Ground state of a spin-1/2 charged particle in a two-dimensional magnetic field

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It is investigated that the structure of the kernel of the Dirac-Weyl operator $\mathbf{D}$ of a charged particle in the magnetic field $B = B_0 + b$, given by the sum of a strongly singular magnetic field $B_0(\cdot) = \sum_j \gamma^j \delta(\cdot - a_j)$ and a magnetic field b with a bounded support. Here the magnetic field b may have some singular points with the order of the singularity less than 2. At a glance, it seems that, following "Aharonov-Casher Theorem" [Phys.Rev.A, **19**, 1979], the dimension of the kernel of $\mathbf{D}$, $\dim\ker\mathbf{D}$, is a function of one variable, the total magnetic flux of B $(= \int_{\mathbb{R}^2} b\,dx\,dy + \sum_j \gamma_j)$. However, since the influence of the strongly singular points occurs, $\dim\ker\mathbf{D}$ indeed is a function of several variables, the total magnetic flux and each of γ_j's.

I. INTRODUCTION

In 1979, Y. Aharonov and A. Casher [1] investigated the dimension of the kernel of the Dirac-Weyl operator $\mathbf{D}$ for a spin-1/2 charged particle in a two-dimensional magnetic field. After them, many authors [2–10] (and references therein) studied it with various situations.

Aharonov and Casher [1] treated the case when the magnetic field B is a smooth function on $\mathbb{R}^2$. In this case, $\mathbf{D}$ is essentially selfadjoint operator on $C_0^\infty(\mathbb{R}^2)$, the space of smooth functions with compact supports and $\dim\ker\mathbf{D}$ is equal to the largest natural number which is less than $|q\Phi|/2\pi - 1$, where q is the charge of the particle and the quantity Φ is the total magnetic flux of B defined by $\Phi = \int_{\mathbb{R}^2} B\,dx\,dy$. This result is well known as "Aharonov-Casher Theorem."

In 1993, A. Arai [2] treated the another case when B is strongly singular, more precisely, B is equal to 0 except on some finite points $\mathbf{a}_\nu$ $(\nu = 1, \ldots, n)$ and can be written as the sum of delta functions with coefficients γ_ν $(\nu = 1, \ldots, n)$ and their derivatives (See, Eq. (1)). The total magnetic flux Φ of B is equal to $\sum_{\nu=1}^n \gamma_\nu$. He have proved that, in this case, $\mathbf{D}$ is not essentially selfadjoint on $C_0^\infty(M)$ with $M = \mathbb{R}^2 \setminus \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and the dimension of the kernel of two selfadjoint extensions of $\mathbf{D}$ is depend on not only Φ but also each of γ_ν's. (He has given an explicit formula [2].) This means that, in this case, "Aharonov-Casher Theorem" is not valid.

In this article, we treat the case when magnetic field is mixed with the Aharonov-Casher type and the Arai type, i.e., both non-zero magnetic field regions with non-zero area and strongly singular points exist, which situation no one has treated yet. The following question arises: whether dose "Aharonov-Casher theorem" or "Arai's theorem" state $\dim\ker \mathbf{D}$? Roughly speaking, we prove that $\dim\ker \mathbf{D}$ can be expressed by a formula similar to Arai's one. (See Theorem 10)

The plan of this article is as follows: In Section II, we define $\mathbf{D}$ and prove some lemmas. In Section III, we prove "Vanishing Theorem." In Section IV, we prove our main theorem, which states the dimension of the kernel of $\mathbf{D}$.

II. PRELIMINARY

The aim of this article is to investigate the structure of the kernel of the Dirac-Weyl operator $\mathbf{D}$ of a charged spin-1/2 particle in the magnetic field $B = B_0 + b$, the sum of a strongly singular magnetic field B_0 (See, Eq. (1) and Ref. [2]) and a magnetic field b with a bounded support. As mentioned above, the magnetic field b may have some singular points with the order of the singularity less than 2.

