Resonances of the Dirac Hamiltonian in the non relativistic limit

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Abstract

For a Dirac operator in $\mathbb{R}^3$ with an electric potential behaving at infinity like a power of $|x|$, we prove the existence of resonances and we study, when $c \to +\infty$, the asymptotic expansion of their real part, and an estimation of their imaginary part, generalizing an old result of Titchmarsh.

1 Introduction

We are interested in the following Dirac operator $D(c)$ in $\mathbb{R}^3$, depending on a parameter $c > 1$

$$D(c) = \begin{pmatrix} V(x) & c\sigma \cdot D_x \\ c\sigma \cdot D_x & V(x) - 2c^2 \end{pmatrix}, \tag{1}$$

Here $\sigma \cdot D_x$ denotes $\sigma_1 D_1 + \sigma_2 D_2 + \sigma_3 D_3$, where the σ_j are the Pauli matrices, and V is a C^∞ real-valued function, satisfying the following hypotheses.

(H1) We assume that V can be extended in an holomorphic function in the following open set of $\mathbb{C}^3$, for some positive constants a and r

$$\Omega = S_a \cup B(0, r) \tag{2}$$

where S_a is the complex sector $\{z \in \mathbb{C}^3,\ |\mathrm{Arg} z_j| < a,\ \forall j = 1,2,3\}$, and $B(0,r)$ be the open complex ball with center 0 and radius r. We assume also that for some positive constants k, m_0 and R, we have

$$|V(z)| \leq m_0(1 + |z|^k) \qquad \forall z \in S_a \tag{3}$$

(H2) We have also, if $x \in \mathbb{R}^3$ and $|x| \geq R$

$$|x|^k \leq m_0 V(x) \tag{4}$$

(H3)We have also, if $x \in \mathbb{R}^3$ and $|x| \geq R$

$$|x|^k \leq m_0 x.\frac{\partial V}{\partial x} \tag{5}$$

We see easily that $D(c)$ is essentially self-adjoint, and Titchmarsh proved, when V is radial, that $D(c)$ has the whole real line as a purely absolutely continuous spectrum (see Thaller [14]). Let H_∞ be the corresponding Schrödinger operator

$$H_\infty = -\frac{1}{2}\Delta + V(x) \tag{6}$$

The spectrum of H_∞ is discrete. We shall prove that, when c is large enough, $D(c)$ has resonances near the eigenvalues of H_∞ and we shall study their asymptotic behaviour when $c \to +\infty$. Recall that, in the semiclassical limit, the asymptotic behaviour of the resonances is studied in Parisse [9] (see also Balslev-Helffer [2]). For the Dirac operator in one dimension, with potential $V(x) = |x|$, Titchmarsh [15] gave an explicit computation of the resonances (see also Veselic [16] and Thaller [14]).

For the definition of resonances, we need the analytic dilations (see Aguilar-Combes [1]). For each $\theta \in \mathbb{C}$ such that $|\Im\theta| < a$, we denote by $H(\theta, c)$ the following Hamiltonian

$$H(\theta, c) = \begin{pmatrix} V(e^\theta x) & e^{-\theta} c\sigma \cdot D_x \\ e^{-\theta} c\sigma \cdot D_x & V(e^\theta x) - 2c^2 \end{pmatrix}, \tag{7}$$

with domain

$$B^1(\mathbb{R}^3, \mathbb{C}^4) = \{u \in H^1(\mathbb{R}^3, \mathbb{C}^4),\ |x|^k u \in L^2(\mathbb{R}^3, \mathbb{C}^4)\} \tag{8}$$

We shall prove in section 2 the following theorem.

Theorem 1 *$H(c,\theta)$ has pure point spectrum for small positive $\Im\theta$. Each eigenvalue $\lambda_j(\theta, c)$ is isolated and of finite even multiplicity, and does not depend on θ.*

The eigenvalues of $H(\theta, c)$, denoted by $E_j(c)$ since they do not depend on θ, will be called resonances. We shall prove in section 3 the following theorem.

Theorem 2 *If $\Im\theta$ is small enough, we have the following properties.*

(i) Let K be a compact set of $\mathbb{C}$ containing no eigenvalue of H_∞. Then, if c is large enough, K contains no resonance.

(ii) Let D be a compact disc centered at an eigenvalue E_0 of H_∞, of multiplicity μ, and containing no other eigenvalue. Then, if c is large enough, D contains a finite number of resonances, and the sum of their multiplicities is 2μ.

Theorem 3 *If $\Im\theta$ is small enough, we have the following property. If D is a disc as in theorem 2, if E_0 is a simple eigenvalue of H_∞, then D contains, for c large enough, one resonance $\lambda(c)$ of multiplicity 2, and there exists a C^∞ function f in a neighborhood of 0 such that $f(0) = E_0$ and, for c large enough*

$$\lambda(c) = f(\frac{1}{c^2}) \tag{9}$$

This theorem is proved in Section 4. Recall that, when $V(x) = \mathcal{O}(< x >^{-s})$ $(s > 0)$, if E_0 is an isolated simple eigenvalue of H_∞, Grigore-Nenciu-Purice [3] proved that for c large enough, $D(c)$ has a double eigenvalue $\lambda(c)$ defined by an equality like (9), but where f is analytic. If V is a polynomial, we may think that the function f in (9) belongs perhaps in some Gevrey class related to the degree of V.

Now, we can study the imaginary part of the resonances. We consider the following Agmon metric ds_c^2 in $\mathbb{R}^3$, depending on c (see Wang [17])

$$ds_c^2 \ = \frac{1}{c^2} V(x)_+ \,(2c^2 - V(x))_+ \, dx^2 \tag{10}$$

where $x_+ = sup(x,0)$. For each $\varepsilon > 0$, we condider the "sea"

$$M(c,\varepsilon) = \{x \in \mathbb{R}^3,\ V(x) \geq (2-\varepsilon)c^2\} \tag{11}$$

We denote by $S(c,\varepsilon)$ the distance, for the metric ds_c^2, of the origin to $M(c,\varepsilon)$.

Theorem 4 *Under the hypothesis of Theorem 2 (point ii), for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that the resonances $E_j(c)$ contained in D satisfy*

$$|\Im E_j(c)| \leq C_\varepsilon e^{-(2-\varepsilon)S(c,\varepsilon)} \tag{12}$$

We are very grateful to X.P. Wang for useful discussions about the exterior scaling, used in section 5.

2 Proof of theorem 1.

We remark first that $D(c)$ is essentially self-adjoint, since we have easily the following implication :

$$u \in L^2(\mathbb{R}^3, \mathbb{C}^4), \qquad \Im\, z < 0, \qquad (D(c) - z)u = 0 \Rightarrow u = 0 \tag{13}$$

Now c is fixed. It can be seen using Cauchy's estimate that (H1) implies

$$|\partial_z^\alpha V(z)| \leq C_\alpha (1+|z|)^{k-|\alpha|} \qquad \forall z \in S_{\frac{a}{2}} \tag{14}$$

¿From the calculus adapted to the harmonic oscillator straightforward modifications are easily made to obtain a calculus for global elliptic pseudo-differential operators adapted to first order systems with a potential behaving like $|x|^k$. Therefore we briefly give the main aspects. See Shubin[12] for more considerations.

For each $m \in \mathbb{R}$ let Γ^m be the space of $d \in C^\infty(\mathbb{R}^6, M_4(\mathbb{C}))$ such that for all α and $\beta \in \mathbb{N}^3$, there exists $C_{\alpha\beta}$ such that, for all $(x,\xi) \in \mathbb{R}^6$,

$$|\partial_x^\alpha \partial_\xi^\beta d(x,\xi)| \leq C_{\alpha\beta}(1+|x|^k+|\xi|)^{m-\frac{|\alpha|}{k}-|\beta|}.$$

For each $d \in \Gamma^m$, let $Op(d)$ be the corresponding operator, associated to d by the standard calculus

$$(Op(d)\varphi)(x) = (2\pi)^{-3} \int e^{i\langle x-y,\xi\rangle} d(y,\xi)\varphi(y)\, dy d\xi \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^4)$$

The operator $Op(d)$ ($d \in \Gamma^m$) is said globally elliptic if, for some positive real number C

$$(|x|^k + |\xi|)^m \leq C(1 + |\mathrm{Det} d(x,\xi)|)^{1/4}$$

for all $(x,\xi) \in \mathbb{R}^6$.

