Refutation of Bell's Theorem

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Bell's theorem is based on a linear combination of spin correlation functions, each of these functions being characterized by a different couple of arguments. The meaning of the simultaneous presence of these different couples of arguments in the same equation can be understood in two radically different ways: either as a strongly objective meaning, that is, all correlation functions are counterfactual properties of the same set of particle pairs, or as a weakly objective meaning, that is, each correlation function is measured on a different (and contextual) set of particle pairs. It is demonstrated that once this meaning is explicated, no discrepancy can appear between local realistic theories and quantum mechanics, and that the discrepancy exhibited by Bell's theorem is due to a meaningless comparison between the local realistic inequality written within strongly objective interpretation (thus relevant to a single set of particle pairs) and the quantum mechanical prediction written within weakly objective interpretation (thus relevant to several different sets of particle pairs).

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1. INTRODUCTION

Bell's theorem[1] is exhibiting a peculiar discrepancy between any local realistic theory and quantum mechanics, the choice between alternatives to be settled by experimental means. The trouble is that neither local realistic conceptions nor quantum mechanics are easy to abandon. Indeed, classical physics and common sense are usually based upon the former, when the latter is often presented as the most successful theory of all times.

Many tests have been performed, all but few[2] showing violations of Bell inequalities. Yet, the ideas brought forth by Bell's theorem are so disconcerting that there is still incredulity, not to mention a certain reluctance, before the verdict. Some physicists, though fewer every day, are still striving to find bias or loopholes capable of explaining the apparent violation of Bell's inequalities[3], but there are very few attempts to refute the theorem itself. The purpose of this article is to provide such a refutation, within a strictly quantum theoretical framework and without the need of any additional assumptions.

Although the mere idea of a refutation might seem very unlikely, as experiments showing violation of Bell's theorem are getting increasingly accurate and loophole-free[4], it must be stressed that experimental tests, however accurate and close to the ideal scheme, cannot prove the validity of the discrepancy exhibited by Bell's theorem but only the validity of quantum mechanics. Actually, it will be assumed in this article that all tests conducted so far are proving with quite a good accuracy the validity of quantum mechanics. In other words, the purpose of this article is not to criticize the numerous experiments, nor quantum mechanics for that matter, but Bell's theorem itself.

2. THE EPRB GEDANKEN EXPERIMENT

2.1 Spin observables and singlet state

Bell's theorem is usually based on an heuristic formulation of the EPR (Einstein, Podolski and Rosen[5]) gedanken experiment, due to David Bohm[6]: in this EPRB gedanken experiment, a pair of spin-$\frac{1}{2}$ particles with total spin zero is produced at a source, each particle moving in opposite directions along the y-axis. Two Stern-Gerlach devices are placed at opposite ends (left and right) of the y-axis, and are oriented respectively along the directions $\mathbf{u}$ and $\mathbf{v}$.

The spin observable associated to a measurement with a Stern-Gerlach device oriented along the unit vector $\boldsymbol{u}$ is $\boldsymbol{\sigma}.\mathbf{u}$, the components of $\boldsymbol{\sigma}$ being the Pauli matrices σ_x, σ_y, and σ_z.

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Let $\mathcal{H}_{\rm L}$ and $\mathcal{H}_{\rm R}$ be the hilbert space respectively associated to each Stern-Gerlach devices. The hilbert space $\mathcal{H}$ associated to the entire EPRB system is the direct product of the hilbert spaces associated to each Stern-Gerlach devices:

$$\mathcal{H} \equiv \mathcal{H}_{\rm L} \otimes \mathcal{H}_{\rm R} \tag{1}$$

The spin observables of spaces $\mathcal{H}_{\rm L}$ and $\mathcal{H}_{\rm R}$ have their respective counterpart in this new product space $\mathcal{H}$ as

$$\boldsymbol{\sigma}_{\rm L}.\mathbf{u} \equiv \boldsymbol{\sigma}.\mathbf{u} \otimes 1\!\!1_{\rm R} \tag{2a}$$
$$\boldsymbol{\sigma}_{\rm R}.\mathbf{v} \equiv 1\!\!1_{\rm L} \otimes \boldsymbol{\sigma}.\mathbf{v} \tag{2b}$$

where $1\!\!1_{\rm L}$ and $1\!\!1_{\rm R}$ are respectively the identity operators of $\mathcal{H}_{\rm L}$ and $\mathcal{H}_{\rm R}$.