In this article, we denote the charge of the particle by $q \in \mathbb{R} \setminus \{0\}$. We assume that B_0 and b may be singular at the finite isolated points $\mathbf{a}_\nu = (a_{\nu 1}, a_{\nu 2}) \in \mathbb{R}^2$, $\nu = 1, \ldots, n$, and we put

$$M = \mathbb{R}^2 \setminus \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}.$$

Remark: By the symbols indexed with 0 (resp. 1), we denote operators P_0 (resp. P_1); vector potentials $\mathbf{A}_0$ (resp. $\mathbf{A}_1$); auxiliary potentials ϕ_0 (reps. ϕ_1); and etc., which are corresponding to the strongly singular part B_0 (resp. the remainder part b) of B. In the case when b is equal to 0, our model we shall examine is the same one treated in Ref. [2].

A. Strongly singular part B_0

We denote by B_0 the strongly singular part of the given magnetic field B. Let

$$B_0(\mathbf{r}) = \sum_{\nu=1}^{n} \sum_{0\le\alpha+\beta\le m} C^{(\nu)}_{\alpha,\beta} D_x^\alpha D_y^\beta \delta(\mathbf{r} - \mathbf{a}_\nu), \quad \mathbf{r} = (x, y) \in \mathbb{R}^2 \tag{1}$$

with a nonnegative integer m and real constants $C^{(\nu)}_{\alpha,\beta}$, where D_x and D_y denote the distributional partial differential operators in x and y, respectively, and $\delta(\mathbf{r})$ is the Dirac delta distribution on $\mathbb{R}^2$. A gauge potential $\mathbf{A}_0(\mathbf{r})$ of B_0 is defined to be on $\mathbb{R}^2$-valued function $\mathbf{A}_0 = (A_{01}, A_{02})$ on M such that

$$B_0(x, y) = D_x A_{02}(x, y) - D_y A_{01}(x, y)$$

in the sense of distribution on $\mathbb{R}^2$.

Let

$$\phi_0(\mathbf{r}) = \sum_{\nu=1}^{n} \sum_{0\le\alpha+\beta\le m} \frac{C^{(\nu)}_{\alpha,\beta}}{2\pi} D_x^\alpha D_y^\beta \log|\mathbf{r} - \mathbf{a}_\nu|,$$

which satisfies

$$\Delta\phi_0(\mathbf{r}) = B_0(\mathbf{r})$$

in the sense of distribution on $\mathbb{R}^2$. Thus, we can choose

$$\mathbf{A}_0 = (A_{01}, A_{02}) = (-D_y\phi_0, D_x\phi_0)$$

as a gauge potential of B_0. For later use, we define some symbols as same as in Ref. [2]. Put

$$C_k^{(\nu)} = (-1)^k k! \sum_{\alpha=0}^{k} C_{\alpha,k-\alpha}^{(\nu)} i^{k-\alpha},\ \gamma_\nu = C_{0,0}^{(\nu)} = C_0^{(\nu)} \text{ and } \Phi_0 = \sum_{\nu=1}^{n} \gamma_\nu. \tag{2}$$

B. The remainder part b

Through this article, we suppose that the remainder part b of B satisfies that

$$b \in C_0^\infty(M) = \{f \in C^\infty(M); \operatorname{supp} f \text{ is bounded}\} \tag{3}$$

and that for arbitrary $\nu = 1, \ldots, n$, there exist constants $c_\nu > 0$ and $\epsilon_\nu < 2$ such that

$$|b(\mathbf{r})| \le \frac{c_\nu}{|\mathbf{r} - \mathbf{a}_\nu|^{\epsilon_\nu}} \text{ near } \mathbf{r} = \mathbf{a}_\nu. \tag{4}$$

Let

$$\phi_1(\mathbf{r}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} b(\mathbf{r}') \log|\mathbf{r} - \mathbf{r}'| d\mathbf{r}' \text{ for } \mathbf{r} = (x, y) \in \mathbb{R}^2.$$

Then we have that $b(\mathbf{r}) = \Delta\phi_1(\mathbf{r})$ on M. Since $b \in L^\epsilon(\mathbb{R}^2)$ with $1 < \epsilon < 2$, using Young's inequality, we can prove the following lemma.