The notation $\langle\cdot,\cdot\rangle$ stands for the inner scalar product of $L^2(\mathbb{R}^3;\mathbb{C}^4)$ and $\|\cdot\|$ denotes the corresponding norm. For $j \in \mathbb{N}$ let

$$B^j(\mathbb{R}^3;\mathbb{C}^4) = \left\{ \phi \in L^2(\mathbb{R}^3;\mathbb{C}^4), \quad x^\alpha D_x^\beta \phi \in L^2(\mathbb{R}^3;\mathbb{C}^4), \text{ for } \frac{|\alpha|}{k} + |\beta| \leq j \right\}$$

In particular, for $d \in \Gamma^m$, $Op(d)$ maps $B^{s-m}(\mathbb{R}^3;\mathbb{C}^4)$ into $B^s(\mathbb{R}^3;\mathbb{C}^4)$ for any $s \in \mathbb{N}$.

It is seen in lemma 3 that for small *positive* $\Im\theta$ the family $H(c,\theta)$ is Kato analytic. The resonances are defined as the eigenvalues of $H(c,\theta)$ for small positive $\Im\theta$.

Lemma 1 *There exists $\tau_0 > 0$ such that if $0 < \Im\theta < \tau_0$ then $H(c,\theta)$ is globally elliptic.*

Proof: The symbol of $H(c,\theta)$ satisfies

$$\mathrm{Det}\ h(x,\xi,c,\theta) = \left(V_\theta(x)\left(V_\theta(x) - 2c^2\right) - c^2 e^{-2\theta}|\xi|^2\right)^2. \tag{15}$$

where $V_\theta(x) = V(e^\theta x)$. We write $\theta = \sigma + i\tau$, $\sigma, \tau \in \mathbb{R}$ and K, C, τ_0 denotes three positives real numbers independant of x and τ. The real numbers K, C (resp. τ_0) may increase (resp. decreases).

Following the analyticity of V, there exists $\tau_0 > 0$ such that for $0 < \Im\theta < \tau_0$, for all $x \in \mathbb{R}^3$,

$$V_\theta(x) = V(xe^\sigma) + i\tau e^\sigma \sum_{j=1}^{3} x_j \frac{\partial V}{\partial x_j}(xe^\sigma) + \tau^2 M(x,\theta).$$

There exists $K, C, \tau_0 > 0$ such that

$$\forall\, \theta \in \mathbb{C} \text{ with } 0 < \Im\theta < \tau_0,\ \forall\, |x| \geq C,\ |M(x,\theta)| \leq K|x|^k. \tag{16}$$

Then, for some $K, C, \tau_0 > 0$, if $0 < \Im\theta < \tau_0$, if $|x| \geq C$ then

$$K^{-1}\tau \leq \mathrm{Arg}V_\theta(x),\ \mathrm{Arg}(V_\theta(x) - 2c^2) \leq K\tau,$$

$$|V_\theta(x)|,\ |V_\theta(x) - 2c^2| \geq K^{-1}|x|^k. \tag{17}$$

¿From (17), there exist $K, C, \tau_0 > 0$ such that for all θ and x such that $0 < \Im\theta < \tau_0$ and $|x| \geq C$,

$$K^{-1}\tau \leq \mathrm{Arg}(V_\theta(x)(V_\theta(x) - 2c^2)) \leq K\tau. \tag{18}$$

Then (18) shows that for some $K, C, \tau_0 > 0$ ($\tau_0 < \pi/2$), if $0 < \Im\theta < \tau_0$, if $|x| \geq C$ then $|V_\theta(x)\,(V_\theta(x) - 2c^2) - c^2 e^{-2\theta}|\xi|^2|$

$$\geq \sin(K^{-1}\tau)|V_\theta(x)(V_\theta(x) - 2c^2)| + c^2\sin(2\tau)|\xi|^2. \tag{19}$$

The proof of Lemma 1 follows from (15),(17),(19). $\square$

Theorem 1 will follow from the two lemma below.

Lemma 2 *There exists $\tau_0 > 0$ such that if $0 < \Im\theta < \tau_0$ then the resolvant set of $H(c,\theta)$ is not empty.*

Proof: For $m \in \mathbb{N}$, let $\widetilde{\Gamma}^m$ be the space of $a(,\cdot,\cdot,\rho) \in C^\infty(\mathbb{R}^6, M_4(\mathbb{C}))$, depending on a parameter $\rho \geq 1$, such that, fo rall α and $\beta \in \mathbb{N}^3$, there exists $C_{\alpha,\beta}$, independent on ρ, such that, for all $(x,\xi,\rho) \in \mathbb{R}^6 \times [1, +\infty[$,

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi,\rho)| \leq C_{\alpha,\beta}(1 + |\xi| + |x|^k + \rho)^{m - \frac{|\alpha|}{k} - |\beta|}.$$

The operator $Op(a(\rho))$ ($a \in \widetilde{\Gamma}^m$), is said globally elliptic with parameter ρ if there exists $C > 0$ such that, for all $(x,\xi,\rho) \in \mathbb{R}^6 \times [1, +\infty[$,

$$(|\xi| + |x|^k + \rho)^m \leq C(1 + |\mathrm{Det}\ a(x,\xi,\rho)|)^{1/4}.$$

As in the proof of Lemma 1, $\theta = \sigma + i\tau$, $\sigma, \tau \in \mathbb{R}$ and K, C, τ_0 are three positives real numbers independant of x and τ which may change. Let $\rho > 0$, $\alpha \in [0, 2\pi)$ and set $P = H(c,\theta) + \rho e^{i\alpha}$. The symbol $p(x, \xi, \rho)$ of P (associated with the standard calculus) belongs to $\widetilde{\Gamma}^1$. Take K, C, τ_0 such that (17) holds and set $\alpha = K\tau$. There exists K, C, τ_0 (possibly different) such that if $0 < \Im\theta < \tau_0$, if $|x| \geq C$ then

$$\begin{aligned} K^{-1}\tau \leq \mathrm{Arg}(V_\theta(x) + \rho e^{i\alpha}),\ \mathrm{Arg}(V_\theta(x) + \rho e^{i\alpha} - 2c^2) \leq K\tau, \\ |V_\theta(x) + \rho e^{i\alpha}| \geq \cos(K\tau)(|V_\theta(x)| + \rho) \geq K^{-1}(|x|^k + \rho) \\ |V_\theta(x) + \rho e^{i\alpha} - 2c^2| \geq \cos(K\tau)(|V_\theta(x) - 2c^2| + \rho) \geq K^{-1}(|x|^k + \rho). \end{aligned} \tag{20}$$

Then (20) shows that for some $K, C, \tau_0 > 0$ ($\tau_0 < \pi/2$), if $0 < \Im\theta < \tau_0$, if $|x| \geq C$ then $|(V_\theta(x) + \rho e^{i\alpha})(V_\theta(x) + \rho e^{i\alpha} - 2c^2) - c^2 e^{-2\theta}|\xi|^2|$

$$\geq \sin(K^{-1}\tau)|(V_\theta(x) + \rho e^{i\alpha})(V_\theta(x) + \rho e^{i\alpha} - 2c^2)| + c^2\sin(2\tau)|\xi|^2. \tag{21}$$

Following (20),(21), P is globally elliptic with parameter ρ. Then, there are $q(\rho)$ and $r(\rho)$ in $\widetilde{\Gamma}^{-1}$ such that

$$(H(c,\theta) + \rho e^{i\alpha})Op(q(\rho)) = I + Op(r(\rho)). \tag{22}$$

Moreover, $\sup_{\rho \geq 1} \rho \|Op(r)\|_{\mathcal{L}(L^2(\mathbb{R}^3))} < \infty$. Thus, the r.h.s. of (22) is invertible for a sufficiently large ρ. This proves Lemma 2. □

Lemma 3 *There exists $\tau_0 > 0$ such that the family of operators $\{H(c,\theta),\ 0 < \Im\theta < \tau_0\}$ is analytic in the sense of Kato.*

Let τ_0 be as in Lemma 1 and set $\theta \in \mathbb{C}$ with $0 < \Im\theta < \tau_0$. The existence of parametrixes for the global elliptic operator $H(c,\theta)$ shows that

$$\exists\, C > 0,\ \forall\, \phi \in B^1,\ \|\phi\|_{B^1} \leq C\left(\|H(c,\theta)\phi\| + \|\phi\|\right).$$

It implies that for all $\theta \in \mathbb{C}$ with $0 < \Im\theta < \tau_0$, $H(c,\theta)$ is closed on B^1.