Contrary to observables $\boldsymbol{\sigma}.\mathbf{u}$ and $\boldsymbol{\sigma}.\mathbf{v}$ which are not commuting with each others if $\mathbf{u} \neq \mathbf{v}$, these new observables $\boldsymbol{\sigma}_{\rm L}.\mathbf{u}$ and $\boldsymbol{\sigma}_{\rm R}.\mathbf{v}$ are commuting observables, thus reflecting the fact that each Stern-Gerlach devices are arbitrary far from each others.

The product of these two observables

$$(\boldsymbol{\sigma}_{\rm L}.\mathbf{u}).(\boldsymbol{\sigma}_{\rm R}.\mathbf{v}) = \boldsymbol{\sigma}.\mathbf{u} \otimes \boldsymbol{\sigma}.\mathbf{v} \tag{3}$$

is therefore also an observable and can be understood as a *spin correlation observable* corresponding to the *joint spin measurement* of both Stern-Gerlach devices.

The product space $\mathcal{H}$ is spanned by the product basis formed by the four eigenvectors of the spin correlation observable $(\boldsymbol{\sigma}_{\rm L}.\mathbf{n})(\boldsymbol{\sigma}_{\rm R}.\mathbf{n})$ where $\mathbf{n}$ is a unitary vector. These eigenvectors are written in the compact notation: $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$.

A system constituted by two spin one half particles has a total spin of either zero or one. In an EPRB gedanken experiment, the source produces particle pairs with zero total spin represented by the singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big[|+-\rangle - |-+\rangle\Big] \tag{4}$$

This singlet state has the important property of being invariant under rotation, allowing not to mention explicitly the chosen vector $\mathbf{n}$ in the writings of the $\mathcal{H}$ basis (see for instance reference [7]).

2.2 Statistical properties of the singlet state

Yet, nothing certain can be said of a single spin measurement, nor of a single spin correlation measurement, performed on a system represented by this singlet state. Only probabilistic predictions, such as expectation values can be made over numerous measurements in the same context, according to Born interpretation of the state vector.

It can be shown (see for instance reference[8], chapter IV), that the expectation value is

$$\langle \hat{A} \rangle_{\phi} = \langle \phi | \hat{A} | \phi \rangle \tag{5}$$

and therefore with equations (2) and (4), that the *expectation value of a spin observable* for the singlet state $|\psi\rangle$ is zero:

$$\langle \boldsymbol{\sigma}_{\rm L}.\mathbf{u} \rangle_{\psi} = \langle \psi | \boldsymbol{\sigma}.\mathbf{u} \otimes 1\!\!1_{\rm R} | \psi \rangle = 0 \tag{6a}$$
$$\langle \boldsymbol{\sigma}_{\rm R}.\mathbf{v} \rangle_{\psi} = \langle \psi | 1\!\!1_{\rm L} \otimes \boldsymbol{\sigma}.\mathbf{v} | \psi \rangle = 0 \tag{6b}$$

whatever $\mathbf{u}$ and $\mathbf{v}$, according to the rotational invariance of the singlet state, and likewise that the *expectation value of spin correlation observable* is:

$$E^{\psi}(\mathbf{u},\mathbf{v}) = \langle \psi | (\boldsymbol{\sigma}_{\rm L}.\mathbf{u})(\boldsymbol{\sigma}_{\rm R}.\mathbf{v}) | \psi \rangle \tag{7a}$$
$$= -\mathbf{u}.\mathbf{v} \tag{7b}$$

(see for instance references [2], [7], or [9]).

2.3 Perfect correlation and hidden-variables

When $\mathbf{u} = \mathbf{v}$, the expectation value of spin correlation observable (7) is equal to -1, meaning that if both Stern-Gerlach devices are oriented along the same direction, then with certainty the outcomes will be found to be opposite.

Since the Stern-Gerlach devices are arbitrary far from each other, thus locals, this perfect correlation can be understood from a realistic point of view as relevant to the idea that somehow the measurement result is *predetermined*.

Yet, this predetermination seems contrary to the stochastic description given by quantum mechanics, which is against all better judgement the most complete description of a physical system. Hence, if measurements on this system are predetermined, then it should be possible to give a more complete description of this system by means of additional hidden-variables.