Lemma 1 *Let $b \in C_0^\infty(M)$ and suppose Eq. (4). Then $\phi_1 \in C(\mathbb{R}^2)$.*

We shall need the local boundedness of ϕ_1 on $\mathbb{R}^2$ as a conclusion of this lemma.

Let

$$\Phi_1 = \int_{\mathbb{R}^2} b(\mathbf{r})\, d\mathbf{r}.$$

Since $b \in C_0^\infty(M)$ and Eq. (4), we have $b \in L^1(\mathbb{R}^2)$. Thus, Φ_1 has a finite value. Considering the boundedness of the support of b, we can prove the following lemma with a little calculating.

Lemma 2 *Suppose $b \in C_0^\infty(M)$. Then*

$$e^{\phi_1(\mathbf{r})} \sim |\mathbf{r}|^{\Phi_1} \textit{ as } \mathbf{r} \to \infty.$$

Remark: We say that $f(x) \sim g(x)$ as $x \to \infty$ if and only if there exist positive constants, c and d, such that for all x enough large, $cf(x) \le g(x) \le df(x)$ holds.

Define an $\mathbb{R}^2$-valued function $\mathbf{a} = (a_1, a_2)$ on M by

$$\mathbf{a}(\mathbf{r}) = (a_1(\mathbf{r}), a_2(\mathbf{r})) = (-\partial_y \phi_1(\mathbf{r}), \partial_x \phi_1(\mathbf{r})).$$

Here, $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$. Then $\mathbf{a}$ is a gauge potential of b, i.e.

$$b(x, y) = D_x a_2(x, y) - D_y a_1(x, y)$$

in the sense of distribution on $\mathbb{R}^2$.

Remark: For simplicity, we assume the smoothness of b in Eq (3). P. Exner and the authors [10] have proven that under the assumptions, $b(x) = \mathcal{O}(|x|^{-2-\delta})$ for some $\delta > 0$ and $b \in L^{1+\epsilon}_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\epsilon > 0$, there exists a positive R such that

$$|\phi_1(x) - \Phi_1 \ln |x|| < \epsilon \ln |x|$$

for all $|x| > R$ and ϕ_1 is continuous. (See, Prop. 2.2 and Prop. 2.5 in [10].) Since we do not need the compact supportness of b, we can remove this restriction.

C. The total magnetic field B

From now on, using the notations in the previous subsections, we rewrite the given (total) magnetic field B as

$$B(\mathbf{r}) = B_0(\mathbf{r}) + b(\mathbf{r})$$

and take

$$\mathbf{A}(\mathbf{r}) = \mathbf{A}_0(\mathbf{r}) + \mathbf{a}(\mathbf{r})$$

as a gauge potential of B. We denote by Φ the total magnetic flux:

$$\Phi = \Phi_0 + \Phi_1.$$

Let

$$\phi(\mathbf{r}) = \phi_0(\mathbf{r}) + \phi_1(\mathbf{r}).$$

Then we have

$$\mathbf{A}(\mathbf{r}) = (-D_y \phi(\mathbf{r}), D_x \phi(\mathbf{r})).$$

D. Dirac-Weyl operators

In this subsection, we define the Dirac-Weyl operators with B_0 and B acting in $\mathbb{C}^2 \otimes L^2(\mathbb{R}^2)$. Let σ_j, $j = 1, 2, 3$, be the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We denote the domain of operator T by $D(T)$.