There exists $\tau_0 > 0$ (with another τ_0), $K > 0$ such that for all $\theta, h \in \mathbb{C}$ satisfying $0 < \Im z, \Im\theta < \tau_0$ and for all $x \in \mathbb{R}^3$,

$$(V(xe^{\theta+h} - V(xe^\theta))/h = e^\theta \sum_{j=1}^{3} x_j \frac{\partial V}{\partial x_j}(xe^\theta) + hN(x,\theta,h)$$

$$|e^\theta \sum_{j=1}^{3} x_j \frac{\partial V}{\partial x_j}(xe^\theta)|, \ |N(x,\theta,h)| \leq K\langle x\rangle^k. \tag{23}$$

Fix $u, v \in L^2(\mathbb{R}^3, \langle x\rangle^{2k}dx)$ and let F be the map: $\theta \mapsto \langle V_\theta u, v\rangle_{L^2(\mathbb{R}^3,\mathbb{C})}$. ¿From (23), if $0 < \Im\theta < \tau_0$ then $(F(\theta+h) - F(\theta))/h$ has a limit as $h \to 0$ $(0 < \Im h < \tau_0)$. For each $u \in L^2(\mathbb{R}^3, \langle x\rangle^{2k}dx)$, $\theta \mapsto V_\theta u$ is a (weakly) analytic vector valued function. Then, for each $\phi \in B^1$, $H(c,\theta)\phi$ is a vector valued analytic function of $\theta \in \{z \in \mathbb{C}, \ 0 < \Im z < \tau_0\}$.

The above closure and analyticity results, added to Lemma 2, imply that $\{H(c,\theta), \, 0 < \Im\theta < \tau_0\}$ is an analytic family of type (A) [7, VII.2]. □

Proof of theorem 1: Using lemma 2 there exists $z \in \mathbb{C}$ such that $(H(c,\theta) - z)^{-1}$ maps $L^2(\mathbb{R}^3;\mathbb{C}^4)$ into $B^1(\mathbb{R}^3;\mathbb{C}^4)$, hence is a compact operator of $L^2(\mathbb{R}^3;\mathbb{C}^4)$. Therefore, the spectrum of $H(c,\theta)$ is a sequence of eigenvalues $\lambda_j(c,\theta)$ of finite multiplicity. It is clear that $H(c,\theta) = U(\Re\theta)H(c,\Im\theta)U(\Re\theta)^{-1}$, that is to say, $H(c,\theta)$ is unitarily equivalent to $H(c,\Im\theta)$. Therefore each $\lambda_j(c,\theta)$ does not depend on $\Re\theta$. In addition, lemma 3 implies that each $\lambda_j(c,\theta)$ depends analytically on θ with at most, algebraic singularities. As [11, pf of th1(i)] it can be proved using Puiseux series, that each $\lambda_j(c,\theta)$ is a constant function of θ. The multiplicity of each of these eigenvalues $\lambda_j(c,\theta)$ is even. This can be proved like in Parisse [9]. This completes the proof. □

3 Proof of theorem 2.

By arguments similar to that of section 2, the spectrum of the following Schrödinger operator

$$H_\theta = -\frac{1}{2}e^{-2\theta}\Delta + V(xe^\theta) \tag{24}$$

is discrete, and the eigenvalues are the same as H_∞, with the same multiplicities. (The only difference with section 2 is that the sign of $\Im\theta$ plays no role, and there is $\tau > 0$ such that the family $(H_\theta)_{|\Im\theta|<\tau}$ is analytic in the sense of Kato). If z is not in this spectrum, we set

$$R_{z\theta\infty} = \begin{pmatrix} (H_\theta - z)^{-1}I_2 & 0 \\ 0 & 0 \end{pmatrix} \tag{25}$$

We set also

$$B^+(\theta_0) = \{\theta \in \mathbb{C},\ \ |\theta| < 1,\ \ \ 0 < \Im\theta < \theta_0\}$$

and

$$V_\theta(x) = V(xe^\theta) \tag{26}$$

Lemma 4 *There exists $\theta_0 > 0$ and $R > 0$ such that, for each $\theta \in B^+(\theta_0)$, there exists $A_\theta > 0$ such that, if $|x| \geq R$*

$$< x >^k \leq A_\theta \Im V_\theta(x) \qquad\qquad < x >^k \leq A_\theta \Im e^{\theta - \overline{\theta}} V_\theta(x) \tag{27}$$

Proof. By the hypotheses (H1) and (H2), we can write, if $\theta = \sigma + i\tau \in B^+(a/2)$

$$V_{\sigma+i\tau}(x) = V_\sigma(x) + i\tau e^\sigma \sum_{j=1}^{3} x_j \frac{\partial V}{\partial x_j}(xe^\sigma) + \mathcal{O}(\tau^2 < x >^k) \tag{28}$$

If $|x|$ is large enough and $0 < \Im\theta < \theta_0$, (where θ_0 depends on the constants of hypotheses (H2) and (H3)), there exists $A_\theta > 0$ such that (27) is valid. □

For each $\varepsilon > 0$, we set

$$\Delta_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \tag{29}$$

The points i) and ii) of Theorem 2 are consequences of the points ii) and iii) of the following Lemma.

Lemma 5 *i) Let K be a compact set of $\mathbb{C}$ and θ such that $0 < \Im\theta < \theta_0$. Then there exists $B_\theta > 0$ (independent of c) such that, if c is large enough, for each $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ in $\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$ ($u_j \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^2)$), for each $z \in K$ and $c \geq 1$, we have*

$$\| < x >^{k/2} u_1\| + \| < x >^{-k} \sigma.Du_1\| + \|u_2\| \leq \dots$$

$$\dots \leq B_\theta \left(\|\Delta_c^{-1}(H(\theta, c) - z)\Delta_c^{-1} u\| + \|u_1\| \right) \tag{30}$$

ii) If K contains no eigenvalue of H_∞, there exists $A_\theta > 0$ (independent of c) such that, if c is large enough

$$\|u\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \leq A_\theta \|\Delta_c^{-1}(H(\theta,c)-z)\Delta_c^{-1}u\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \tag{31}$$

for all $u \in \mathcal{S}(\mathbb{R}^3,\mathbb{C}^4)$, and $z \in K$, and therefore,

$$\|u\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \leq A_\theta \|(H(\theta,c)-z)u\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \tag{32}$$

iii) If D is a disc centered at an eigenvalue of H_∞, and containing no other eigenvalue, then, if $0 < \Im\theta < \theta_0$,

$$\lim_{c\to+\infty} \sup_{z\in\partial D} \|(H(c,\theta)-z)^{-1} - R_{z\theta\infty}\| = 0 \tag{33}$$

Proof of point i). The equality $\Delta_c^{-1}(H(\theta,c)-z)\Delta_c^{-1}u = \begin{pmatrix} f \\ g \end{pmatrix}$ is equivalent to

$$f = (V_\theta - z)u_1 + e^{-\theta}\sigma.Du_2 \tag{34}$$

$$g = e^{-\theta}\sigma.Du_1 + \left(\frac{V_\theta - z}{c^2} - 2\right)u_2 \tag{35}$$

It follows from the two last equalities that

$$\langle u_1, e^{\theta-\overline{\theta}}(V_\theta - z)u_1\rangle - \langle\left(\frac{V_\theta - z}{c^2} - 2\right)u_2, u_2\rangle = \langle u_1, e^{\theta-\overline{\theta}}f\rangle - \langle g, u_2\rangle \tag{36}$$

and therefore, taking the imaginary parts in the last equality and applying Lemma 4,

$$\begin{aligned} \| < x >^{k/2} u_1\|^2 + c^{-2}\| < x >^{k/2} u_2\|^2 \leq \dots \\ \dots \leq B_\theta\left[\|f\|^2 + \|u_1\|^2 + \|g\|\|u_2\| + c^{-2}\|u_2\|^2\right] \end{aligned} \tag{37}$$

Taking now the real parts in (36), we obtain, with another B_θ

$$\|u_2\|^2 \leq B_\theta\left[\|f\|^2 + \|u_1\|^2 + \|g\|\|u_2\| + c^{-2}\|u_2\|^2\right]$$

The inequality (30) (with another B_θ) follows easily, if c is large enough, from the two last ones.

