Uniqueness of the Invariant Measure for a Stochastic PDE Driven by Degenerate Noise

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J.-P. Eckmann[1,2]**, M. Hairer**[1]

[1] Département de Physique Théorique, Université de Genève
[2] Section de Mathématiques, Université de Genève
Email: `Jean-Pierre.Eckmann@physics.unige.ch`
Email: `Martin.Hairer@physics.unige.ch`

Abstract

We consider the stochastic Ginzburg-Landau equation in a bounded domain. We assume the stochastic forcing acts only on high spatial frequencies. The low-lying frequencies are then only connected to this forcing through the non-linear (cubic) term of the Ginzburg-Landau equation. Under these assumptions, we show that the stochastic PDE has a *unique* invariant measure. The techniques of proof combine a controllability argument for the low-lying frequencies with an infinite dimensional version of the Malliavin calculus to show positivity and regularity of the invariant measure. This then implies the uniqueness of that measure.

Contents

1 Introduction

In this paper, we study a stochastic variant of the Ginzburg-Landau equation on a finite domain with periodic boundary conditions. The deterministic equation is

$$\dot{u} = \Delta u + u - u^3 \;, \quad u(0) = u^{(0)} \in \mathcal{H} \;, \tag{1.1}$$

where $\mathcal{H}$ is the real Hilbert space $\mathcal{W}^1_{\text{per}}([-\pi,\pi])$, *i.e.*, the closure of the space of smooth periodic functions $u : [-\pi,\pi] \to \mathbf{R}$ equipped with the norm

$$\|u\|^2 = \int_{-\pi}^{\pi} \bigl(|u(x)|^2 + |u'(x)|^2\bigr)\, dx \;.$$

(The restriction to the interval $[-\pi,\pi]$ is irrelevant since other lengths of intervals can be obtained by scaling space, time and amplitude u in (1.1).) While we work exclusively with the real Ginzburg-Landau equation (1.1) our methods generalize immediately to the complex Ginzburg-Landau equation

$$\dot{u} = (1+ia)\Delta u + u - (1+ib)|u|^2 u \;, \quad a,b \in \mathbf{R} \;, \tag{1.2}$$

which has a more interesting dynamics than (1.1). But the notational details are slightly more involved because of the complex values of u and so we stick with (1.1).

While a lot is known about existence and regularity of solutions of (1.1) or (1.2), only very little information has been obtained about the attractor of such systems, and in particular, nothing seems to be known about invariant measures on the attractor.

On the other hand, when (1.1) is replaced by a stochastic differential equation, more can be said about the invariant measure, see [DPZ96] and references therein. Since the problem (1.1) involves only functions with periodic boundary conditions, it can be rewritten in terms of the Fourier series for u:

$$u(x,t) \;=\; \sum_{k\in\mathbf{Z}} e^{ikx} u_k(t) \;, \quad u_k \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx} u(x)\, dx \;.$$

We call k the momenta, u_k the modes, and, since $u(x,t)$ is real we must always have $u_k(t) = \bar{u}_{-k}(t)$, where $\bar{z}$ is the complex conjugate of z. With these notations (1.1) takes the form

$$\dot{u}_k = (1-k^2)u_k - \sum_{k_1+k_2+k_3=k} u_{k_1}u_{k_2}u_{k_3} ,$$

for all $k \in \mathbf{Z}$ and the initial condition satisfies $\{(1+|k|)u_k(0)\} \in \ell^2$. In the sequel, we will use the symbol $\mathcal{H}$ indifferently for the space $\mathcal{W}^1_{\text{per}}([-\pi,\pi])$ and for its counterpart in Fourier space. In the earlier literature on uniqueness of the invariant measure for stochastic differential equations, see *e.g.*, [DPZ96], the authors are mostly interested in systems where each of the u_k is forced by some external noise term. The main aim of our work is to study forcing by noise which *acts only on the high-frequency part* of u, namely on the u_k with $|k| \ge k_*$ for some finite $k_* \in \mathbf{N}$. The low-frequency amplitudes u_k with $|k| < k_*$ are then only *indirectly* forced through the noise, namely through the nonlinear coupling of the modes. In this respect, our approach is reminiscent of the work done on thermally driven chains in [EPR99a, EPR99b, EH00], where the chains were only stochastically driven at the ends.

In the context of our problem, the *existence* of an invariant measure is a classical result for the noise we consider [DPZ96], and the main novelty of our paper is a proof of *uniqueness* of that measure. To prove uniqueness we begin by proving controllability of the equations, *i.e.*, to show that the high-frequency noise together with non-linear coupling effectively drives the low-frequency modes. Using this, we then use Malliavin calculus in infinite dimensions, to show regularity of the transition probabilities. This then implies uniqueness of the invariant measure.

We will study the system of equations

$$du_k = -k^2 u_k\,dt + \big(u_k - (u^3)_k\big)\,dt + \frac{q_k}{\sqrt{4\pi(1+k^2)}}\,dw_k(t) , \tag{1.3}$$

with $u \in \mathcal{H}$. The above equations hold for $k \in \mathbf{Z}$, and it is always understood that

$$(u^3)_k = \sum_{\substack{k_1+k_2+k_3=k \\ k_1,k_2,k_3 \in \mathbf{Z}}} u_{k_1}u_{k_2}u_{k_3} , \tag{1.4}$$

with $u_{-k} = \bar{u}_k$. *To avoid inessential notational problems we will work with even periodic functions, so that* $u_k = u_{-k} \in \mathbf{R}$. We will work with the basis

$$e_k(x) = \frac{1}{\sqrt{\pi(1+k^2)}}\cos(kx) . \tag{1.5}$$

Note that this basis is orthonormal w.r.t. the scalar product in $\mathcal{H}$, but the u_k are actually given by $u_k = (4\pi(1+k^2))^{-1/2}\langle u, e_k\rangle$. (We choose this to make the cubic term (1.4) look simple.)

The noise is supposed to act only on the high frequencies, but there we need it to be strong enough in the following way. Let $a_k = k^2 + 1$. Then we require that there exist constants $c_1, c_2 > 0$ such that for $k \ge k_*$,

$$c_1 a_k^{-\alpha} \le q_k \le c_2 a_k^{-\beta} , \qquad \alpha \ge 2 , \quad \alpha - 1/8 < \beta \le \alpha . \tag{1.6}$$

These conditions imply

$$\sum_{k=0}^{\infty}(1+k^{4\alpha-3/2})q_k^2 < \infty ,$$

$$\sup_{k\geq k_*} k^{-2\alpha}q_k^{-1} < \infty .$$

We formulate the problem in a more general setting: Let $F(u)$ be a polynomial of odd degree with negative leading coefficient. Let A be the operator of multiplication by $1+k^2$ and let Q be the operator of multiplication by q_k. Then (1.3) is of the form

$$d\Phi^t \;=\; -A\Phi^t\,dt + F(\Phi^t)\,dt + Q\,dW(t) , \tag{1.7}$$

where $dW(t) = \sum_{k=0}^{\infty} e_k dw_k(t)$ is the cylindrical Wiener process on $\mathcal{H}$ with the w_k mutually independent real Brownian motions.[1] We define $\Phi^t(\xi)$ as the solution of (1.7) with initial condition $\Phi^0(\xi) = \xi$. Clearly, the conditions on Q can be formulated as

$$\|A^{\alpha-3/8}Q\|_{\mathrm{HS}} < \infty , \tag{1.8a}$$

$$q_k^{-1}k^{-2\alpha} \text{ is bounded for } k \geq k_* , \tag{1.8b}$$

where $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm on $\mathcal{H}$. Note that for each k, (1.3) is obtained by multiplying (1.7) by $(4\pi(1+k^2))^{-1/2}\langle\cdot, e_k\rangle$.

Important Remark. The crucial aspect of our conditions is the possibility of choosing $q_k = 0$ for all $k < k_*$, *i.e.*, the noise drives only the high frequencies. But we also allow any of the q_k with $k < k_*$ to be different from 0, which corresponds to long wavelength forcing. Furthermore, as we are allowing α to be arbitrarily large, this means that the forcing at high frequencies has an amplitude which can decay like any power. The point of this paper is to show that these conditions are sufficient to ensure the existence of a unique invariant measure for (1.7).

Theorem 1.1 *The process (1.7) has a unique invariant Borel measure on $\mathcal{H}$.*

There are two main steps in the proof of Theorem 1.1. First, the nature of the nonlinearity F implies that the modes with $k \geq k_*$ couple in such a way to those with $k < k_*$ as to allow *controllability*. Intuitively, this means that any point in phase space can be reached to arbitrary precision in any given time, by a suitable choice of the high-frequency controls.

Second, we show that a version of the Malliavin calculus can be implemented in our infinite-dimensional context. This will be the hard part of our study, and the main result of that part is a proof that the strong Feller property holds. This means that for any measurable function $\varphi \in \mathcal{B}_b(\mathcal{H})$, the function

$$\big(\mathcal{P}^t\varphi\big)(\xi) \;\equiv\; \mathbf{E}\Big(\big(\varphi\circ\Phi^t\big)(\xi)\Big) \tag{1.9}$$

[1] It is convenient to have, in the case of (1.3), $A = 1-\Delta$ and $F(u) = 2u - u^3$ rather than $A = -1-\Delta$ and $F(u) = -u^3$.

is *continuous*.[2] We show this by proving that a cutoff version of (1.7) (modifying the dynamics at large amplitudes by a parameter ϱ) makes $\mathcal{P}_\varrho^t \varphi$ a *differentiable* map.

The interest in such highly degenerate stochastic PDE's is related to questions in hydrodynamics where one would ask how "energy" is transferred from high to low frequency modes, and vice versa when only some of the modes are driven. This could then shed some light on the entropy-enstrophy problem in the (driven) Navier-Stokes equation.

To end this introduction, we will try to compare the results of our paper to current work of others. These groups consider the 2-D Navier Stokes equation without deterministic external forces, also in bounded domains. In these equations, any initial condition eventually converges to zero, as long as there is no stochastic forcing. First there is earlier work by Flandoli-Maslovski [FM95] dealing with noise whose amplitude is bounded below by $|k|^{-c}$. In the work of [BKL00], the stochastic forcing acts on modes with low k, and they get uniqueness of the invariant measure and analyticity, with probability 1. In the work of Kuksin and Shirikyan [KA00] the bounded noise is quite general acts on all Fourier modes, and acts at definite times with "noise-less" intervals in-between. Again, the measure is unique. It is supported by $\mathcal{C}^\infty$ functions, is mixing and has a Gibbs property. In the work of [ESM00], uniqueness is shown for NS by forcing only 3 modes.

The main difference between those results and the present paper is our control of a situation which is already unstable at the deterministic level. Thus, in this sense, it comes closer to a description of a deterministically turbulent fluid (*e.g.*, obtained by an external force). On the other hand, in our work, we need to actually force all high spatial frequencies. Perhaps, this could be eliminated by a combination with ideas from the papers above.

2 Some Preliminaries on the Dynamics

Here, we summarize some facts about deterministic and stochastic GL equations from the literature which we need to get started.

We will consider the dynamics on the following space:

Definition 2.1 *We define $\mathcal{H}$ as the subspace of even functions in $\mathcal{W}^1_{\mathrm{per}}([-\pi,\pi])$. The norm on $\mathcal{H}$ will be denoted by $\|\cdot\|$, and the scalar product by $\langle\cdot,\cdot\rangle$.*

We consider first the deterministic equation

$$\dot{u} = \Delta u + u - u^3 \;, \quad u(0) = u^{(0)} \in \mathcal{H} \;, \tag{2.1}$$

Due to its dissipative character the solutions are, for positive times, analytic in a strip around the real axis. More precisely, denote by $\|\cdot\|_{\mathscr{A}_\eta}$ the norm

$$\|f\|_{\mathscr{A}_\eta} = \sup_{|\mathrm{Im} z| \le \eta} |f(z)| \;,$$

and by $\mathscr{A}_\eta$ the corresponding Banach space of analytic functions. Then the following result holds.

[2] Throughout the paper, **E** denotes expectation and **P** denotes probability for the random variables.

Lemma 2.2 *For every initial value $u^{(0)} \in \mathcal{H}$, there exist a time T and a constant C such that for $0 < t \le T$, the solution $u(t, u^{(0)})$ of (2.1) belongs to $\mathscr{A}_{\sqrt{t}}$ and satisfies $\|u(t, u^{(0)})\|_{\mathscr{A}_{\sqrt{t}}} \le C$.*

Proof. The statement is proven in [Col94] for the case of the infinite line. Since the periodic functions form an invariant subspace under the evolution, the result applies to our case. □

We next collect some useful results for the stochastic equation (1.7):

Proposition 2.3 *For every $t > 0$ and every $p \ge 1$ the solution of (1.7) with initial condition $\Phi^0(\xi) = \xi \in \mathcal{H}$ exists in $\mathcal{H}$ up to time t. It defines by (1.9) a Markovian transition semigroup on $\mathcal{H}$. One has the bound*

$$\mathbf{E}\Big(\sup_{s\in[0,t]} \|\Phi^s(\xi)\|^p\Big) \le C_{t,p}(1 + \|\xi\|)^p \ .$$

Furthermore, the process (1.7) has an invariant measure.

These results are well-known and in Section 8.6 we sketch where to find them in the literature.

3 Controllability

In this section we show the "approximate controllability" of (1.3). The control problem under consideration is

$$\dot u = \Delta u + u - u^3 + Q\, f(t)\ , \qquad u(0) = u^{(\mathrm{i})} \in \mathcal{H}\ , \tag{3.1}$$

where f is the control. Using Fourier series' and the hypotheses on Q, we see that by choosing $f_k \equiv 0$ for $|k| < k_*$, (3.1) can be brought to the form

$$\dot u_k = \begin{cases} -k^2 u_k + u_k - \displaystyle\sum_{\ell+m+n=k} u_\ell u_m u_n + \frac{q_k}{\sqrt{4\pi(1+k^2)}} f_k(t)\ , & |k| \ge k_*\ , \\ -k^2 u_k + u_k - \displaystyle\sum_{\ell+m+n=k} u_\ell u_m u_n\ , & |k| < k_*\ , \end{cases} \tag{3.2}$$

with $\{u_k\} \in \mathcal{H}$ and $t \mapsto \{f_k(t)\} \in \mathrm{L}^\infty([0,\tau],\mathcal{H})$. We will refer in the sequel to $\{u_k\}_{|k|<k_*}$ as the *low-frequency modes* and to $\{u_k\}_{|k|\ge k_*}$ as the *high-frequency modes*. We also introduce the projectors Π_{L} and Π_{H} which project onto the low (resp. high) frequency modes. Let $\mathcal{H}_{\mathrm{L}}$ and $\mathcal{H}_{\mathrm{H}}$ denote the ranges of Π_{L} and Π_{H} respectively. Clearly $\mathcal{H}_{\mathrm{L}}$ is finite dimensional, whereas $\mathcal{H}_{\mathrm{H}}$ is a separable Hilbert space.

The main result of this section is approximate controllability in the following sense:

Theorem 3.1 *For every time $\tau > 0$ the following is true: For every $u^{(\mathrm{i})}, u^{(\mathrm{f})} \in \mathcal{H}$ and every $\varepsilon > 0$, there exists a control $f \in \mathrm{L}^\infty([0,\tau],\mathcal{H})$ such that the solution $u(t)$ of (3.1) with $u(0) = u^{(\mathrm{i})}$ satisfies $\|u(\tau) - u^{(\mathrm{f})}\| \le \varepsilon$.*

Proof. The construction of the control proceeds in 4 different phases, of which the third is the actual controlling of the low-frequency part by the high-frequency controls. In the construction, we will encounter a time $\tau(R, \varepsilon')$ which depends on the norm R of $u^{(\mathrm{f})}$ and some precision ε'. Given this function, we split the given time τ as $\tau = \sum_{i=1}^{4} \tau_i$, with $\tau_4 \le \tau(\|u^{(\mathrm{f})}\|, \varepsilon/2)$ and all $\tau_i > 0$. We will use the cumulated times $t_j = \sum_{i=1}^{j} \tau_i$.

Step 1. In this step we choose $f \equiv 0$, and we define $u^{(1)} = u(t_1)$, where $t \mapsto u(t)$ is the solution of (3.1) with initial condition $u(0) = u^{(\mathrm{i})}$. Since there is no control, we really have (2.1) and hence, by Lemma 2.2, we see that $u^{(1)} \in \mathscr{A}_\eta$ for some $\eta > 0$.

Step 2. We will construct a smooth control $f : [t_1, t_2] \to \mathcal{H}$ such that $u^{(2)} = u(t_2)$ satisfies:

$$\Pi_{\mathrm{H}} u^{(2)} = 0 .$$

In other words, in this step, we drive the high-frequency part to 0. To construct f, we choose a $\mathcal{C}^\infty$ function $\varphi : [t_1, t_2] \to \mathbf{R}$, interpolating between 1 and 0 with vanishing derivatives at the ends. Define $u_{\mathrm{H}}(t) = \varphi(t) \Pi_{\mathrm{H}} u^{(1)}$ for $t \in [t_1, t_2]$. This will be the evolution of the high-frequency part. We next define the low-frequency part $u_{\mathrm{L}} = u_{\mathrm{L}}(t)$ as the solution of the ordinary differential equation

$$\dot{u}_{\mathrm{L}} = \Delta u_{\mathrm{L}} + u_{\mathrm{L}} - \Pi_{\mathrm{L}}\big((u_{\mathrm{L}} + u_{\mathrm{H}})^3\big) ,$$

with $u_{\mathrm{L}}(t_1) = \Pi_{\mathrm{L}} u^{(1)}$. We then set $u(t) = u_{\mathrm{L}}(t) \oplus u_{\mathrm{H}}(t)$ and substitute into (3.1) which we need to solve for the control $Qf(t)$ for $t \in [t_1, t_2]$.

Since $u_{\mathrm{L}}(t) \oplus u_{\mathrm{H}}(t)$ as constructed above is in $\mathscr{A}_\eta$ and since $Qf = \dot{u} - \Delta u - u + u^3$, and Δ maps $\mathscr{A}_\eta$ to $\mathscr{A}_{\eta/2}$ we conclude that $Qf \in \mathscr{A}_{\eta/2}$. By construction, the components q_k of Q decay polynomially with k and do not vanish for $k \ge k_*$. Therefore, Q^{-1} is a bounded operator from $\mathscr{A}_{\eta/2} \cap \mathcal{H}_{\mathrm{H}}$ to $\mathcal{H}_{\mathrm{H}}$. Thus, we can solve for f in this step.

Step 3. As mentioned before, this step really exploits the coupling between high and low frequencies. Here, we start from $u^{(2)}$ at time t_2 and we want to reach $\Pi_{\mathrm{L}} u^{(\mathrm{f})}$ at time t_3. In fact, we will instead reach a point $u^{(3)}$ with $\|\Pi_{\mathrm{L}} u^{(3)} - \Pi_{\mathrm{L}} u^{(\mathrm{f})}\| < \varepsilon/2$.

The idea is to choose for every low frequency $|k| < k_*$ a set of three[3] high frequencies that will be used to control u_k.

Definition 3.2 *We assign to every k with $|k| < k_*$ a set $\mathscr{I}_k \subset \{k : |k| \ge k_*\}$ of three indices. The sets $\mathscr{I}_k$ are disjoint for different k. We also define $\mathscr{I}_{\mathrm{L}} = \{k : |k| < k_*\}$ and*

$$\mathscr{I} = \mathscr{I}_{\mathrm{L}} \cup \Big(\bigcup_{|k| < k_*} \mathscr{I}_k \Big) .$$

The $\mathscr{I}_k$ will be constructed in such a way that they satisfy the following conditions.

(A) *Let $\mathscr{I}_k = \{k_1, k_2, k_3\}$. Then, $k_1 + k_2 + k_3 = k$, and $|k_i| \ge k_*$ for $i = 1, 2, 3$.*

[3] The number 3 is the highest power of the nonlinearity F in the GL equation.

(B) $\mathscr{I}_{-k} = -\mathscr{I}_k$, *where* $-\mathscr{I}_k = \{k \ : \ -k \in \mathscr{I}_k\}$ *(we do not require this for* $k = 0$*).*

(C) *Let* S *be a collection of three indices in* $\mathscr{I}$, $S = \{k_1, k_2, k_3\}$ *and assume* $k_1 + k_2 + k_3 = k$ *and* $|k| < k_*$, *then either* $S = \mathscr{I}_k$, *or* $S \subset \mathscr{I}_{\mathrm{L}}$, *or* S *is of the form* $S = \{k, k', -k'\}$.