Let

$$p_1 = -iD_x, \quad p_2 = -iD_y$$

be distributional differential operators acting in $L^2(\mathbb{R}^2)$. The velocity operator $\mathbf{P}_0 = (P_{01}, P_{02})$ with the gauge potential $\mathbf{A}_0$ is given by

$$P_{0j} = p_j - qA_{0j} \text{ with } D(P_{0j}) = C_0^\infty(M).$$

Define the Dirac-Weyl operator $Q_{\min}$ with the strongly singular magnetic field B_0 by

$$Q_{\min} = \sigma_1 P_{01} + \sigma_2 P_{02} \text{ with } D(Q_{\min}) = \mathbb{C}^2 \otimes C_0^\infty(M).$$

As the above equation, in this article, we often omit the tensorial sign, $\otimes$, between two operators. The velocity operator $\mathbf{P} = (P_1, P_2)$ with the gauge potential $\mathbf{A}$ is given by

$$P_j = p_j - q(A_{0j} + a_j) \text{ with } D(P_j) = C_0^\infty(M).$$

Define the Dirac-Weyl operator $D_{\min}$ with the total magnetic field B by

$$D_{\min} = \sigma_1 P_1 + \sigma_2 P_2 \text{ with } D(D_{\min}) = \mathbb{C}^2 \otimes C_0^\infty(M).$$

Note that $Q_{\min}$ is written in the form

$$Q_{\min} = \begin{pmatrix} 0 & Q_- \\ Q_+ & 0 \end{pmatrix},$$

where

$$Q_\pm = P_{01} \pm iP_{02} \text{ with } D(Q_\pm) = C_0^\infty(M),$$

and $D_{\min}$ is written in

$$D_{\min} = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix},$$

where

$$D_\pm = P_1 \pm iP_2 \text{ with } D(D_\pm) = C_0^\infty(M).$$

The operators $Q_{\min}$ and $D_{\min}$ are symmetric, and thus, $Q_{\min}$ and $D_{\min}$ are closable. We denote by $\bar{Q}_{\min}$ and $\bar{D}_{\min}$ the closure of $Q_{\min}$ and $D_{\min}$, respectively.

Let

$$\mathbf{D}_1 = \begin{pmatrix} 0 & D_+^* \\ \bar{D}_+ & 0 \end{pmatrix} \text{ and } \mathbf{D}_2 = \begin{pmatrix} 0 & \bar{D}_- \\ D_-^* & 0 \end{pmatrix}$$

be Dirac-Weyl operators with B acting in $\mathbb{C}^2 \otimes L^2(\mathbb{R}^2)$. These are selfadjoint extensions of $D_{\min}$. In Section IV, it is proven that these do not coincide under some conditions.

Remark: In Ref. [2], for the symmetric operator $Q_{\min}$, two selfadjoint extensions $Q_{\min}^{(1)}$ and $Q_{\min}^{(2)}$ of it have been constructed in the same fashion as above. Under some conditions, these selfadjoint extensions are not coincided with each other.

E. Generalized kernel

In this subsection, we preliminarily examine the generalized kernel (the space of generalized eigenvectors with eigenvalue 0) of $D_{\min}$ to identify the kernels of $\mathbf{D}_1$ and $\mathbf{D}_2$. We use the following natural identification of $\mathbb{R}^2$ and $\mathbb{C}$: each $(x, y) \in \mathbb{R}^2$ corresponds to $x + iy \in \mathbb{C}$ with the the imaginary unit i. We set that $\partial = (\partial/\partial x - i\partial/\partial y)/2$, $\bar{\partial} = (\partial/\partial x + i\partial/\partial y)/2$, and $a_\nu = a_{\nu_1} + ia_{\nu_2}$ for $\mathbf{a}_\nu = (a_{\nu 1}, a_{\nu 2})$.

The following proposition shall be needed later.

Proposition 3 *The following identity as differential operators*

$$D_\pm = e^{\mp q\phi_1} Q_\pm e^{\pm q\phi_1}$$

holds on $C^\infty(M)$.

