{A}_{\Lambda_0}$,

$$HA\Omega = \sum_{X\cap\Lambda_0\neq\varnothing} [\Phi(X), A]\Omega,$$

where Ω is the cyclic vector of the state ω.

The relation between spontaneous symmetry breaking and the absence of a gap in the energy spectrum in the ground state was originally put forward by Goldstone [1]. For short range interactions in many-body systems, it is proved [2, 3] that spontaneous symmetry breaking implies the absence of an energy gap in the excitation spectrum. We refer here to [28] where the Goldstone theorem is proved rigorously for quantum lattice systems.

Theorem 14 (Goldstone Theorem [28]). *If Φ is translation invariant and satisfies*

$$\sum_{X \ni 0} |X| \|\Phi(X)\| < \infty \tag{21}$$

then

(i) At $T = 0$: If the system has an energy gap then there is no spontaneous symmetry breakdown.

(ii) At $T > 0$: If the system has L^1 clustering then there is no spontaneous symmetry breakdown.

The L^1 clustering means here that for each observable A, one has:

$$\sum_{x \in \mathbb{Z}^\nu} \left| \omega(A \tau_x A) - \omega(A)^2 \right| < \infty.$$

The first step is to construct something like a *canonical order parameter operator*. Denote

$$L(A) = [H, A].$$

The Duhamel two-point function becomes now:

$$(A, B)_\sim \equiv \frac{1}{\beta} \int_0^\beta \omega(A^* \alpha_{iu} B) du = \omega \left(A^* \frac{1 - e^{-\beta L}}{\beta L} B \right).$$

The KMS-condition

$$\omega\left(AB\right) = \omega\left(B \alpha_{i\beta} A\right)$$

yields

$$\omega\big([A, B]\big) = \omega\Big(A(1 - e^{-\beta L})B\Big),$$

for A, B in a dense domain of $\mathcal{A}$, and hence if $B \in \mathrm{Dom}(L^{-1})$ then

$$\beta(A, B)_\sim = \omega\Big(\big[A, L^{-1}B\big]\Big).$$

The Duhamel two-point function appears in the Bogoliubov inequality [29] for KMS-states given by:

$$\left|\omega\big([A^*, B]\big)\right|^2 \leq \beta \omega\big([A^*, L(A)]\big)(B, B)_\sim.$$

Finally denote $\bar{A} = A - \omega(A)$.

Assumption 3. *Assume that there are no long range correlations in the fluctuations of the symmetry generator density, i.e. assume*

$$\sum_{z\in\mathbb{Z}^\nu}\left|\omega(q\tau_z q)-\omega(q)^2\right|<\infty.$$

Then also the uniform susceptibility c_β defined by

$$c_\beta\equiv\lim_{n\to\infty}\frac{\beta}{2|\Lambda_n|}\big(\overline{Q}_n,\overline{Q}_n\big)_\sim$$

is finite, i.e. $c_\beta<\infty$.

Proposition 15. *Under Assumption 2 and 3 we have*

$$c_\beta=\lim_{n\to\infty}\frac{\beta}{2|\Lambda_n|}\big(\overline{Q}_n,\alpha_t\overline{Q}_n\big)_\sim>0 \tag{22}$$

and c_β is independent of t, and also given by

$$c_\beta=\lim_{n\to\infty}\frac{\beta}{2}\omega\Big(\big[\overline{Q}_n,L^{-1}(q)\big]\Big).$$

Proof. Let

$$c_\beta(t)=\lim_{n\to\infty}\frac{\beta}{2|\Lambda_n|}\big(\overline{Q}_n,\alpha_t\overline{Q}_n\big)_\sim=\lim_{n\to\infty}\frac{1}{2|\Lambda_n|}\omega\Big(\overline{Q}_n\frac{1-e^{-\beta L}}{L}e^{itL}\overline{Q}_n\Big)$$

First we show $c_\beta(t=0)>0$. Let A be an arbitrary order parameter operator. Translation invariance and the Bogoliubov inequality yield

$$\begin{aligned}0&<\lim_{n\to\infty}\left|\omega\Big(\Big[\overline{Q}_n,\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n}\tau_xA\Big]\Big)\right|^2\\&\le\lim_{n\to\infty}\frac{\beta}{|\Lambda_n|^2}\omega\Big(\Big[\sum_{x\in\Lambda_n}\tau_xA,L\Big(\sum_{y\in\Lambda_n}\tau_yA\Big)\Big]\Big)\big(\overline{Q}_n,\overline{Q}_n\big)_\sim.\end{aligned}$$

In [28] it is shown that (21) also implies

$$\sum_{z\in\mathbb{Z}^\nu}\omega\Big(\big[\tau_zA,L(A)\big]\Big)=\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\omega\Big(\Big[\sum_{x\in\Lambda_n}\tau_xA,L\Big(\sum_{y\in\Lambda_n}\tau_yA\Big)\Big]\Big)<\infty$$

for each local observable A. Hence

$$0<\sum_{z\in\mathbb{Z}^\nu}\omega\Big(\big[\tau_zA,L(A)\big]\Big)\lim_{n\to\infty}\frac{\beta}{|\Lambda_n|}\big(\overline{Q}_n,\overline{Q}_n\big)_\sim$$

yielding $c_\beta(t=0) > 0$.

The proof of the time invariance of c_β is based on [27] and goes as follows:

$$\begin{aligned}\frac{d}{idt}c_\beta(t) &= \lim_{n\to\infty}\frac{\beta}{2|\Lambda_n|}\big(\overline{Q}_n, \alpha_t L(\overline{Q}_n)\big)_\sim = \lim_{n\to\infty}\frac{1}{2|\Lambda_n|}\omega\left(\overline{Q}_n(1-e^{-\beta L})e^{itL}\overline{Q}_n\right)\\ &= \lim_{n\to\infty}\frac{1}{2|\Lambda_n|}\omega\Big(\big[\overline{Q}_n, e^{itL}\overline{Q}_n\big]\Big).\end{aligned}$$

With $\overline{Q}_n = \sum_{x\in\Lambda_n}\bar{q}_x$, translation invariance and (20):

$$\begin{aligned}\frac{d}{idt}c_\beta(t) &= \lim_{n\to\infty}\frac{1}{2}\omega\Big(\big[\overline{Q}_n, e^{itL}\bar{q}\big]\Big) = \frac{1}{2}\left.\frac{d}{d\phi}\right|_{\phi=0}\omega\big(\gamma_s(\alpha_t\bar{q})\big) = \frac{1}{2}\left.\frac{d}{d\phi}\right|_{\phi=0}\omega\big(\alpha_t(\gamma_s\bar{q})\big)\\ &= \frac{1}{2}\left.\frac{d}{d\phi}\right|_{\phi=0}\omega(\bar{q}) = 0.\end{aligned}$$

□

From the proposition it follows that if $L^{-1}\bar{q}$ exists, it is an order parameter operator. We call it the canonical order parameter operator, it is an order parameter constructed directly from the two given quantities, the Hamiltonian and the symmetry generator. However it can not be expected in general that $\bar{q} \in \mathrm{Dom}(L^{-1})$, especially not for systems without an energy gap, because of problems at zero energy. Expressions like $(1-e^{-\beta L})L^{-1}\bar{q}$ on the contrary are well defined. The bulk of our efforts below consists of mastering the difficulties with the canonical order parameter by considering the k-mode fluctuations and by afterwards taking the limit $k \to 0$. This method has already been used in [15], where the Goldstone coordinates are constructed for models of interacting bose gases.