Bell's idea is thence to write down mathematical requirements of such a local hidden-variables theory, and to confront it both to quantum mechanics and experiments. A single particle pair is thus supposed to be entirely characterized by means of a set of hidden-variables, which are represented altogether by a parameter λ, so that the measurement result A on the left along $\mathbf{u}$ is $A(\lambda, \mathbf{u})$, and the result B on the right along $\mathbf{v}$ is $B(\lambda, \mathbf{v})$. Hence, the perfect correlation condition is written $B(\lambda, \mathbf{u}) = -A(\lambda, \mathbf{u})$ whatever the particle pair and its corresponding parameter .

Although the hidden-variable theory is supposed to be deterministic, it must be capable of reproducing the stochastic nature of the EPRB gedanken experiment expressed in equations (6) and (7). For that purpose, the complete state λ_i of any particle pair i is a random variable: it is supposed to be drawn randomly according to a probability distribution ρ (see references [1] and [10]): the probability of having λ_i equal to a particular λ is equal to $\rho(\lambda)$.

Consider a set of N particle pairs $\{i = 1, \ldots, N\}$, the *mean value of joint measurements* over this set of N particle pairs is

$$E^{\rho}(\mathbf{u}, \mathbf{v}) = \frac{1}{N} \sum_{i=1}^{N} -A(\mathbf{u}, \lambda_i).A(\mathbf{v}, \lambda_i) \tag{8}$$

The probability distribution ρ is supposed to grant the equality between this equation and its quantum mechanical counterpart (7a) when N is arbitrary large.

3. THE CHSH FUNCTION

Thence, Bell's idea is to study linear combinations of correlation functions given by quantum mechanics (equation (7)) and by local hidden-variables theories (equation (8)) with *different arguments*[11], and to compare the results.

A well known choice of such a linear combination is the CHSH [12] combination, written with four different arguments $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$ and $\mathbf{b}'$:

$$S \equiv |E(\mathbf{a}, \mathbf{b}) - E(\mathbf{a}, \mathbf{b}') + E(\mathbf{a}', \mathbf{b}) + E(\mathbf{a}', \mathbf{b}')| \tag{9}$$

The meaning of the simultaneous presence of these *different arguments* in this CHSH function must be explicated.

Basically, there are two possible interpretation, the *strongly objective* interpretation and the *weakly objective* interpretation [13, 14]:

strongly objective interpretation These four correlation functions have a counterfactual meaning. They are not relevant to real experiments but rather with what result *would have been* obtained if measured on the same N particle pairs along different directions.

weakly objective interpretation These four correlation functions have a contextual meaning. Each correlation function is actually to be measured on a distinct set of N particle pairs, that is, 4 correlation functions with *alternative settings of the instruments*[10] measured on 4 respective sets of N particle pairs.

Actually, the CHSH function was designed for testing purpose[12], and many experiments where thus conducted (the most famous being undoubtedly Aspects's test[15]), so that the most likely interpretation of the linear combination of expectation values is the weakly objective one. Nevertheless, the strongly objective interpretation must be surveyed as well, since it remains a possible interpretation and since this choice between strong and weak objectivity is not at all explicit in many papers, including John Bell's.

It must be stressed that these interpretations are radically different, not only epistemologically, but also physically. Indeed, the strongly objective interpretation requires a single set of N particle pairs characterized by the corresponding set of parameters $\{\lambda_i\}$, whereas the weakly objective interpretation requires no less than 4 sets of N particle pairs (in this article, it is assumed that the experimenter can control the number N of particle pairs constituting a distinct set).

The trouble is that a set of N particle pairs characterized by $\{\lambda_i\}$ cannot be reproduced to identical, neither theoretically (for each complete state λ_i of any particle pair i is a random variable, as defined in section 2.3), nor empirically (for the experimenter has no control over the complete state of a particle pair).

Of course, if N is arbitrary large, these four sets of N particle pairs have necessarily the same statistical properties (described by the probability distribution ρ), but they are nonetheless *four different sets of particle pairs* (see reference [9], page 348) respectively characterized by four different sets of hidden-variables parameters $\{\lambda_{1,i}\}$, $\{\lambda_{2,i}\}$, $\{\lambda_{3,i}\}$ and $\{\lambda_{4,i}\}$.