Proof of point ii). Suppose that the inequality () were false. Then there would exist a sequence (u_n) in $\mathcal{S}(\mathbb{R}^3,\mathbb{C}^4)$, a sequence (z_n) in K, and a sequence $c_n \to +\infty$ such that

$$\|u_n\| = 1 \qquad \|\Delta_{c_n}^{-1}(H(c_n,\theta) - z_n)\Delta_{c_n}^{-1}u_n\| \to 0 \tag{38}$$

Taking a subsequence, we can assume that $z_n \to z \in K$. Let us set $u_n = \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix}$ and $\Delta_{c_n}^{-1}H(c_n,\theta)\Delta_{c_n}^{-1}u_n = \begin{pmatrix} f_n \\ g_n \end{pmatrix}$. If we set $V_\theta(x) = V(xe^\theta)$, we have the relations (34) and (35) with f, u_1, u_2 replaced by f_n, φ_n, ψ_n. By (30) the sequences $< x >^{k/2} \varphi_n$ and $< x >^{-k} \sigma.D\varphi_n$ are bounded in $L^2(\mathbb{R}^3,\mathbb{C}^2)$. By these properties, we may assume (after taking subsequences) that there exist φ and ψ in $L^2(\mathbf{R}^3,\mathbb{C}^2)$ such that $\varphi_n \to \varphi$ (strongly) and $\psi_n \to \psi$ (weakly) in $L^2(\mathbf{R}^3,\mathbb{C}^2)$. We have

$$(V_\theta - z)\varphi + e^{-\theta}\sigma.D\psi = 0 \tag{39}$$

and

$$e^{-\theta}\sigma.D\varphi - 2\psi = 0 \tag{40}$$

and therefore $(H_\theta - z)\varphi = 0$. If $\varphi = 0$, it follows that $\|\varphi_n\| \to 0$, and, since $\|f_n\| + \|g_n\| \to 0$, the point i) shows that $\|\psi_n\| \to 0$, and this gives a contradiction since $\|\varphi_n\|^2 + \|\psi_n\|^2 = 1$. Therefore, there exists $\varphi \neq 0$ in $L^2(\mathbb{R}^3,\mathbb{C}^2)$ such that $(H_\theta - z)\varphi = 0$, and there is a contradiction since $z \in K$ and K contains no eigenvalue of H_θ. The inequality (31) is proved, and (32) follows easily.

Proof of point iii) Suppose that there exist θ such that $0 < \Im\theta < \theta_0$, a sequence (F_n) in $L^2(\mathbb{R}^3,\mathbb{C}^4)$, a sequence (z_n) in ∂D, a sequence $c_n \to +\infty$, and $\delta > 0$ such that

$$\|F_n\| = 1 \qquad \|(H(c_n,\theta) - z_n)^{-1}F_n - R_{z_n\theta\infty}F_n\| \geq \delta \tag{41}$$

Let us set

$$F_n = \begin{pmatrix} f_n \\ g_n \end{pmatrix} \qquad U_n = (H(c_n,\theta) - z_n)^{-1}F_n = \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} \tag{42}$$

By the point ii) (applied to the compact ∂D), the sequence $\|U_n\|$ is bounded. By the point i) (applied to the function $\Delta_{c_n} U_n$), we have

$$\| < x >^{k/2} \varphi_n\| + \| < x >^{-k} \sigma.D\varphi_n\| + c_n\|\psi_n\| \leq B_\theta[\|F_n\| + \|\varphi_n\|] = \mathcal{O}(1) \tag{43}$$

Therefore $\|\psi_n\| \to 0$, which implies, together with (41), that, for n large enough

$$\|\varphi_n - (H(\theta) - z_n)^{-1} f_n\| \geq \frac{\delta}{2} \tag{44}$$

By (43), we may assume, (after taking a subsequence), that there exist φ and ψ in $L^2(\mathbf{R}^3,\mathbb{C}^2)$ such that $\varphi_n \to \varphi$ (strongly) and $c_n\psi_n \to \psi$ (weakly) in $L^2(\mathbf{R}^3,\mathbb{C}^2)$. We may assume also that $z_n \to z \in \partial D$ and that f_n weakly converges to $f \in L^2(\mathbb{R}^3,\mathbb{C}^4)$. It follows that

$$(V_\theta(x) - z)\varphi + e^{-\theta}\sigma.D\psi = f$$

$$e^{-\theta}\sigma.D\varphi - 2\psi = 0$$

and therefore $(H_\theta - z)\varphi = f$. Since the operator $(H_\theta - z)^{-1}$ is compact, we may assume also that

$$(H_\theta - z_n)^{-1} f_n \to \widetilde{\varphi} \in L^2(\mathbb{R}^3,\mathbb{C}^2) \tag{45}$$

(strong convergence). We have $(H_\theta - z)\widetilde{\varphi} = (H_\theta - z)\varphi = f$, and there is a contradiction with (44) since $\|\varphi - \widetilde{\varphi}\| \geq \delta/2$ and z is not in the spectrum of H_θ.

Proof of of theorem 2. The point i) is a consequence of Lemma 5 (point ii). For the point ii), let E_0 be a simple eigenvalue of H_∞. Let D be a disc, centered at E_0, with radius $\rho > 0$, containing no other eigenvalue of H_∞ inside it, and Γ be the boundary of D. By the point i), we know that, for c large enough, $H(\theta, c) - z$ is invertible for all $z \in \Gamma$. We define then an operator $\Pi_{\theta c}$ by

$$\Pi_{\theta c} = \frac{1}{2i\pi}\int_\Gamma (H(\theta, c) - z)^{-1} dz \tag{46}$$

Similarly we define $\Pi_{\theta\infty}$ by

$$\Pi_{\theta\infty} = \frac{1}{2i\pi}\int_\Gamma R_{z\theta\infty} dz \tag{47}$$

where $R_{z\theta\infty}$ is defined in (25). It follows from Lemma 5 (point iii) that

$$\lim_{c\to+\infty} \|\Pi_{\theta c} - \Pi_{\theta\infty}\| = 0 \tag{48}$$

The point ii) follows easily. □

4 Proof of theorem 3.

If $D = B(E_0, \rho)$ is a disc like in the theorems 2 and 3, and if E_0 is a simple eigenvalue of H_∞, we know, by theorem 2, that, for c large enough, $H(\theta, c)$ has only one eigenvalue $\lambda(c)$ of multiplicity 2 in $B(E_0, \rho)$. Since E_0 is also a simple eigenvalue of the dilated Schrödinger operator H_θ defined in (24) (section 3), let φ_θ be a normalized eigenvector ($H_\theta\varphi_\theta = E_0\varphi_\theta$, $\|\varphi_\theta\| = 1$). By the global ellipticity of H_θ, we know that φ_θ is in $\mathcal{S}(\mathbb{R}^3)$. Let

$$\psi_\theta = \begin{pmatrix} \varphi_\theta \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{49}$$

If $\Pi_{\theta c}$ is defined in (46), (where Γ is the boundary of D), $\Pi_{\theta c}\psi_\theta$ is in the eigenspace of $H(\theta, c)$ corresponding to the eigenvalue $\lambda(c)$ and, by (48), if c is large enough, $\Pi_{\theta c}\psi_\theta \neq 0$. Therefore

$$\lambda(c) = \frac{(H(\theta,c)\Pi_{\theta c}\psi_\theta, \Pi_{\theta c}\psi_\theta)}{\|\Pi_{\theta c}\psi_\theta\|^2} \qquad E_0 = \frac{(H_\theta\Pi_{\theta\infty}\psi_\theta, \Pi_{\theta\infty}\psi_\theta)}{\|\Pi_{\theta\infty}\psi_\theta\|^2} \tag{50}$$

(since $\Pi_{\theta\infty}\psi_\theta = \psi_\theta$).