4.2 Fluctuations

By the Goldstone theorem, spontaneous symmetry breaking implies that the system does not have exponential or L^1 clustering. In particular the variances of local fluctuations

$$F_n(A) = \frac{1}{|\Lambda_n|^{1/2}}\sum_{x\in\Lambda_n}\big(\tau_x(A) - \omega(A)\big)$$

may not be convergent in the thermodynamic limit for certain A (in particular for A an order parameter operator) because of long range order correlations. The central limit as in Theorem 1 no longer holds. However one can study the k-mode fluctuations, i.e. one considers for $k = (k_1, k_2, \ldots, k_\nu) \in \mathbb{R}^\nu$, with $k_j \neq 0$ for $j = 1, 2, \ldots, \nu$:

$$F_{n,k}(A) = \frac{1}{|\Lambda_n|^{1/2}}\sum_{x\in\Lambda_n}\big(\tau_x(A) - \omega(A)\big)\cos k.x.$$

It is believed that the central limit theorem holds for the k-mode fluctuations for every extremal translation invariant state, even at criticality. This is essentially because one stays away from the singularity at $k = 0$. A completely rigorous proof of this statement is found in [30]. The idea can be described qualitatively as follows. See also [31] for a similar line of reasoning.

For $A \in \mathcal{A}_L$, denote the Fourier transforms of the l-point correlation functions $\omega(\tau_{x_1} A \tau_{x_2} A \cdots \tau_{x_l} A)$ by $\mu(k_1, k_2, \cdots, k_l)$ (i.e. k_j are different vectors in $\mathbb{R}^\nu$ here, not the components of a particular k). In general μ is a measure. By translation invariance it can be written as a function of $k_1, k_1 + k_2, \ldots, k_1 + k_2 + \cdots + k_l$. As in [31], assume that the only singularities in μ are of the type $\delta(k_1 + \cdots + k_i)$.

We show now that the truncated correlation functions $\omega_T\left(F_{n,k}(A)^l\right)$ vanish for $l \geq 3$ and remain finite for $l = 2$. Let $\omega(A) = 0$, then

$$\begin{aligned}
&\omega_T\left(F_{n,k}(A)^l\right) \\
&= \frac{1}{|\Lambda_n|^{l/2}} \sum_{x_1, x_2, \ldots, x_l} \omega_T(\tau_{x_1} A \tau_{x_2} A \cdots \tau_{x_l} A) \cos k.x_1 \cos k.x_2 \cdots \cos k.x_l \\
&= \frac{1}{|\Lambda_n|^{l/2}} \sum_{x_1, y_1 \cdots, y_{l-1}} \omega_T(A \tau_{y_1} A \cdots \tau_{y_{l-1}} A) \cos k.x_1 \cos k.(y_1 + x_1) \cdots \cos k.(y_{l-1} + x_1).
\end{aligned}$$

The expansion of the cosines into exponentials yields two types of terms, namely terms which do not depend on x_1 and terms which do depend on x_1. The first kind of terms do not appear for l odd and for l even they are exactly the ones which are cancelled out by the truncation. The second kind of terms tend to zero because of the scaling factors. Let us illustrate this by means of an example. First let $l = 2$:

$$\omega_T\left(F_{n,k}(A)^2\right) = \omega\left(F_{n,k}(A)^2\right) = \frac{1}{|\Lambda_n|} \sum_{x,y} \omega(A \tau_{y-x} A) \cos k.x \cos k.y.$$

Since

$$\mu(k) = \sum_z \omega(A \tau_z A) e^{-ik.z}$$

can at most have a singularity at $k = 0$, $\mu(k) < \infty$ for $k \neq 0$. Also

$$\frac{1}{|\Lambda_n|} \sum_z e^{ik.z} \to \delta_{k,0}.$$

Hence the only terms contributing in the two-point correlation function are the terms containing the factor $e^{\pm ik.(y-x)}$, i.e. the terms of the first kind. In the limit we find

$$\lim_n \omega_T\left(F_{n,k}(A)^2\right) = \frac{1}{2}[\mu(k) + \mu(-k)] < \infty.$$

Now let $l = 4$ and consider a typical term:

$$\frac{1}{|\Lambda_n|^2} \sum_{x_1, y_1, y_2, y_3} \omega\left(A\tau_{y_1} A\tau_{y_2} A\tau_{y_3} A\right) e^{-ik.(y_1 - y_2 + y_3)}.$$

Ignoring boundary effects in the sums, this becomes

$$\begin{aligned} &\frac{1}{|\Lambda_n|} \sum_{y_1, y_2, y_3} \omega\left(A\tau_{y_1} A\tau_{y_2} A\tau_{y_3} A\right) e^{-ik.(y_1 - y_2 + y_3)} \\ &= \frac{1}{|\Lambda_n|} \sum_{y_1, y_2, y_3} \omega\left(A\tau_{y_1} A\tau_{y_2} [A\tau_{y_3 - y_2} A]\right) e^{-ik.(y_1 - y_2 + y_3)} \\ &= \sum_{x,z} \omega\Big(A\tau_x A \frac{1}{|\Lambda_n|} \sum_y \tau_y [A\tau_z A]\Big) e^{-ik.(x+z)}. \end{aligned}$$

In the limit we get

$$\sum_x \omega\left(A\tau_x A\right) e^{-ik.x} \sum_z \omega\left(A\tau_z A\right) e^{-ik.z}$$

cancelling out against two-point correlations in the 4-point truncated correlation function.