The difference between each interpretation can therefore be embodied in the degrees of freedom of the whole system. Let f be the degrees of freedom of a single particle pair. In the strongly objective interpretation the degrees of freedom of the whole system is Nf, whereas in the weakly objective interpretation the degrees of freedom is 4 times as large, that is equal to $4Nf$, which is something completely different.

That is why before establishing Bell's theorem, one has to choose explicitly one interpretation and stick to it. Unfortunately, this is not what has been done. It will be shown here that the discrepancy exhibited by Bell's theorem is due to a meaningless comparison between strongly objective and weakly objective results, which means comparing the numerical value of the CHSH function for two systems, one with Nf degrees of freedom, the other with $4Nf$ degrees of freedom, hence the discrepancy.

4. STRONGLY OBJECTIVE INTERPRETATION: COUNTERFACTUAL PROPERTIES OF N PARTICLE PAIRS

4.1 Local realistic inequality within strongly objective interpretation

Counterfactuality being a corollary to realism [14], the local realistic formulation of the CHSH function within strongly objectivity is perfectly meaningful:

$$S^{\rho}_{\text{strong}} = \Big| E^{\rho}(\mathbf{a},\mathbf{b}) - E^{\rho}(\mathbf{a},\mathbf{b}') + E^{\rho}(\mathbf{a}',\mathbf{b}) + E^{\rho}(\mathbf{a}',\mathbf{b}') \Big| \tag{10}$$

which is explicitly (using equation (8))

$$\begin{aligned} S^{\rho}_{\text{strong}} = \Bigg| \frac{1}{N}\sum_{i=1}^{N} & A(\mathbf{a},\lambda_i)A(\mathbf{b},\lambda_i) - A(\mathbf{a},\lambda_i)A(\mathbf{b}',\lambda_{\mathbf{i}}) \\ & + A(\mathbf{a}',\lambda_i)A(\mathbf{b},\lambda_i) + A(\mathbf{a}',\lambda_i)A(\mathbf{b}',\lambda_i)\Bigg| \end{aligned} \tag{11}$$

and after factorisation

$$\begin{aligned} S^{\rho}_{\text{strong}} = \Bigg| \frac{1}{N}\sum_{i=1}^{N} & A(\mathbf{a},\lambda_i)\Big[A(\mathbf{b},\lambda_i) - A(\mathbf{b}',\lambda_{\mathbf{i}})\Big] \\ & - A(\mathbf{a}',\lambda_i)\Big[A(\mathbf{b},\lambda_i) + A(\mathbf{b}',\lambda_i)\Big]\Bigg| \end{aligned} \tag{12}$$

with two possible values for each term of the summation[2, 9]:

$$\begin{aligned} & A(\mathbf{a},\lambda_i)\Big[A(\mathbf{b},\lambda_i) - A(\mathbf{b}',\lambda_{\mathbf{i}})\Big] \\ & - A(\mathbf{a}',\lambda_i)\Big[A(\mathbf{b},\lambda_i) + A(\mathbf{b}',\lambda_i)\Big] = \pm 2 \end{aligned} \tag{13}$$

so that the narrowest local realistic local inequality within strongly objective interpretation is :

$$S^{\rho}_{\text{strong}} \leq 2 \tag{14}$$

which is the generalized formulation of Bell inequality due to CHSH [12]. It must be stressed however once more that this inequality has been established within strongly objective interpretation, so that each expectation value is relevant to the same set of N particle pairs. Hence, this result cannot be applied directly to real experimental tests, where mean values are measured upon four distinct sets of N particle pairs. The question whether the same inequality can be derived from the idea that the different arguments have only a weakly objective meaning will be discussed in section 5.1.

4.2 Quantum mechanical prediction within strongly objective interpretation

The quantum prediction for the CHSH function within strongly objective interpretation is written

$$S^{\psi}_{\mathrm{strong}} = |E^{\psi}(\mathbf{a},\mathbf{b}) - E^{\psi}(\mathbf{a},\mathbf{b}') + E^{\psi}(\mathbf{a}',\mathbf{b}) + E^{\psi}(\mathbf{a}',\mathbf{b}')| \tag{15}$$

This equation is usually directly evaluated by replacing each expectation value by the result of equation (7b). This is unfortunately all too hasty.