Lemma 6 *Let ψ be a function in $\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$ and Γ be the boundary of $D = B(E_0, \rho)$. Let $F(\varepsilon, z)$ be the function defined, for ε small enough and $z \in \Gamma$ by*

$$F(\varepsilon, z) = (H(\theta, 1/\varepsilon) - z)^{-1}\psi \qquad if \quad \varepsilon \neq 0 \tag{51}$$

$$F(\varepsilon, z) = R_{z\theta\infty}\psi \qquad if \quad \varepsilon = 0 \tag{52}$$

where $R_{z\theta\infty}$ is defined in (25). Then $\varepsilon \to F(\varepsilon, z)$ is C^∞ from some neighborhood of 0 *to $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$, and depends continuously of z in Γ.*

Proof. If Δ_ε is the operator defined in (29), we can write, by (34) and (35)

$$\Delta_c^{-1}(H(\theta, c) - z)\Delta_c^{-1} = A + c^{-2}B \tag{53}$$

where

$$A = \begin{pmatrix} V_\theta - z & e^{-\theta}\sigma.D \\ e^{-\theta}\sigma.D & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & V_\theta - z \end{pmatrix}$$

By Lemma 5, there is $t_0 > 0$ such that $A + tB : B^1 \to \mathcal{H}$ is invertible if $0 < t \leq t_0$, and there exists $K > 0$ such that

$$\|(A + tB)^{-1} f\| \leq K\|f\| \qquad 0 < t \leq t_0 \qquad \forall f \in \mathcal{H} \tag{54}$$

Moreover, if we set $(A + tB)^{-1} f = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$, we have, by Lemma 5

$$\| < x >^{k/2} u(t)\| + \| < x >^{-k} \sigma.Du(t)\| + \|v(t)\| \leq \dots$$

$$\dots \leq K(\|f\| + \|u(t)\|) \qquad 0 < t \leq t_0 \qquad \forall f \in \mathcal{H}$$

In the other hand, if H_θ is the operator defined in (24), and $z \in \Gamma$, the operators $D^\alpha(H_\theta - z)^{-1}D^\beta$ are bounded in $L^2(\mathbb{R}^3)$ if $|\alpha + \beta| \leq 2$ (we construct easily a parametrix of this operator in a suitable class). Therefore, the following operator S is bounded in $\mathcal{H}$

$$S = \begin{pmatrix} (H_\theta - z)^{-1} & \frac{e^{-\theta}}{2}(H_\theta - z)^{-1}\sigma.D \\ \frac{e^{-\theta}}{2}\sigma.D(H_\theta - z)^{-1} & \frac{e^{-2\theta}}{4}\sigma.D(H_\theta - z)^{-1}\sigma.D - \frac{I}{2} \end{pmatrix}$$

and it satisfies $AS = I$. Moreover $u \in \mathcal{H}$ and $(A + tB)u = 0$ imply $u = 0$ $(0 \leq t \leq t_0)$. It follows easily from these properties that, if $f \in \mathcal{H}$, the function $G(t)f$ defined by

$$G(t)f = (A + tB)^{-1}f \qquad \text{if} \quad 0 < t \leq t_0 \qquad G(0)f = Sf \tag{55}$$

is continuous in $[0, t_0]$ to $\mathcal{H}$. Let E be the space of $f \in \mathcal{H}$ such that, for each m, $< x >^m u$ is in $\mathcal{H}$. Using the commutation relation

$$x_j(A + tB)^{-1} = (A + tB)^{-1}x_j - ie^{-\theta}(A + tB)^{-1}\alpha_j(A + tB)^{-1}$$

where $\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$, it follows that, for each integer m, there is K_m such that

$$\| < x >^m (A+tB)^{-1} f\| \leq K_m \| < x >^m f\| \qquad \forall f \in E \qquad 0 \leq t \leq t_0$$

and that, for each $f \in E$, the function $< x >^m G(t)f$ is continuous in $[0, t_0]$ to $\mathcal{H}$. It follows that, for each $f \in E$, the function $G(t)f$ is C^∞ on $[0, t_0]$ to $\mathcal{H}$, and that

$$G^{(p)}(t)f = (-1)^p (A+tB)^{-1} \left(B(A+tB)^{-1}\right)^p \qquad \text{if} \quad 0 < t \leq t_0 \qquad (56)$$

and $G^{(p)}(0)f = (-1)^p S(BS)^p f$. This property can be proved, by induction on p, using the previous remarks. The Lemma follows easily since $F(\varepsilon, z) = \Delta_\varepsilon G(\varepsilon^2)\Delta_\varepsilon \psi$.

Proof of Theorem 3. Since ψ_θ defined in (49) is in $\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$, (this can be proved by using a parametrix of H_θ), it follows from (50) and Lemma 6 that the function g defined in some neighborhood of 0 by

$$g(\varepsilon) = \lambda(1/\varepsilon) \qquad if \quad \varepsilon \neq 0 \qquad (57)$$

$$g(0) = E_0 \qquad (58)$$

is C^∞. We remark that

$$JH(\theta, c)J = D_{\theta,-c} \qquad J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad (59)$$

Since ψ_θ defined in (49) satisfies $J\psi_\theta = \psi_\theta$, it follows that g is an even function of ε, and there exists a C^∞ function f in a neighborhood of 0 such that $g(\varepsilon) = f(\varepsilon^2)$, which proves theorem 3.

5 Imaginary part of the resonances.

In this section, we need another definition of the resonances, using the exterior scaling. We are very grateful to X.P. Wang for this suggestion. For each $\varepsilon > 0$ and $c > 1$, we have to introduce two auxiliary Hamiltonians : one of them (denoted by $D(\theta,c)$) is obtained from $D(c)$ by an exterior complex scaling (cf. Hunziker [6]), and the other one, denoted by $H_0(c)$, is obtained from $D(c)$ by a modification of the potential (cf. Wang [17] and Parisse [9]).

For the construction of the distorted operator $D(\theta,c)$, we use, for each $\varepsilon \in]0,1[$, a function $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi(t) = 0$ if $t \leq 2 - \frac{\varepsilon}{2}$ and $\varphi(t) = 1$ if $t \geq 2$. For each $\theta \in \mathbb{C}$ and $x \in \mathbb{R}^3$, we set

$$\varphi_\theta(x) = x + \theta X_c(x) \qquad X_c(x) = x\varphi\left(\frac{V(x)}{c^2}\right) \tag{60}$$

If $|\theta|$ is small enough, we can define a system $p_\theta = (p_{\theta,1}, p_{\theta,2}, p_{\theta,3})$ of differential operators by

$$p_\theta = {}^t\, (\varphi_\theta'(x))^{-1} D_x - \frac{i}{2}\nabla(\ln J_\theta(x)) \qquad J_\theta(x) = \det \varphi_\theta'(x) \tag{61}$$

and a distorted Dirac operator $D(\theta,c)$ by

$$D(\theta,c) = \begin{pmatrix} V(\varphi_\theta(x)) & c\sigma\cdot p_\theta \\ c\sigma\cdot p_\theta & V(\varphi_\theta(x)) - 2c^2 \end{pmatrix} \tag{62}$$

Proposition 1 *With the previous notations, if $|\theta|$ is small enough, if D is a disc as in theorem 2 (point ii), and if c is large enough, the spectrum of $D(\theta,c)$ in D is the same sequence of eigenvalues $E_j(c)$ as for the operator $H(\theta,c)$ defined in (7), with the same multiplicities.*

For the proof of this proposition, we shall use the following Lemma.

Lemma 7 *There exist $A > 0$ and $\theta_0 > 0$ with the following properties. If $z \in \mathbb{C}$, $\Im\, z < 0$, $c \geq 1$, if $\theta \in \Omega$, where*

$$\Omega = \{\theta \in \mathbb{C}, \quad |\theta| < \theta_0, \qquad 0 < \Im\,\theta < \frac{|\Im\, z|}{A(c^2 + |Re\; z|)}\} \tag{63}$$

then $z - H(\theta,c) : B^1 \to \mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ is invertible and

$$\|(z - H(\theta,c))^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{A}{|\Im\, z|} \tag{64}$$

Moreover, for each $f \in \mathcal{H}$, the function $\theta \to (z - H(\theta, c))^{-1} f$ $(\theta \in \Omega)$, extended by $(z - D(c))^{-1} f$ for real θ, is holomorphic in Ω and weakly continuous in $\overline{\Omega}$.