Finally, take $l = 3$, then all terms are of the second kind, e.g.

$$\begin{aligned} &\frac{1}{|\Lambda_n|^{3/2}} \sum_{x_1, y_1, y_2} \omega(A\tau_{y_1} A\tau_{y_2} A) e^{ik.(x_1 + y_1 - y_2)} \\ =&\frac{1}{|\Lambda_n|^{3/2}} \sum_{y_1, y_2} \omega(A\tau_{y_1} [A\tau_{y_2 - y_1} A]) e^{ik.(y_1 - y_2)} \sum_{x_1} e^{ik.x_1}. \end{aligned}$$

The sum over x_1 is bounded by $\prod_{j=1}^{\nu} |\sin \frac{k_j}{2}|^{-1}$, yielding

$$\sum_y \omega\Big(A \frac{1}{|\Lambda_n|} \sum_x \tau_x [A\tau_y A]\Big) e^{-ik.y}$$

which converges to

$$\omega(A) \sum_y \omega\left(A\tau_y A\right) e^{-ik.y} = 0.$$

Hence this term vanishes.

Using the formula

$$\omega\left(e^{i\lambda Q}\right) = \exp\Big\{ \sum_{l=1}^{\infty} \frac{(i\lambda)^l}{l!} \omega_T\Big(\underbrace{Q, \ldots, Q}_{l\ times}\Big) \Big\}$$

one arrives at the central limit theorem

$$\lim_n \omega\left(e^{iF_{n,k}(A)}\right) = e^{-\frac{1}{2}s_k(A,A)},$$

with $s_k(A,A) = \lim_{n\to\infty} \omega\left(F_{n,k}(A)^2\right)$.

In [30] one can find a rigorous proof of the central limit theorem for the k-mode fluctuations, $k = (k_j \neq 0)_{j=1}^{\nu}$, for states satisfying a certain clustering condition, expressed as a condition on the function α_ω (see (4)). Although this condition is much weaker than for the $k = 0$ fluctuations, it is not clear whether it is always satisfied for any extremal transation invariant state. The arguments above however suggests that this clustering condition on the state is merely technical and that a general rigorous proof of the central limit theorem along the lines of [30] is possible for $k = (k_j \neq 0)_{j=1}^{\nu}$ without any condition. We continue on the basis of the arguments above.

Theorem 16 (Central limit theorem). *For all $A \in \mathcal{A}_{L,sa}$ and $k = (k_j \neq 0)_{j=1}^{\nu}$*

(i) $\lim_{n\to\infty} \omega\left(F_{n,k}(A)^2\right) < \infty$

(ii) $\lim_{n\to\infty} \omega\left(e^{iF_{n,k}(A)}\right) = e^{-\frac{1}{2}s_k(A,A)}$ *with* $s_k(A,B) = \lim_{n\to\infty} \mathrm{Re}\,\omega\left(F_{n,k}(A)^* F_{n,k}(B)\right)$.

Because of (i), the limit

$$\lim_{n\to\infty} \omega\big(F_{n,k}(A)^* F_{n,k}(B)\big) \equiv \langle A, B\rangle_k$$

defines a positive sesquilinear form which satisfies the Cauchy-Schwarz inequality

$$|\langle A, B\rangle_k|^2 \leq \langle A, A\rangle_k \langle B, B\rangle_k.$$

More explicitly

$$\langle A, B\rangle_k = \frac{1}{2}\sum_{z\in\mathbb{Z}^\nu}\big(\omega(A^*\tau_z B) - \omega(A^*)\omega(B)\big)\cos k.z.$$

Let

$$\sigma_k(A,B) = 2\ \mathrm{Im}\ \langle A, B\rangle_k,$$

then

$$\text{strong} - \lim_{n\to\infty} \pi\big([F_{n,k}(A), F_{n,k}(B)]\big) = i\sigma_k(A,B).$$

The identification of the central limit with bose fields is again on the basis of a reconstruction theorem [30]:

Theorem 17. *Let the state ω satisfy Theorem 16. The state $\tilde{\omega}_k$ is the quasi free state on the CCR-algebra $\mathcal{W}(\mathcal{A}_{L,sa}, \sigma_k)$ which is determined by the bilinear form s_k*

$$\tilde{\omega}_k\left(W_k(A)\right) = e^{-\frac{1}{2}s_k(A,A)}.$$

The $W_k(A)$, $A \in \mathcal{A}_{L,sa}$ are the Weyl operators generating $\mathcal{W}(\mathcal{A}_{L,sa}, \sigma_k)$.

The theorem states that for all $A_1, A_2, \dots, A_l \in \mathcal{A}_{L,sa}$,

$$\lim_{n\to\infty} \omega\Big(e^{iF_{n,k}(A_1)}e^{iF_{n,k}(A_2)}\dots e^{iF_{n,k}(A_l)}\Big) = \tilde{\omega}_k\Big(W_k(A_1)W_k(A_2)\dots W_k(A_l)\Big).$$

The state $\tilde{\omega}_k$ is regular and hence for every $A \in \mathcal{A}_{L,sa}$ there exists a self-adjoint bosonic field $F_k(A)$ in the GNS representation $(\tilde{\mathcal{H}}_k, \tilde{\pi}_k, \tilde{\Omega}_k)$ of $\tilde{\omega}_k$ such that

$$\tilde{\pi}_k(W_k(A)) = e^{iF_k(A)}.$$

This implies that in the sense of the central limit, the local fluctuations converge to the bosonic fields associated with the system $\Big(\mathcal{W}(\mathcal{A}_{L,sa}, \sigma_k), \tilde{\omega}_k\Big)$,

$$\lim_{n\to\infty} F_{n,k}(A) = F_k(A).$$

4.3 Goldstone modes for finite wavelengths

As in section 2.5 we assume that the state ω is such that there is a spectrum ϵ_k^β such that

$$HA_+\Omega = \int \epsilon_k^\beta dP(k)A_+\Omega, \tag{23}$$

with $\epsilon_0^\beta = 0$ and $\epsilon_k^\beta > 0$ for $k \neq 0$ (Assumption 1). In fact, for our purposes, it is sufficient that (23) is satisfied for $A = q$.

We adopt the same notations as in section 2, i.e. the spectral resolution of the Hamiltonian H is denoted by

$$H = \int_{-\infty}^{\infty} \lambda dE_\lambda,$$

A_+ is the 'positive energy' part of the self adjoint observable $A \in \mathcal{A}$, namely

$$A_+ = \int_{\lambda\in\mathbb{R}} \int_{\lambda'\in(-\infty,\lambda)} dE_\lambda A dE_{\lambda'}$$

and $\{dP(k)|k \in [0, 2\pi)\}$ is the spectral resolution of the unitary operator U_x implementing the translations in the GNS-representation of ω, i.e.

$$\pi(\tau_x A) = U_x\pi(A)U_{-x}$$

and

$$U_x = |\Omega)(\Omega| + \int e^{ik.x} dP(k).$$

Again

$$L(A) = [H, A] = \iint (\lambda - \lambda') dE_\lambda A dE_{\lambda'}$$

and if $\mathcal{D}$ is the set of functions on $\mathbb{R}$ with compact support, then:

Proposition 18. *For $f \in \mathcal{D}$*

$$\lim_{n\to\infty} \omega\left(F_{n,k}(q)\hat{f}(H)F_{n,k}(q)\right) = \frac{c_k^\beta \epsilon_k^\beta}{2(1 - e^{-\beta\epsilon_k^\beta})}\left(\hat{f}(\epsilon_k^\beta) + \hat{f}(-\epsilon_k^\beta)e^{-\beta\epsilon_k^\beta}\right),$$

where

$$c_k^\beta = \lim_{n\to\infty} \beta\Big(F_{n,k}(q), F_{n,k}(q)\Big)_\sim.$$

Proof. Let $\bar{q} = q - \omega(q)$ and for convenience denote again $q = \bar{q}$. Let

$$a(z) = \beta(q, \tau_z q)_\sim$$

then $a(z) = a(-z)$ and by assumption $\sum_z |a(z)| < \infty$. Hence

$$\hat{a}(k) = \sum_z a(z)e^{-ik.z} = \sum_z a(z)\cos k.z = 2c_k^\beta.$$

For $h \in \mathcal{D}_{\mathbb{Z}^\nu}$, the set of sequences on $\mathbb{Z}^\nu$ with compact support,

$$\sum_z h(z)\overline{a(z)} = \sum_z h(z) \int e^{-ik.z}\beta(q, dP(k)q)_\sim = \int \hat{h}(k)(q, dP(k)q)_\sim.$$

By the Parsival equality

$$\sum_z h(z)\overline{a(z)} = \int \frac{dk}{(2\pi)^\nu}\hat{h}(k)\overline{\hat{a}(k)}.$$

Hence

$$\int \hat{h}(k)\beta(q, dP(k)q)_\sim = 2\int \frac{dk}{(2\pi)^\nu}\hat{h}(k)c_k^\beta.$$

On the other hand

$$
\begin{aligned}
\beta(q,\tau_z q)_\sim &= \omega\Big(q\frac{1-e^{-\beta H}}{H}\tau_z q\Big)\\
&= \int_{-\infty}^{0}\frac{1-e^{-\beta\lambda}}{\lambda}\omega(qdE_\lambda\tau_z q) + \int_0^\infty \frac{1-e^{-\beta\lambda}}{\lambda}\omega(qdE_\lambda\tau_z q)\\
&= \int_0^\infty \frac{1-e^{\beta\lambda}}{-\lambda}e^{-\beta\lambda}\omega(qdE_\lambda\tau_{-z}q) + \int_0^\infty \frac{1-e^{-\beta\lambda}}{\lambda}\omega(qdE_\lambda\tau_z q)\\
&= 2\omega\Big(q_-\frac{1-e^{-\beta H}}{H}\tau_{-z}q_+\Big) = 2\int \frac{1-e^{-\beta\epsilon_k^\beta}}{\epsilon_k^\beta}e^{-ik.z}\omega\left(q_- dP(k)q_+\right),
\end{aligned}
$$

and hence

$$
\sum_z h(z)(q,\tau_z q)_\sim = 2\int \hat{h}(k)\frac{1-e^{-\beta\epsilon_k^\beta}}{\epsilon_k^\beta}\omega\left(q_- dP(k)q_+\right).
$$

Finally

$$
\int \hat{h}(k)\frac{1-e^{-\beta\epsilon_k^\beta}}{\epsilon_k^\beta}\omega\left(q_- dP(k)q_+\right) = \int \frac{dk}{(2\pi)^\nu}\hat{h}(k)c_k^\beta,
$$

or

$$
\int \hat{h}(k)\omega\left(q_- dP(k)q_+\right) = \int \frac{dk}{(2\pi)^\nu}\hat{h}(k)\frac{c_k^\beta\epsilon_k^\beta}{1-e^{-\beta\epsilon_k^\beta}}.
$$

By a similar calculation one finds

$$
\omega\left(q\hat{f}(H)\tau_{y-x}q\right) = \int \frac{dk}{(2\pi)^\nu}\left\{\hat{f}(-\epsilon_k^\beta)e^{-\beta\epsilon_k^\beta}+\hat{f}(\epsilon_k^\beta)\right\}e^{ik.(x-y)}\frac{c_k^\beta\epsilon_k^\beta}{1-e^{-\beta\epsilon_k^\beta}}.
$$

Let

$$
\delta_n(k) = \frac{1}{|\Lambda_n|}\sum_{x,y\in\Lambda_n} e^{ik.(x-y)},
$$

then

$$
\lim_{n\to\infty}\int \frac{dk'}{(2\pi)^\nu}\hat{h}(k')\delta_n(k'-k) = \hat{h}(k).
$$

Hence

$$
\begin{aligned}
&\lim_{n\to\infty}\omega\left(F_{n,k}(q)\hat{f}(H)F_{n,k}(q)\right)\\
&= \lim_{n\to\infty}\frac{1}{2|\Lambda_n|}\sum_{x,y\in\Lambda_n}\omega\left(q\hat{f}(H)\tau_{y-x}q\right)\cos k.(y-x)\\
&= \lim_{n\to\infty}\frac{1}{2}\int\frac{dk'}{(2\pi)^\nu}\left\{\hat{f}(-\epsilon_{k'}^\beta)e^{-\beta\epsilon_{k'}^\beta}+\hat{f}(\epsilon_{k'}^\beta)\right\}\frac{c_{k'}^\beta\epsilon_{k'}^\beta}{1-e^{-\beta\epsilon_{k'}^\beta}}\frac{1}{2}(\delta_n(k'-k)+\delta_n(k'+k))\\
&= \frac{c_k^\beta\epsilon_k^\beta}{2(1-e^{-\beta\epsilon_k^\beta})}\Big(\hat{f}(\epsilon_k^\beta)+\hat{f}(-\epsilon_k^\beta)e^{-\beta\epsilon_k^\beta}\Big).
\end{aligned}
$$

□

Corollary 19.

$$\tilde{\omega}_k\left(F_k(q)^2\right) = \lim_{n\to\infty} \omega\left(F_{n,k}(q)^2\right) = \frac{c_k^\beta \epsilon_k^\beta}{2} \coth \frac{\beta\epsilon_k^\beta}{2}.$$