Indeed, in order to understand the quantum mechanical meaning of equation (15), it is better to take a step backward using equation (7a):

$$\begin{aligned} S^{\psi}_{\mathrm{strong}} = \Big| &\langle\psi|(\boldsymbol{\sigma}_{\mathrm{L}}.\mathbf{a})(\boldsymbol{\sigma}_{\mathrm{R}}.\mathbf{b})|\psi\rangle - \langle\psi|(\boldsymbol{\sigma}_{\mathrm{L}}.\mathbf{a})(\boldsymbol{\sigma}_{\mathrm{R}}.\mathbf{b}')|\psi\rangle \\ &+ \langle\psi|(\boldsymbol{\sigma}_{\mathrm{L}}.\mathbf{a}')(\boldsymbol{\sigma}_{\mathrm{R}}.\mathbf{b})|\psi\rangle + \langle\psi|(\boldsymbol{\sigma}_{\mathrm{L}}.\mathbf{a}')(\boldsymbol{\sigma}_{\mathrm{R}}.\mathbf{b}')|\psi\rangle \Big| \end{aligned} \tag{16}$$

Now, it can be shown (by calculating the commutator of $(\boldsymbol{\sigma}_{\mathrm{L}}.\mathbf{u})(\boldsymbol{\sigma}_{\mathrm{R}}.\mathbf{v})$ and $(\boldsymbol{\sigma}_{\mathrm{L}}.\mathbf{u})(\boldsymbol{\sigma}_{\mathrm{R}}.\mathbf{v}')$ with $\mathbf{v} \neq \mathbf{v}'$) that the four spin correlation observables in this equation are *non commuting observables.*

According to Von Neumann [16], the linear combination of expectation values of different observables $\hat{R}$, $\hat{S}, \ldots$ is always meaningful in quantum mechanics:

$$\langle \hat{R} + \hat{S} + \ldots \rangle_{\phi} = \langle \hat{R} \rangle_{\phi} + \langle \hat{S} \rangle_{\phi} + \ldots \tag{17}$$

even if $\hat{R}$, $\hat{S}, \ldots$ are non commuting observables,

Yet, it is important restate that quantum mechanics is a weakly objective theory, and that expectation values given by quantum mechanics are as well weakly objective statements [13, 17], that is to say statements relevant to observations.

Hence, when $\hat{R}$, $\hat{S}, \ldots$ are non commuting observables, the expectation values cannot be simultaneously relevant to the same set of N systems. The only possible meaning of equation (17) for non commuting observables is that each expectation value is relevant to a distinct set of N systems (all systems being represented by the quantum state $|\phi\rangle$). Likewise, the meaning of the linear combination of expectation values of equation (16) is therefore but contextual, that is to say weakly objective.

A remark might be necessary here, since these expectation values are known with certainty, one could be tempted to claim the right to consider them as counterfactuals. This would be a mistake, for conterfactuality requires measurement compatibility, that is commuting observables. The certainty of a contextual prediction is not sufficient to make it a counterfactual prediction, or in other words, *weakly objective results known with certainty are not strongly objective results* (incidentally, this is also true in case of perfect correlation).

Therefore, $S^{\psi}_{\mathrm{strong}}$ is meaningless and deceitful, for its only possible meaning is weakly objective when it was intended to be strongly objective. Hence, $S^{\rho}_{\mathrm{strong}}$ cannot be compared with any strongly objective prediction given by quantum mechanics. Bell's theorem cannot be established within strongly objective interpretation of the CHSH function.

This denial is the first part of the refutation of Bell's theorem, though maybe not the most conclusive (at least for the CHSH formulation of the theorem), since the strength of this theorem is mainly its ability to be confronted to experimental tests with apparent success. Yet, this step had to be overcome, for now that strongly objective interpretation is right out, there is no choice but to rely on the weakly objective interpretation in order to compare hidden-variables theories and quantum mechanics.

5. WEAKLY OBJECTIVE INTERPRETATION : CONTEXTUAL MEASUREMENTS ON 4 DISTINCT SETS OF N PARTICLE PAIRS

5.1 Quantum mechanical prediction within weakly objective interpretation

In last section, it was shown that quantum mechanical formalism could be deceitful regarding the meaning of linear combination of expectation values of non commuting observables. In this section, a simple method will be provided in order to avoid such misleading writings.