Proof of the Lemma. If we set $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, the equality $(H(\theta, c) - z)u = f$ implies

$$\Im \left[e^{\theta} \langle f, u \rangle \right] = \Im \left[e^{\theta} \int V_{\theta}(x) |u(x)|^2 dx \right] - \Im \left(e^{\theta} z \right) \|u\|^2 - 2c^2 \Im \left(e^{\theta} \right) \|u_2\|^2$$

By the hypotheses on the potential V, there exist R, A and ε_0, independent on all the parameters, such that

$$\Im\theta < x >^k \leq A \Im \left[e^{\theta} V_{\theta}(x) \right] \qquad if \quad |\theta| \leq 1, \ \ 0 < \Im\theta < \varepsilon_0, \ \ |x| \geq R$$

and

$$|\Im \left(e^{\theta} V_{\theta}(x) \right) | \leq A \Im\theta \qquad if \quad |\theta| \leq 1, \ \ 0 < \Im\theta < \varepsilon_0, \ \ |x| \leq R$$

It follows that, with other constants A and ε_0, if $\Im z < 0$, $|\theta| < 1$, $0 < \Im\theta < \varepsilon_0$, and if $(D(\theta, c) - z)u = f$, we have

$$|\Im z| \|u\|^2 \leq A \left[\|f\| \|u\| + |\Im\theta| (c^2 + |Re\ z|) \|u\|^2 \right]$$

If moreover $0 \leq \Im\theta \leq |\Im z| / (2A(c^2 + |Re\ z|))$, then

$$\|u\|_{\mathcal{H}} \leq \frac{2A}{|\Im\ z|} \|(z - H(\theta, c))u\|_{\mathcal{H}} \tag{65}$$

By the results of section 2, it follows that, for each $\theta \in \Omega$ (with another A), $z - H(\theta, c) : B^1 \to \mathcal{H}$ is invertible and that the inverse depends holomorphically on θ in Ω. The result about weak continuity follows from (64), using the implication (13). □

End of the proof of the Proposition. Once the lemma 7 is established, the proof of Proposition 1 follows the classical proof of the Aguilar-Balslev-Combes theorem [1] (see Hislop-Sigal [5] or Laguel [8] for more details). For real θ, small enough, we define an operator $U_{\theta} : \mathcal{H} \to \mathcal{H}$ by

$$(U_{\theta} f)(x) = e^{3\theta/2} f(x e^{\theta}) \tag{66}$$

and an operator $\widetilde{U}_\theta : \mathcal{H} \to \mathcal{H}$ by

$$(\widetilde{U}_\theta f)(x) = J_\theta(x)^{1/2} f(\varphi_\theta(x)) \tag{67}$$

Then U_θ and $\widetilde{U}_\theta$ are unitary, and we have

$$H(\theta, c) = U_\theta D(c) U_\theta^{-1} \qquad D(\theta, c) = \widetilde{U}_\theta D(c) \widetilde{U}_\theta^{-1} \tag{68}$$

There exists a subspace $\mathcal{A}$ in $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ and $\theta_0 > 0$ such that, for each $f \in \mathcal{A}$, the functions $\theta \to U_\theta f$ and $\theta \to \widetilde{U}_\theta f$ extend to holomorphic functions from $B(0, \theta_0)$ to $\mathcal{H}$, and such that, for each $\theta \in B(0, \theta_0)$, $U_\theta \mathcal{A}$ and $\widetilde{U}_\theta \mathcal{A}$ are dense in $\mathcal{H}$. If $f, g \in \mathcal{A}$, $|\theta| < \theta_0$ and $\Im \theta > 0$, we set

$$F_{fg}(z, \theta) = < U_{\overline{\theta}} f, \, (z - H(\theta, c))^{-1} U_\theta g > \tag{69}$$

$$\widetilde{F}_{fg}(z, \theta) = < \widetilde{U}_{\overline{\theta}} f, \, (z - D(\theta, c))^{-1} \widetilde{U}_\theta g > \tag{70}$$

By the results of section 2 and their analogous for $D(\theta, c)$, we know that, if $c \geq 1$, these functions of z are meromorphic in D. Let A and θ_0 be the constants of Lemma 7. There is an analogous of Lemma 7 with $H(\theta, c)$ replaced by $D(\theta, c)$, and we may assume that the constants A and θ_0 are the same. If E_0 is the center of D and ρ its radius, let

$$\omega = \{\theta \in \mathbb{C}, \qquad |\theta| < \theta_0, \qquad 0 < \Im\, \theta < \frac{\rho}{2A(c^2 + |E_0| + \rho)}\}$$

By Lemma 7, if $z \in D$ and $\Im z < -\frac{\rho}{2}$, the functions $\theta \to F_{fg}(z, \theta)$ and $\theta \to \widetilde{F}_{fg}(z, \theta)$ are holomorphic in ω and continuous in $\overline{\omega}$. By (68), they are equal in $\overline{\omega} \cap \mathbb{R}$, and therefore they are equal in ω. Now, if $\theta \in \omega$, the functions $z \to F_{fg}(z, \theta)$ and $z \to \widetilde{F}_{fg}(z, \theta)$ are meromorphic in D and equal in $\{z \in D, \quad \Im z < -\frac{\rho}{2}\}$, and therefore they are equal on D. A point $z_0 \in D$ is an eigenvalue of $H(\theta, c)$ (resp. of $D(\theta, c)$) iff there are f and $g \in \mathcal{A}$ such that z_0 is a pole of $z \to F_{fg}(z, \theta)$ (resp. of $z \to \widetilde{F}_{fg}(z, \theta)$). Therefore, these eigenvalues are the same. □

Therefore, under the hypotheses of theorem 2, if D is a disc centered at E_0, of radius ρ, and containing no other eigenvalue of H_∞, if $E_j(c)$ $(1 \leq j \leq 2\mu)$ are the resonances in D, there exists an orthonormal system of functions ψ_j in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ $(1 \leq j \leq 2\mu)$, such that, if c is large enough,

$$D(\theta, c)\psi_j = E_j(c)\psi_j \tag{71}$$

Now we shall define a modified real-valued potential, like in Wang [17] and Parisse [9] in the semiclassical study of multiple wells or resonances for the Dirac operator. For that, we can choose a function $\psi \in C^\infty(\mathbb{R})$, nondecreasing, such that $\psi(t) = t$ if $t \leq 2 - \frac{\varepsilon}{2}$, $\psi(t) \leq t$ for all t, and $\psi(t) = 2 - \frac{\varepsilon}{4}$ if $t \geq 2$. Using this function, we define a modified potential V_0 (depending on ε and c) by

$$V_0(x) = c^2 \psi\left(\frac{V(x)}{c^2}\right) \tag{72}$$

Let $d(x, V_0, c)$ be the distance from $x \in \mathbb{R}^3$ to the origin for the Agmon metric defined as in section 1, but with the potential V_0 instead of V. We set

$$\Sigma(c, \varepsilon) = \inf_{V(x) \geq (2 - \frac{\varepsilon}{2})c^2} d(x, V_0, c) \tag{73}$$