Proposition 20. *Let c_β be the constant defined by (22), i.e.*

$$c_\beta(t) = \lim_{n\to\infty} \beta\Big(F_n(q), \alpha_t F_n(q)\Big)_\sim,$$

then

$$\frac{d}{dt}c_\beta = 0 \Leftrightarrow \lim_{k\to 0} \epsilon_k^\beta = 0.$$

Hence by Proposition 15, the Hamiltonian H is gapless in the sense that

$$\lim_{k\to 0} \epsilon_k^\beta = 0.$$

Proof. Let $\epsilon^\beta = \lim_{k\to 0} \epsilon_k^\beta = 0$ and

$$c_\beta(t) = \lim_{n\to\infty} \Big(F_n(q), \alpha_t F_n(q)\Big)_\sim.$$

Then by Proposition 18

$$\begin{aligned} c_\beta(t) &= \lim_{n\to\infty} \omega\left(F_n(q)\frac{1-e^{-\beta L}}{L}e^{itL}F_n(q)\right) \\ &= c_\beta(t=0)\cos\epsilon^\beta t. \end{aligned}$$

□

The positive sesquilinear form $\langle\cdot,\cdot\rangle_k$ on the algebra $\mathcal{A}_{L,sa}$ induces a positive sesquilinear form on the k-fluctuation algebra, which we again denote by $\langle\cdot,\cdot\rangle_k$:

$$\langle F_k(A), F_k(B)\rangle_k \equiv \langle A, B\rangle_k = \tilde{\omega}_k\Big(F_k(A)^* F_k(B)\Big).$$

Similarly the notations s_k and σ_k are transported from $\mathcal{A}_{L,sa}$ onto the k-fluctuation algebra.

Now let f be any function such that $\hat{f}(L)\bar{q} \in \mathcal{A}_{L,sa}$ and $\|\hat{f}(L)\bar{q}\| < \infty$. Then $F_k(\hat{f}(L)\bar{q})$ is a well defined element of the k-fluctuation algebra. In particular

$$\begin{aligned} \Big\|F_k(\hat{f}(L)q)\Big\|_k^2 &= s_k\left(\hat{f}(L)\bar{q}, \hat{f}(L)\bar{q}\right) \\ &= \lim_{n\to\infty} \omega\Big(F_{n,k}(q)|\hat{f}(H)|^2 F_{n,k}(q)\Big) \\ &= |\hat{f}(\epsilon_k^\beta)|^2 \frac{c_k^\beta \epsilon_k^\beta}{2} \coth \frac{\beta\epsilon_k^\beta}{2}. \end{aligned}$$

(We used that $\hat{f}(L)q$ is self adjoint iff $\bar{\hat{f}}(\lambda) = \hat{f}(-\lambda)$.) Since the k-fluctuation algebra is by definition closed for the $\langle\cdot,\cdot\rangle_k$ topology we can define elements $\widetilde{\hat{f}(L)F_k(q)}$ for a much wider class of functions, namely all those functions for which $|\hat{f}(\pm\epsilon_k^\beta)| < \infty$: first approximate f by a sequence of functions $\{f_m\}$ for which $\hat{f}_m(L)\bar{q}$ is well defined and then let $\widetilde{\hat{f}(L)F_k(q)} =$ strong$-\lim_m F_k(\hat{f}_m(L)\bar{q})$, with respect to the $\langle\cdot,\cdot\rangle_k$ norm. It is sufficient that $\hat{f}_m(\pm\epsilon_k^\beta) \to \hat{f}(\pm\epsilon_k^\beta)$.

Let $\mathcal{F}$ be the set

$$\mathcal{F} = \left\{ \widetilde{f(L)}F_k(q) \mid |f(\epsilon_k)| < \infty \right\}.$$

It is obvious that we can define $\widetilde{L_k}$, the generator of the dynamics, mapping $\mathcal{F}$ into $\mathcal{F}$ by

$$\widetilde{L_k}\widetilde{f(L)}F_k(q) = \widetilde{g_f(L)}F_k(q)$$

where

$$g_f(\pm\epsilon_k) = \pm\epsilon_k f(\pm\epsilon_k).$$

If $F_k(f_m(L)\bar{q})$ is a sequence of k-fluctuation operators converging strongly to $\widetilde{f(L)}F_k(q)$, then this definition is equivalent to

$$i\widetilde{L_k}\widetilde{f(L)}F_k(q) \equiv \text{strong}-\lim_m F_k(iLf_m(L)\bar{q}).$$

We can also define $i\widetilde{L_k}^{-1}F_k(q)$, which can formally be interpreted as $F_k\big(iL^{-1}\bar{q}\big)$, the k-fluctuation of the canonical order parameter, even if $iL^{-1}\bar{q}$ does not exist. We have

$$\tilde{\omega}_k\left(\left(i\widetilde{L_k}^{-1}F_k(q)\right)^2\right) = \frac{c_k^\beta}{2\epsilon_k^\beta}\coth\frac{\beta\epsilon_k^\beta}{2}$$

or

$$\tilde{\omega}_k\left(F_k(q)^2\right) = (\epsilon_k^\beta)^2\tilde{\omega}_k\left(\left(i\widetilde{L_k}^{-1}F_k(q)\right)^2\right).$$

Moreover

$$\begin{aligned}\left[F_k(q), i\widetilde{L_k}^{-1}F_k(q)\right] &= \tilde{\omega}_k\left(\left[F_k(q), i\widetilde{L_k}^{-1}F_k(q)\right]\right) = i\lim_{n\to\infty}\omega\left(F_{n,k}(q)\frac{1-e^{-\beta L}}{L}F_{n,k}(q)\right)\\ &= ic_k^\beta.\end{aligned}$$

For ease of notation, denote

$$Q_k = F_k(q) \qquad P_k = i\widetilde{L_k}^{-1}F_k(q),$$

and denote by $\widetilde{\mathcal{B}_k}$ the algebra generated by Q_k and P_k. Remark that for $k \to 0$, $\tilde{\omega}_k\left(P_k^2\right)$ diverges as $(\epsilon_k^\beta)^{-2}$. This divergence corresponds to the well known phenomenon of long range correlations in the order parameter fluctuations.

Theorem 21. *The spontaneous breaking of the symmetry γ implies the existence of a canonical pair of fluctuations (Q_k, P_k), constructed directly from the symmetry generator density and satisfying*

$$[Q_k, P_k] = ic_k^\beta \quad , \quad c_k^\beta = \lim_{n\to\infty} \beta\Big(F_{n,k}(q), F_{n,k}(q)\Big)_\sim > 0$$
$$\tilde{\omega}_k\left(Q_k^2\right) = (\epsilon_k^\beta)^2 \tilde{\omega}_k\left(P_k^2\right).$$