The construction of the $\mathscr{I}_k$ is easy and there are many possibilities. For example, assume $k_* = 1000$. Then we choose $\mathscr{I}_0 = \{3'000'000, 2'000'000, -5'000'000\}$. Next choose $\mathscr{I}_1 = \{3'001'001, 2'001'000, -5'002'000\}$. Clearly, these choices satisfy (A) above, and all combination frequencies other than 0 or 1 lie outside the set $\{|k| < 1000\}$. Defining $\mathscr{I}_k = \{3'000'000+1001k, 2'000'000+1000k, -5'000'000-2000k\}$, for $0 \le k < 1000$, and then $\mathscr{I}_{-k} = -\mathscr{I}_k$ for $k \neq 0$, we complete the construction. The generalization to arbitrary k_* is left to the reader.

We are going to construct a control which, in addition to driving the low frequency part as indicated, also implies $u_k(t) \equiv 0$ for $k \notin \mathscr{I}$ for $t \in [t_2, t_3]$. By the conditions on $\mathscr{I}$, the low-frequency part of (3.2) is then equal to (having chosen the controls equal to 0 for $k < k_*$):

$$\dot{u}_k = \Big(1 - k^2 - 3 \sum_{n \in \mathscr{I} \setminus \mathscr{I}_{\mathrm{L}}} |u_n|^2\Big) u_k - \sum_{\substack{\ell+m+n=k \\ \{\ell,m,n\} \subset \mathscr{I}_{\mathrm{L}}}} u_\ell u_m u_n - \prod_{n \in \mathscr{I}_k} u_n \ . \tag{3.3}$$

(For $k = 0$ there is a factor 2 in front of the last product.) This expression is not easy to work with. To simplify the combinatorial problem, we choose the controls of the 3 amplitudes u_n with $n \in \mathscr{I}_k$ in such a way that these u_n are all equal to a fixed real[4] function $z_k(t)$ which we will determine below. With this particular choice, (3.3) reduces for $|k| < k_*$ to

$$0 = -\dot{u}_k + \Big(1 - k^2 - 3 \sum_{|n| < k_*} |z_n|^2\Big) u_k - \big((\Pi_{\mathrm{L}} u)^3\big)_k - z_k^3 \ . \tag{3.4}$$

We now claim that for every path $\gamma \in \mathcal{C}^\infty([t_2, t_3]; \mathcal{H}_{\mathrm{L}})$ and every $\varepsilon > 0$, *we can find a set of bounded functions* $t \mapsto z_k(t)$ *such that the solution of (3.4) shadows* γ *at a distance at most* ε.

To prove this statement, consider the map $F : \mathbf{R}^{k_*} \to \mathbf{R}^{k_*}$ of the form (obtained when substituting the path γ into (3.4))

$$F : \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{k_*-1} \end{pmatrix} \mapsto \begin{pmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{k_*-1}(z) \end{pmatrix} = \begin{pmatrix} 2z_0^3 \\ z_1^3 \\ \vdots \\ z_{k_*-1}^3 \end{pmatrix} + \begin{pmatrix} \mathcal{P}_0(z) \\ \mathcal{P}_1(z) \\ \vdots \\ \mathcal{P}_{k_*-1}(z) \end{pmatrix} ,$$

where the $\mathcal{P}_m$ are polynomials of degree at most 2. We want to find a solution to $F = 0$. The F_m form a Gröbner basis for the ideal of the ring of polynomials they generate. As an immediate consequence, the equation $F(z) = 0$ possesses exactly 3^{k_*} complex

[4] It is here that our choice of even functions u somewhat simplifies the discussion. In the general case, one would have to argue with complex functions z_k, but there are no difficulties with this.

solutions, if they are counted with multiplicities [MS95b]. Since the coefficients of the $\mathcal{P}_m$ are real this implies that there exists at least one real solution.

Having found a (possibly discontinuous) solution for the z_k, we find nearby smooth functions $\tilde{z}_k$ with the following properties:

- The equation (3.4) with $\tilde{z}_k$ replacing z_k and initial condition $u_k(t_2) = u_k^{(2)}$ leads to a solution u with $\|u(t_3) - \Pi_{\rm L} u^{({\rm f})}\| \le \varepsilon/2$.
- One has $\tilde{z}_k(t_3) = 0$.

Having found the $\tilde{z}_k$ we construct the f_k in such a way that for $n \in \mathscr{I}_k$ one has $u_n(t) = \tilde{z}_k(t)$. Finally, for $k \notin \mathscr{I}$ we choose the controls in such a way that $u_k(t) \equiv 0$ for $t \in [t_2, t_3]$. We define $u^{(3)}$ as the solution obtained in this way for $t = t_3$.

Step 4. Starting from $u^{(3)}$ we want to reach $u^{({\rm f})}$. Note that $u^{(3)}$ is in $\mathscr{A}_\eta$ (for every $\eta > 0$) since it has only a finite number of non-vanishing modes. By construction we also have $\|\Pi_{\rm L} u^{(3)} - \Pi_{\rm L} u^{({\rm f})}\| \le \varepsilon/2$. We only need to adapt the high frequency part without moving the low-frequency part too much.

Since $\mathscr{A}_\eta$ is dense in $\mathcal{H}$, there is a $u^{(4)} \in \mathscr{A}_\eta$ with $\|u^{(4)} - u^{({\rm f})}\| \le \varepsilon/4$. By the reasoning of Step 2 there is for every $\tau' > 0$ a control for which $\Pi_{\rm H} u(t_3 + \tau') = \Pi_{\rm H} u^{(4)}$ when starting from $u(t_3) = u^{(3)}$. Given ε there is a τ_* such that if $\tau' < \tau_*$ then $\|\Pi_{\rm L} u(t_3 + \tau') - \Pi_{\rm L} u(t_3)\| < \varepsilon/4$. *This τ_* depends only on $\|u^{({\rm f})}\|$ and ε*, as can be seen from the following argument: Since $\Pi_{\rm H} u^{(3)} = 0$, we can choose the controls in such a way that $\|\Pi_{\rm H} u(t_3 + t)\|$ is an increasing function of t and is therefore bounded by $\|\Pi_{\rm H} u^{({\rm f})}\|$. The equation for the low-frequency part is then a finite dimensional ODE in which all high-frequency contributions can be bounded in terms of $R = \|u^{({\rm f})}\|$.

Combining the estimates we see that

$$
\begin{aligned}
\|u(t_4) - u^{({\rm f})}\| &= \|\Pi_{\rm L}(u(t_4) - u^{({\rm f})})\| + \|\Pi_{\rm H}(u(t_4) - u^{({\rm f})})\| \\
&\le \|\Pi_{\rm L}(u(t_4) - u(t_3))\| + \|\Pi_{\rm L}(u(t_3) - u^{({\rm f})})\| \\
&\quad + \|\Pi_{\rm H}(u^{(4)} - u^{({\rm f})})\| \le \varepsilon \;.
\end{aligned}
$$

The proof of Theorem 3.1 is complete. □

4 Strong Feller Property and Proof of Theorem 1.1

The aim of this section is to show the strong Feller property of the process defined by (1.3) resp. (1.7).

Theorem 4.1 *The Markov semigroup $\mathcal{P}^t$ defined in (1.9) is strong Feller.*

Proof of Theorem 1.1. This proof follows a well-known strategy, see *e.g.*, [DPZ96]. First of all, there is at least one invariant measure for the process (1.7), since for a problem in a finite domain, the semigroup $t \mapsto e^{-At}$ is compact, and therefore [DPZ96, Theorem 6.1.2] applies.

By the controllability Theorem 3.1, we deduce, see [DPZ96, Theorem 7.4.1], that the transition probability from any point in $\mathcal{H}$ to any open set in $\mathcal{H}$ cannot vanish, *i.e.*, the Markov process is irreducible. Furthermore, by Theorem 4.1 the process is strong Feller. Therefore we can use Doob's theorem [DPZ96, pp.42–43] to conclude that the invariant measure is unique. This completes the proof of Theorem 1.1. □

Before we start with the proof of Theorem 4.1, we explain our strategy. Since the nonlinearity F in (1.3) is unbounded, we consider first a cutoff version F_ϱ of F:

$$F_\varrho(x) \;=\; \bigl(1 - \chi\bigl(\|x\|/(3\varrho)\bigr)\bigr)F(x) \;,$$

where χ is a smooth, non-negative function satisfying

$$\chi(z) \;=\; \begin{cases} 1 & \text{if } z > 2, \\ 0 & \text{if } z < 1. \end{cases}$$

Similarly, we define

$$Q_\varrho(x) \;=\; Q + \chi(\|x\|/\varrho)\Pi_{k_*} \;, \tag{4.1}$$

where Π_{k_*} is the projection onto the frequencies below k_*.

Remark 4.2 These cutoffs have the following effect as a function of $\|x\|$:

- When $\|x\| \le \varrho$ then $Q_\varrho(x) = Q$ and $F_\varrho(x) = F(x)$.
- When $\varrho < \|x\| \le 2\varrho$ then $Q_\varrho(x)$ depends on x and $F_\varrho(x) = F(x)$.
- When $2\varrho < \|x\| \le 6\varrho$ then all Fourier components of $Q_\varrho(x)$ *including the ones below* k_* are non-zero and $F_\varrho(x)$ is proportional to a $F(x)$ times a factor ≤ 1.
- When $6\varrho < \|x\|$ then all Fourier components of $Q_\varrho(x)$ *including the ones below* k_* are non-zero and $F_\varrho(x) = 0$.

Thus, at high amplitudes, the nonlinearity is truncated to 0 and the stochastic forcing extends to *all* degrees of freedom.

Instead of (1.7) we then consider the modified problem

$$d\Phi^t_\varrho = -A\Phi^t_\varrho\,dt + \bigl(F_\varrho \circ \Phi^t_\varrho\bigr)\,dt + \bigl(Q_\varrho \circ \Phi^t_\varrho\bigr)\,dW(t) \;, \tag{4.2}$$

with $\Phi^0_\varrho(\xi) = \xi \in \mathcal{H}$. Note that the cutoffs are chosen in such a way that the dynamics of $\Phi^t_\varrho(\xi)$ *coincides* with that of $\Phi^t(\xi)$ as long as $\|\Phi^t(\xi)\| < \varrho$. Furthermore, as will be seen later, the definition of Q_ϱ has been made in such a way as to *preserve controllability*. We will show that the solution of (4.2) defines a Markov semigroup

$$\mathcal{P}^t_\varrho\varphi(\xi) = \mathbf{E}\bigl(\varphi \circ \Phi^t_\varrho\bigr)(\xi) \;,$$

with the following smoothing property:

Theorem 4.3 *There exist exponents $\mu, \nu > 0$, and for all $\varrho > \varrho_0$ there is a constant C_ϱ such that for every $\varphi \in \mathcal{B}_b(\mathcal{H})$, for every $t > 0$ and for every $\xi \in \mathcal{H}$, the function $\mathcal{P}^t_\varrho\varphi$ is differentiable and its derivative satisfies*

$$\|D\mathcal{P}^t_\varrho\varphi(\xi)\| \le C_\varrho t^{-\mu}(1 + \|\xi\|^\nu)\|\varphi\|_{\mathrm{L}^\infty} \;. \tag{4.3}$$

Using this theorem, the proof of Theorem 4.1 follows from a limiting argument.

Proof of Theorem 4.1. Choose $x \in \mathcal{H}$, $t > 0$, and $\varepsilon > 0$. We denote by $\mathcal{B}$ the ball of radius $2\|x\|$ centered around the origin in $\mathcal{H}$. Using Proposition 2.3 we can find a constant ϱ (sufficiently large) such that for every $y \in \mathcal{B}$, the inequality

$$\mathbf{P}\Bigl(\sup_{s\in[0,t]}\bigl\|\Phi^s(y)\bigr\| > \varrho\Bigr) \le \frac{\varepsilon}{8}$$

holds. Choose $\varphi \in \mathcal{B}_b(\mathcal{H})$ with $\|\varphi\|_{\mathrm{L}^\infty} \le 1$. We have by the triangle inequality

$$\begin{aligned}\bigl|\mathcal{P}^t\varphi(x) - \mathcal{P}^t\varphi(y)\bigr| \le \bigl|\mathcal{P}^t\varphi(x) - \mathcal{P}^t_\varrho\varphi(x)\bigr| &+ \bigl|\mathcal{P}^t_\varrho\varphi(x) - \mathcal{P}^t_\varrho\varphi(y)\bigr| \\ &+ \bigl|\mathcal{P}^t\varphi(y) - \mathcal{P}^t_\varrho\varphi(y)\bigr| \;.\end{aligned}$$

Since the dynamics of the cutoff equation and the dynamics of the original equation coincide on the ball of radius ϱ, we can write, for every $z \in \mathcal{B}$,

$$\begin{aligned}\bigl|\mathcal{P}^t\varphi(z) - \mathcal{P}^t_\varrho\varphi(z)\bigr| &= \mathbf{E}\bigl|\bigl(\varphi\circ\Phi^t\bigr)(z) - \bigl(\varphi\circ\Phi^t_\varrho\bigr)(z)\bigr| \\ &\le 2\|\varphi\|_{\mathrm{L}^\infty}\; \mathbf{P}\Bigl(\sup_{s\in[0,t]}\bigl\|\Phi^s(z)\bigr\| > \varrho\Bigr) \le \frac{\varepsilon}{4}\;.\end{aligned}$$

This implies that

$$\bigl|\mathcal{P}^t\varphi(x) - \mathcal{P}^t\varphi(y)\bigr| \le \frac{\varepsilon}{2} + \bigl|\mathcal{P}^t_\varrho\varphi(x) - \mathcal{P}^t_\varrho\varphi(y)\bigr|\;.$$

By Theorem 4.3 we see that if y is sufficiently close to x then

$$\bigl|\mathcal{P}^t_\varrho\varphi(x) - \mathcal{P}^t_\varrho\varphi(y)\bigr| \le \frac{\varepsilon}{2}\;.$$

Since ε is arbitrary we conclude that $\mathcal{P}^t\varphi$ is continuous when $\|\varphi\|_{\mathrm{L}^\infty} \le 1$. The generalization to any value of $\|\varphi\|_{\mathrm{L}^\infty}$ follows by linearity in φ. The proof of Theorem 4.1 is complete. □

5 Regularity of the Cutoff Process

In this section, we start the proof of Theorem 4.3. If the cutoff problem were finite dimensional, a result like Theorem 4.3 could be derived easily using, *e.g.*, the works of Hörmander [Hör67, Hör85] or Norris [Nor86]. In the present infinite-dimensional context we need to modify the corresponding techniques, but the general idea retained is Norris'. The main idea will be to treat the (infinite number of) high-frequency modes by a method which is an extension of [DPZ96, Cer99], while the low-frequency part is handled by a variant of the Malliavin calculus adapted from [Nor86]. It is at the juncture of these two techniques that we need a cutoff in the nonlinearity.

5.1 Splitting and Interpolation Spaces

Throughout the remainder of this paper, we will again denote by $\mathcal{H}_{\rm L}$ and $\mathcal{H}_{\rm H}$ the spaces corresponding to the low (resp. high)-frequency parts. We slightly change the meaning of "low-frequency" by including in the low-frequency part all those frequencies that are driven by the noise which are in $\mathscr{I}$ as defined above. More precisely, the low-frequency part is now $\{k \;:\; |k| \le L-1\}$, where $L = \max\{k \;:\; k \in \mathscr{I}\}+1$. The important fact is that

- $\mathcal{H}_{\rm L}$ is finite dimensional and
- for every unforced frequency k, there exist three different forced frequencies k_1, k_2, k_3 in the low-frequency part for which $k_1+k_2+k_3 = k$, and, as explained above, none of these frequencies is in the triplet for another k.

Since $A = 1-\Delta$ is diagonal with respect to this splitting, we can define its low (resp. high)-frequency parts $A_{\rm L}$ and $A_{\rm H}$ as operators on $\mathcal{H}_{\rm L}$ and $\mathcal{H}_{\rm H}$. From now on, L will always denote the dimension of $\mathcal{H}_{\rm L}$, which will therefore be identified with $\mathbf{R}^L$.[5] We also allow ourselves to switch freely between equivalent norms on $\mathbf{R}^L$, when deriving the various bounds.

In the sequel, we will always use the notations $D_{\rm L}$ and $D_{\rm H}$ to denote the derivatives with respect to $\mathcal{H}_{\rm L}$ (resp. $\mathcal{H}_{\rm H}$) of a differentiable function defined on $\mathcal{H}$. The words "derivative" and "differentiable" will always be understood in the strong sense, *i.e.*, if $f : \mathscr{B}_1 \to \mathscr{B}_2$ with $\mathscr{B}_1$ and $\mathscr{B}_2$ some Banach spaces, then $Df : \mathscr{B}_1 \to \mathscr{L}(\mathscr{B}_1, \mathscr{B}_2)$, *i.e.*, it is bounded from $\mathscr{B}_1$ to $\mathscr{B}_2$.

We introduce the interpolation spaces $\mathcal{H}^\gamma$ (for every $\gamma \ge 0$) defined as being equal to the domain of A^γ equipped with the graph norm

$$\|x\|_\gamma^2 = \|A^\gamma x\|^2 = \|(1-\Delta)^\gamma x\|^2 \;.$$

Clearly, the $\mathcal{H}^\gamma$ are Hilbert spaces and we have the inclusions

$$\mathcal{H}^\gamma \subset \mathcal{H}^\delta \quad \text{if} \quad \gamma \ge \delta \;.$$

Note that in usual conventions, $\mathcal{H}^\gamma$ would be the Sobolev space of index $2\gamma+1$. Our motivation for using non-standard notation comes from the fact that our basic space is that with *one* derivative, which we call $\mathcal{H}$, and that γ measures additional smoothness in terms of powers of the generator of the linear part.

5.2 Proof of Theorem 4.3

The proof of Theorem 4.3 is based on Proposition 5.1 and Proposition 5.2 which we now state.

Proposition 5.1 *Assume that the noise satisfies condition (1.6). Then (4.2) defines a stochastic flow Φ_ϱ^t on $\mathcal{H}$ with the following properties which hold for any $p \ge 1$:*

(A) *If $\xi \in \mathcal{H}^\gamma$ with some γ satisfying $0 \le \gamma \le \alpha$, the solution of (4.2) stays in $\mathcal{H}^\gamma$, with a bound*

$$\mathbf{E}\Big(\sup_{0<t<T} \|\Phi_\varrho^t(\xi)\|_\gamma^p\Big) \le C_{T,p,\varrho}(1+\|\xi\|_\gamma)^p \;. \tag{5.1a}$$

[5] The choice of L above is dictated by the desire to obtain a dimension equal to L and not $L+1$.

If $\gamma \geq 1$ the solution exists in the strong sense in $\mathcal{H}$.

(B) *The quantity $\Phi_\varrho^t(\xi)$ is in $\mathcal{H}^\alpha$ with probability 1 for every time $t > 0$ and every $\xi \in \mathcal{H}$. Furthermore, for every $T > 0$ there is a constant $C_{T,p,\varrho}$ for which*

$$\mathbf{E}\Big(\sup_{0<t<T} t^{\alpha p}\|\Phi_\varrho^t(\xi)\|_\alpha^p\Big) \leq C_{T,p,\varrho}(1+\|\xi\|)^p \;. \tag{5.1b}$$

(C) *The mapping $\xi \mapsto \Phi_\varrho^t(\xi)$ (for ω and t fixed) has a.s. bounded partial derivatives with respect to ξ. Furthermore, we have for every $\xi, h \in \mathcal{H}$ the bound*

$$\mathbf{E}\Big(\sup_{0<t<T}\big\|\big(D\Phi_\varrho^t(\xi)\big)h\big\|^p\Big) \leq C_{T,p,\varrho}\|h\|^p \;, \tag{5.1c}$$

for every $T > 0$.

(D) *For every $h \in \mathcal{H}$ and $\xi \in \mathcal{H}^\alpha$, the quantity $\big(D\Phi_\varrho^t(\xi)\big)h$ is in $\mathcal{H}^\alpha$ with probability 1 for every $t > 0$. Furthermore, for a ν depending only on α the bound*

$$\mathbf{E}\Big(\sup_{0<t<T} t^{\alpha p}\big\|\big(D\Phi_\varrho^t(\xi)\big)h\big\|_\alpha^p\Big) \leq C_{T,p,\varrho}(1+\|\xi\|_\alpha)^{\nu p}\|h\|^p \;, \tag{5.1d}$$

holds for every $T > 0$.

(E) *For every $\xi \in \mathcal{H}^\gamma$ with $\gamma \leq \alpha$, we have the small-time estimate*

$$\mathbf{E}\Big(\sup_{0<t<\varepsilon}\big\|\Phi_\varrho^t(\xi) - e^{-At}\xi\big\|_\gamma^p\Big) \leq C_{T,p,\varrho}(1+\|\xi\|_\gamma)^p\varepsilon^{p/16} \;, \tag{5.1e}$$

which holds for every $\varepsilon \in (0,T]$ and every $T > 0$.