Proof: By direct computations, we have

$$\begin{aligned} Q_+ &= -2ie^{-q\phi_0}\bar{\partial}e^{+q\phi_0}, \\ Q_- &= -2ie^{+q\phi_0}\partial e^{-q\phi_0}, \\ D_+ &= -2ie^{-q\phi}\bar{\partial}e^{+q\phi}, \\ D_+ &= -2ie^{+q\phi}\partial e^{-q\phi} \end{aligned}$$

on $C^\infty(M)$. Thus, we obtain the desired result.

Let $D(M) = C_0^\infty(M)$ and $D'(M)$ be the space of distributions on M. We say that $f \in D'(M)$ is a generalized eigenvector with eigenvalue 0 of the operators T on $C_0^\infty(M)$ if

$$\langle f, (Tg)^* \rangle = 0 \text{ for all } g \in C_0^\infty(M),$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical bilinear form on $D'(M) \times C_0^\infty(M)$. We denote by $S(T)$ the space of all the generalized eigenvectors with eigenvalue 0 of T. Note that $\ker \bar{D}_\pm \subset S(D_\pm)$.

For a function ψ on M, let

$$\begin{aligned} H_\psi(M) &= \{e^{-q\psi} f | f \text{ is holomorphic on } M\}, \\ H_\psi^a(M) &= \{e^{q\psi} f | f \text{ is antiholomorphic on } M\}. \end{aligned}$$

The following lemma is Lemma 4.4 in Ref. [2].

Lemma 4 *We have*

$$S(Q_+) = H_{\phi_0}(M) \text{ and } S(Q_-) = H^a_{\phi_0}(M).$$

The following lemma can be proven in the same way as in the proof of Lemma 4.4 in Ref. [2].

Lemma 5 *We have*

$$S(D_+) = H_{\phi}(M) \text{ and } S(D_-) = H^a_{\phi}(M).$$

Note that $S(Q_{\min}) = S(Q_+) \oplus S(Q_-)$ and $S(D_{\min}) = S(D_+) \oplus S(D_-)$.

III. VANISHING THEOREM

In this section, we examine the vanishing theorem for Dirac-Weyl operators $\mathbf{D}_1$ and $\mathbf{D}_2$. Suppose that $b \in C_0^\infty(M)$ and that Eq. (4) holds. We need the following lemma proved by Arai.

Lemma 6 (Arai, Lemma 4.3 in Ref. [2]) *We have* $\ker \bar{Q}_\pm = \{0\}$ *i.e.* $\ker \bar{Q}_{\min} = \{0\}$.

The following is the main theorem in this section.

Theorem 7 (Vanishing Theorem) *Suppose that the multiplication operator $e^{q\phi_1}$ (resp. $e^{-q\phi_1}$) acting in $L^2(\mathbb{R}^2)$ is bounded. Then*

$$\ker \bar{D}_+ = \{0\}. \ (\textit{resp.} \ \ker \bar{D}_- = \{0\}.)$$

Remark: The following three statements are equivalent. (i) The operator $e^{q\phi_1}$ is bounded; (ii) $\limsup_{|\mathbf{r}|\to\infty} q\phi_1(\mathbf{r}) < \infty$; (iii) $q\Phi_1 < 0$.

Proof: Consider the case when $e^{q\phi_1}$ is bounded. By Proposition 3, we have

$$e^{q\phi_1} S(D_+) = S(Q_+). \tag{5}$$

Let $\Phi \in \ker \bar{D}_+$. Then there exists a sequence $\{\Phi_n; n \in \mathbb{N}\} \subset C_0^\infty(M)$ such that $\Phi_n \to \Phi$ and $D_+\Phi_n \to 0$ in $L^2(\mathbb{R}^2)$ as $n \to \infty$. Since $\Phi \in S(D_+)$, by Eq. (5) and the assumption,

$$e^{q\phi_1}\Phi \in S(Q_+) \cap L^2(\mathbb{R}^2).$$

We have $e^{q\phi_1}\Phi_n \in C_0^\infty(M)$ for all $n \in \mathbb{N}$ and, since $e^{q\phi_1}$ is bounded, we have