Lemma 8 *If $\varepsilon < 1/2$, there exists $K_\varepsilon > 0$ such that*

i)
$$V(x) \geq \frac{3}{2}c^2 \Rightarrow c \leq K_\varepsilon d(x, V_0, c)$$

ii)
$$< x > \leq K_\varepsilon (1 + d(x, V_0, c)) \qquad \forall x \in \mathbb{R}^3$$

iii)
$$S(c, \varepsilon) \leq \Sigma(c, \varepsilon)$$

Proof. Let $x \in \mathbb{R}^3$, and $t \to x(t)$ be a C^1 curve such that $x(0) = 0$ and $x(1) = x$. Suppose that $V(x) \geq (3/2)c^2$. Let t_0 and t_1 such that

$$0 < t_0 < t_1 < 1 \qquad V(x(t_0)) = \frac{1}{2}c^2 \qquad V(x(t_1)) = c^2$$

and

$$\frac{1}{2}c^2 \leq V(x(t)) \leq c^2 \qquad \forall t \in [t_0, t_1]$$

For each $t \in [t_0, t_1]$, we have $V_0(x(t)) \geq \frac{1}{2}c^2$ and $2c^2 - V_0(x(t)) \geq \frac{\varepsilon}{4}c^2$, and therefore

$$\frac{1}{c}\int_0^1 \left[V_0(x(t)_+(2c^2 - V_0(x(t))\right]^{1/2} |x'(t)|dt \geq \frac{c\sqrt{\varepsilon}}{4}|x(t_1) - x(t_0)|$$

By the hypotheses on the potential V, there exists $K > 0$ and $K' > 0$ such that, if c is large enough,

$$\frac{1}{2}c^2 \leq |V(x(t_0)) - V(x(t_1))| \leq K|x(t_0) - x(t_1)| \left[< x(t_0) > + < x(t_1) >\right]^{k-1}$$

$$\ldots \leq K'|x(t_0) - x(t_1)|V(x(t_1))^{(k-1)/k} \leq K'|x(t_0) - x(t_1)|c^{2-2/k}$$

The point i) follows from the last inequalities. For the point ii), we can find $R > 0$ such that $V_0(x) \geq 1$ if $|x| \geq R$. If $|x| \geq R$ and if $x(t)$ is a curve as above, there exists $t_0 \in [0, 1]$ such that $|x(t_0)| \leq R$ and $|x(t)| \geq R$ if $t \in [t_0, 1]$. It follows that

$$\frac{1}{c}\int_0^1 \left[V_0(x(t)_+(2c^2 - V_0(x(t))\right]^{1/2} |x'(t)|dt \geq \frac{\varepsilon}{2}|x - x(t_0)|$$

and therefore $|x| \leq R + \frac{2}{\varepsilon}d(x, V_0, c)$. The proof of the point iii) is straightforward. □

We denote by $H_0(c)$ the modified Hamiltonian corresponding to the modified potential V_0

$$H_0(c) = \begin{pmatrix} V_0(x) & c\sigma \cdot D_x \\ c\sigma \cdot D_x & V_0(x) - 2c^2 \end{pmatrix}, \tag{74}$$

We see easily that $H_0(c)$ is essentially self-adjoint and, using the arduments of section 3, we see that, if D is a neighborhood of E_0 like in the theorem 2 (point ii), $D \cap \mathbb{R}$ contains, for c large enough, 2μ eigenvalues $\lambda_j(c)$ $(1 \leq j \leq 2\mu)$ of $H_0(c)$ (if they are repeated according to their multiplicities). Let $\varphi_j = \varphi_j(c)$ $(1 \leq j \leq 2\mu)$ be an orthonormal system of corresponding eigenfunctions.

$$H_0(c)\varphi_j = \lambda_j(c)\varphi_j \qquad \|\varphi_j\| = 1 \tag{75}$$

and we have, if ρ is the radius of D and if c is large enough

$$|\lambda_j(c) - E_0| \leq \frac{\rho}{2} \tag{76}$$

The following result about the exponential decay at infinity of the functions $\varphi_j(c)$ is well-known (see Wang [17]).

Proposition 2 *With the previous notations, for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$, independent of c such that the functions φ_j $(1 \le j \le 2\mu)$ satisfy*

$$\|e^{(1-\varepsilon)d(.,V_0,c)}\varphi_j\|^2 + \frac{1}{c^2}\|e^{(1-\varepsilon)d(.,V_0,c)}\nabla\varphi_j\|^2 \le C_\varepsilon \tag{77}$$

Proof. The proof is the same as in Wang [17] but, since it is written in [17] in the semiclassical context, we give a sketch of the proof here. By a direct calculus, we see, like in Wang [17] (Proposition 2.1) that, for each real-valued function Φ, bounded, uniformly lipschitzian on $\mathbb{R}^3$, we have

$$c^2\int_{\mathbb{R}^3}|\nabla(e^\Phi\varphi_j)|^2\,dx + \int_{\mathbb{R}^3}\delta(x,c)|e^\Phi\varphi_j|^2\,dx = 0 \tag{78}$$

where

$$\delta(x,c) = [V_0(x) - \lambda_j(c)]\left[2c^2 - V_0(x) + \lambda_j(c)\right] - c^2|\nabla\Phi(x)|^2 \tag{79}$$

There exists $R_\varepsilon > 0$ such that, if $0 \le \varepsilon \le 1$

$$|x| \ge R_\varepsilon \Rightarrow V_0(x) \ge \frac{8(|E_0| + (\rho/2)) + 4}{2\varepsilon^2 - \varepsilon^3}$$

If Φ satisfies $\Phi(0) = 0$ and

$$c^2|\nabla\Phi|^2 \le V_0(x)_+\,(2c^2 - V_0(x))\,(1-\varepsilon)^2 \tag{80}$$

using (76), we see that

$$\delta(x,c) \ge c^2 \qquad if\ \ |x| \ge R_\varepsilon \tag{81}$$

We can find $K_\varepsilon > 0$, independent on c, such that

$$c^{-2}|\delta(x,c)| + |\Phi(x)| \le K_\varepsilon \qquad if\ \ |x| \le R_\varepsilon \tag{82}$$

It follows that

$$\int_{\mathbb{R}^3}|\nabla(e^{\Phi(x)}\varphi_j(x))|^2\,dx + \int_{|x|\ge R_\varepsilon}|e^{\Phi(x)}\varphi_j(x)|^2\,dx \le \dots \tag{83}$$

$$\dots \le K_\varepsilon\int_{|x|\le R_\varepsilon}|e^{\Phi(x)}\varphi_j(x)|^2\,dx \le K_\varepsilon e^{K_\varepsilon} \tag{84}$$

Since, for c large enough, $|\nabla\Phi(x)|^2 \leq 6c^2$, it follows from (84) and (82) that

$$\int_{|x|\geq R_\varepsilon} |e^{\Phi(x)}\nabla\varphi_j(x)|^2\, dx \leq (2+12c^2)K_\varepsilon e^{K_\varepsilon} \tag{85}$$

Since φ_j satisfies (75), we remark also that

$$\int_{|x|\leq R_\varepsilon} |e^{\Phi(x)}\nabla\varphi_j(x)|^2\, dx \leq 3\frac{e^{2K_\varepsilon}}{c^2}\Big[\|H_0(c)\varphi_j\|^2 + \|V_0\varphi_j\|^2 + \|c^2\varphi_j\|^2\Big] \tag{86}$$

$$\leq K'_\varepsilon c^2 \tag{87}$$

where K'_ε is independent on c. We used $|\lambda_j(c)| \leq |E_0| + (\rho/2)$ and $V_0(x) \leq (2-(\varepsilon/4))c^2$. Therefore, with $K''_\varepsilon > 0$ independent on c, and on the function Φ satisfying (80)

$$\frac{1}{c^2}\|e^\Phi\nabla\varphi_j\|^2 + \|e^\Phi\varphi_j\|^2 \leq K''_\varepsilon \tag{88}$$

The proposition follows by the argument of [17]. $\square$

Now we shall study the decay at infinity of the orthonormal system of functions ψ_j satisfying (71), following the technique of Sigal [13]. For that, we set

$$\widetilde{d}(x, V_0, c) = \inf(d(x, V_0, c),\ \Sigma(c,\varepsilon)) \tag{89}$$

Proposition 3 *With the previous notations, for each $\varepsilon > 0$, there exists $K_\varepsilon > 0$, independent of c such that the functions ψ_j $(1 \leq j \leq 2\mu)$ satisfy*

$$\|e^{(1-\varepsilon)\widetilde{d}(.,V_0,c)}\psi_j\| \leq K_\varepsilon c^{(1-2/k)_+} \tag{90}$$