It was stressed in section 3 that strong objectivity and weak objectivity are bound to physically different systems. This difference should therefore appear in our equations. Indeed, the correlation expressed in equation (7b) is relevant to spin measurements performed on particles that were once constituting a single particle pair. Yet, two particles issued from two distinct particle pairs never interact with each others, so that spin measurements performed on these particles should not be correlated. Hence, if left and right spin measurements are performed on two distinct sets of N particle pairs, instead of the same set, there should not be any correlation either, and this property should appear in a generalized spin correlation function (i.e. generalized to the case of spin measurements performed on different sets of particle pairs).

This can be easily done by means of a distinct EPRB space for each set of N particle pairs. Let $\mathcal{H}_j$ be the EPRB hilbert space associated with the jth set of particle pairs. In this hilbert space, the EPRB gedanken experiment is represented by the singlet state $|\psi_j\rangle$ (see section 2),

$$|\psi_j\rangle = \frac{1}{\sqrt{2}}\Big[|+-\rangle_j - |-+\rangle_j\Big] \tag{18}$$

The spin observables and spin correlation observables are respectively similar to equations (2) and (3).

The whole experiment with the four sets of particle pairs can be expressed in a new direct product space $\mathcal{H}_{1234}$

$$\mathcal{H}_{1234} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \tag{19}$$

by the state vector

$$|\psi_{1234}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle \otimes |\psi_4\rangle \tag{20}$$

The counterparts of observables in $\mathcal{H}_{1234}$ are obtained akin to what was done in section 2.1. For instance, the observables pertaining to the right Stern-Gerlach device for the 1st, 2nd, 3rd and 4th set of particle pairs are respectively

$$\boldsymbol{\sigma}_{1,\mathrm{R}}.\mathbf{u} \equiv (\boldsymbol{\sigma}_{\mathrm{R}}.\mathbf{u}) \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 \otimes \mathbb{1}_4 \tag{21a}$$
$$\boldsymbol{\sigma}_{2,\mathrm{R}}.\mathbf{u} \equiv \mathbb{1}_1 \otimes (\boldsymbol{\sigma}_{\mathrm{R}}.\mathbf{u}) \otimes \mathbb{1}_3 \otimes \mathbb{1}_4 \tag{21b}$$
$$\boldsymbol{\sigma}_{3,\mathrm{R}}.\mathbf{u} \equiv \mathbb{1}_1 \otimes \mathbb{1}_2 \otimes (\boldsymbol{\sigma}_{\mathrm{R}}.\mathbf{u}) \otimes \mathbb{1}_4 \tag{21c}$$
$$\boldsymbol{\sigma}_{4,\mathrm{R}}.\mathbf{u} \equiv \mathbb{1}_1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 \otimes (\boldsymbol{\sigma}_{\mathrm{R}}.\mathbf{u}) \tag{21d}$$

where $\mathbf{1}_j$ is the identity operator of the EPRB space $\mathcal{H}_j$, the counterparts for the left Stern-Gerlach device being obtained by simply replacing the letter R by the letter L in these equations.

Hence, the expectation value of the product of two spin observables, the first belonging to the kth set and the second to the lth set, is

$$E^{\psi}_{kl}(\mathbf{u},\mathbf{v}) \equiv \langle\psi_{1234}|(\boldsymbol{\sigma}_{k,L}.\mathbf{u})(\boldsymbol{\sigma}_{l,R}.\mathbf{v})|\psi_{1234}\rangle \tag{22}$$

and this is the *generalized expectation value of spin correlation observables* that was looked for.

The expectation value for measurements performed on the same set ($k = l$) of particle pairs is already known (see equations (7)), and $E^{\psi}_{kk}(\mathbf{u},\mathbf{v})$ should provide the same result. Indeed, using equations (20) and (21) leads to

$$\begin{aligned} E^{\psi}_{kk}(\mathbf{u},\mathbf{v}) &= \langle\psi_k|(\boldsymbol{\sigma}_L.\mathbf{u}).(\boldsymbol{\sigma}_R.\mathbf{v})|\psi_k\rangle \\ &= -\mathbf{u}.\mathbf{v} \end{aligned} \tag{23}$$

but, if $k \neq l$ the result is quite different:

$$\begin{aligned} \underset{k\neq l}{E^{\psi}_{kl}}(\mathbf{u},\mathbf{v}) &= \langle\psi_k|(\boldsymbol{\sigma}_L.\mathbf{u})|\psi_k\rangle.\langle\psi_l|(\boldsymbol{\sigma}_R.\mathbf{v})|\psi_l\rangle \\ &= \langle\psi_k|\boldsymbol{\sigma}.\mathbf{u}\otimes\mathbb{1}_{\mathrm{R}}|\psi_k\rangle.\langle\psi_l|\mathbb{1}_{\mathrm{L}}\otimes\boldsymbol{\sigma}.\mathbf{v}|\psi_l\rangle \\ &= 0 \end{aligned} \tag{24}$$

this according to equation (6). There are indeed no correlations between two sets of particle pairs, as requested in the beginning of this section.