Moreover the infinitesimal generator L of the microscopic dynamics induces a map $\widetilde{L_k}$ on $\widetilde{\mathcal{B}_k}$ which leaves $\widetilde{\mathcal{B}_k}$ invariant:

$$\widetilde{L_k} Q_k = -i(\epsilon_k^\beta)^2 P_k$$
$$\widetilde{L_k} P_k = iQ_k,$$

indicating that the canonical pair (Q_k, P_k) behaves dynamically like a quantum harmonic oscillator pair, independent of all other degrees of freedom of the system. With $\widetilde{\alpha_t^k}$ the time evolution generated by $\widetilde{L_k}$, we get the equations of motion

$$\widetilde{\alpha_t^k} Q_k = Q_k \cos \epsilon_k^\beta t + \epsilon_k^\beta P_k \sin \epsilon_k^\beta t$$
$$\widetilde{\alpha_t^k} P_k = -\frac{Q_k}{\epsilon_k^\beta} \sin \epsilon_k^\beta t + P_k \cos \epsilon_k^\beta t.$$

The pair (Q_k, P_k) is called the canonical pair of the Goldstone mode. The Goldstone boson creation and annihilation operators are defined by

$$a_k^* = \frac{Q_k - i\epsilon_k^\beta P_k}{\sqrt{2\epsilon_k^\beta}} \qquad a_k = \frac{Q_k + i\epsilon_k^\beta P_k}{\sqrt{2\epsilon_k^\beta}}$$

satisfying

$$[a_k, a_k^*] = c_k^\beta.$$

The quasi-free state $\tilde{\omega}_k$ is a β-KMS state for the evolution $\widetilde{\alpha_t^k}$ which is expressed by the fact that the Goldstone bosons have a Bose-Einstein distribution, i.e.

$$\tilde{\omega}_k\left(a_k^* a_k\right) = \frac{c_k^\beta}{e^{\beta\epsilon_k^\beta} - 1},$$

which is equivalent to

$$\tilde{\omega}_k\left(Q_k^2\right) = \frac{c_k^\beta \epsilon_k^\beta}{2} \coth \frac{\beta\epsilon_k^\beta}{2}.$$

The state $\tilde{\omega}_k$ is gauge invariant:

$$\tilde{\omega}_k\left(a_k^* a_k^*\right) = 0 = \tilde{\omega}_k\left(a_k^*\right).$$

Proof. All there is left to prove is the dynamics of the pair (Q_k, P_k). We have trivially

$$i\widetilde{L_k} P_k = -Q_k.$$

Now we calculate $i\widetilde{L_k} Q_k$. We have

$$\tilde{\omega}_k \left((i\widetilde{L_k} Q_k)^2 \right) = (\epsilon_k^\beta)^2 \tilde{\omega}_k \left(F_k(q)^2 \right)$$

and

$$\langle i\widetilde{L_k} Q_k, P_k \rangle_k = \left\langle i\widetilde{L_k} F_k(q), \widetilde{iL_k}^{-1} F_k(q) \right\rangle_k = \tilde{\omega}_k \left(F_k(q)^2 \right).$$

Hence

$$\begin{aligned}
&\left\langle i\widetilde{L_k} Q_k - (\epsilon_k^\beta)^2 P_k, i\widetilde{L_k} Q_k - (\epsilon_k^\beta)^2 P_k \right\rangle_k \\
&= \langle i\widetilde{L_k} Q_k, i\widetilde{L_k} Q_k \rangle_k + (\epsilon_k^\beta)^4 \langle P_k, P_k \rangle_k - (\epsilon_k^\beta)^2 \langle i\widetilde{L_k} Q_k, P_k \rangle_k - (\epsilon_k^\beta)^2 \langle P_k, i\widetilde{L_k} Q_k \rangle_k \\
&= 0.
\end{aligned}$$

By the equivalence relation (Theorem 3)

$$i\widetilde{L_k} Q_k = (\epsilon_k^\beta)^2 P_k.$$

□

4.4 Goldstone mode for infinite wavelength

Finally, we look for the Goldstone mode operators in the limit of k tending to zero, i.e. in the long wavelength limit. We take the results of section 4.3 and study the limit $k \to 0$. Among other results, we show that the long wavelength Goldstone mode survives in this limit only in the ground state. This shows also that no long wavelength quantum Goldstone modes are present for temperatures $T > 0$. For $T > 0$, the spontaneous symmetry breakdown does not show any quantum behaviour, only classical modes are present.

Let $\epsilon_k = \lim_{\beta \to \infty} \epsilon_k^\beta$, the ground state spectrum. Because of the Goldstone theorem, we have that

$$\lim_{k \to 0} \epsilon_k = 0,$$

For the sake of clarity, we treat explicitly the case

$$\epsilon_k = \epsilon |k|^\delta \quad , \quad \epsilon, \delta > 0$$

of powerlaw behaviour as a function of k, for small momenta k. The essentials of our result however are independent of this special choice.

Let $c_\beta = \lim_{k \to 0} c_k^\beta$ and $c_k = \lim_{\beta \to \infty} c_k^\beta$.

Assumption 4. *Assume*

$$\lim_{k\to 0} c_k = \lim_{\beta\to\infty} c_\beta = c < \infty.$$

First let $\beta < \infty$. The variances

$$\tilde{\omega}_k\left(Q_k^2\right) = \frac{c_k^\beta \epsilon_k^\beta}{2} \coth \frac{\beta\epsilon_k^\beta}{2}$$
$$\tilde{\omega}_k\left(P_k^2\right) = \frac{c_k^\beta}{2\epsilon_k^\beta} \coth \frac{\beta\epsilon_k^\beta}{2}$$

behave as follows for $k \to 0$

$$\tilde{\omega}_k\left(Q_k^2\right) \approx \frac{c_k^\beta}{\beta} \to \frac{c_\beta}{\beta}\,, \quad \text{finite,}$$
$$\tilde{\omega}_k\left(P_k^2\right) \approx \frac{c_k^\beta}{\beta(\epsilon_k^\beta)^2} \to \infty.$$

Since observable fluctuation operators are allways characterized by a finite, non-zero variance, it is clear that we have to renormalize P_k before taking a limit $k \to 0$:

$$\tilde{P}_k \propto \epsilon_k^\beta P_k.$$

This however implies that the commutator

$$[Q_k, \tilde{P}_k] \propto i c_k^\beta \epsilon_k^\beta$$

vanishes in the limit $k \to 0$. In other words the quantum character and hence also the harmonic oscillation of the Goldstone mode disappears in the appropriate limit $k \to 0$, at least at non-zero temperature.