$$e^{q\phi_1}\Phi_n \to e^{q\phi_1}\Phi \quad \text{in} \quad L^2(\mathbb{R}^2)$$

as $n \to \infty$. Moreover,

$$\|Q_+ e^{q\phi_1}\Phi_n\| \leq \|e^{q\phi_1}\| \, \|e^{-q\phi_1} Q_+ e^{q\phi_1}\Phi_n\| = \|e^{q\phi_1}\| \, \|D_+\Phi_n\| \to 0$$

as $n \to \infty$. Thus, $e^{q\phi_1}\Phi \in D(\bar{Q}_+)$ and $\bar{Q}_+ e^{q\phi_1}\Phi = 0$. Therefore, by Lemma 6, $e^{q\phi_1}\Phi = 0$. Thus, we have $\Phi = 0$, which means $\ker \bar{D}_+ = \{0\}$. In the case when $e^{-q\phi_1}$ is bounded, we can obtain the desired result in the same way as above.

IV. KERNEL OF DIRAC-WEYL OPERATOR

In the previous section, in the case when $e^{q\phi_1}$ (resp. $e^{-q\phi_1}$) is bounded, we proved that $\ker \bar{D}_+ = \{0\}$ (resp. $\ker \bar{D}_- = \{0\}$). In this section, we examine $\ker \bar{D}_-$ and $\ker \bar{D}_+$, and we identify the kernels of the two selfadjoint extensions $\mathbf{D}_1$ and $\mathbf{D}_2$ in restricted cases, respectively.

Let $\mathbb{Z}_+$ be the set of nonnegative integers. We introduce the set

$$W_\pm(\Phi) = \left\{ (p, k_1, \ldots, k_n) \in \mathbb{Z}_+ \times \mathbb{Z}^n; p + \sum_{\nu=1}^{n} k_\nu < \pm\frac{q\Phi}{2\pi} - 1, \pm\frac{q\gamma_\nu}{2\pi} - 1 < k_\nu, \nu = 1, \ldots, n \right\},$$

and put

$$N_\pm(\Phi; n; q) = \# W_\pm(\Phi),$$

the number of the elements of $W_\pm(\Phi)$. Note that $N_-(\Phi; n; q) = N_+(\Phi; n; -q)$. Let

$$F(z) = -\frac{1}{2\pi} \sum_{\nu=1}^{n} \sum_{k=1}^{m} \frac{C_k^{(\nu)}}{k(z - a_\nu)^k}.$$

Here, the constants $C_k^{(\nu)}$ were defined by Eq. (2). For $(p_\pm, k_1, \ldots, k_n) \in \mathbb{Z}_+ \times \mathbb{Z}^n$, define functions on M by

$$\Omega^+_{p_+, k_1, \ldots, k_n}(\mathbf{r}) = \left(\prod_{\nu=1}^{n} |\mathbf{r} - \mathbf{a}_\nu|^{-q\gamma_\nu/2\pi} (z - a_\nu)^{k_\nu} \right) P_+(z) e^{iq\Im F(z)},$$

$$\Omega^-_{p_-, k_1, \ldots, k_n}(\mathbf{r}) = \left(\prod_{\nu=1}^{n} |\mathbf{r} - \mathbf{a}_\nu|^{q\gamma_\nu/2\pi} (\bar{z} - \bar{a}_\nu)^{k_\nu} \right) P_-(\bar{z}) e^{-iq\Im \bar{F}(z)},$$

with $P_\pm$ a polynomial of order $p_\pm$ such that $P_+(a_\nu) \neq 0$, $P_-(\bar{a}_\nu) \neq 0$, $\nu = 1, \ldots, n$.

Remark: The sets $W_\pm(\Phi_0)$ and the numbers $N_\pm(\Phi_0; n; q)$ are equal to $W_\pm$ and $N_\pm(n; q)$ given in Ref. [2] respectively, and these functions $\Omega^\pm_{p_\pm, k_1, \ldots, k_n}$ are the same ones given in Ref. [2].