Now, contrary to what was done in section 4.2, it is possible to proceed much more according to quantum mechanical postulates, for the spin correlation observables of equations (21) are commuting with each others, so that it is possible to define a new observable describing the whole experiment:

$$\begin{aligned} \hat{S}_{\mathrm{weak}} \equiv\ & (\boldsymbol{\sigma}_{1,L}.\mathbf{a}).(\boldsymbol{\sigma}_{1,R}.\mathbf{b}) - (\boldsymbol{\sigma}_{2,L}.\mathbf{a}).(\boldsymbol{\sigma}_{2,R}.\mathbf{b}') \\ & +(\boldsymbol{\sigma}_{3,L}.\mathbf{a}').(\boldsymbol{\sigma}_{3,R}.\mathbf{b}) + (\boldsymbol{\sigma}_{4,L}.\mathbf{a}').(\boldsymbol{\sigma}_{4,R}.\mathbf{b}') \end{aligned} \tag{25}$$

and the quantum prediction for the CHSH function within weakly objective interpretation is therefore

$$S^{\psi}_{\text{weak}} = \left| \langle \psi_{1234} | \hat{S}_{\text{weak}} | \psi_{1234} \rangle \right| \tag{26}$$

which using equations (20) and(25) is

$$\begin{aligned} S^{\psi}_{\text{weak}} = \Big| & \langle \psi_1 | (\boldsymbol{\sigma}_L.\mathbf{a})(\boldsymbol{\sigma}_R.\mathbf{b}) | \psi_1 \rangle - \langle \psi_2 | (\boldsymbol{\sigma}_L.\mathbf{a}')(\boldsymbol{\sigma}_R.\mathbf{b}) | \psi_2 \rangle \\ & + \langle \psi_3 | (\boldsymbol{\sigma}_L.\mathbf{a})(\boldsymbol{\sigma}_R.\mathbf{b}') | \psi_3 \rangle + \langle \psi_4 | (\boldsymbol{\sigma}_L.\mathbf{a}')(\boldsymbol{\sigma}_R.\mathbf{b}') | \psi_4 \rangle \Big| \end{aligned} \tag{27}$$

that is, using equation (23),

$$S^{\psi}_{\text{weak}} = \left| E^{\psi}_{11}(\mathbf{a},\mathbf{b}) - E^{\psi}_{22}(\mathbf{a},\mathbf{b}') + E^{\psi}_{33}(\mathbf{a}',\mathbf{b}) + E^{\psi}_{44}(\mathbf{a}',\mathbf{b}') \right| \tag{28}$$

This equation is not ambiguous (as was equation (16)): it is a linear combination of expectation values, each of these being relevant to a distinct and contextual set of N particle pairs. This equation is therefore weakly objective, as requested.

Finally, using equation (23),

$$S^{\psi}_{\text{weak}} = \left| \mathbf{a}.\mathbf{b} - \mathbf{a}.\mathbf{b}' + \mathbf{a}'.\mathbf{b} + \mathbf{a}'.\mathbf{b}' \right| \tag{29}$$

with a maximum equal to

$$\max(S^{\psi}_{\text{weak}}) = 2\sqrt{2} \tag{30}$$

Hence, not surprisingly, quantum mechanics, which is a weakly objective theory [13], provides a clear answer to the CHSH function understood as a weakly objective question.