At zero temperature ($\beta = \infty$), in the ground state, the situation is completely different. The variances behave now for $k \to 0$ as follows:

$$\tilde{\omega}_k\left(Q_k^2\right) = \frac{c_k\epsilon_k}{2} \to 0$$
$$\tilde{\omega}_k\left(P_k^2\right) = \frac{c_k}{2\epsilon_k} \to \infty,$$

but their product

$$\tilde{\omega}_k\left(Q_k^2\right)\tilde{\omega}_k\left(P_k^2\right) = \frac{c_k^2}{4} \to \frac{c^2}{4}$$

remains finite. This means that the divergence of the order parameter operator fluctuations due to long range correlations is exactly compensated by a proportional squeezing of the symmetry generator fluctuations. Therefore one can find a renormalized Q_k and

P_k, denoted by $\tilde{Q}_k$ and $\tilde{P}_k$, having both a finite, non-zero variance, with a finite non zero commutator; indeed take:

$$\tilde{Q}_k \equiv |k|^{-\delta/2} Q_k \qquad \tilde{P}_k \equiv |k|^{\delta/2} P_k,$$

then

$$\begin{aligned}
\tilde{\omega}_k\left(\tilde{Q}_k^2\right) &= \frac{c_k \epsilon}{2} \to \frac{c\epsilon}{2} \\
\tilde{\omega}_k\left(\tilde{P}_k^2\right) &= \frac{c_k}{2\epsilon} \to \frac{c}{2\epsilon} \\
[\tilde{Q}_k, \tilde{P}_k] &= ic_k \to ic.
\end{aligned}$$

Remark that this scaling transformation has no effect on the creation and annihilation operators, in particular:

$$a_k^* = \frac{Q_k - i\epsilon_k P_k}{\sqrt{2\epsilon_k}} = \frac{|k|^{\delta/2}\tilde{Q}_k - i\epsilon|k|^{\delta}|k|^{-\delta/2}\tilde{P}_k}{\sqrt{2\epsilon|k|^{\delta}}} = \frac{\tilde{Q}_k - i\epsilon\tilde{P}_k}{\sqrt{2\epsilon}}.$$

On the other hand, the equations of motion (Theorem 21)

$$\begin{aligned}
\widetilde{L_k} Q_k &= -i\epsilon_k^2 P_k \\
\widetilde{L_k} P_k &= iQ_k.
\end{aligned}$$

are transformed into

$$\begin{aligned}
\frac{\widetilde{L_k}}{|k|^{\delta}} \tilde{Q}_k &= -i\epsilon^2 \tilde{P}_k \\
\frac{\widetilde{L_k}}{|k|^{\delta}} \tilde{P}_k &= i\tilde{Q}_k.
\end{aligned}$$

In a natural way, the rescaling of Q_k and P_k induces a rescaling of the time evolution

$$\widetilde{L_k} \to \tilde{L} \equiv \frac{\widetilde{L_k}}{|k|^{\delta}}.$$

This rescaling can be interpreted as a rescaling of time t to $\tau = |k|^{\delta} t$. This rescaled time evolution is still correctly diagonalised by the creation and annihilation operators, as it should be since the rescaling has no effect on them:

$$\tilde{\alpha}_\tau a_k^* \equiv \widetilde{\alpha_t^k} a_k^* = e^{i\epsilon_k t} a_k^* = e^{i\epsilon\tau} a_k^*.$$

Finally let $\tilde{\mathcal{B}}$ be the CCR-algebra generated by a pair of boson creation and annihilation operators (a^*, a), satisfying the commutation relation:

$$[a, a^*] = c;$$

and let $\tilde{\omega}$ by definition be the gauge invariant, quasi-free Fock state annihilated by a:

$$\begin{aligned}
&\tilde{\omega}(a^* a) = \tilde{\omega}(a^* a^*) = \tilde{\omega}(a^*) = 0 \\
&\tilde{\omega}(a a^*) = c;
\end{aligned}$$

and let $\tilde{\alpha}_\tau$ be the dynamics on $\tilde{\mathcal{B}}$ defined by

$$\tilde{\alpha}_\tau(a^*) = e^{i\epsilon\tau} a^*.$$

Then, $\tilde{\omega}$ is a ground state for $\tilde{\alpha}_\tau$ in the sense $(X \in \tilde{\mathcal{B}})$ [16]

$$\tilde{\omega}(X^* \tilde{L}(X)) \geq 0.$$

In particular one has:

$$\tilde{\omega}(a^* \tilde{L}(a)) = 0 \ , \ \ \tilde{\omega}(a \tilde{L}(a^*)) = \epsilon c > 0;$$

Finally, define (Q, P) as the usual position and momentum operators of a quantum harmonic oscillator

$$Q = \sqrt{\frac{\epsilon}{2}}(a^* + a) \qquad P = \frac{i}{\sqrt{2\epsilon}}(a^* - a)$$

or

$$a^* = \frac{Q - i\epsilon P}{\sqrt{2\epsilon}}.$$

Then the results of this section can then be summarized in the following theorem:

Theorem 22. *In the ground state, $\beta = \infty$, under Assumption 4, the dynamical system $(\widetilde{\mathcal{B}}, \widetilde{\alpha_t^k}, \tilde{\omega}_k)$ converges in the limit $k \to 0$ to the dynamical system $(\tilde{\mathcal{B}}, \tilde{\alpha}_t, \tilde{\omega})$ in the sense that*

$$\lim_{k\to 0} \tilde{\omega}_k \left(a_k^\sharp \widetilde{\alpha_t^k} a_k^\sharp \right) = \tilde{\omega}\left(a^\sharp \tilde{\alpha}_\tau a^\sharp \right),$$

where the notation $\sharp$ means any combination of creation and annihilation operators and $\tau = |k|^\delta t$. Therefore we can identify

$$Q = \lim_{k\to 0} \tilde{Q}_k \qquad P = \lim_{k\to 0} \tilde{P}_k$$

and the quantum harmonic oscillation of the canonical pair (Q_k, P_k) with frequency ϵ_k becomes a quantum harmonic oscillation of the canonical pair (Q, P) with frequency ϵ:

$$\tilde{L}Q = -i\epsilon^2 P \quad \textit{and} \quad \tilde{L}P = iQ,$$

with as solution

$$\begin{aligned}
\tilde{\alpha}_\tau Q &= Q\cos\epsilon\tau + \epsilon P \sin\epsilon\tau \\
\tilde{\alpha}_\tau P &= -\frac{Q}{\epsilon}\sin\epsilon\tau + P\cos\epsilon\tau.
\end{aligned}$$

The pair (Q, P) is called the canonical pair of the collective Goldstone mode.

Remark that formally this situation is very similar to the case where there is a gap in the energy spectrum (see section 2, Theorem 12, and section 3). In both cases we have a quantum harmonic oscillator pair, oscillating with a well-defined frequency. This pair is in both cases canonically constructed from the only two a priori given operators, i.e. the Hamiltonian and the symmetry generator.

It is however important to remember the rescaling that has been done, in fact the previous theorem tells us that in the ground state the long range correlations in the order parameter fluctuations are exactly compensated by a squeezing of the generator fluctuations. Both operators continue to form a harmonic oscillator pair in the limit $k \to 0$, although the frequency becomes infinitesimally small and hence the period of oscillation infinitely (or macroscopically) large.