This proposition will be proved in Section 8.4.

Proposition 5.2 *There exist a time $T^* > 0$ and exponents $\mu, \nu > 0$ such that for every $\varphi \in \mathcal{C}_b^2(\mathcal{H})$, every $\xi \in \mathcal{H}^\alpha$ and every $t \leq T^*$,*

$$\|D\mathcal{P}_\varrho^t\varphi(\xi)\| \leq Ct^{-\mu}\big(1+\|\xi\|_\alpha^\nu\big)\|\varphi\|_{\mathrm{L}^\infty} \;. \tag{5.2}$$

Proof of Theorem 4.3. We first prove the bound for the case $\varphi \in \mathcal{C}_b^2(\mathcal{H})$. Let $h \in \mathcal{H}$. Using the definition (1.9) of $\mathcal{P}_\varrho^t\varphi$ and the Markov property of the flow we write

$$\begin{aligned}\|D\mathcal{P}_\varrho^{2t}\varphi(\xi)h\| &= \big\|D\mathbf{E}\big(\mathcal{P}_\varrho^t\varphi\circ\Phi_\varrho^t\big)(\xi)h\big\| = \Big\|\mathbf{E}\Big(\big(D\mathcal{P}_\varrho^t\varphi\circ\Phi_\varrho^t\big)(\xi)D\Phi_\varrho^t(\xi)h\Big)\Big\| \\ &\leq \sqrt{\mathbf{E}\big\|\big(D\mathcal{P}_\varrho^t\varphi\circ\Phi_\varrho^t\big)(\xi)\big\|^2}\sqrt{\mathbf{E}\big\|D\Phi_\varrho^t(\xi)h\big\|^2} \;.\end{aligned}$$

Bounding the first square root by Proposition 5.2 and then applying Proposition 5.1, we get a bound

$$\begin{aligned}\|D\mathcal{P}_\varrho^{2t}\varphi(\xi)h\| &\leq C\|\varphi\|_{\mathrm{L}^\infty}t^{-\mu}\sqrt{\mathbf{E}\big(1+\|\Phi_\varrho^t(\xi)\|_\alpha^\nu\big)^2}\sqrt{\mathbf{E}\big\|D\Phi_\varrho^t(\xi)h\big\|^2} \\ &\leq C\|\varphi\|_{\mathrm{L}^\infty}t^{-\mu}t^{-\alpha\nu}(1+\|\xi\|)^\nu\|h\| \;.\end{aligned}$$

Thus, we have shown (4.3) when $\varphi \in \mathcal{C}_b^2(\mathcal{H})$. The method of extension to arbitrary $\varphi \in \mathcal{B}_b(\mathcal{H})$ can be found in [DPZ96, Lemma 7.1.5]. The proof of Theorem 4.3 is complete. □

5.3 Smoothing Properties of the Transition Semigroup

In this subsection we prove the smoothing bound Proposition 5.2. Thus, we will no longer be interested in smoothing in position space as shown in Proposition 5.1 but in smoothing properties of the transition semigroup associated to (4.2).

Important remark. In this section and up to Section 8.6 we always tacitly assume that we are considering the cutoff equation (4.2) and we will omit the index ϱ.

Thus, we will write Eq.(4.2) as

$$d\Phi^t = -A\Phi^t \, dt + \big(F \circ \Phi^t\big) \, dt + \big(Q \circ \Phi^t\big) \, dW(t) \; . \tag{5.3}$$

The solution of (5.3) generates a semigroup on the space $\mathcal{B}_b(\mathcal{H})$ of bounded Borel functions over $\mathcal{H} = \mathcal{H}_{\mathrm{L}} \oplus \mathcal{H}_{\mathrm{H}}$ by

$$\mathcal{P}^t \varphi = \mathbf{E}\big(\varphi \circ \Phi^t\big) \; , \qquad \varphi \in \mathcal{B}_b(\mathcal{H}) \; .$$

Our goal will be to show that the mixing properties of the nonlinearity are strong enough to make $\mathcal{P}^t\varphi$ differentiable, even if φ is only measurable.

We will need a separate treatment of the high and low frequencies, and so we reformulate (5.3) as

$$\begin{aligned} d\Phi^t_{\mathrm{L}} &= -A_{\mathrm{L}}\Phi^t_{\mathrm{L}} \, dt + \big(F_{\mathrm{L}} \circ \Phi^t\big) \, dt + \big(Q_{\mathrm{L}} \circ \Phi^t\big) \, dW_{\mathrm{L}}(t) \; , \qquad && \Phi^t_{\mathrm{L}} \in \mathcal{H}_{\mathrm{L}} \; , && (5.4\mathrm{a}) \\ d\Phi^t_{\mathrm{H}} &= -A_{\mathrm{H}}\Phi^t_{\mathrm{H}} \, dt + \big(F_{\mathrm{H}} \circ \Phi^t\big) \, dt + Q_{\mathrm{H}} \, dW_{\mathrm{H}}(t) \; , && \Phi^t_{\mathrm{H}} \in \mathcal{H}_{\mathrm{H}} \; , && (5.4\mathrm{b}) \end{aligned}$$

where $\mathcal{H}_{\mathrm{L}}$ and $\mathcal{H}_{\mathrm{H}}$ are defined in Section 5.1 and the cutoff version of Q was defined in (4.1). Note that $Q_{\mathrm{H}}\big(\Phi^t(\xi)\big)$ is independent of ξ and t by construction, which is why we can use Q_{H} in (5.4b).

The proof of Proposition 5.2 is based on the following two results dealing with the low-frequency part and the cross-terms between low and high frequencies, respectively.

Proposition 5.3 *There exist a time $T^* > 0$ and exponents $\mu, \nu > 0$ such that for every $\varphi \in \mathcal{C}^2_b(\mathcal{H})$, every $\xi \in \mathcal{H}^\alpha$ and every $t \le T^*$,*

$$\Big\| \mathbf{E}\Big(\big(D_{\mathrm{L}}\varphi \circ \Phi^t\big)(\xi)\big(D_{\mathrm{L}}\Phi^t_{\mathrm{L}}\big)(\xi) \Big) \Big\| \le C t^{-\mu} \big(1 + \|\xi\|^\nu_\alpha\big) \|\varphi\|_{\mathrm{L}^\infty} \; .$$

Lemma 5.4 *For every $T > 0$ and every $p \ge 1$, there are constants $C_1, C_2 > 0$ such that for every $t \le T$, one has the estimates (valid for $h_{\mathrm{L}} \in \mathcal{H}_{\mathrm{L}}$ and $h_{\mathrm{H}} \in \mathcal{H}_{\mathrm{H}}$):*

$$\mathbf{E} \sup_{0<s<t} \big\| \big(D_{\mathrm{L}}\Phi^s_{\mathrm{H}}\big)(\xi) h_{\mathrm{L}} \big\|^p \le C_1 t^p \|h_{\mathrm{L}}\|^p \; , \tag{5.5a}$$

$$\mathbf{E} \sup_{0<s<t} \big\| \big(D_{\mathrm{H}}\Phi^s_{\mathrm{L}}\big)(\xi) h_{\mathrm{H}} \big\|^p \le C_2 t^{p/4} \|h_{\mathrm{H}}\|^p \; . \tag{5.5b}$$

These bounds are independent of $\xi \in \mathcal{H}$.

Remark 5.5 In the absence of the cutoff ϱ one can prove inequalities like (5.5), but with an additional factor of $(1 + \|\xi\|^2)^p$ on the right. This is not good enough for our strategy and is the reason for introducing a cutoff.

The proof of Proposition 5.3 will be given in Section 6 and the proof of Lemma 5.4 will be given in Section 8.5.

Proof of Proposition 5.2. The proof will be performed in the spirit of [DPZ96] and [Cer99], using a modified version of the Bismut-Elworthy formula. Take a function $\varphi \in \mathcal{C}_b^2(\mathcal{H})$. We consider $Q_{\rm L}$ and $Q_{\rm H}$ as acting on and into $\mathcal{H}_{\rm L}$ and $\mathcal{H}_{\rm H}$ respectively. It is possible to write as a consequence of Itô's formula:

$$\begin{aligned}\big(\varphi\circ\Phi^t\big)(\xi) &= \mathcal{P}^t\varphi(\xi) + \int_0^t \big((D\mathcal{P}^{t-s}\varphi)\circ\Phi^s\big)(\xi)\,\big(Q\circ\Phi^s\big)(\xi)\,dW(s)\\ &= \mathcal{P}^t\varphi(\xi) + \int_0^t \big((D_{\rm L}\mathcal{P}^{t-s}\varphi)\circ\Phi^s\big)(\xi)\,\big(Q_{\rm L}\circ\Phi^s\big)(\xi)\,dW_{\rm L}(s)\\ &\qquad + \int_0^t \big((D_{\rm H}\mathcal{P}^{t-s}\varphi)\circ\Phi^s\big)(\xi)\,Q_{\rm H}\,dW_{\rm H}(s)\;. \end{aligned}\tag{5.6}$$

Choose some $h\in\mathcal{H}_{\rm H}$. By Proposition 5.1, $\big(D_{\rm H}\Phi^t_{\rm H}\big)(\xi)h$ is in $\mathcal{H}^\alpha$ for positive times and is bounded by (5.1d). Using condition (1.8b) we see that $Q_{\rm H}^{-1}$ maps to $\mathcal{H}_{\rm H}$ and so we can multiply both sides of (5.6) by

$$\int_{t/4}^{3t/4}\Big\langle Q_{\rm H}^{-1}\big(D_{\rm H}\Phi^s_{\rm H}\big)(\xi)h\,,\,dW_{\rm H}(s)\Big\rangle\;,$$

where the scalar product is taken in $\mathcal{H}_{\rm H}$. Taking expectations on both sides, the first two terms on the right vanish because $dW_{\rm L}$ and $dW_{\rm H}$ are independent and of mean zero. Thus, we get

$$\begin{aligned}&\mathbf{E}\Big(\big(\varphi\circ\Phi^t\big)(\xi)\int_{t/4}^{3t/4}\Big\langle Q_{\rm H}^{-1}\big(D_{\rm H}\Phi^s_{\rm H}\big)(\xi)h\,,\,dW_{\rm H}(s)\Big\rangle\Big)\\ &\qquad = \mathbf{E}\int_{t/4}^{3t/4}\big((D_{\rm H}\mathcal{P}^{t-s}\varphi)\circ\Phi^s\big)(\xi)\,\big(D_{\rm H}\Phi^s_{\rm H}\big)(\xi)h\,ds\;,\end{aligned}\tag{5.7}$$

We add to both sides of (5.7) the term

$$\mathbf{E}\int_{t/4}^{3t/4}\big((D_{\rm L}\mathcal{P}^{t-s}\varphi)\circ\Phi^s\big)(\xi)\,\big(D_{\rm H}\Phi^s_{\rm L}\big)(\xi)h\,ds\;,$$

and note that the r.h.s. can be rewritten as

$$\int_{t/4}^{3t/4} D_{\rm H}\mathbf{E}\big((\mathcal{P}^{t-s}\varphi)\circ\Phi^s\big)(\xi)h\,ds = \frac{t}{2}D_{\rm H}\mathbf{E}\big(\varphi\circ\Phi^t\big)(\xi)h\;,$$

since by the Markov property, $\mathbf{E}\big(\mathcal{P}^{t-s}\varphi\circ\Phi^s\big)(\xi) = \mathbf{E}\big(\varphi\circ\Phi^t\big)(\xi)$. Therefore, (5.7) leads to

$$\begin{aligned}\big(D_{\rm H}\mathcal{P}^t\varphi\big)(\xi)h &= \frac{2}{t}\mathbf{E}\Big(\big(\varphi\circ\Phi^t\big)(\xi)\int_{t/4}^{3t/4}\big\langle Q_{\rm H}^{-1}\big(D_{\rm H}\Phi^s_{\rm H}\big)(\xi)h, dW_{\rm H}(s)\big\rangle\Big)\\ &\qquad + \frac{2}{t}\mathbf{E}\int_{t/4}^{3t/4}\big((D_{\rm L}\mathcal{P}^{t-s}\varphi)\circ\Phi^s\big)(\xi)\,\big(D_{\rm H}\Phi^s_{\rm L}\big)(\xi)h\,ds\;.\end{aligned}\tag{5.8}$$

For the low-frequency part, we use the equality

$$\begin{aligned}\big(D_{\mathrm{L}}\mathcal{P}^t\varphi\big)(\xi) = \mathbf{E}\Big(\big(D_{\mathrm{L}}\mathcal{P}^{t/2}\varphi\circ\Phi^{t/2}\big)(\xi)\big(D_{\mathrm{L}}\Phi_{\mathrm{L}}^{t/2}\big)(\xi)\Big)&\\ +\mathbf{E}\Big(\big(D_{\mathrm{H}}\mathcal{P}^{t/2}\varphi\circ\Phi^{t/2}\big)(\xi)\big(D_{\mathrm{L}}\Phi_{\mathrm{H}}^{t/2}\big)(\xi)\Big)\;.&\end{aligned}\tag{5.9}$$

We introduce the Banach spaces $\mathscr{B}_{T,\mu_*,\nu_*}$ of measurable functions $f:(0,T)\times\mathcal{H}^\alpha\to\mathcal{H}$, for which

$$|||f|||_{T,\mu_*,\nu_*}\equiv\sup_{0<t<T}\sup_{\xi\in\mathcal{H}^\alpha}\frac{t^{\mu_*}\|f(t,\xi)\|}{1+\|\xi\|_\alpha^{\nu_*}}\tag{5.10}$$

is finite. Choose μ_* as the maximum of the constants α and the μ appearing in Proposition 5.3. Similarly ν_* is the maximum of the ν of Proposition 5.1 (D) and the one in Proposition 5.3.

We will show that there exists a $T>0$ such that $f_\varphi:(t,\xi)\mapsto\big(D\mathcal{P}^t\varphi\big)(\xi)$ belongs to $\mathscr{B}_{T,\mu_*,\nu_*}$ and that $|||f_\varphi|||_{T,\mu_*,\nu_*}\le C\|\varphi\|_{\mathrm{L}^\infty}$, thus proving Proposition 5.2. The fact that $f_\varphi\in\mathscr{B}_{T,\mu_*,\nu_*}$ for every T if $\varphi\in\mathcal{C}_b^2(\mathcal{H})$ is shown in [DPZ96], so we only have to show the bound on its norm.

By Proposition 5.1, Eqs.(5.5b) and (5.8), and the Cauchy-Schwarz inequality, we have the estimate, valid for $h\in\mathcal{H}_{\mathrm{H}}$:

$$\begin{aligned}\big|\big(D_{\mathrm{H}}\mathcal{P}^t\varphi\big)(\xi)h\big| &\le \|\varphi\|_{\mathrm{L}^\infty}\frac{2}{t}\left(\mathbf{E}\int_{t/4}^{3t/4}\big\|Q_{\mathrm{H}}^{-1}\big(D_{\mathrm{H}}\Phi_{\mathrm{H}}^s\big)(\xi)h\big\|^2\,ds\right)^{1/2}\\
&\quad+\frac{2}{t}|||f_\varphi|||_{t,\mu_*,\nu_*}\,\mathbf{E}\int_{t/4}^{3t/4}\frac{1+\|\Phi^s(\xi)\|_\alpha^{\nu_*}}{(t-s)^{\mu_*}}\big\|\big(D_{\mathrm{H}}\Phi_{\mathrm{L}}^s\big)(\xi)h\big\|\,ds\\
&\le Ct^{-\alpha}\|\varphi\|_{\mathrm{L}^\infty}\big(1+\|\xi\|_\alpha^{\nu_*}\big)\|h\|\\
&\quad+Ct^{-\mu_*}|||f_\varphi|||_{t,\mu_*,\nu_*}\left(\mathbf{E}\sup_{s\in[\frac{t}{4},\frac{3t}{4}]}\big(1+\|\Phi^s(\xi)\|_\alpha^{\nu_*}\big)^2\right)^{1/2}\\
&\qquad\times\left(\mathbf{E}\sup_{s\in[\frac{t}{4},\frac{3t}{4}]}\big\|\big(D_{\mathrm{H}}\Phi_{\mathrm{L}}^s\big)(\xi)h\big\|^2\right)^{1/2}\\
&\le Ct^{-\alpha}\|\varphi\|_{\mathrm{L}^\infty}\big(1+\|\xi\|_\alpha^{\nu_*}\big)\|h\|+Ct^{-\mu_*+1/4}|||f_\varphi|||_{t,\mu_*,\nu_*}(1+\|\xi\|_\alpha^{\nu_*})\|h\|\;.\end{aligned}$$

For the low-frequency part, we use Proposition 5.3, Eqs.(5.5a) and (5.9), and the property that $\|\mathcal{P}^t\varphi\|_{\mathrm{L}^\infty}\le\|\varphi\|_{\mathrm{L}^\infty}$ and find for $h\in\mathcal{H}_{\mathrm{L}}$,

$$\begin{aligned}\big|\big(D_{\mathrm{L}}\mathcal{P}^t\varphi\big)(\xi)h\big| &\le Ct^{-\mu_*}\|\varphi\|_{\mathrm{L}^\infty}\big(1+\|\xi\|_\alpha^{\nu_*}\big)\,\|h\|\\
&\quad+Ct^{-\mu_*}|||f_\varphi|||_{t,\mu_*,\nu_*}\mathbf{E}\Big(\big(1+\|\Phi^{t/2}(\xi)\|_\alpha^{\nu_*}\big)\big\|\big(D_{\mathrm{L}}\Phi_{\mathrm{H}}^{t/2}\big)(\xi)h\big\|\Big)\\
&\le Ct^{-\mu_*}\|\varphi\|_{\mathrm{L}^\infty}\big(1+\|\xi\|_\alpha^{\nu_*}\big)\,\|h\|\\
&\quad+Ct^{-\mu_*}|||f_\varphi|||_{t,\mu_*,\nu_*}\sqrt{\mathbf{E}\big(1+\|\Phi^{t/2}(\xi)\|_\alpha^{\nu_*}\big)^2}\sqrt{\mathbf{E}\big\|\big(D_{\mathrm{L}}\Phi_{\mathrm{H}}^{t/2}\big)(\xi)h\big\|^2}\\
&\le Ct^{-\mu_*}\|\varphi\|_{\mathrm{L}^\infty}\big(1+\|\xi\|_\alpha^{\nu_*}\big)\,\|h\|+Ct^{-\mu_*+1}|||f_\varphi|||_{t,\mu_*,\nu_*}\big(1+\|\xi\|_\alpha^{\nu_*}\big)\,\|h\|\;.\end{aligned}$$

Combining the above expressions we get for every $T < T^*$ a bound of the type

$$|\!|\!| f_\varphi |\!|\!|_{T,\mu_*,\nu_*} \le C_1 \|\varphi\|_{\mathrm{L}^\infty} + C_2 T^{1/4} |\!|\!| f_\varphi |\!|\!|_{T,\mu_*,\nu_*} .$$

Choosing $T^{1/4} < 1/(2C_2)$, we find

$$|\!|\!| f_\varphi |\!|\!|_{T,\mu_*,\nu_*} \le C \|\varphi\|_{\mathrm{L}^\infty} . \tag{5.11}$$

Since $f_\varphi(t,\xi) = \big(D\mathcal{P}^t\varphi\big)(\xi)$, inspection of (5.10) shows that (5.11) is equivalent to (5.2). The proof of Proposition 5.2 is complete. □

6 Malliavin Calculus

To prove Proposition 5.3 we will apply a modification of Norris' version of the Malliavin calculus. This modification takes into account some new features which are necessary due to our splitting of the problem in high and low frequencies (which in turn was done to deal with the infinite dimensional nature of the problem).

Consider first the deterministic PDE for a flow:

$$\frac{d\Psi^t(\xi)}{dt} = -A\Psi^t(\xi) + \big(F \circ \Psi^t\big)(\xi) . \tag{6.1}$$

This is really an abstract reformulation for the flow defined by the GL equation, and ξ belongs to a space $\mathcal{H}$, which for our problem is a suitable Sobolev space. The linear operator A is chosen as $1 - \Delta$, while the non-linear term F corresponds to $2u - u^3$ in the GL equation. Below, we will work with approximations to the GL equation, and all we need to know is that $A : \mathcal{H} \to \mathcal{H}$ is the generator of a strongly continuous semigroup, and F will be seen to be bounded with bounded derivatives.

For each fixed $\xi \in \mathcal{H}$ we consider the following stochastic variant of (6.1):

$$d\Psi^t(\xi) = -A\Psi^t(\xi)\, dt + \big(F \circ \Psi^t\big)(\xi)\, dt + \big(Q \circ \Psi^t\big)(\xi)\, dW(t) . \tag{6.2}$$

with initial condition $\Psi^0(\xi) = \xi$. Furthermore, W is the cylindrical Wiener process on a separable Hilbert space $\mathcal{W}$ and Q is a strongly differentiable map from $\mathcal{H}$ to $\mathscr{L}^2(\mathcal{W},\mathcal{H})$, the space of bounded linear Hilbert-Schmidt operators from $\mathcal{W}$ to $\mathcal{H}$.