In the proof, and also later, we shall use a cut-off function defined as follows. We can choose a function $h \in C^\infty(\mathbb{R})$ such that $0 \leq h(t) \leq 1$ for all t, $h(t) = 1$ if $t \leq 2-\varepsilon$ and $h(t) = 0$ if $t \geq 2 - \frac{\varepsilon}{2}$. We set

$$\chi(x) = h\left(\frac{V(x)}{c^2}\right) \qquad \forall x \in \mathbb{R}^3 \tag{91}$$

We remark that, with A_ε independent on c

$$|\nabla\chi(x)| \leq A_\varepsilon c^{-2/k} \tag{92}$$

We remark also that

$$\chi D(\theta,c) = \chi H_0(c) \tag{93}$$

and therefore

$$D(\theta,c)\chi - \chi H_0(c) = [H_0(c),\chi] = c(D\chi).\alpha \tag{94}$$

where

$$(D\chi).\alpha = \begin{pmatrix} 0 & \sigma\cdot(D\chi) \\ \sigma\cdot(D\chi) & 0 \end{pmatrix},$$

Proof of Proposition 3. Let γ be the boundary of D (a circle with center E_0, and with radius ρ). If c is large enough, all the resonances $E_j(c)$ $(1 \le j \le 2\mu)$ are contained in $B(E_0,\rho/2)$. The same arguments as for Lemma 5 (point ii) show that, for c large enough

$$\|(z - D(\theta,c))^{-1}\| \le K \tag{95}$$

for all $z \in \gamma$, where K is independent on c. Let P be the projection defined, for c large enough, by

$$Pf = \frac{1}{2i\pi}\int_\gamma (z - D(\theta,c))^{-1} f\, dz \tag{96}$$

First, we shall prove that the functions $P\varphi_j$ satisfy the estimations of the proposition. It follows from (94) that, for each $z \in \gamma$, and for all $f \in L^2(\mathbb{R}^3,\mathbb{C}^4)$

$$(z - D(\theta,c))^{-1}(\chi f) = \Big[\chi + c(z - D(\theta,c))^{-1}(D\chi).\alpha\Big]\,(z - H_0(c))^{-1} f \tag{97}$$

Applying this equality with $f = \varphi_j$ and integrating over γ, we obtain, by (96)

$$P(\chi\varphi_j) = \chi\varphi_j + g_j \qquad g_j = \frac{c}{2i\pi}\int_\gamma \frac{(z - D(\theta,c))^{-1}(D\chi).\alpha\varphi_j}{z - \lambda_j(c)}\, dz \tag{98}$$

We can write

$$\|e^{(1-\varepsilon)\widetilde{d}(.,V_0,c)}P\varphi_j\| \le e^{(1-\varepsilon)\Sigma(c,\varepsilon)}\|P((1-\chi)\varphi_j)\| + \|e^{(1-\varepsilon)\widetilde{d}(.,V_0,c)}\chi\varphi_j)\| + \dots \tag{99}$$

$$\ldots + \|e^{(1-\varepsilon)\widetilde{d}(.,V_0,c)} g_j\| \tag{100}$$

By (95), the L^2 norm of the projector P is bounded by some constant K independent of c. By the definition of $\Sigma(c,\varepsilon)$ and by the proposition 2,

$$e^{(1-\varepsilon)\Sigma(c,\varepsilon)} \|P((1-\chi)\varphi_j)\| \leq K_\varepsilon \tag{101}$$

for some constant K_ε, independent on c. If c is large enough, using (95) and (76), we see that

$$\|e^{(1-\varepsilon)\widetilde{d}(.,V_0,c)} g_j\| \leq K_0 c e^{(1-\varepsilon)\Sigma(c,\varepsilon)} \|(\nabla\chi)\varphi_j\| \tag{102}$$

with some other constant K_0. Therefore, using also (92) and the definition of $\Sigma(c,\varepsilon)$, we obtain,

$$\|e^{(1-\varepsilon)\widetilde{d}(.,V_0,c)} g_j\| \leq K'_\varepsilon c^{1-(2/k)} \|e^{(1-\varepsilon)d(.,V_0,c)} \varphi_j\| \leq K''_\varepsilon c^{1-(2/k)} \tag{103}$$

where K'_ε and K''_ε are independent on c. We used Proposition 2, which shows also that

$$\|e^{(1-\varepsilon)\widetilde{d}(.,V_0,c)} (\chi\varphi_j)\| \leq \|e^{(1-\varepsilon)d(.,V_0,c)} \varphi_j\| \leq C_\varepsilon \tag{104}$$

Summing up, we proved that, for some other K_ε independent on c

$$\|e^{(1-\varepsilon)\widetilde{d}(.,V_0,c)} P\varphi_j\| \leq K_\varepsilon c^{(1-(2/k))+} \tag{105}$$

Now we shall orthogonalize the system $(P\varphi_j)$ $(1 \leq j \leq 2\mu)$. We remark that

$$P\varphi_j - \varphi_j = \frac{1}{2i\pi} \int_\gamma \frac{(z - D(\theta,c))^{-1}(D(\theta,c) - H_0(c))\varphi_j}{z - \lambda_j(c)} \, dz \tag{106}$$

It follows that

$$\|P\varphi_j - \varphi_j\| \leq K_0 \|(D(\theta,c) - H_0(c))\varphi_j\| \tag{107}$$

where K_0 is independent of c. We have, if V_θ is defined in (26) and V_0 in (72)

$$\|(D(\theta,c) - H_0(c))\varphi_j\| \leq K \int_{V(x)\geq(2-\varepsilon/2)c^2} |\nabla\varphi_j(x)|^2 \, dx + \ldots$$

$$\dots + K \int_{V(x) \geq (2-\varepsilon/2)c^2} [1 + |V_\theta(x) - V_0(x)|^2|]\varphi_j(x)|^2 \, dx \tag{108}$$

for some constant K, and we have also $|V_\theta(x) - V_0(x)| \leq K < x >^k$. By Lemma 8 and proposition 2, it follows that, for some K_ε

$$\|P\varphi_j - \varphi_j\| \leq K_\varepsilon e^{-\Sigma(c,\varepsilon)}$$

By Lemma 8, $\|P\varphi_j - \varphi_j\| \to 0$ when $c \to +\infty$. Hence the Gram matrix $S = (P\varphi_j, P\varphi_k)_{1 \leq j,k \leq 2\mu}$ tends to identity when $c \to +\infty$. Therefore, if c is large enough, $T = S^{-1/2}$ is defined, and bounded independently of c. If we set $T = (a_{jk})$, the system of functions $\psi_j = \sum a_{jk} P\varphi_k$ is an orthonormal basis of ImP, which satisfies the estimations (90). □

End of the proof of theorem 4. We consider again the function χ defined in (91) and an orthonormal system of eigenfunctions ψ_j satisfying (71). By Proposition 3, we can write

$$\int_{\text{supp}\,(1-\chi)} |\psi_j(x)|^2 \, dx \leq K_\varepsilon^2 c^2 e^{-2(1-\varepsilon)\Sigma(c,\varepsilon)} \tag{109}$$

It follows by Lemma 8 (point i)) that, if c is large enough

$$\int (1 - \chi(x))|\psi_j(x)|^2 \, dx \leq \frac{1}{2} \tag{110}$$

If we write the imaginary part of the scalar product of both sides of (71) with $\chi\psi_j$, we obtain, using (93)

$$(\Im E_j(c)) \int_{\mathbb{R}^3} \chi(x)|\psi_j(x)|^2 \, dx = \Im \langle D(\theta, c)\psi_j, \chi\psi_j \rangle = \dots$$

$$\dots = \Im \langle H_0(c)\psi_j, \chi\psi_j \rangle = -\frac{1}{2}\langle [H_0(c), \chi]\psi_j, \psi_j \rangle \tag{111}$$

Using (110) and (92), we have, for some constants K, K' and K''_ε

$$|\Im E_j(c)| \leq |([H_0(c), \chi]\psi_j, \psi_j)| \leq Kc \int |\nabla\chi(x)||\psi_j(x)|^2 \, dx \leq \dots$$

$$\dots \leq K' c^{1-(2/k)} \int_{\text{supp}(1-\chi)} |\psi_j(x)|^2 \, dx \leq K''_\varepsilon c^3 e^{-2(1-\varepsilon)\Sigma(c,\varepsilon)}$$

The estimation (12) of Theorem 4 follows, with another ε, using Lemma 8.

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