The rescaling of the k-mode fluctuations actually gives information about the size of the 0-mode fluctuations. In a finite box Λ_n of length $L = 2n+1$, the smallest non-zero wave vector has length $|k| \propto L^{-1}$. Therefore the rescaling of Q_k with a factor $|k|^{-\delta/2}$ suggests a rescaling by $L^{\delta/2} = |\Lambda_n|^{\delta/2\nu}$ of the fluctuation, i.e.

$$F_n(q) = \frac{1}{|\Lambda_n|^{\frac{1}{2}-\frac{\delta}{2\nu}}} \sum_{x\in\Lambda_n} \big(q_x - \omega(q)\big),$$

in order that its variance is non-zero and finite. This means that the fluctuations of the symmetry generator are of order $|\Lambda_n|^{\frac{1}{2}-\frac{\delta}{2\nu}}$, i.e. *subnormal fluctuations*. Similarly the fluctuations of the order parameter are of order $|\Lambda_n|^{\frac{1}{2}+\frac{\delta}{2\nu}}$, i.e. *abnormal fluctuations*. This requires $\frac{\delta}{2\nu} \leq \frac{1}{2}$, or $\delta \leq \nu$. This condition is undoubtly related to the condition $c < \infty$ (Assumption 4). Remark also that if SSB disappears, i.e. if $c = 0$, then the Goldstone boson disappears.

If the energy spectrum has a gap (see section 3), on the contrary, both the fluctuation of the generator and the fluctuation of the order parameter are normal (of the order $|\Lambda_n|^{1/2}$). The frequency is strictly positive and the period of oscillation remains finite (or of microscopic order).

References

[1] Goldstone J. *Il Nuovo Cimento*, 19:154, 1961.

[2] Kastler D., Robinson D.W., and Swieca A. Conserved currents and associated symmetries; Goldstone's theorem. *Communications in Mathematical Physics*, 2:108 – 120, 1966.

[3] Swieca J.A. Range of forces and broken symmetries in many-body systems. *Communications in Mathematical Physics*, 4:1 – 7, 1967.

[4] Martin P.A. A remark on the Goldstone theorem in statistical mechanics. *Il Nuovo Cimento*, 68 B(2):302 – 313, 1982.

[5] Fannes M., Pulè J.V., and Verbeure A. Goldstone theorem for Bose systems. *Letters in Mathematical Physics*, 6:385 – 389, 1982.

[6] Goderis D., Verbeure A., and Vets P. Fluctuation oscillations and symmetry breaking: the BCS-model. *Il Nuovo Cimento*, 106 B(4):375 – 383, 1991.

[7] Broidioi M., Nachtergaele B., and Verbeure A. The Overhauser model: Equilibrium fluctuation dynamics. *Journal of Mathematical Physics*, 32(10):2929 –2935, 1991.

[8] Broidioi M. and Verbeure A. Plasmon frequency for a spin-density wave model. *Helvetica Physica Acta*, 64:1093 – 1112, 1991.

[9] Verbeure A. and Zagrebnov V.A. Phase transition and algebra of fluctuation operators in an exactly soluble model of a quantum anharmonic crystal. *Journal of Statistical Physics*, 69:329, 1992.

[10] Broidioi M. and Verbeure A. The plasmon in the one component plasma. *Helvetica Physica Acta*, 66:155 – 180, 1993.

[11] Goderis D. and Vets P. Central limit theorem for mixing quantum systems and the CCR-algebra of fluctuations. *Communications in Mathematical Physics*, 122:249, 1989.

[12] Goderis D., Verbeure A., and Vets P. Dynamics of fluctuations for quantum lattice systems. *Communications in Mathematical Physics*, 128:533 – 549, 1990.

[13] Anderson P.W. Random-phase approximation in the theory of superconductivity. *Physical Review*, 112(6):1900 – 1916, 1958.

[14] Stern H. Broken symmetry, sum rules, and collective modes in many-body systems. *Physical Review*, 147(1):94 – 101, 1966.

[15] Michoel T. and Verbeure A. Goldstone boson normal coordinates in interacting Bose gases. *Journal of Statistical Physics*, 96(5/6):1125 – 1162, 1999.

[16] Bratteli O. and Robinson D.W. *Operator Algebras and Quantum Statistical Mechanics 2*. Springer Berlin, Heidelberg, New York, 1996.

[17] Wreszinski W.F. Charges and symmetries in quantum theories without locality. *Fortschritte der Physik*, 35(5):379 – 413, 1987.

[18] Goderis D., Verbeure A., and Vets P. Non-commutative central limits. *Probability Theory and Related Fields*, 82:527 – 544, 1989.

[19] Arveson W. On groups of automorphisms of operator algebras. *Journal of Functional Analysis*, 15(3):217 – 243, 1974.

[20] Hörmander L. Estimates for translation invariant operators in L^p spaces. *Acta Mathematica*, 104:93 – 140, 1960.

[21] Naudts J., Verbeure A., and Weder R. Linear response theory and the KMS condition. *Communications in Mathematical Physics*, 44:87, 1975.

[22] Fannes M. and Verbeure A. Correlation inequalities and equilibrium states. *Communications in Mathematical Physics*, 55:125 – 131, 1977.

[23] Fannes M. and Verbeure A. Correlation inequalities and equilibrium states. II. *Communications in Mathematical Physics*, 57:165 – 171, 1977.

[24] Fannes M., Spohn H., and Verbeure A. Equilibrium states for mean-field models. *Journal of Mathematical Physics*, 21:355, 1980.

[25] Thirring W. and Wehrl A. On the mathematical structure of the B.C.S.-model. *Communications in Mathematical Physics*, 4:303 – 314, 1967.

[26] Thirring W. On the mathematical structure of the B.C.S.-model II. *Communications in Mathematical Physics*, 7:181 – 189, 1968.

[27] Requardt M. About the poor decay of certain cross-correlation functions in the statistical mechanics of phase transitions in the static and dynamical regime. *Journal of Statistical Physics*, 29(3):117 – 127, 1982.

[28] Landau L., Fernando Perez J., and Wreszinski W.F. Energy gap, clustering, and the Goldstone theorem in statistical mechanics. *Journal of Statistical Physics*, 26(4):755 – 766, 1981.

[29] Bogoliubov N.N. *Phys. Abh. S.U.*, 1:229, 1962.

[30] Michoel T., Momont B., and Verbeure A. CCR-algebra structure of normal k-mode fluctuations. *Reports on Mathematical Physics*, 41(3):361 – 395, 1998.

[31] Narnhofer H., Requardt M., and Thirring W. Quasi-particles at finite temperatures. *Communications in Mathematical Physics*, 92:247 – 268, 1983.