We next introduce the notion of directional derivative (in the direction of the noise) and the reader familiar with this concept can pass directly to (6.3). To understand this concept consider first the case of a function $t \mapsto v_i^t \in \mathcal{W}$. Then the variation $\mathscr{D}_{v_i}\Psi^t$ of Ψ^t in the direction v_i is obtained by replacing $dW(t)$ by $dW(t) + \varepsilon v_i^t\, dt$ and it satisfies the equation

$$\begin{aligned} d\mathscr{D}_{v_i}\Psi^t \;=\; & \big(-A\mathscr{D}_{v_i}\Psi^t + (DF \circ \Psi^t)\mathscr{D}_{v_i}\Psi^t\big)\, dt + \big((DQ \circ \Psi^t)\mathscr{D}_{v_i}\Psi^t\big)\, dW(t) \\ & + (Q \circ \Psi^t) v_i^t\, dt . \end{aligned}$$

Intuitively, the first line comes from varying Ψ^t with respect to the noise and the second comes from varying the noise itself.

We will need a finite number L of directional derivatives, and so we introduce some more general notation. We combine L vectors v_i as used above into a matrix called v

which is an element of $\Omega \times [0,\infty) \to \mathcal{W}^L$. We identify $\mathcal{W}^L$ with $\mathscr{L}(\mathbf{R}^L, \mathcal{W})$. Note that we now allow v to depend on Ω, and to make things work, we require v to be a predictable stochastic process, *i.e.*, v^t only depends on the noise before time t. The stochastic process $G_v^t \in \mathcal{H}^L$ (corresponding to $\mathscr{D}_v \Psi^t$) is then defined as the solution of the equation

$$\begin{aligned} dG_v^t h &= \Big(-AG_v^t + \big(DF \circ \Psi^t\big)G_v^t + \big(Q \circ \Psi^t\big)v^t\Big) h\, dt \\ &\quad + \Big(\big(DQ \circ \Psi^t\big)\, G_v^t h\Big)\, dW(t)\;, \\ G_v^0 &= 0\;, \end{aligned} \tag{6.3}$$

which has to hold for all $h \in \mathbf{R}^L$.

Having given the detailed definition of G_v^t, we will denote it henceforth by the more suggestive

$$G_v^t(\xi) \;=\; \mathscr{D}_v \Psi^t(\xi)\;,$$

to make clear that it is a directional derivative. We use the notation $\mathscr{D}_v$ to distinguish this derivative from the derivative D with respect to the initial condition ξ.

For (6.2) and (6.3) to make sense, two assumptions on F, Q and v are needed:

A1 $F : \mathcal{H} \to \mathcal{H}$ and $Q : \mathcal{H} \to \mathscr{L}^2(\mathcal{W}, \mathcal{H})$ are of at most linear growth and have bounded first and second derivatives.

A2 The stochastic process $t \mapsto v^t$ is predictable, has a continuous version, and satisfies

$$\mathbf{E}\Big(\sup_{s\in[0,t]} \|v^s\|^p\Big) < \infty\;,$$

for every $t > 0$ and every $p \geq 1$. (The norm being the norm of $\mathcal{W}^L$.)

It is easy to see that these conditions imply the hypotheses of Theorem 8.9 for the problems (6.2) and (6.3). Therefore G_v^t is a well-defined strongly Markovian stochastic process.

With these notations one has the well-known Bismut integration by parts formula [Nor86].

Proposition 6.1 *Let Ψ^t and $\mathscr{D}_v \Psi^t$ be defined as above and assume* **A1** *and* **A2** *are satisfied. Let $\mathcal{B} \subset \mathcal{H}$ be an open subset of $\mathcal{H}$ such that $\Psi^t \in \mathcal{B}$ almost surely and let $\varphi : \mathcal{B} \to \mathbf{R}$ be a differentiable function such that*

$$\mathbf{E}\|\varphi(\Psi^t)\|^2 + \mathbf{E}\|D\varphi(\Psi^t)\|^2 < \infty\;.$$

Then we have for every $h \in \mathbf{R}^L$ the following identity in $\mathbf{R}$:

$$\mathbf{E}\big(D\varphi(\Psi^t)\mathscr{D}_v\Psi^t h\big) = \mathbf{E}\Big(\varphi(\Psi^t)\int_0^t \big\langle v^s h, dW(s)\big\rangle\Big)\;, \tag{6.4}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product of $\mathcal{W}$.

Remark 6.2 The Eq.(6.4) is useful because it relates the expectation of $D\varphi$ to that of φ. In order to fully exploit (6.4) we will need to get rid of the factor $\mathscr{D}_v\Psi^t$. This will be possible by a clever choice of v. This procedure is explained for example in [Nor86] but we will need a new variant of his results because of the high-frequency part. In the sequel, we will proceed in two steps. *We need only bounds on $D_{\mathrm{L}}\varphi$, since the smoothness of the high-frequency part follows by other means.* Thus, it *suffices* to construct $\mathscr{D}_v\Psi^t$ in such a way that $\Pi_{\mathrm{L}}\mathscr{D}_v\Psi^t$ is invertible, where Π_{L} is the orthogonal projection onto $\mathcal{H}_{\mathrm{L}}$. The construction of $\Pi_{\mathrm{L}}\mathscr{D}_v\Psi^t$ follows closely the presentation of [Nor86]. However, we also want $\Pi_{\mathrm{H}}\mathscr{D}_v\Psi^t = 0$ and this elimination of the high-frequency part seems to be new.

Proof. The finite dimensional case is stated (with slightly different assumptions on F) in [Nor86]. The extension to the infinite-dimensional setting can be done without major difficulty. By **A1–A2** and Theorem 8.9, we ensure that all the expressions appearing in the proof and the statement are well-defined. By **A2**, we can use Itô's formula to ensure the validity of the assumptions for the infinite-dimensional version of Girsanov's theorem [DPZ96]. □

6.1 The Construction of v

In order to use Proposition 6.1 we will construct $v = (v_{\mathrm{L}}, v_{\mathrm{H}})$ in such a way that the high-frequency part of $\mathscr{D}_v\Phi^t = (\mathscr{D}_v\Phi^t_{\mathrm{L}}, \mathscr{D}_v\Phi^t_{\mathrm{H}})$ vanishes. This construction is new and will be explained in detail in this subsection.

Notation. The equations which follow are quite involved. To keep the notation at a reasonable level without sacrificing precision we will adopt the following conventions:

$$\begin{aligned}(D_{\mathrm{L}}F_{\mathrm{L}})^t &\equiv (D_{\mathrm{L}}F_{\mathrm{L}}) \circ \Phi^t \;,\\ (D_{\mathrm{L}}Q_{\mathrm{L}})^t &\equiv (D_{\mathrm{L}}Q_{\mathrm{L}}) \circ \Phi^t \;,\end{aligned}$$

and similarly for other derivatives of the Q and the F. Furthermore, the reader should note that $D_{\mathrm{L}}Q_{\mathrm{L}}$ is a linear map from $\mathcal{H}_{\mathrm{L}}$ to the linear maps $\mathcal{H}_{\mathrm{L}} \to \mathcal{H}_{\mathrm{L}}$ and therefore, below, $(D_{\mathrm{L}}Q_{\mathrm{L}})h$ with $h \in \mathcal{H}_{\mathrm{L}}$ is a linear map $\mathcal{H}_{\mathrm{L}} \to \mathcal{H}_{\mathrm{L}}$. The dimension of $\mathcal{H}_{\mathrm{L}}$ is $L < \infty$.

Inspired by [Nor86], we define the $L \times L$ matrix-valued stochastic processes U^t_{L} and V^t_{L} by the following SDE's, which must hold for every $h \in \mathcal{H}_{\mathrm{L}}$:

$$\begin{aligned}dU^t_{\mathrm{L}}h &= -A_{\mathrm{L}}U^t_{\mathrm{L}}h\,dt + (D_{\mathrm{L}}F_{\mathrm{L}})^tU^t_{\mathrm{L}}h\,dt + \big((D_{\mathrm{L}}Q_{\mathrm{L}})^tU^t_{\mathrm{L}}h\big)\,dW_{\mathrm{L}}(t)\;,\\ U^0_{\mathrm{L}} &= I \in \mathscr{L}(\mathcal{H}_{\mathrm{L}},\mathcal{H}_{\mathrm{L}})\;,\end{aligned}\tag{6.5a}$$

$$\begin{aligned}dV^t_{\mathrm{L}}h &= V^t_{\mathrm{L}}A_{\mathrm{L}}h\,dt - V^t_{\mathrm{L}}(D_{\mathrm{L}}F_{\mathrm{L}})^th\,dt - V^t_{\mathrm{L}}\big((D_{\mathrm{L}}Q_{\mathrm{L}})^th\big)\,dW_{\mathrm{L}}(t)\\ &\qquad + \sum_{i=0}^{L-1} V^t_{\mathrm{L}}\Big((D_{\mathrm{L}}Q_{\mathrm{L}})^t\big((D_{\mathrm{L}}Q_{\mathrm{L}})^th\big)e_i\Big)e_i\,dt\;,\\ V^0_{\mathrm{L}} &= I \in \mathscr{L}(\mathcal{H}_{\mathrm{L}},\mathcal{H}_{\mathrm{L}})\;.\end{aligned}\tag{6.5b}$$

The last term in the definition of V_{L}^t will be written as

$$\sum_{i=0}^{L-1} V_{\mathrm{L}}^t\big((D_{\mathrm{L}}Q_{\mathrm{L}}^i)^t\big)^2 h\,dt\;,$$

where Q_{L}^i is the i^{th} column of the matrix Q_{L}.

For small times, the process U_{L}^t is an approximation to the partial Jacobian $D_{\mathrm{L}}\Phi_{\mathrm{L}}^t$, and V_{L}^t is an approximation to its inverse.

We first make sure that the objects in (6.5) are well-defined. The following lemma summarizes the properties of U_{L} and V_{L} which we need later.

Lemma 6.3 *The processes U_{L}^t and V_{L}^t satisfy the following bounds. For every $p \geq 1$ and all $T > 0$ there is a constant $C_{T,p,\varrho}$ independent of the initial data (for Φ^t) such that*

$$\mathbf{E}\sup_{t\in[0,T]}\big(\|U_{\mathrm{L}}^t\|^p + \|V_{\mathrm{L}}^t\|^p\big) \leq C_{T,p,\varrho}\;, \tag{6.6a}$$

$$\mathbf{E}\big(\sup_{t\in[0,\varepsilon]}\|V_{\mathrm{L}}^t - I\|^p\big) \leq C_{T,p,\varrho}\varepsilon^{p/2}\;, \tag{6.6b}$$

for all $\varepsilon < T$. Furthermore, V_{L} is the inverse of U_{L} in the sense that $V_{\mathrm{L}}^t = (U_{\mathrm{L}}^t)^{-1}$ almost surely

Proof. The bound (6.6a) is a straightforward application of Theorem 8.9 whose conditions are easily checked. (Note that we are here in a finite-dimensional, linear setting.) To prove (6.6b), note that I is the initial condition for V_{L}. One writes (6.5b) in its integral form and then the result follows by applying (6.6a). The last statement can be shown easily by applying Itô's formula to the product $V_{\mathrm{L}}^tU_{\mathrm{L}}^t$. (In fact, the definition of V_{L} was precisely made with this in mind.) □

We continue with the construction of v. Since A and Q are diagonal with respect to the splitting $\mathcal{H} = \mathcal{H}_{\mathrm{L}} \oplus \mathcal{H}_{\mathrm{H}}$, we can write (6.3) as

$$\begin{aligned} d\,\mathscr{D}_v\Phi_{\mathrm{L}}^t = \Big(&-A_{\mathrm{L}}\,\mathscr{D}_v\Phi_{\mathrm{L}}^t + (D_{\mathrm{L}}F_{\mathrm{L}})^t\,\mathscr{D}_v\Phi_{\mathrm{L}}^t \\ &+ (D_{\mathrm{H}}F_{\mathrm{L}})^t\,\mathscr{D}_v\Phi_{\mathrm{H}}^t + Q_{\mathrm{L}}^t v_{\mathrm{L}}^t\Big)\,dt \\ &+ \Big((D_{\mathrm{L}}Q_{\mathrm{L}})^t\,\mathscr{D}_v\Phi_{\mathrm{L}}^t\Big)\,dW_{\mathrm{L}}(t) \\ &+ \Big((D_{\mathrm{H}}Q_{\mathrm{L}})^t\,\mathscr{D}_v\Phi_{\mathrm{H}}^t\Big)\,dW_{\mathrm{L}}(t)\,, \end{aligned} \tag{6.7a}$$

$$\begin{aligned} d\,\mathscr{D}_v\Phi_{\mathrm{H}}^t = \Big(&-A_{\mathrm{H}}\,\mathscr{D}_v\Phi_{\mathrm{H}}^t + (D_{\mathrm{L}}F_{\mathrm{H}})^t\,\mathscr{D}_v\Phi_{\mathrm{L}}^t \\ &+ (D_{\mathrm{H}}F_{\mathrm{H}})^t\,\mathscr{D}_v\Phi_{\mathrm{H}}^t + Q_{\mathrm{H}}v_{\mathrm{H}}^t\Big)\,dt\;, \end{aligned} \tag{6.7b}$$

with zero initial condition. Since we want to consider derivatives with respect to the low-frequency part, we would like to define (implicitly) v_{H}^t as

$$v_{\mathrm{H}}^t = -Q_{\mathrm{H}}^{-1}(D_{\mathrm{L}}F_{\mathrm{H}})^t\,\mathscr{D}_v\Phi_{\mathrm{L}}^t\;.$$

In this way, the solution of (6.7b) would be $\mathscr{D}_v \Phi_{\rm H}^t \equiv 0$. We next would define the "directions" $v_{\rm L}$ and $v_{\rm H}$ by

$$\begin{aligned} v_{\rm L}^t &= \bigl(V_{\rm L}^t Q_{\rm L}^t\bigr)^* , \\ v_{\rm H}^t &= -Q_{\rm H}^{-1} (D_{\rm L} F_{\rm H})^t \, \mathscr{D}_v \Phi_{\rm L}^t , \end{aligned}$$

where $\mathscr{D}_v \Phi_{\rm L}^t$ is the solution to (6.7a) with $\mathscr{D}_v \Phi_{\rm H}^t$ replaced by 0 and $v_{\rm L}$ replaced by $\bigl(V_{\rm L}^t Q_{\rm L}^t\bigr)^*$. Here, X^* denotes the transpose of the real matrix X. *This program needs some care because of the implicit nature of the definition of v.*

Since we are constructing a solution of (6.7) whose high-frequency part is going to vanish, we consider instead the simpler equation for $y^t \in \mathscr{L}(\mathcal{H}_{\rm L}, \mathcal{H}_{\rm L})$:

$$dy^t = \Bigl(-A_{\rm L} y^t + (D_{\rm L} F_{\rm L})^t y^t + Q_{\rm L}^t \bigl(V_{\rm L}^t Q_{\rm L}^t\bigr)^*\Bigr)\, dt + \bigl((D_{\rm L} Q_{\rm L})^t y^t\bigr)\, dW_{\rm L}(t) \ , \quad (6.8)$$

with $y^0 = 0$, and where we use again the notation $F^t = F \circ \Phi^t$, and similar notation for Q.

The verification that (6.8) is well-defined and can be bounded is again a consequence of Theorem 8.9 and is left to the reader. *Given the solution of (6.8) we proceed to make our definitive choice of $v_{\rm L}^t$ and $v_{\rm H}^t$:*

Definition 6.4 *Given an initial condition $\xi \in \mathcal{H}^\alpha$ (for Φ^t) and a cutoff $\varrho < \infty$ we define $v^t = v_{\rm L}^t \oplus v_{\rm H}^t$ by*

$$\begin{aligned} v_{\rm L}^t &\equiv \ \bigl(V_{\rm L}^t Q_{\rm L}^t\bigr)^* &&= \ \bigl(V_{\rm L}^t (Q_{\rm L} \circ \Phi^t)\bigr)^* , \\ v_{\rm H}^t &\equiv -Q_{\rm H}^{-1} (D_{\rm L} F_{\rm H})^t \, y^t &&= -Q_{\rm H}^{-1} \bigl((D_{\rm L} F_{\rm H}) \circ \Phi^t\bigr)\, y^t , \end{aligned} \quad (6.9)$$

where Φ^t solves (5.3), $V_{\rm L}^t$ is the solution of (6.5b), and y^t solves (6.8).

Lemma 6.5 *The process v^t satisfies for all $p \geq 1$ and all $t > 0$:*

$$\mathbf{E}\Bigl(\sup_{s\in[0,t]} \|v^s\|^p\Bigr) < C_{t,p,\varrho} \bigl(1 + \|\xi\|_\alpha\bigr)^p ,$$

i.e., it satisfies assumption **A2** *of Proposition 6.1.*

Proof. By Proposition 5.1, Φ^t is in $\mathcal{H}^\alpha$ for all $t \geq 0$. In Lemma 8.1 **P6**, it will be checked that $D_{\rm L} F_{\rm H}$ maps $\mathcal{H}^\alpha$ into $\mathscr{L}(\mathcal{H}_{\rm L}, \mathcal{H}^\alpha \cap \mathcal{H}_{\rm H})$ and that this map has linear growth. By the *lower bound* (1.6) on the amplitudes q_k, we see that $Q_{\rm H}^{-1}$ is bounded from $\mathcal{H}^\alpha \cap \mathcal{H}_{\rm H}$ to $\mathcal{H}_{\rm H}$ and thus the assertion follows. □

We now verify that $\mathscr{D}_v \Phi_{\rm H}^t \equiv 0$. Indeed, consider the equations (6.7). This is a system for two unknowns, $Y^t = \mathscr{D}_v \Phi_{\rm L}^t$ and $X^t = \mathscr{D}_v \Phi_{\rm H}^t$. For our choice of $v_{\rm L}^t$ and $v_{\rm H}^t$ this system takes the form

$$\begin{aligned} d\,Y^t = \Bigl(&-A_{\rm L}\, Y^t + (D_{\rm L} F_{\rm L})^t\, Y^t \\ &+ (D_{\rm H} F_{\rm L})^t\, X^t + Q_{\rm L}^t (V_{\rm L}^t Q_{\rm L}^t)^*\Bigr)\, dt \end{aligned} \quad (6.10a)$$

$$
\begin{aligned}
&+ \Big((D_{\mathrm{L}} Q_{\mathrm{L}})^t \, Y^t \Big) \, dW_{\mathrm{L}}(t) \\
&+ \Big((D_{\mathrm{H}} Q_{\mathrm{L}})^t \, X^t \Big) \, dW_{\mathrm{L}}(t) \,,
\end{aligned}
$$

$$
\begin{aligned}
d\,X^t = \Big(&-A_{\mathrm{H}}\, X^t + (D_{\mathrm{L}} F_{\mathrm{H}})^t \, Y^t \\
&+ (D_{\mathrm{H}} F_{\mathrm{H}})^t \, X^t - (D_{\mathrm{L}} F_{\mathrm{H}})^t y^t \Big) \, dt \;.
\end{aligned} \tag{6.10b}
$$

By inspection, we see that $X^t \equiv 0$ and

$$
dY^t = -A_{\mathrm{L}} Y^t + (D_{\mathrm{L}} F_{\mathrm{L}})^t Y^t + \big((D_{\mathrm{L}} Q_{\mathrm{L}})^t \, Y^t \big) \, dW_{\mathrm{L}}(t) + Q^t_{\mathrm{L}} (V^t_{\mathrm{L}} Q^t_{\mathrm{L}})^* \, dt \tag{6.11}
$$

solve the problem, *i.e.*, $Y^t = y^t$, by the construction of y^t. Applying the Itô formula to the product $V^t_{\mathrm{L}} Y^t$ and using Eqs.(6.5b) and (6.11), we see immediately that we have defined $Y^t = \mathscr{D}_v \Phi^t_{\mathrm{L}}$ in such a way that

$$
d\big(V^t_{\mathrm{L}} \mathscr{D}_v \Phi^t_{\mathrm{L}} \big) = V^t_{\mathrm{L}} Q^t_{\mathrm{L}} (Q^t_{\mathrm{L}})^* (V^t_{\mathrm{L}})^* \, dt \;,
$$

because all other terms cancel. Thus we finally have shown

Theorem 6.6 *Given an initial condition $\xi \in \mathcal{H}^\alpha$ (for Φ^t) and a cutoff $\varrho < \infty$, the following is true: If v^t is given by Definition 6.4 then*

$$
\begin{aligned}
&\mathscr{D}_v \Phi^t_{\mathrm{L}} = U^t_{\mathrm{L}} \int_0^t V^s_{\mathrm{L}} \big((Q_{\mathrm{L}} Q^*_{\mathrm{L}}) \circ \Phi^s \big) \big(V^s_{\mathrm{L}} \big)^* \, ds \equiv U^t_{\mathrm{L}} C^t_{\mathrm{L}} \,, \\
&\mathscr{D}_v \Phi^t_{\mathrm{H}} \equiv 0 \;.
\end{aligned} \tag{6.12}
$$

Definition 6.7 *We will call the matrix C^t_{L} the* partial Malliavin matrix *of our system.*

7 The Partial Malliavin Matrix

In this section, we estimate the partial Malliavin matrix C^t_{L} from below. We fix some time $t > 0$ and denote by $\mathcal{S}^L$ the unit sphere in $\mathbf{R}^L$. Our bound is

Theorem 7.1 *There are constants $\mu, \nu \geq 0$ such that for every $T > 0$ and every $p \geq 1$ there is a $C_{T,p,\varrho}$ such that for all initial conditions $\xi \in \mathcal{H}^\alpha$ for the flow Φ^t and all $t < T$, one has*

$$
\mathbf{E}\Big(\big(\det C^t_{\mathrm{L}} \big)^{-p} \Big) \leq C_{T,p,\varrho} t^{-\mu p} \big(1 + \|\xi\|_\alpha \big)^{\nu p} \;.
$$

Corollary 7.2 *There are constants $\mu, \nu \geq 0$ such that for every $T > 0$ and every $p \geq 1$ there is a $C_{T,p,\varrho}$ such that for all initial conditions $\xi \in \mathcal{H}^\alpha$ for the flow Φ^t and all $t < T$, one has, with v given by Definition 6.4:*

$$
\mathbf{E} \big\| \big(\mathscr{D}_v \Phi^t_{\mathrm{L}} \big)^{-p} \big\| \leq C_{T,p,\varrho} t^{-\mu p} \big(1 + \|\xi\|_\alpha \big)^{\nu p} \;.
$$

This corollary follows from $(\mathscr{D}_v \Phi^t_{\mathrm{L}})^{-1} = (C^t_{\mathrm{L}})^{-1} V^t_{\mathrm{L}}$ and Eq.(6.6a).