First of all, we solve the differential equations $D_\pm \Omega = 0$ on M and examine when the solutions Ω of it are in $L^2(M)$.

Lemma 8 *(i) The functions $e^{\mp q\phi_1} \Omega^\pm_{p_\pm, k_1, \ldots, k_n}$ satisfy the partial differential equations*

$$D_\pm e^{\mp q\phi_1} \Omega^\pm_{p_\pm, k_1, \ldots, k_n} = 0 \text{ on } M,$$

respectively.

(ii) The functions $e^{\mp q\phi_1} \Omega^\pm_{p_\pm, k_1, \ldots, k_n}$ are in $L^2(\mathbb{R}^2)$ if and only if $(p_\pm, k_1, \ldots, k_n) \in W_\pm(\Phi)$, respectively.

Proof: By Lemma 4.6 (i) in Ref. [2], we have

$$Q_\pm \Omega^\pm_{p_\pm,k_1,\ldots,k_n} = 0 \text{ on } M.$$

Thus, with Proposition 3, we obtain part (i). We prove part (ii). We have

$$|e^{-q\phi_1}\Omega^+_{p_+,k_1,\ldots,k_n}(\mathbf{r})| \sim \text{const.}|\mathbf{r}|^{-(q\Phi/2\pi)+p_+ + \sum_{\nu=1}^n k_\nu}$$

as $|\mathbf{r}| \to \infty$, and

$$|e^{-q\phi_1}\Omega^+_{p_+,k_1,\ldots,k_n}(\mathbf{r})| \sim \text{const.}|\mathbf{r}-\mathbf{a}_\nu|^{-(q\gamma_\nu/2\pi)+p_+}$$

as $\mathbf{r} \to \mathbf{a}_\nu$. Therefore, the desired assertion on $e^{-q\phi_1}\Omega^+_{p_+,k_1,\ldots,k_n}$ follows. Similarly we can prove the assertion on $e^{q\phi_1}\Omega^-_{p_-,k_1,\ldots,k_n}$.

Using this lemma, we can identify the kernels of $D^*_\pm$.

Theorem 9 *We have*

$$\ker D^*_- = \left\{ e^{-q\phi_1}\Omega^+_{p,k_1,\ldots,k_n}; (p,k_1,\ldots,k_n) \in W_+(\Phi) \right\}, \tag{6}$$

where $\ker D^*_- = \{0\}$ *if* $W_+(\Phi) = \emptyset$, *and*

$$\ker D^*_+ = \left\{ e^{q\phi_1}\Omega^-_{p,k_1,\ldots,k_n}; (p,k_1,\ldots,k_n) \in W_-(\Phi) \right\} \tag{7}$$

where $\ker D^*_+ = \{0\}$ *if* $W_-(\Phi) = \emptyset$.

Proof: By Lemma 8, the sets on the RHS of Eq. (6) and Eq. (7) are included in $\ker D^*_-$ and $\ker D^*_+$, respectively. To prove the converse inclusion relations, let

$$\Omega = (\Omega_+, \Omega_-) \in \ker D^*_- \oplus \ker D^*_+.$$

Note that

$$\ker D^*_- \oplus \ker D^*_+ = \ker D^*_{\min} = \mathbb{C}^2 \otimes L^2(\mathbb{R}^2) \cap S(D_{\min}).$$