5.2 Local realistic inequality within weakly objective interpretation

The last step consists in comparing S^{ψ}_{weak} with S^{ρ}_{weak}. It was stressed in section 3 that the jth set of particle pairs must be characterized by a distinct set of hidden-variables parameters $\{\lambda_{j,i}\, ;\, i = 1, \ldots, N\}$. Hence, to the generalized expectation value of spin correlation observable (equation (22)) is responding the *generalized mean value of joint spin measurements*:

$$E^{\rho}_{kl}(\mathbf{u},\mathbf{v}) \equiv \frac{1}{N} \sum_{i=1}^{N} -A(\mathbf{u},\lambda_{k,i}).A(\mathbf{v},\lambda_{l,i}) \tag{31}$$

so that the local realistic CHSH function within weakly objective interpretation is

$$S^{\rho}_{\text{weak}} = \left| E^{\rho}_{11}(\mathbf{a},\mathbf{b}) - E^{\rho}_{22}(\mathbf{a},\mathbf{b}') + E^{\rho}_{33}(\mathbf{a}',\mathbf{b}) + E^{\rho}_{44}(\mathbf{a}',\mathbf{b}') \right| \tag{32}$$

and that is explicitly

$$\begin{aligned} S^{\rho}_{\text{weak}} = \Big| \frac{1}{N} \sum_{i=1}^{N} \big[& A(\mathbf{a},\lambda_{1,i}).A(\mathbf{b},\lambda_{1,i}) - A(\mathbf{a},\lambda_{2,i}).A(\mathbf{b}',\lambda_{2,i}) \\ & + A(\mathbf{a}',\lambda_{3,i}).A(\mathbf{b},\lambda_{3,i}) + A(\mathbf{a}',\lambda_{4,i}).A(\mathbf{b}',\lambda_{4,i}) \big] \Big| \end{aligned} \tag{33}$$

This expression should be compare with the one pertaining to the strongly objective interpretation (equation (8)) which contained terms that could be factorized. Here, since each terms are different from the others, no factorisation is possible. *There is no way to derive Bell's inequality.* This, of course, is of extreme importance. Surprisingly, this fact is not pointed out for the first time, as Arno BOHM already has (page 351,352 of his book[9]). Unfortunately, he remarked this in a matter-of-fact way.

Yet, this impossibility cannot be ignored, for it has been shown (in section 4.2) that Bell's theorem could not be established within strongly objective interpretation.

The only local realistic inequality that can be drawn is obtained by considering (as was done with equation (13)) the possible numerical values of each term of the summation in equation (33), that is

$$\begin{aligned} A(\mathbf{a},\lambda_{1,i}).A(\mathbf{b},\lambda_{1,i}) - A(\mathbf{a},\lambda_{2,i}).A(\mathbf{b}',\lambda_{2,i}) + A(\mathbf{a}',\lambda_{3,i}).A(\mathbf{b},\lambda_{3,i}) \\ +A(\mathbf{a}',\lambda_{4,i}).A(\mathbf{b}',\lambda_{4,i}) = +4, +2, 0, -2, -4 \end{aligned} \tag{34}$$

which extrema are +4 and -4, so that the narrowest local realistic inequality that can be derived from equation (33) is nothing but

$$S^{\rho}_{\text{weak}} \leq 4 \tag{35}$$

This narrowest local realistic inequality cannot be violated by quantum mechanics, as the maximum of S^{ψ}_{weak} is $2\sqrt{2}$. Therefore, Bell's theorem cannot be established within weakly objective interpretation of the CHSH function.

It must be stressed that the local realistic inequality (35) for S^{ρ}_{weak} would have been impossible to reckon with the integral formulation of the expectation value

$$E(\mathbf{a},\mathbf{b}) \equiv \int \rho(\lambda)A(\mathbf{a},\lambda)B(\mathbf{b},\lambda)\mathrm{d}\lambda \tag{36}$$

instead of the discrete formulation used in this article. With this integral writing (which can be found in most demonstrations of Bell's theorem), no differences can be made between expectation values of two distinct sets of N particles, for λ is but a mute variable. A linear combination of such integral expectation value can therefore only be strongly objective, not weakly objective.

6. CONCLUSION : QUANTUM MECHANICS SHOULD BE APPROACHED WITH PRUDENCE

The apparent discrepancy exhibited by John Bell is therefore only due to a meaningless comparison between S^{ρ}_{strong} and S^{ψ}_{weak}, the former being relevant to a system with Nf degrees of freedom, the latter to a system with $4Nf$ degrees of freedom (see section 3).

Bell's theorem could not be established, neither within strongly objective interpretation of the CHSH function, for quantum mechanics could not provide a strongly objective result about non commuting observables (see section 4.2), nor within a weakly interpretation, for the only local realistic inequality that could be written was wide enough to avoid any violation by quantum mechanics (see section 5.2). Bell's theorem is therefore refuted.

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