As a first step, we formulate a bound from which Theorem 7.1 follows easily.

Theorem 7.3 *There are a $\mu > 0$ and a $\nu > 0$ such that for every $p \geq 1$, every $t < T$ and every $\xi \in \mathcal{H}^2$, one has*

$$\mathbf{P}\Big(\inf_{h \in \mathcal{S}^L} \int_0^t \big\| Q^s_{\mathrm{L}} \big(V^s_{\mathrm{L}}\big)^* h \big\|^2 \, ds < \varepsilon \Big) \;\leq\; C_{T,p,\varrho} \varepsilon^p t^{-\mu p} (1 + \|\xi\|_2)^{\nu p} \,,$$

with $C_{T,p,\varrho}$ independent of ξ.

Proof of Theorem 7.1. Note that $\int_0^t \|Q^s_{\mathrm{L}} (V^s_{\mathrm{L}})^* h\|^2 \, ds$ is, by Eq.(6.12), nothing but the quantity $\langle h, C^t_{\mathrm{L}} h\rangle$. Then, Theorem 7.1 follows at once. □

The proof of Theorem 7.3 is largely inspired from [Nor86, Sect. 4], but we need some new features to deal with the infinite dimensional high-frequency part. This will take up the next three subsections.

Our proof needs a modification of the Lie brackets considered when we study the Hörmander condition. We explain first these identities in a finite dimensional setting.

7.1 Finite Dimensional Case

Assume in this subsection that both $\mathcal{H}_{\mathrm{L}}$ and $\mathcal{H}_{\mathrm{H}}$ are finite dimensional.

The operator Q maps $\mathcal{H}$ to $\mathscr{L}(\mathcal{W}, \mathcal{H})$, where $\mathcal{W}$ should be thought of as the subspace of $\mathcal{H}$ which is actually driven by the noise. We assume it is spanned by the first W basis vectors of $\mathcal{H}$. We denote by $Q_i : \mathcal{H} \to \mathcal{H}$ the i^{th} column of Q, $(i = 0, \ldots, W-1)$.[6] Finally, $\widehat{F}$ is the drift (in this section, we absorb the linear part of the SDE into $\widehat{F} = -A + F$, to simplify the expressions). The equation for Φ^t is

$$\Phi^t(\xi) = \xi + \int_0^t \big(\widehat{F} \circ \Phi^s\big)(\xi) \, ds + \int_0^t \sum_{i=0}^{W-1} \big(Q_i \circ \Phi^s\big)(\xi) \, dw_i(s) \,.$$

Let $K : \mathcal{H} \to \mathcal{H}_{\mathrm{L}}$ be a smooth function whose derivatives are all bounded and define $K^t = K \circ \Phi^t$, $\widehat{F}^t = \widehat{F} \circ \Phi^t$, and $Q^t_i = Q_i \circ \Phi^t$. We then have by Itô's formula

$$dK^t = (DK)^t \widehat{F}^t \, dt + \sum_{i=0}^{W-1} (DK)^t Q^t_i \, dw_i(t) + \tfrac{1}{2} \sum_{i=0}^{W-1} (D^2 K)^t (Q^t_i ; Q^t_i) \, dt \,. \tag{7.1}$$

We next rewrite the equation (6.5) for V^t_{L} as:

$$dV^t_{\mathrm{L}} = -V^t_{\mathrm{L}} (D_{\mathrm{L}} \widehat{F}_{\mathrm{L}})^t \, dt - \sum_{i=0}^{L-1} V^t_{\mathrm{L}} (D_{\mathrm{L}} Q_i)^t \, dw_i(t) + \sum_{i=0}^{L-1} V^t_{\mathrm{L}} \big((D_{\mathrm{L}} Q_i)^t\big)^2 \, dt \,.$$

The following definition is useful. Let $A : \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{H} \to \mathcal{H}_{\mathrm{L}}$ be two functions with continuous bounded derivatives. We define the *projected Lie bracket* $[A, B]_{\mathrm{L}} : \mathcal{H} \to \mathcal{H}_{\mathrm{L}}$ by

$$[A, B]_{\mathrm{L}}(x) = \Pi_{\mathrm{L}} [A, B](x) = \big(DB(x)\big) A(x) - \big(D_{\mathrm{L}} A_{\mathrm{L}}(x)\big) B(x) \,.$$

[6] There is a slight ambiguity of notation here, since Q_i really means $Q_{\varrho,i}$ which is not the same as Q_ϱ.

By Itô's formula, we have therefore the following equation for the product $V_{\mathrm{L}}^t K^t$:

$$
\begin{aligned}
d\big(V_{\mathrm{L}}^t K^t\big) = -V_{\mathrm{L}}^t (D_{\mathrm{L}} \widehat{F}_{\mathrm{L}})^t K^t \, dt - V_{\mathrm{L}}^t \sum_{i=0}^{L-1} (D_{\mathrm{L}} Q_i)^t K^t \, dw_i(t) \\
+ V_{\mathrm{L}}^t \sum_{i=0}^{L-1} \big((D_{\mathrm{L}} Q_i)^t\big)^2 K^t \, dt + V_{\mathrm{L}}^t (DK)^t \widehat{F}^t \, dt \\
+ V_{\mathrm{L}}^t \sum_{i=0}^{W-1} (DK)^t Q_i^t \, dw_i(t) \\
+ \tfrac{1}{2} V_{\mathrm{L}}^t \sum_{i=0}^{W-1} (D^2 K)^t (Q_i^t; Q_i^t) \, dt \\
- V_{\mathrm{L}}^t \sum_{i=0}^{L-1} (D_{\mathrm{L}} Q_i)^t (DK)^t Q_i^t \, dt \;.
\end{aligned}
$$

By construction, $D_{\mathrm{L}} Q_i = 0$ for $i > L$ and therefore we extend all the sums above to W. A straightforward calculation then leads to

$$
\begin{aligned}
d\big(V_{\mathrm{L}}^t K^t\big) = V_{\mathrm{L}}^t \Big([\widehat{F}, K]_{\mathrm{L}}^t + \tfrac{1}{2} \sum_{i=0}^{W-1} \big[Q_i, [Q_i, K]_{\mathrm{L}}\big]_{\mathrm{L}}^t \Big) \, dt \\
+ V_{\mathrm{L}}^t \sum_{i=0}^{W-1} [Q_i, K]_{\mathrm{L}}^t \; dw_i(t) \\
+ \tfrac{1}{2} V_{\mathrm{L}}^t \sum_{i=0}^{W-1} \Big(\big((D_{\mathrm{L}} Q_i)^t\big)^2 K^t - (DK)^t (DQ_i)^t Q_i^t \\
+ (D D_{\mathrm{L}} Q_i)^t (Q_i^t; K^t) \Big) \;.
\end{aligned}
\tag{7.2}
$$

Note next that for $i < L$, both K and Q_i map to $\mathcal{H}_{\mathrm{L}}$ and therefore $D D_{\mathrm{L}} Q_i (Q_i; K) = D_{\mathrm{L}}^2 Q_i (Q_i; K)$ when $i < L$ and it is 0 otherwise. Similarly, $(DK)(DQ_i)Q_i$ equals $(DK)(D_{\mathrm{L}} Q_i) Q_i$ when $i < L$ and vanishes otherwise. Thus, the last sum in (7.2) only extends to $L - 1$.

In order to simplify (7.2) further, we define the vector field $\widetilde{F} : \mathcal{H} \to \mathcal{H}$ by

$$
\widetilde{F} = \widehat{F} - \tfrac{1}{2} \sum_{i=0}^{L-1} (D_{\mathrm{L}} Q_i) Q_i \;.
$$

Then we get

$$
d\big(V_{\mathrm{L}}^t K^t\big) = V_{\mathrm{L}}^t \Big([\widetilde{F}, K]_{\mathrm{L}}^t + \tfrac{1}{2} \sum_{i=0}^{W-1} \big[Q_i, [Q_i, K]_{\mathrm{L}}\big]_{\mathrm{L}}^t \Big) \, dt + V_{\mathrm{L}}^t \sum_{i=0}^{W-1} [Q_i, K]_{\mathrm{L}}^t \, dw_i(t) \;.
$$

This is very similar to [Nor86, p. 128], who uses the conventional Lie bracket instead of $[\cdot, \cdot]_{\mathrm{L}}$.

7.2 Infinite Dimensional Case

In this case, some additional care is needed when we transcribe (7.1). The problem is that the stochastic flow Φ^t solves (5.4) in the mild sense but not in the strong sense. Nevertheless, this technical difficulty will be circumvented by choosing the initial condition in $\mathcal{H}^\alpha$. We have indeed by Proposition 5.1 (A) that if the initial condition is in $\mathcal{H}^\gamma$ with $\gamma \in [1, \alpha]$, then the solution of (5.4) is in the same space. Thus, Proposition 5.1 allows us to use Itô's formula also in the infinite dimensional case.

For any two Banach (or Hilbert) spaces $\mathscr{B}_1$, $\mathscr{B}_2$, we denote by $P(\mathscr{B}_1, \mathscr{B}_2)$ the set of all $\mathcal{C}^\infty$ functions $\mathscr{B}_1 \to \mathscr{B}_2$, which are polynomially bounded together with all their derivatives. Let $K \in P(\mathcal{H}, \mathcal{H}_\mathrm{L})$ and $X \in P(\mathcal{H}, \mathcal{H})$. We define as above $[X, K]_\mathrm{L} \in P(\mathcal{H}, \mathcal{H}_\mathrm{L})$ by

$$[X, K]_\mathrm{L}(x) = \big(DK(x)\big)X(x) - \big(D_\mathrm{L} X_\mathrm{L}(x)\big)K(x) \;.$$

Furthermore, we define $[A, K]_\mathrm{L} \in P(\mathcal{D}(A), \mathcal{H}_\mathrm{L})$ by the corresponding formula, *i.e.*,

$$[A, K]_\mathrm{L}(x) = \big(DK(x)\big)Ax - A_\mathrm{L} K(x) \;,$$

where $A = 1 - \Delta$. Notice that if K is a constant vector field, *i.e.*, $DK = 0$, then $[A, K]_\mathrm{L}$ extends uniquely to an element of $P(\mathcal{H}, \mathcal{H}_\mathrm{L})$.

We choose again the basis $\{e_i\}_{i=0}^\infty$ of Fourier modes in $\mathcal{H}$ (see Eq.(1.5)) and define $dw_i(t) = \langle e_i, dW(t)\rangle$. We also define the stochastic process $K^t(\xi) = (K \circ \Phi^t)(\xi)$ and

$$\widetilde{F} = F - \tfrac{1}{2} \sum_{i=0}^{L-1} (D_\mathrm{L} Q_i) Q_i \;,$$

where $Q_i(x) = Q(x)e_i$. Then one has

Proposition 7.4 *Let $\xi \in \mathcal{H}^1$ and $K \in P(\mathcal{H}, \mathcal{H}_\mathrm{L})$. Then the equality*

$$\begin{aligned} V_\mathrm{L}^t(\xi)K^t(\xi) = K(\xi) &+ \int_0^t V_\mathrm{L}^s(\xi) \sum_{i=0}^\infty [Q_i, K]_\mathrm{L}^s(\xi)\, dw_i(s) \\ &+ \int_0^t V_\mathrm{L}^s(\xi)\Big(-[A, K]_\mathrm{L}^s(\xi) + [\widetilde{F}, K]_\mathrm{L}^s(\xi)\Big)\, ds \\ &+ \tfrac{1}{2} \int_0^t V_\mathrm{L}^s(\xi) \sum_{i=0}^\infty \big[Q_i, [Q_i, K]_\mathrm{L}\big]_\mathrm{L}^s(\xi)\, ds \;, \end{aligned}$$

holds almost surely. Furthermore, the same equality holds if $\xi \in \mathcal{H}^2$ and $K \in P(\mathcal{H}^1, \mathcal{H}_\mathrm{L})$.

Note that by $[A, K]_\mathrm{L}^s(\xi)$ we mean $\Big(DK\big(\Phi^s(\xi)\big)\Big)\big(A\Phi^s(\xi)\big) - A_\mathrm{L} K\big(\Phi^s(\xi)\big)$.

Proof. Itô's formula. □

7.3 The Restricted Hörmander Condition

The condition for having appropriate mixing properties is the following Hörmander-like condition.

Definition 7.5 *Let $\mathcal{K} = \{K^{(i)}\}_{i=1}^M$ be a collection of functions in $P(\mathcal{H}, \mathcal{H}_{\mathrm{L}})$. We say that $\mathcal{K}$ satisfies the* restricted Hörmander condition *if there exist constants $\delta, R > 0$ such that for every $h \in \mathcal{H}_{\mathrm{L}}$ and every $y \in \mathcal{H}$ one has*

$$\sup_{K\in\mathcal{K}} \inf_{\|x-y\|\le R} \langle h, K(x)\rangle^2 \ge \delta\|h\|^2 . \tag{7.3}$$

We now construct the set $\mathcal{K}$ for our problem. We define the operator $[X^0, \cdot\ ]_{\mathrm{L}} : P(\mathcal{H}^\gamma, \mathcal{H}_{\mathrm{L}}) \to P(\mathcal{H}^{\gamma+1}, \mathcal{H}_{\mathrm{L}})$ by

$$[X^0, K]_{\mathrm{L}} = -[A, K]_{\mathrm{L}} + [F, K]_{\mathrm{L}} + \tfrac{1}{2}\sum_{i=0}^{\infty}\big[Q_i, [Q_i, K]_{\mathrm{L}}\big]_{\mathrm{L}} - \tfrac{1}{2}\sum_{i=0}^{L-1}\big[(D_{\mathrm{L}}Q_i)Q_i, K\big]_{\mathrm{L}} .$$

This is a well-defined operation since Q is Hilbert-Schmidt and DQ is finite rank and we can write

$$\sum_{i=0}^{\infty}\big[Q_i, [Q_i, K]_{\mathrm{L}}\big]_{\mathrm{L}} = \sum_{i=0}^{\infty}\big(D^2K\big)(Q_i; Q_i) + r ,$$

with r a finite sum of bounded terms.

Definition 7.6 *We define*

- $\mathcal{K}_0 = \{Q_i,$ *with* $i = 0, \dots, L-1\}$,
- $\mathcal{K}_1 = \{[X^0, Q_i]_{\mathrm{L}},$ *with* $i = k_*, \dots, L-1\}$,
- $\mathcal{K}_\ell = \{[Q_i, K]_{\mathrm{L}},$ *with* $K \in \mathcal{K}_{\ell-1}$ *and* $i = k_*, \dots, L-1\}$, *when* $\ell > 1$.

Finally,

$$\mathcal{K} = \mathcal{K}_0 \cup \dots \cup \mathcal{K}_3 .^7$$

Remark 7.7 Since for $i \ge k_*$ the Q_i are constant vector fields, the quantity $[X^0, K]$ is in $P(\mathcal{H}, \mathcal{H}_{\mathrm{L}})$ and not only in $P(\mathcal{H}^1, \mathcal{H}_{\mathrm{L}})$. Furthermore, if $K \in \mathcal{K}$ then D^jK is bounded for all $j \ge 0$.

We have

Theorem 7.8 *The set $\mathcal{K}$ constructed above satisfies the restricted Hörmander condition for the cutoff GL equation if ϱ is chosen sufficiently large. Furthermore, the inequality (7.3) holds for $R = \varrho/2$. Finally, $\delta > \delta_0 > 0$ for all sufficiently large ϱ.*

Proof. The basic idea of the proof is as follows: The leading term of F is the cubic term u^m with $m = 3$. Clearly, if i_1, i_2, i_3 are any 3 modes, we find

$$\big[e_{i_1}, \big[e_{i_2}, [u \mapsto u^3, e_{i_3}]_{\mathrm{L}}\big]_{\mathrm{L}}\big]_{\mathrm{L}} = \sum_{k=\pm i_1 \pm i_2 \pm i_3} C_k \Pi_{\mathrm{L}} e_k , \tag{7.4}$$

[7] The number 3 is the power 3 in u^3.

where the e_ℓ are the basis vectors of $\mathcal{H}$ defined in (1.5), and the C_k are *non-zero* combinatorial constants. By our Definition 3.2 of the set $\mathscr{I}$, the following is true: For every choice of a fixed k the three numbers i_1, i_2, and i_3 of $\mathscr{I}_k$ satisfy

- For $j = 1, 2, 3$ one has $i_j \in \{k_*, \dots, L-1\}$.
- If $|k| < k_*$ exactly one of the six sums $\pm i_1 \pm i_2 \pm i_3$ lies in the set $\{0, \dots, k_* - 1\}$ and exactly one lies in $\{-(k_* - 1), \dots, 0\}$.

In particular, the expression (7.4) does not depend on u. If instead of u^3 we take a lower power, the triple commutator will vanish.

The basic idea has to be slightly modified because of the cutoff ϱ. First of all, the constant R in the definition of the Hörmander condition is set to $R = \varrho/2$. Consider first the case where $\|x\| \geq 5\varrho/2$. In that case we see from (4.1) that the $Q_{\varrho,i}$, viewed as vector fields, are of the form

$$Q_{\varrho,i}(x) = \begin{cases} (q_i + 1)e_i, & \text{if } i < k_*, \\ q_i e_i, & \text{if } i \geq k_*. \end{cases}$$

Since these vectors span a basis of $\mathcal{H}_{\mathrm{L}}$ the inequality (7.3) follows in this case (already by choosing only $K \in \mathcal{K}_0$).

Consider next the more delicate case when $\|x\| \leq 5\varrho/2$.

Lemma 7.9 *For all $\|x\| \leq 3\varrho$ one has for $\{i_1, i_2, i_3\} = \mathscr{I}_k$ the identity*

$$\big[e_{i_1}, [e_{i_2}, [X^0, e_{i_3}]_{\mathrm{L}}]_{\mathrm{L}}\big]_{\mathrm{L}}(x) = \sum_{k=\pm i_1 \pm i_2 \pm i_3} C_k \Pi_{\mathrm{L}} e_k + r_\varrho(x) \,, \tag{7.5}$$

where r_ϱ satisfies a bound

$$\|r_\varrho(x)\| \leq C\varrho^{-1} \,,$$

with the constant C independent of x and of $k < k_*$.