Hence $\Omega \in S(D)$ and $\Omega_\pm \in L^2(\mathbb{R}^2)$. Therefore, by Lemma 5, there exists a holomorphic function f and an anti-holomorphic function g on M such that

$$\Omega_+ = e^{-q(\phi_1+\phi_0)}f, \quad \Omega_- = e^{q(\phi_1+\phi_0)}g.$$

We note that

$$\phi_0(\mathbf{r}) = \Re F(z) + \sum_{\nu=1}^n \frac{\gamma_\nu}{2\pi}\log|\mathbf{r}-\mathbf{a}_\nu|$$

and $\phi_1 \in C(\mathbb{R}^2)$. Thus, using the condition, $\Omega_+ \in L^2(\mathbb{R}^2)$, we see that f must be of the form

$$f(z) = e^{qF(z)}h(z)$$

with a meromorphic function h on $\mathbb{C} \cup \{\infty\}$ with possible poles at $z = a_\nu$, $\nu = 1, \ldots, n$. Thus, Ω_+ has to take the form

$$\Omega_+ = e^{-q\phi_1} \Omega^+_{p,k_1,\ldots,k_n}$$

with some $(p, k_1, \ldots, k_n) \in \mathbb{Z}_+ \times \mathbb{Z}^n$ and a polynomial P_+ of degree p such that $P_+(a_\nu) \neq 0$, $\nu = 1, \ldots, n$. By Lemma 8 (ii), $(p, k_1, \ldots, k_n)$ must be in $W_+(\Phi)$. Thus, Ω_+ is in the set on the RHS of Eq. (6). In the same way, we can prove that Ω_- is in the set on the RHS of Eq. (7).

Consequently, we obtain our main theorem.

Theorem 10 *(i) If $e^{q\phi_1}$ is bounded, we have*

$$\ker D_1 = \left\{ \begin{pmatrix} 0 \\ e^{q\phi_1} \Omega^-_{p,k_1,\ldots,k_n} \end{pmatrix} ; (p, k_1, \ldots, k_n) \in W_-(\Phi) \right\}, \tag{8}$$

where $\ker D_1 = \{0\}$ *if* $W_-(\Phi) = \emptyset$. *In particular,*

$$\dim \ker D_1 = N_-(\Phi; n; q). \tag{9}$$

(ii) If $e^{-q\phi_1}$ is bounded, we have

$$\ker D_2 = \left\{ \begin{pmatrix} e^{-q\phi_1} \Omega^+_{p,k_1,\ldots,k_n} \\ 0 \end{pmatrix} ; (p, k_1, \ldots, k_n) \in W_+(\Phi) \right\}, \tag{10}$$

where $\ker D_2 = \{0\}$ *if* $W_+(\Phi) = \emptyset$. *In particular,*

$$\dim \ker D_2 = N_+(\Phi; n; q). \tag{11}$$

Proof: Consider the part (i). By Theorem 7, we have

$$\ker D_1 = \left\{ \begin{pmatrix} 0 \\ \Omega \end{pmatrix} ; \Omega \in \ker D_+^* \right\}.$$

Hence, by Lemma 8, the set on the RHS of Eq. (8) is included in $\ker D_1$. By Theorem 9, we obtain the converse inclusion relation. Therefore, we obtain Eq. (8). In the same way, we can obtain Eq. (10). We write $(p, k_1, \ldots, k_n) = (p, \mathbf{k})$. Thus, $\{e^{\mp q\phi_1} \Omega^\pm_{p_j,\mathbf{k}_j}\}_{j=1}^l$ are linearly independent if and only if $(p_j, \mathbf{k}_j) \neq (p_i, \mathbf{k}_i)$, $i \neq j$, $i, j = 1, \ldots, l$. Thus, Eq. (9) and Eq. (11) follow.

As a corollary of Theorem 10, we have the following.

Corollary 11 *Let $n \geq 2$ and suppose that $N_+(\Phi; n; q) + N_+(\Phi; n; q) \geq 1$. Then $D_{\min}$ is not essentially selfadjoint.*

Proof: In this case, the two selfadjoint extensions $\mathbf{D}_1$ and $\mathbf{D}_2$ are not coincided with each other. Thus, $D_{\min}$ is not essentially selfadjoint.

Remark: In the same way as in the proof of Corollary 4.9 in Ref. [2], we can prove under some conditions that $N_\pm(\Phi_1; n; q) \geq 1$ and, thus, $D_{\min}$ is not essentially selfadjoint.

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