Proof. In $[X^0, \cdot]_{\mathrm{L}}$ there are 4 terms. The first, A, leads successively to $[A, e_{i_3}]_{\mathrm{L}} = (1 + i_3^2)e_{i_3}$, which is constant, and hence the Lie bracket with e_{i_2} vanishes. The second term contains the non-linear interaction F_ϱ. Since $\|x\| \leq 3\varrho$ one has $F_\varrho(x) = F(x)$. Thus, (7.4) yields the leading term of (7.5). The two remaining terms will contribute to $r_\varrho(x)$. We just discuss the first one. We have, using (4.1),

$$[Q_{\varrho,i}, e_{i_3}]_{\mathrm{L}}(x) = -DQ_{\varrho,i}(x)e_{i_3} = -\frac{1}{\varrho}\chi'(\|x\|/\varrho)\frac{\langle x, e_{i_3}\rangle}{\|x\|}\Pi_{k_*} e_i \,.$$

This gives clearly a bound of order ϱ^{-1} for this Lie bracket, and the further ones are handled in the same way. □

We continue the proof of Theorem 7.8. When $k < k_*$, we consider the elements of $\mathcal{K}_3$. They are of the form

$$\big[Q_{\varrho,i_1}, [Q_{\varrho,i_2}, [X^0, Q_{\varrho,i_3}]_{\mathrm{L}}]_{\mathrm{L}}\big]_{\mathrm{L}}(x) = q_{i_1} q_{i_2} q_{i_3} \Big(\sum_{k=\pm i_1 \pm i_2 \pm i_3} C_k \Pi_{\mathrm{L}} e_k + r_\varrho(x) \Big) \,.$$

Thus, for $\varrho = \infty$ these vectors together with the Q_i with $i \in \{k_*, \ldots, L-1\}$ span $\mathcal{H}_\mathrm{L}$ (independently of y with $\|y\| \le 3\varrho$) and therefore (7.4) holds in this case, if $\|x\| \le 5\varrho/2$ and $R = \varrho/2$. The assertion for finite, but large enough ϱ follows immediately by a perturbation argument. This completes the case of $\|x\| \le 5\varrho/2$ and hence the proof of Theorem 7.8. □

Proof of Theorem 7.3. The proof is very similar to the one in [Nor86], but we have to keep track of the x, t-dependence of the estimates. First of all, choose $x \in \mathcal{H}^2$ and $t \in (0, t_0]$.

From now on, we will use the notation $\mathcal{O}(\nu)$ as a shortcut for $C(1 + \|x\|_2^\nu)$, where the constant C may depend on t_0 and p, but is independent of x and t. Denote by R the constant found in Theorem 7.8 and define the subset $\mathcal{B}_x$ of $\mathcal{H}^2$ by

$$\mathcal{B}_x = \bigl\{ y \in \mathcal{H}^2 \ : \ \|y - x\| \le R \quad \text{and} \quad \|y\|_\gamma \le \|x\|_\gamma + 1 \text{ for } \gamma = 1, 2 \bigr\} \;.$$

We also denote by $\mathcal{B}(I)$ a ball of (small) radius $\mathcal{O}(1/L)$ centered at the identity in the space of all $L \times L$ matrices. (Recall that L is the dimension of $\mathcal{H}_\mathrm{L}$, and that $K \in \mathcal{K}$ maps to $\mathcal{H}_\mathrm{L}$.) We then have a bound of the type

$$\sup_{y \in \mathcal{B}_x} \sup_{K \in \mathcal{K}} \sum_{i=0}^{\infty} \bigl\| [Q_i, K]_\mathrm{L}(y) \bigr\|^2 \le \mathcal{O}(0) \;. \tag{7.6}$$

This is a consequence of the fact that QQ^* is trace class and thus the sum converges and its principal term is equal to

$$\begin{aligned} \mathrm{Tr}\bigl(Q^*(y) \,\bigl(DK\bigr)^*(y)\, \bigl(DK\bigr)(y)\, Q(y)\bigr) \\ &= \mathrm{Tr}\bigl(\bigl(DK\bigr)(y) Q(y)\, Q^*(y)\, \bigl(DK\bigr)^*(y)\bigr) \\ &= \sum_{i=0}^{L-1} \|Q^*(y)\, \bigl(DK\bigr)^*(y) e_i\|^2 \le C_\varrho \;. \end{aligned}$$

The last inequality follows from Remark 7.7. The other terms form a finite sum containing derivatives of the Q_i and are bounded in a similar way.

We have furthermore bounds of the type

$$\begin{aligned} \sup_{y \in \mathcal{B}_x} \sup_{K \in \mathcal{K}} \bigl\| [X^0, K]_\mathrm{L}(y) \bigr\|^2 &\le \mathcal{O}(\nu) \;, \\ \sup_{y \in \mathcal{B}_x} \sup_{K \in \mathcal{K}} \bigl\| \bigl[X^0, [X^0, K]_\mathrm{L}\bigr]_\mathrm{L}(y) \bigr\|^2 &\le \mathcal{O}(\nu) \;, \\ \sup_{y \in \mathcal{B}_x} \sup_{K \in \mathcal{K}} \sum_{i=0}^{\infty} \bigl\| \bigl[Q_i, [X^0, K]_\mathrm{L}\bigr]_\mathrm{L}(y) \bigr\|^2 &\le \mathcal{O}(\nu) \;, \end{aligned} \tag{7.7}$$

where $\nu = 1$.

Let $\mathcal{S}_\mathrm{L}$ be the unit sphere in $\mathcal{H}_\mathrm{L}$. By the assumptions on $\mathcal{K}$ and the choice of $\mathcal{B}(I)$ we see that:

(A) *For every $h_0 \in \mathcal{S}_{\rm L}$, there exist a $K \in \mathcal{K}$ and a neighborhood $\mathcal{N}$ of h_0 in $\mathcal{S}_{\rm L}$ such that*

$$\inf_{y\in\mathcal{B}_x} \inf_{V\in\mathcal{B}(I)} \inf_{h\in\mathcal{N}} \langle\, VK(y), h\rangle^2 \geq \frac{\delta}{2}\;,$$

with δ the constant appearing in (7.3).

Next, we define a stopping time τ by $\tau = \min\{t, \tau_1, \tau_2\}$ with

$$\begin{aligned}\tau_1 &= \inf\bigl\{s \geq 0 \;:\; \Phi^s(x) \notin \mathcal{B}_x\bigr\}\;,\\ \tau_2 &= \inf\bigl\{s \geq 0 \;:\; V_{\rm L}^s(x) \notin \mathcal{B}(I)\bigr\}\;,\\ t &< T \text{ as chosen in the statement of Theorem 7.3 .}\end{aligned}$$

It follows easily from Proposition 5.1 (E) that the probability of τ_1 being small (meaning that in the sequel we will always assume $\varepsilon \leq 1$) can be bounded by

$$\mathbf{P}(\tau_1 < \varepsilon) \leq C_p(1 + \|x\|_2)^{16p}\varepsilon^p\;,$$

with C_p independent of x. Similarly, using Lemma 6.3, we see that

$$\mathbf{P}(\tau_2 < \varepsilon) \leq C_p\varepsilon^p\;.$$

Observing that $\mathbf{P}(t < \varepsilon) < t^{-p}\varepsilon^p$ and combining this with the two estimates, we get for every $p \geq 1$:

$$\mathbf{P}\bigl(\tau < \varepsilon\bigr) \leq \mathcal{O}(16p)t^{-p}\varepsilon^p\;.$$

From this and (A) we deduce

(B) *for every $h_0 \in \mathcal{S}_{\rm L}$ there exist a $K \in \mathcal{K}$ and a neighborhood $\mathcal{N}$ of h_0 in $\mathcal{S}_{\rm L}$ such that for $\varepsilon < 1$,*

$$\sup_{h\in\mathcal{N}} \mathbf{P}\biggl(\int_0^\tau \bigl\langle V_{\rm L}^s(x)K^s(x), h\bigr\rangle^2\, ds \leq \varepsilon\biggr) \leq \mathbf{P}(\tau < 2\varepsilon/\delta) \leq \mathcal{O}(16p)t^{-p}\varepsilon^p\;,$$

with δ the constant appearing in (7.3).

Following [Nor86], we will show below that (B) implies:

(C) *for every $h_0 \in \mathcal{S}_{\rm L}$ there exist an $i \in \{k_*, \ldots, L-1\}$, a neighborhood $\mathcal{N}$ of h_0 in $\mathcal{S}_{\rm L}$ and constants ν, $\mu > 0$ such that for $\varepsilon < 1$ and $p > 1$ one has*

$$\sup_{h\in\mathcal{N}} \mathbf{P}\biggl(\int_0^\tau \bigl\langle V_{\rm L}^s(x)Q_i^s(x), h\bigr\rangle^2\, ds \leq \varepsilon\biggr) \leq \mathcal{O}(\nu p)t^{-\mu p}\varepsilon^p\;.$$

Remark 7.10 Note that for small $\|x\|$, $Q_i(x) = Q_{i,\varrho}(x)$ may be 0 when $i < k_*$, but the point is that then we can find another i for which the inequality holds.

By a straightforward argument, given in detail in [Nor86, p. 127], one concludes that (C) implies Theorem 7.3.

It thus only remains to show that (B) implies (C). We follow closely Norris and choose a $K \in \mathcal{K}$ such that (B) holds. If K happens to be in $\mathcal{K}_0$ then it is equal to a Q_i, and thus we already have (C). Otherwise, assume $K \in \mathcal{K}_j$ with $j \geq 1$. Then we use a Martingale inequality.

Lemma 7.11 *Let $\mathcal{H}$ be a separable Hilbert space and $W(t)$ be the cylindrical Wiener process on $\mathcal{H}$. Let β^t be a real-valued predictable process and γ^t, ζ^t be predictable $\mathcal{H}$-valued processes. Define*

$$\begin{aligned} a^t &= a^0 + \int_0^t \beta^s\, ds + \int_0^t \langle \gamma^s, dW(s)\rangle\;, \\ b^t &= b^0 + \int_0^t a^s\, ds + \int_0^t \langle \zeta^s, dW(s)\rangle\;. \end{aligned}$$

Suppose $\tau \le t_0$ is a bounded stopping time such that for some constant $C_0 < \infty$ we have

$$\sup_{0<s<\tau} \bigl\{|\beta^s|,\, |a^s|,\, \|\zeta^s\|,\, \|\gamma^s\|\bigr\} \le C_0\;.$$

Then, for every $p > 1$, there exists a constant C_{p,t_0} such that

$$\mathbf{P}\Bigl(\int_0^\tau (b^s)^2\, ds < \varepsilon^{20} \quad\text{and}\quad \int_0^\tau \bigl(|a^s|^2 + \|\zeta^s\|^2\bigr)\, ds \ge \varepsilon\Bigr) \le C_{p,t_0}\,(1 + C_0^6)^p \varepsilon^p\;,$$

for every $\varepsilon \le 1$.

Proof. The proof is given in [Nor86], but without the explicit dependence on C_0. If we follow his proof carefully we get an estimate of the type

$$\mathbf{P}\Bigl(\int_0^\tau (b^s)^2\, ds < \varepsilon^{10} \text{ and } \int_0^\tau \bigl(|a^s|^2 + \|\zeta^s\|^2\bigr)\, ds \ge (1 + C_0^3)\varepsilon\Bigr) \le C_1\,(1 + C_0^{12})^p \varepsilon^p\;.$$

Replacing ε by ε^2 and making the assumption $\varepsilon < 1/(1 + C_0^3)$, we recover our statement. The statement is trivial for $\varepsilon > 1/(1 + C_0^3)$, since any probability is always smaller than 1. □

We apply this inequality as follows: Define, for $K_0 \in \mathcal{K}$,

$$\begin{aligned} a^t(x) &= \bigl\langle V_{\rm L}^t\bigl([X^0, K_0]_{\rm L}^t\bigr)(x), h\bigr\rangle\;, \\ b^t(x) &= \bigl\langle V_{\rm L}^t K_0^t(x), h\bigr\rangle\;, \\ \beta^t(x) &= \bigl\langle V_{\rm L}^t\bigl(\bigl[X^0, [X^0, K_0]_{\rm L}\bigr]_{\rm L}^t\bigr)(x), h\bigr\rangle\;, \\ (\gamma^t)^i(x) &= \bigl\langle V_{\rm L}^t\bigl(\bigl[Q_i, [X^0, K_0]_{\rm L}\bigr]_{\rm L}^t\bigr)(x), h\bigr\rangle\;, \\ (\zeta^t)^i(x) &= \bigl\langle V_{\rm L}^t\bigl([Q_i, K_0]_{\rm L}^t\bigr)(x), h\bigr\rangle\;. \end{aligned}$$

In this expression, $\zeta^t(x) \in \mathcal{H}$, $(\zeta^t)^i(x) = \langle \zeta^t(x), e_i\rangle$ and similarly for the γ^t. It is clear by Proposition 7.4, Eq.(7.6), and Eq.(7.7) that the assumptions of Lemma 7.11 are satisfied with $C_0 = \mathcal{O}(\nu)$ for some $\nu > 0$.

We continue the proof that (B) implies (C) in the case when $K \in \mathcal{K}_j$, with $j = 1$. Then, by the construction of $\mathcal{K}_j$ with $j \ge 1$, there is a $K_0 \in \mathcal{K}_{j-1}$ such that we have either $K = [Q_i, K_0]_{\rm L}$ for some $i \in \{k_*, \ldots, L-1\}$, or $K = [X^0, K_0]_{\rm L}$. In fact, for $j = 1$ only the second case occurs and $K_0 = Q_i$ for some i, but we are already preparing an inductive step. Applying Lemma 7.11, we have for every $\varepsilon \leq 1$:

$$\mathbf{P}\Big(\int_0^\tau \big\langle V_{\rm L}^s K_0^s(x), h\big\rangle^2\,ds < \varepsilon \quad\text{and}\quad \int_0^\tau \Big(\big\langle V_{\rm L}^s [X^0, K_0]_{\rm L}^s(x), h\big\rangle^2 + \sum_{i=0}^\infty \big\langle V_{\rm L}^s [Q_i, K_0]_{\rm L}^s(x), h\big\rangle^2\Big)\,ds \geq \varepsilon^{1/20}\Big) \leq \mathcal{O}(6\nu p)\varepsilon^{p/20}\;.$$

Since the second integral above is always larger than $\int_0^\tau \big\langle V_{\rm L}^t K^t(x), h\big\rangle^2\,dt$, the probability for it to be smaller than $\varepsilon^{1/20}$ is, by (B), bounded by $\mathcal{O}(16p)t^{-p}\varepsilon^{p/20}$. This implies (replacing ν by $\max\{6\nu, 16\}$) that

$$\mathbf{P}\Big(\int_0^\tau \big\langle V_{\rm L}^s K_0^s(x), h\big\rangle^2\,ds < \varepsilon\Big) \leq \mathcal{O}(\nu p)t^{-p}\varepsilon^{p/20}\;.$$

Since for $j = 1$ we have $K_0 = Q_i$ with $i \in \{k_*, \ldots, L-1\}$, we have shown (C) in this case. The above reasoning is repeated for $j = 2$ and $j = 3$, by iterating the above argument. For example, if $K = [Q_{i_1}, [X^0, Q_{i_2}]_{\rm L}]_{\rm L}$,with $i_1, i_2 \in \{k_*, \ldots, L-1\}$, we apply Lemma 7.11 twice, showing the first time that $\langle [X^0, Q_{i_2}]_{\rm L}, h\rangle^2$ is unlikely to be small and then again to show that $\langle Q_{i_2}, h\rangle^2$ is also unlikely to be small (with other powers of ε), which is what we wanted. Finally, since every K used in (B) is in $\mathcal{K}$, at most 3 such invocations of Lemma 7.11 will be sufficient to conclude that (C) holds. The proof of Theorem 7.3 is complete. □

7.4 Estimates on the Low-Frequency Derivatives (Proof of Proposition 5.3)

Having proven the crucial bound Theorem 7.1 on the reduced Malliavin matrix, we can now proceed to prove Proposition 5.3, *i.e.*, the smoothing properties of the dynamics in the low-frequency part. For convenience, we restate it here.

Proposition 7.12 *There exist a time $T^* > 0$ and exponents $\mu, \nu > 0$ such that for every $\varphi \in \mathcal{C}_b^2(\mathcal{H})$, every $\xi \in \mathcal{H}^\alpha$, and every $t \leq T^*$, one has*

$$\Big\|\mathbf{E}\Big(\big(D_{\rm L}\varphi \circ \Phi^t\big)(\xi)\big(D_{\rm L}\Phi_{\rm L}^t\big)(\xi)\Big)\Big\| \leq Ct^{-\mu}\big(1 + \|\xi\|_\alpha^\nu\big)\|\varphi\|_{\rm L^\infty}\;. \tag{7.8}$$

Proof. The proof will use the integration by parts formula (6.4) together with Theorem 7.1. Fix $\xi \in \mathcal{H}^\alpha$ and $t > 0$. In this proof, we omit the argument ξ to gain legibility, but it will be understood that the formulas do generally only hold if evaluated at some $\xi \in \mathcal{H}^\alpha$. We extend our phase space to include $D_{\rm L}\Phi^t$, $V_{\rm L}^t$ and $\mathscr{D}_v\Phi_{\rm L}^t$. We define a new stochastic process Ψ^t by

$$\Psi^t = \big(\Phi^t, \mathscr{D}_v\Phi_{\rm L}^t, D_{\rm L}\Phi^t, V_{\rm L}^t\big) \in \widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathbf{R}^{L\cdot L} \oplus \mathcal{H}^L \oplus \mathbf{R}^{L\cdot L}\;.$$

Applying the definitions of these processes, we see that Ψ^t is defined by the autonomous SDE given by

$$d\Phi^t = -A\Phi^t\,dt + F(\Phi^t)\,dt + Q(\Phi^t)\,dW(t)\;,$$

$$\begin{aligned}
dD_{\mathrm{L}}\Phi^t &= -AD_{\mathrm{L}}\Phi^t\,dt + DF(\Phi^t)D_{\mathrm{L}}\Phi^t\,dt + DQ(\Phi^t)D_{\mathrm{L}}\Phi^t\,dW(t)\;,\\
dV_{\mathrm{L}}^t &= V_{\mathrm{L}}^t A_{\mathrm{L}}\,dt - V_{\mathrm{L}}^t\,D_{\mathrm{L}}F_{\mathrm{L}}(\Phi^t)\,dt - V_{\mathrm{L}}^t D_{\mathrm{L}}Q_{\mathrm{L}}(\Phi^t)\,dW_{\mathrm{L}}(t)\\
&\quad + V_{\mathrm{L}}^t\sum_{i=0}^{L-1}\big(D_{\mathrm{L}}Q_{\mathrm{L}}^i(\Phi^t)\big)^2\,dt\;,\\
d\mathscr{D}_v\Phi_{\mathrm{L}}^t &= -A_{\mathrm{L}}\mathscr{D}_v\Phi_{\mathrm{L}}^t\,dt + D_{\mathrm{L}}F_{\mathrm{L}}(\Phi^t)\mathscr{D}_v\Phi_{\mathrm{L}}^t\,dt + Q_{\mathrm{L}}(\Phi^t)^2(V_{\mathrm{L}}^t)^*\,dt\\
&\quad + D_{\mathrm{L}}Q_{\mathrm{L}}(\Phi^t)\,\mathscr{D}_v\Phi_{\mathrm{L}}^t\,dW_{\mathrm{L}}(t)\;.
\end{aligned}$$

This expression will be written in the short form

$$d\Psi^t = -\widetilde{A}\Psi^t\,dt + \widetilde{F}(\Psi^t)\,dt + \widetilde{Q}(\Psi^t)\,dW(t)\;,$$

with $\Psi^t \in \widetilde{\mathcal{H}}$ and $dW(t)$ the cylindrical Wiener process on $\mathcal{H}$. It can easily be verified that this equation satisfies assumption **A1** of Proposition 6.1. We consider again the stochastic process $v^t \in \mathcal{H}$ defined in (6.9). It is clear from Lemma 6.5 that v^t satisfies **A2**. With this particular choice of v, the first component of $\mathscr{D}_v\Psi^t$ (the one in $\mathcal{H}$) is equal to $\mathscr{D}_v\Phi_{\mathrm{L}}^t \oplus 0$.

We choose a function $\varphi \in \mathcal{C}_b^2(\mathcal{H})$ and fix two indices $i, k \in \{0,\ldots,L-1\}$. Define $\tilde\varphi_{i,k} : \widetilde{\mathcal{H}} \to \mathbf{R}$ by

$$\tilde\varphi_{i,k}(\Psi^t) = \sum_{j=0}^{L-1}\varphi(\Phi^t)\big((\mathscr{D}_v\Phi_{\mathrm{L}}^t)^{-1}\big)_{i,j}\big(D_{\mathrm{L}}\Phi_{\mathrm{L}}^t\big)_{j,k}\;,$$

where the inverse has to be understood as the inverse of a square matrix. By Theorem 7.1, $\tilde\varphi_{i,k}$ satisfies the assumptions of Proposition 6.1. A simple computation gives for every $h \in \mathbf{R}^L$ the identity:

$$\begin{aligned}
D\tilde\varphi_{i,k}(\Psi^t)\mathscr{D}_v\Psi^t h &= D_{\mathrm{L}}\varphi(\Phi^t)\big(\mathscr{D}_v\Phi_{\mathrm{L}}^t h\big)\big((\mathscr{D}_v\Phi_{\mathrm{L}}^t)^{-1}\big)_{i,j}\big(D_{\mathrm{L}}\Phi_{\mathrm{L}}^t\big)_{j,k}\\
&\quad + \varphi(\Phi^t)\big((\mathscr{D}_v\Phi_{\mathrm{L}}^t)^{-1}(\mathscr{D}_v^2\Phi_{\mathrm{L}}^t h)(\mathscr{D}_v\Phi_{\mathrm{L}}^t)^{-1}\big)_{i,j}\big(D_{\mathrm{L}}\Phi_{\mathrm{L}}^t\big)_{j,k}\\
&\quad + \varphi(\Phi^t)\big((\mathscr{D}_v D_{\mathrm{L}}\Phi_{\mathrm{L}}^t)h\big)_{i,j}\big((\mathscr{D}_v\Phi_{\mathrm{L}}^t)^{-1}\big)_{j,k}\;,
\end{aligned}\tag{7.9}$$

where summation over j is implicit. We now apply the integration by parts formula in the form of Proposition 6.1. This gives the identity

$$\mathbf{E}\big(D\tilde\varphi_{i,k}(\Psi^t)\mathscr{D}_v\Psi^t h\big) = \mathbf{E}\Big(\tilde\varphi_{i,k}(\Psi^t)\int_0^t\big\langle v^s h, dW(s)\big\rangle\Big)\;.$$

Substituting (7.9), we find

$$\begin{aligned}
&\mathbf{E}\Big(D_{\mathrm{L}}\varphi(\Phi^t)\big(\mathscr{D}_v\Phi_{\mathrm{L}}^t h\big)\big((\mathscr{D}_v\Phi_{\mathrm{L}}^t)^{-1}\big)_{i,j}\big(D_{\mathrm{L}}\Phi_{\mathrm{L}}^t\big)_{j,k}\Big) =\\
&\qquad - \mathbf{E}\Big(\varphi(\Phi^t)\big((\mathscr{D}_v\Phi_{\mathrm{L}}^t)^{-1}(\mathscr{D}_v^2\Phi_{\mathrm{L}}^t h)(\mathscr{D}_v\Phi_{\mathrm{L}}^t)^{-1}\big)_{i,j}\big(D_{\mathrm{L}}\Phi_{\mathrm{L}}^t\big)_{j,k}\Big)\\
&\qquad - \mathbf{E}\Big(\varphi(\Phi^t)\big((\mathscr{D}_v D_{\mathrm{L}}\Phi_{\mathrm{L}}^t)h\big)_{i,j}\big((\mathscr{D}_v\Phi_{\mathrm{L}}^t)^{-1}\big)_{j,k}\Big)
\end{aligned}$$

$$+\,\mathbf{E}\Big(\varphi(\Phi^t)\big((\mathscr{D}_v\Phi_{\rm L}^t)^{-1}\big)_{i,j}\big(D_{\rm L}\Phi_{\rm L}^t\big)_{j,k}\int_0^t\big\langle v^s h, dW(s)\big\rangle\Big)\;.$$

The summation over the index j is implicit in every term. We now choose $h = e_i$ and sum over the index i. The left-hand side is then equal to

$$\mathbf{E}\Big(\big(D_{\rm L}\varphi(\Phi^t)\big)D_{\rm L}\Phi_{\rm L}^t e_k\Big)\;,$$

which is precisely the expression we want to bound. The right-hand side can be bounded in terms of $\|\varphi\|_{{\rm L}^\infty}$ and of $\mathbf{E}\big((\mathscr{D}_v\Phi_{\rm L}^t)^{-4}\big)$ (at worst). The other factors are all given by components of $\mathscr{D}_v\Psi^t$ and can therefore be bounded by means of Theorem 8.9. Therefore, (7.8) follows. The proof of Proposition 7.12 is complete. □

8 Existence Theorems

In this section, we prove existence theorems for several PDE's and SDE's, in particular we prove Proposition 5.1 and Lemma 5.4. Much of the material here relies on well-known techniques, but we include the details for completeness.

We consider again the problem

$$d\Phi^t = -A\Phi^t\,dt + F(\Phi^t)\,dt + Q(\Phi^t)\,dW(t)\;, \tag{8.1}$$

with $\Phi^0 = \xi$ given. The initial condition ξ will be taken in one of the Hilbert spaces $\mathcal{H}^\gamma$. We will show that, after some time, the solution lies in some smaller Hilbert space. Note that we are working here with the *cutoff* equations, but we omit the index ϱ.

We will of course require that all stochastic processes are predictable. This means that if we write ${\rm L}^p(\Omega,\mathscr{Y})$, with $\mathscr{Y}$ some Banach space of functions of the interval $[0,T]$, we really mean that the only functions we consider are those that are measurable with respect to the predictable σ-field when considered as functions over $\Omega\times[0,T]$.

We first state precisely what is known about the ingredients of (8.1).

Lemma 8.1 *The following properties hold for A, F and Q.*

P1 *The space $\mathcal{H}$ is a real separable Hilbert space and $A : \mathcal{D}(A)\to\mathcal{H}$ is a self-adjoint strictly positive operator.*

P2 *The map $F:\mathcal{H}\to\mathcal{H}$ has bounded derivatives of all orders.*

P3 *For every $\gamma\ge 0$, F maps $\mathcal{H}^\gamma$ into itself. Furthermore, there exists a constant $n>0$ independent of γ and constants $C_{F,\gamma}$ such that F satisfies the bounds*

$$\|F(x)\|_\gamma \le C_{F,\gamma}\big(1+\|x\|_\gamma\big)\;, \tag{8.2a}$$

$$\|F(x)-F(y)\|_\gamma \le C_{F,\gamma}\|x-y\|_\gamma\big(1+\|x\|_\gamma+\|y\|_\gamma\big)^n\;, \tag{8.2b}$$

for all x and y in $\mathcal{H}^\gamma$.

P4 *There exists an $\alpha>0$ such that for every $x, x_1, x_2\in\mathcal{H}$ the map $Q:\mathcal{H}\to\mathscr{L}(\mathcal{H},\mathcal{H})$ satisfies*

$$\big\|A^{\alpha-3/8}Q(x)\big\|_{\rm HS}\le C\;,\quad \big\|A^{\alpha-3/8}\big(Q(x_1)-Q(x_2)\big)\big\|_{\rm HS}\le C\|x_1-x_2\|\;,$$

where $\|\cdot\|_{\rm HS}$ denotes the Hilbert-Schmidt norm in $\mathcal{H}$.

P5 *The derivative of Q satisfies*

$$\big\|A^\alpha\big(DQ(x)\big)h\big\|_{\mathrm{HS}} \le C\|h\| , \tag{8.3}$$

for every $x, h \in \mathcal{H}$.

P6 *The derivative of F satisfies*

$$\big\|\big(DF(x)\big)y\big\|_\gamma \le C(1+\|x\|_\gamma)\|y\|_\gamma ,$$

for every $x, y \in \mathcal{H}^\gamma$.

Proof. The points **P1**, **P2** are obvious. The point **P4** follows from the definition (1.6) of Q and the construction of Q_ϱ in (4.1). To prove **P3**, recall that the map $F = F_\varrho$ of the GL equation is of the type

$$F_\varrho(u) = \chi\big(\|u\|/(3\varrho)\big)P(u) ,$$

with P some polynomial and $\chi \in \mathcal{C}_0^\infty(\mathbf{R})$. The key point is to notice that the estimate

$$\|uv\|_\gamma \le C_\gamma\big(\|u\|\,\|v\|_\gamma + \|u\|_\gamma\|v\|\big)$$

holds for every $\gamma \ge 0$, where uv denotes the multiplication of two functions. In particular, we have

$$\|u^n\|_\gamma \le C\|u\|_\gamma\|u\|^{n-1} ,$$

which, together with the fact that χ has compact support, shows (8.2a). This also shows that the derivatives of F in $\mathcal{H}^\gamma$ are polynomially bounded and so (8.2b) holds. **P6** follows by the same argument.

The point **P5** immediately follows from the fact that the image of the operator $\big(DQ(x)\big)h$ is contained in $\mathcal{H}_{\mathrm{L}}$ for every $x, h \in \mathcal{H}$. □

Remark 8.2 The condition **P1** implies that e^{-At} is an analytic semigroup of contraction operators on $\mathcal{H}$. We will use repeatedly the bound

$$\big\|e^{-At}x\big\|_\gamma \;\le\; C_\gamma t^{-\gamma}\|x\| .$$

We begin the study of (8.1) by considering the equation for the mild solution

$$\begin{aligned}\Psi(t,\xi,\omega) \;=\; e^{-At}\xi &+ \int_0^t e^{-A(t-s)}F\big(\Psi(s,\xi,\omega)\big)\,ds \\ &+ \int_0^t e^{-A(t-s)}Q\big(\Psi(s,\xi,\omega)\big)\,dW(s,\omega) .\end{aligned} \tag{8.4}$$

The study of this equation is in several steps. We will consider first the noise term, then the equation for a fixed instance of ω, and finally prove existence and bounds.

We need some more notation:

Definition 8.3 *Let $\mathcal{H}^\alpha$ be as above the domain of A^α with the graph norm. We fix, once and for all, a maximal time T. We denote by $\mathcal{H}_T^\alpha$ the space $\mathcal{C}([0,T],\mathcal{H}^\alpha)$ equipped with the norm*

$$\|y\|_{\mathcal{H}_T^\alpha} = \sup_{t\in[0,T]}\|y(t)\|_\alpha .$$

We write $\mathcal{H}_T$ instead of $\mathcal{H}_T^0$.

8.1 The Noise Term

Let $y \in \mathrm{L}^p(\Omega, \mathcal{H}_T)$. (One should think of y as being $y(t) = \Phi^t$.) The noise term in (8.4) will be studied as a function on $\mathrm{L}^p(\Omega, \mathcal{H}_T)$. It is given by the function Z defined as

$$\big(Z(y)\big)(\omega) = t \mapsto \int_0^t e^{-A(t-s)} Q\big(y(\omega)(s)\big)\, dW(s,\omega)\ . \tag{8.5}$$

We will show that $Z(y)$ is in $\mathrm{L}^p(\Omega, \mathcal{H}_T^\alpha)$ when y is in $\mathrm{L}^p(\Omega, \mathcal{H}_T)$. The natural norm here is the L^p norm defined by

$$|\!|\!| Z(y) |\!|\!|_{\mathcal{H}_T^\alpha, p} = \left(\mathbf{E}_\omega \sup_{t\in[0,T]} \|(Z(y))_t(\omega)\|_\alpha^p\right)^{1/p}\ .$$

Proposition 8.4 *Let $\mathcal{H}$, A and Q be as above and assume* **P1** *and* **P4** *are satisfied. Then, for every $p \geq 1$ and every $T < T_0$ one has*

$$|\!|\!| Z(y) |\!|\!|_{\mathcal{H}_T^\alpha, p} \leq C_{T_0} T^{p/16}\ . \tag{8.6}$$

Proof. Choose an element $y \in \mathrm{L}^p(\Omega, \mathcal{H}_T)$. In the sequel, we will consider y as a function over $[0,T] \times \Omega$ and we will not write explicitly the dependence on Ω.

In order to get bounds on Z, we use the factorization formula and the Young inequality. Choose $\gamma \in (1/p, 1/8)$. The factorization formula [DPZ92] then gives the equality

$$\big(Z(y)\big)(t) = C \int_0^t (t-s)^{\gamma-1} e^{-A(t-s)} \int_0^s (s-r)^{-\gamma} e^{-A(s-r)} Q(y(r))\, dW(r)\, ds\ .$$

Since A commutes with e^{-At}, the Hölder inequality leads to

$$\begin{aligned}
&\big\|\big(Z(y)\big)(t)\big\|_\alpha^p \qquad (8.7)\\
&= C\Big\|\int_0^t (t-s)^{\gamma-1} e^{-A(t-s)} \int_0^s (s-r)^{-\gamma} A^\alpha e^{-A(s-r)} Q(y(r))\, dW(r)\, ds\Big\|^p \\
&\leq C t^\nu \int_0^t \Big\|\int_0^s (s-r)^{-\gamma} A^\alpha e^{-A(s-r)} Q(y(r))\, dW(r)\Big\|^p\, ds\ ,
\end{aligned}$$

with $\nu = (p\gamma - 1)/(p-1)$. For the next bound we need the following result:

Lemma 8.5 *[DPZ92, Thm. 7.2]. Let $r \mapsto \Psi^r$ be an arbitrary predictable $\mathscr{L}^2(\mathcal{H})$-valued process. Then, for every $p \geq 2$, there exists a constant C such that*

$$\mathbf{E}\Big(\Big\|\int_0^s \Psi^r\, dW(r)\Big\|^p\Big) \leq C\mathbf{E}\Big(\int_0^s \|\Psi^r\|_{\mathrm{HS}}^2\, dr\Big)^{p/2}\ .$$

This lemma, the Young inequality applied to (8.7), and **P4** above imply

$$|\!|\!| Z(y) |\!|\!|_{\mathcal{H}_T^\alpha, p}^p = \mathbf{E}\Big(\sup_{0\leq t\leq T}\Big\|\int_0^t A^\alpha e^{-A(t-s)} Q\big(y(s)\big)\, dW(s)\Big\|^p\Big)$$

$$
\begin{aligned}
&\le CT^{\nu}\mathbf{E}\int_0^T \Big\|\int_0^s (s-r)^{-\gamma}A^{\alpha}e^{-A(s-r)}Q\big(y(r)\big)\,dW(r)\Big\|^p\,ds\\
&\le CT^{\nu}\mathbf{E}\int_0^T \Big(\int_0^s (s-r)^{-2\gamma}\big\|A^{\alpha}e^{-A(s-r)}Q\big(y(r)\big)\big\|_{\mathrm{HS}}^2\,dr\Big)^{p/2}\,ds\\
&\le CT^{\nu}\mathbf{E}\int_0^T \Big(\int_0^s (s-r)^{-2\gamma}\big\|A^{3/8}e^{-A(s-r)}\big\|^2\big\|A^{\alpha-3/8}Q\big(y(r)\big)\big\|_{\mathrm{HS}}^2\,dr\Big)^{p/2}\,ds\\
&\le CT^{\nu}\mathbf{E}\int_0^T \Big(\int_0^s (s-r)^{-2\gamma-3/4}\big\|A^{\alpha-3/8}Q\big(y(r)\big)\big\|_{\mathrm{HS}}^2\,dr\Big)^{p/2}\,ds\\
&\le CT^{\nu}\Big(\int_0^T s^{-2\gamma-3/4}\,ds\Big)^{p/2}\mathbf{E}\int_0^T \big\|A^{\alpha-3/8}Q(y(s))\big\|_{\mathrm{HS}}^p\,ds\\
&\le CT^{1+\nu}\Big(\int_0^T s^{-2\gamma-3/4}\,ds\Big)^{p/2}\,,
\end{aligned}
\tag{8.8}
$$

provided $\gamma < 1/8$. We choose $\gamma = 1/16$ (which thus imposes the condition $p > 16$), and we find

$$
|||Z(y)|||_{\mathcal{H}_T^{\alpha},p}^p \le CT_0^{1+\nu}T^{p/16}\,.
$$

Thus, we have shown (8.6) for $p > 16$. Since we are working in a probability space the case of $p \ge 1$ follows. This completes the proof of Proposition 8.4. □

8.2 A Deterministic Problem

The next step in our study of (8.4) is the analysis of the problem for a *fixed* instance of the noise ω. Then (8.4) is of the form

$$
h(t,\xi,z) = e^{-At}\xi + \int_0^t e^{-A(t-s)}F\big(h(s,\xi,z)\big)\,ds + z(t)\ ,
$$

where we assume that $z \in \mathcal{H}_T^{\alpha}$. One should think of this as an instance of $Z(\Phi)$, but at this point of our proof, the necessary bounds are not yet available.

We find it more convenient to study instead of h the quantity g defined by $g(t,\xi,z) = h(t,\xi,z) - z(t)$. Then g satisfies

$$
g(t,\xi,z) = e^{-At}\xi + \int_0^t e^{-A(t-s)}F\big(g(s,\xi,z)+z(s)\big)\,ds\ . \tag{8.9}
$$

We consider the solution (assuming it exists) as a map from the initial condition ξ and the deterministic noise term z. More precisely, we define

$$
G(\xi,z)_t \;=\; g(t,\xi,z)\ .
$$

This is a map defined on $\mathcal{H}\times\mathcal{H}_T^{\alpha}$. Clearly, (8.9) reads:

$$
G(\xi,z)_t \;=\; e^{-At}\xi + \int_0^t e^{-A(t-s)}F\big(G(\xi,z)_s + z(s)\big)\,ds\ . \tag{8.10}
$$

To formulate the bounds on G, we need some more spaces that take into account the regularizing effect of the semigroup $t\mapsto e^{-At}$.

Definition 8.6 *For $\gamma \geq 0$ the spaces $\mathcal{G}_T^\gamma$ are defined as the closures of $\mathcal{C}([0,T],\mathcal{H}^\gamma)$ under the norm*

$$\|y\|_{\mathcal{G}_T^\gamma} = \sup_{t\in(0,T]} t^\gamma \|y(t)\|_\gamma + \sup_{t\in[0,T]} \|y(t)\| \;.$$

Note that

$$\|y\|_{\mathcal{G}_T^\gamma} \;\leq\; C_{\gamma,T}\|y\|_{\mathcal{H}_T^\gamma} \;.$$

With these definitions, one has:

Proposition 8.7 *Assume the conditions* **P1–P4** *are satisfied. Assume $\xi \in \mathcal{H}$ and $z \in \mathcal{H}_T^\alpha$. Then, there exists a map $G : \mathcal{H}\times\mathcal{H}_T^\alpha \to \mathcal{H}_T$ solving (8.10). One has the following bounds:*

(A) *If $\xi \in \mathcal{H}^\gamma$ with $\gamma \leq \alpha$ one has for every $T > 0$ the bound*

$$\|G(\xi,z)\|_{\mathcal{H}_T^\gamma} \leq C(1 + \|\xi\|_\gamma + \|z\|_{\mathcal{H}_T^\gamma}) \;. \tag{8.11}$$

(B) *If $\xi \in \mathcal{H}$ one has for every $T > 0$ the bound*

$$\|G(\xi,z)\|_{\mathcal{G}_T^\alpha} \leq C(1 + \|\xi\| + \|z\|_{\mathcal{H}_T^\alpha}) \;. \tag{8.12}$$

Before we start with the proof proper we note the following regularizing bound: Define

$$\big(\mathcal{N}f\big)(t) \;=\; \int_0^t e^{-A(t-s)} f(s)\,ds \;. \tag{8.13}$$

Then one has:

Lemma 8.8 *For every $\varepsilon \in [0,1)$ and every $\gamma > \varepsilon$ there is a constant $C_{\varepsilon,\gamma}$ such that*

$$\|\mathcal{N}f\|_{\mathcal{G}_T^\gamma} \;\leq\; C_{\varepsilon,\gamma} T \|f\|_{\mathcal{G}_T^{\gamma-\varepsilon}} \;,$$

for all $f \in \mathcal{G}_T^{\gamma-\varepsilon}$.

Proof. We start with

$$\begin{aligned}
\big\|\big(\mathcal{N}f\big)(t)\big\|_\gamma \;&\leq\; \int_0^{t/2} \big\|A^\gamma e^{-A(t-s)} f(s)\big\|\,ds + \int_{t/2}^t \big\|A^\varepsilon e^{-A(t-s)} A^{\gamma-\varepsilon} f(s)\big\|\,ds \\
&\leq\; \int_0^{t/2} (t-s)^{-\gamma}\|f(s)\|\,ds + \int_{t/2}^t (t-s)^{-\varepsilon}\|f(s)\|_{\gamma-\varepsilon}\,ds \\
&\leq\; \int_0^{t/2} (t-s)^{-\gamma}\|f\|_{\mathcal{G}_T^{\gamma-\varepsilon}}\,ds + \int_{t/2}^t (t-s)^{-\varepsilon} s^{\varepsilon-\gamma}\|f\|_{\mathcal{G}_T^{\gamma-\varepsilon}}\,ds \\
&\leq\; C t^{1-\gamma}\|f\|_{\mathcal{G}_T^{\gamma-\varepsilon}} + C t^{1-\varepsilon} t^{\varepsilon-\gamma}\|f\|_{\mathcal{G}_T^{\gamma-\varepsilon}} \;.
\end{aligned}$$

Therefore, $t^\gamma\big\|\big(\mathcal{N}f\big)(t)\big\|_\gamma \leq CT\|f\|_{\mathcal{G}_T^{\gamma-\varepsilon}}$. Similarly, we have

$$\big\|\big(\mathcal{N}f\big)(t)\big\| \;\leq\; \int_0^t \big\|e^{-A(t-s)} f(s)\big\|\,ds \;\leq\; Ct\|f\|_{\mathcal{G}_T^{\gamma-\varepsilon}} \;.$$

Combining the two inequalities, the result follows. □

Proof of Proposition 8.7. We first choose an initial condition $\xi \in \mathcal{H}^\gamma$ and a function $z \in \mathcal{H}_T^\gamma$. The local existence of the solutions in $\mathcal{H}^\gamma$ is a well-known result. Thus there exists, for a possibly small time $\widetilde{T} > 0$, a function $u \in \mathcal{C}([0,\widetilde{T}],\mathcal{H}^\gamma)$ satisfying

$$u(t) = e^{-At}\xi + \int_0^t e^{-A(t-s)} F\big(u(s)+z(s)\big)\,ds\ .$$

In order to get an *a priori* bound on $\|u(t)\|_\gamma$ we use assumption **P3** and find

$$\begin{aligned}\|u(t)\|_\gamma &\le \|\xi\|_\gamma + C_{F,\gamma}\int_0^t \big(1+\|u(s)+z(s)\|_\gamma\big)\,ds \\ &\le C\big(1+\|\xi\|_\gamma+\|z\|_{\mathcal{H}_T^\gamma}\big) + C_{F,\gamma}\int_0^t \|u(s)\|_\gamma\,ds\ .\end{aligned}$$

By Gronwall's lemma we get for $t < T$,

$$\|u(t)\|_\gamma \le C_1(1+\|\xi\|_\gamma+\|z\|_{\mathcal{H}_T^\gamma})\ . \tag{8.14}$$

Note that (8.14) tells us that if the initial condition ξ is in $\mathcal{H}^\gamma$ and if z is in $\mathcal{H}_T^\gamma$, then $u(t)$ is, for small enough t, again in $\mathcal{H}^\gamma$ with the above bound. Therefore, we can iterate the above reasoning and show the global existence of the solutions up to time T, with bounds. Thus, G is well-defined and satisfies the bound (8.11).

We turn to the proof of the estimate (8.12). Define for $z \in \mathcal{H}_T$ the map $\mathcal{M}_z$ by

$$\big(\mathcal{M}_z(x)\big)(t) = e^{-At}\xi + \int_0^t e^{-A(t-s)} F\big(x(s)+z(s)\big)\,ds\ . \tag{8.15}$$

Taking $\xi \in \mathcal{H}$ we see from (8.14) with $\gamma = 0$ that there exists a fixed point u of $\mathcal{M}_z$ which satisfies

$$\|u\|_{\mathcal{H}_T} \;=\; \sup_{t\in[0,T]}\|u(t)\| \;\le\; C_1(1+\|\xi\|+\|z\|_{\mathcal{H}_T})\ .$$

Assume next that $z \in \mathcal{H}_T^\alpha$ and hence *a fortiori* $z \in \mathcal{G}_T^\alpha$. Then, by **P3** one has

$$\|F(x+z)\|_{\mathcal{G}_T^\gamma} \;\le\; C\big(1+\|x\|_{\mathcal{G}_T^\gamma}+\|z\|_{\mathcal{G}_T^\gamma}\big)\ .$$

Since u is a fixed point and (8.15) contains a term of the form of (8.13) we can apply Lemma 8.8 and obtain for every $\gamma \le \alpha$ and $\varepsilon \in [0,1)$:

$$\begin{aligned}\|u\|_{\mathcal{G}_T^{\gamma+\varepsilon}} \;&=\; \|\mathcal{M}_z(u)\|_{\mathcal{G}_T^{\gamma+\varepsilon}} \;\le\; C\|\xi\| + CT\|F(u+z)\|_{\mathcal{G}_T^\gamma} \\ &\le\; C\|\xi\| + C\big(1+\|u\|_{\mathcal{G}_T^\gamma}+\|z\|_{\mathcal{G}_T^\gamma}\big)\ .\end{aligned} \tag{8.16}$$

Thus, as long as $\|z\|_{\mathcal{G}_T^\gamma}$ is finite, we can apply repeatedly (8.16) until reaching $\gamma = \alpha$, and this proves (8.12). The proof of Proposition 8.7 is complete. □

8.3 Stochastic Differential Equations in Hilbert Spaces

Before we can start with the final steps of the proof of Proposition 5.1 we state in the next subsection a general existence theorem for stochastic differential equations in Hilbert spaces. The symbol $\mathcal{H}$ denotes a separable Hilbert space. We are interested in solutions to the SDE

$$dX^t = (-AX^t + N(t,\omega,X^t) + M^t)\,dt + B(t,\omega,X^t)\,dW(t)\;, \tag{8.17}$$

where $W(t)$ is the cylindrical Wiener process on a separable Hilbert space $\mathcal{H}_0$. We assume $B(t,\omega,X^t) : \mathcal{H}_0 \to \mathcal{H}$ is Hilbert-Schmidt. We will denote by Ω the underlying probability space and by $\{\mathscr{F}_t\}_{t\geq 0}$ the associated filtration.

The exact conditions spell out as follows:

C1 The operator $A : \mathcal{D}(A) \to \mathcal{H}$ is the generator of a strongly continuous semigroup in $\mathcal{H}$.

C2 There exists a constant $C > 0$ such that for arbitrary $x, y \in \mathcal{H}$, $t \geq 0$ and $\omega \in \Omega$ the estimates

$$\|N(t,\omega,x) - N(t,\omega,y)\| + \|B(t,\omega,x) - B(t,\omega,y)\|_{\mathrm{HS}} \leq C\|x-y\|\;,$$
$$\|N(t,\omega,x)\|^2 + \|B(t,\omega,x)\|_{\mathrm{HS}}^2 \leq C^2(1 + \|x\|^2)\;,$$

hold.

C3 For arbitrary $x, h \in \mathcal{H}$ and $h_0 \in \mathcal{H}_0$, the stochastic processes $\langle N(\cdot,\cdot,x), h\rangle$ and $\langle B(\cdot,\cdot,x)h_0, h\rangle$ are predictable.

C4 The $\mathcal{H}$-valued stochastic process M^t is predictable, has continuous sample paths, and satisfies

$$\sup_{t\in[0,T]} \mathbf{E}\,\|M^t\|^p < \infty\;,$$

for every $T > 0$ and every $p \geq 1$.

C5 For arbitrary $t > 0$ and $\omega \in \Omega$, the maps $x \mapsto N(t,\omega,x)$ and $x \mapsto B(t,\omega,x)$ are twice continuously differentiable with their derivatives bounded by a constant independent of t, x and ω.

We have the following existence theorem.

Theorem 8.9 *Assume that $\xi \in \mathcal{H}$ and that* **C1** – **C4** *are satisfied.*

- *For any $T > 0$, there exists a mild solution X_ξ^t of (8.17) with $X_\xi^0 = \xi$. This solution is unique among the $\mathcal{H}$-valued processes satisfying*

$$\mathbf{P}\left(\int_0^T \|X_\xi^t\|^2\,dt < \infty\right) = 1\;.$$

  Furthermore, X_ξ has a continuous version and is strongly Markov.

- *For every $p \geq 1$ and $T > 0$, there exists a constant $C_{p,T}$ such that*

$$\mathbf{E}\big(\sup_{t\in[0,T]} \|X_\xi^t\|^p\big) \leq C_{p,T}(1 + \|\xi\|^p)\;. \tag{8.18}$$

– *If, in addition,* **C5** *is satisfied, the mapping $\xi \mapsto X^t_\xi(\omega)$ has a.s. bounded partial derivatives with respect to the initial condition ξ. These derivatives satisfy the SDE's obtained by formally differentiating (8.17) with respect to X.*

Proof. The proof of this theorem for the case $M^t \equiv 0$ can be found in [DPZ96]. The same proof carries through for the case of non-vanishing M^t satisfying **C4**. □

8.4 Bounds on the Cutoff Dynamics (Proof of Proposition 5.1)

With the tools from stochastic analysis in place, we can now prove Proposition 5.1. We start with the

Proof of (A). In this case we identify the equation (8.17) with (4.2) and apply Theorem 8.9. The condition **C1** of Theorem 8.9 is obviously true, and the condition **C3** is redundant in this case. The condition **C2** is satisfied because F and Q of (8.17) satisfy **P2–P4**. Therefore, (8.18) holds and hence we have shown (5.1a) for the case of $\gamma = 0$. In particular, Φ^t_ϱ exists and satisfies

$$\begin{aligned}\Phi^t_\varrho(\xi,\omega) \;=\; e^{-At}\xi + \int_0^t e^{-A(t-s)} F\big(\Phi^s_\varrho(\xi,\omega)\big)\,ds \\ + \int_0^t e^{-A(t-s)} Q\big(\Phi^s_\varrho(\xi,\omega)\big)\,dW(s)\;.\end{aligned} \tag{8.19}$$

We can extend (5.1a) to arbitrary $\gamma \le \alpha$ as follows. We set as in (8.5),

$$\big(Z(\Phi_\varrho)\big)_t(\omega) \;=\; \int_0^t e^{-A(t-s)} Q\big(\Phi^s_\varrho(\xi,\omega)\big)\,dW(s)\;. \tag{8.20}$$

By Proposition 8.4, we find that for all $p \ge 1$ one has

$$\left(\mathbf{E}_\omega \sup_{t\in[0,T]} \|(Z(\Phi_\varrho))_t(\omega)\|^p_\alpha\right)^{1/p} < C_{T,p} \tag{8.21}$$

for all ξ. From this, we conclude that, almost surely,

$$\sup_{t\in[0,T]} \|(Z(\Phi_\varrho))_t(\omega)\|_\alpha < \infty\;. \tag{8.22}$$

Subtracting (8.20) from (8.19) we get

$$\begin{aligned}\Phi^t_\varrho(\xi,\omega) - \big(Z(\Phi_\varrho)\big)_t(\omega) \;=\; e^{-At}\xi + \int_0^t e^{-A(t-s)} F\big(\Phi^s_\varrho(\xi,\omega)\big)\,ds \\ = e^{-At}\xi + \int_0^t e^{-A(t-s)} F\Big(\Phi^s_\varrho(\xi,\omega) - \big(Z(\Phi_\varrho)\big)_s(\omega) + \big(Z(\Phi_\varrho)\big)_s(\omega)\Big)\,ds\;.\end{aligned} \tag{8.23}$$

Comparing (8.23) with (8.10) we see that, a.s.,

$$\Phi^t_\varrho(\xi,\omega) - \big(Z(\Phi_\varrho)\big)_t(\omega) = G\big(\xi, Z\big(\Phi_\varrho(\xi,\cdot)\big)(\omega)\big)\;.$$

We now use z as a shorthand:

$$z(t) = \Big(Z\big(\Phi_\varrho(\xi,\cdot)\big)\Big)_t(\omega)\;.$$

Assume now $\xi \in \mathcal{H}^\gamma$. Note that by (8.22), $z(t)$ is in $\mathcal{H}^\alpha$. If $\gamma \le \alpha$, we can apply Proposition 8.7 and from (8.11) we conclude that almost surely,

$$\sup_{t\in[0,T]} \|G(\xi,z)\|_\gamma \le C\big(1 + \|\xi\|_\gamma + \sup_{t\in[0,T]} \|z\|_\gamma\big)\;.$$

Finally, since $\gamma \le \alpha$, we find

$$\begin{aligned}
\mathbf{E}\Big(\sup_{t\in[0,T]} \|\Phi_\varrho^t(\xi)\|_\gamma^p\Big) &\le C\mathbf{E}\Big(\sup_{t\in[0,T]} \|G(\xi,z)_t\|_\gamma^p\Big) + C\mathbf{E}\Big(\sup_{t\in[0,T]} \|z(t)\|_\gamma^p\Big)\\
&\le C(1+\|\xi\|_\gamma)^p + C\mathbf{E}\Big(\sup_{t\in[0,T]} \|z(t)\|_\gamma^p\Big)\\
&\le C(1+\|\xi\|_\gamma)^p\;,
\end{aligned}\tag{8.24}$$

where we applied (8.21) to get the last inequality. Thus, we have shown (5.1a) for all $\gamma \le \alpha$. The fact that the solution is strong if $\gamma \ge 1$ is an immediate consequence of [Lun95, Lemma 4.1.6] and [DPZ92, Thm. 5.29].

Proof of (B). This bound can be shown in a similar way, using the bound (8.12) of Proposition 8.7: Take $\xi \in \mathcal{H}$. By the above, we know that there exists a solution to (8.19) satisfying the bound (5.1b) with $\gamma = 0$. We define $z(t)$ and $G(\xi,z)_t$ as above. But now we apply the bound (8.12) of Proposition 8.7 and we conclude that almost surely,

$$\sup_{t\in[0,T]} t^\alpha \|G(\xi,z)\|_\alpha \le C\big(1 + \|\xi\| + \sup_{t\in[0,T]} \|z\|_\alpha\big)\;.$$

Following a procedure similar to (8.24), we conclude that (5.1b) holds.

Proof of (C). The existence of the partial derivatives follows from Theorem 8.9. To show the bound, choose $\xi \in \mathcal{H}$ and $h \in \mathcal{H}$ with $\|h\| = 1$, and define the process $\Psi^t = \big(D\Phi_\varrho^t(\xi)\big)h$. It is by Theorem 8.9 a mild solution to the equation

$$d\Psi^t = -A\Psi^t\,dt + \Big(\big(DF\circ\Phi_\varrho^t\big)(\xi)\Psi^t\Big)\,dt + \Big(\big(DQ\circ\Phi_\varrho^t\big)(\xi)\Psi^t\Big)\,dW(t)\;. \tag{8.25}$$

By **P3** and **P5**, this equation satisfies conditions **C1–C3** of Theorem 8.9, so we can apply it to get the desired bound (5.1c). (The constant term drops since the problem is linear in h.)

Proof of (D). Choose $h \in \mathcal{H}$ and $\xi \in \mathcal{H}^\alpha$ and define as above $\Psi^t = \big(D\Phi_\varrho^t(\xi)\big)h$, which is the mild solution to (8.25) with initial condition h. We write this as

$$\begin{aligned}
\Psi^t &= e^{-At}h + \int_0^t e^{-A(t-s)}\Big(\big(DF\circ\Phi_\varrho^s\big)(\xi)\Psi^s\Big)\,ds\\
&\quad + \int_0^t e^{-A(t-s)}\Big(\big(DQ\circ\Phi_\varrho^s\big)(\xi)\Psi^s\Big)\,dW(s)\\
&\equiv S_1^t + S_2^t + S_3^t\;.
\end{aligned}$$

The term S_1^t satisfies

$$\sup_{t\in(0,T]} t^\alpha \|S_1^t\|_\alpha \;\le\; C\|h\|\;. \tag{8.26}$$

The term S_3^t is very similar to what is found in (8.5), with $Q\bigl(y(s)\bigr)$ replaced by $(DQ\circ\Phi_\varrho^s)\Psi^s$. Repeating the steps of (8.8) for a sufficiently large p, we obtain now with $\gamma=\frac14$, some $\mu>0$ and writing $X^s=\bigl(DQ\circ\Phi_\varrho^s\bigr)(\xi)\Psi^s$:

$$\begin{aligned}
\mathbf{E}\sup_{t\in[0,T]}\|S_3^t\|_\alpha^p &= \mathbf{E}\Bigl(\sup_{0\le t\le T}\Bigl\|\int_0^t A^\alpha e^{-A(t-s)}X^s\,dW(s)\Bigr\|^p\Bigr)\\
&\le CT^\mu\mathbf{E}\int_0^T\Bigl\|\int_0^s (s-r)^{-\gamma}A^\alpha e^{-A(s-r)}X^r\,dW(r)\Bigr\|^p\,ds\\
&\le CT^\mu\mathbf{E}\int_0^T\Bigl(\int_0^s (s-r)^{-2\gamma}\bigl\|A^\alpha e^{-A(s-r)}X^r\bigr\|_{\mathrm{HS}}^2\,dr\Bigr)^{p/2}\,ds\\
&\le CT^\mu\mathbf{E}\int_0^T\Bigl(\int_0^s (s-r)^{-2\gamma}\bigl\|A^\alpha X^r\bigr\|_{\mathrm{HS}}^2\,dr\Bigr)^{p/2}\,ds\\
&\le CT^\mu\Bigl(\int_0^T s^{-2\gamma}\,ds\Bigr)^{p/2}\mathbf{E}\int_0^T\bigl\|A^\alpha X^s\bigr\|_{\mathrm{HS}}^p\,ds\\
&\le CT^{\mu+p/4}\mathbf{E}\int_0^T\bigl\|A^\alpha\bigl(DQ\circ\Phi_\varrho^s\bigr)(\xi)\Psi^s\bigr\|_{\mathrm{HS}}^p\,ds\;.
\end{aligned}$$

We now use **P5**, *i.e.*, (8.3) and then (5.1c) and get

$$\mathbf{E}\sup_{t\in[0,T]}\|S_3^t\|_\alpha^p \;\le\; CT^{\mu+p/4}\mathbf{E}\int_0^T\|\Psi^s\|^p\,ds \;\le\; CT^{\mu+p/4+1}\|h\|^p\;. \tag{8.27}$$

To treat the term S_2^t, we fix a realization $\omega\in\Omega$ of the noise and use Lemma 8.8. This gives for $\varepsilon\in[0,1)$ the bound

$$\sup_{t\in(0,T]} t^\gamma\|S_2^t\|_\gamma \le CT\sup_{t\in(0,T]} t^{\gamma-\varepsilon}\bigl\|\bigl(DF\circ\Phi_\varrho^t\bigr)(\xi)\Psi^t\bigr\|_{\gamma-\varepsilon}\;.$$

By **P6**, this leads to the bound, a.s.,

$$\sup_{t\in(0,T]} t^\gamma\|S_2^t\|_\gamma \le C_T\Bigl(1+\sup_{t\in(0,T]}\bigl\|\Phi_\varrho^t(\xi)\bigr\|_{\gamma-\varepsilon}\Bigr)\sup_{t\in(0,T]} t^{\gamma-\varepsilon}\bigl\|\Psi^t\bigr\|_{\gamma-\varepsilon}\;.$$

Taking expectations we have

$$\mathbf{E}\sup_{t\in(0,T]} t^{\gamma p}\|S_2^t\|_\gamma^p \le C_T^p\mathbf{E}\left(\Bigl(1+\sup_{t\in(0,T]}\bigl\|\Phi_\varrho^t(\xi)\bigr\|_{\gamma-\varepsilon}\Bigr)^p\sup_{t\in(0,T]} t^{(\gamma-\varepsilon)p}\bigl\|\Psi^t\bigr\|_{\gamma-\varepsilon}^p\right)\;.$$

By the Schwarz inequality and (5.1a) we get

$$\mathbf{E}\sup_{t\in(0,T]} t^{\gamma p}\|S_2^t\|_\gamma^p \le C\bigl(1+\|\xi\|_{\gamma-\varepsilon}^p\bigr)\Bigl(\mathbf{E}\sup_{t\in(0,T]} t^{(\gamma-\varepsilon)2p}\bigl\|\Psi^t\bigr\|_{\gamma-\varepsilon}^{2p}\Bigr)^{1/2}\;. \tag{8.28}$$

Since $\Psi^t = \big(D\Phi_\varrho^t(\xi)\big)h = S_1^t + S_2^t + S_3^t$, combining (8.26)–(8.28) leads to

$$\begin{aligned}&\mathbf{E} \sup_{t\in(0,T]} t^{\gamma p}\|\big(D\Phi_\varrho^t(\xi)\big)h\|_\gamma^p \\ &\qquad \le C\|h\|^p + C\big(1+\|\xi\|_{\gamma-\varepsilon}^p\big)\Big(\mathbf{E}\sup_{t\in(0,T]} t^{(\gamma-\varepsilon)2p}\big\|\big(D\Phi_\varrho^t(\xi)\big)h\big\|_{\gamma-\varepsilon}^{2p}\Big)^{1/2} .\end{aligned}$$

Thus, we have gained ε in regularity. Choosing $\varepsilon = \frac{1}{2}$ and iterating sufficiently many times we obtain (5.1d) for sufficiently large p. The general case then follows from the Hölder inequality.

Proof of (E). We estimate this expression by

$$\big\|\Phi_\varrho^t(\xi) - e^{-At}\xi\big\|_\gamma \le \int_0^t \big\|F\big(\Phi_\varrho^s(\xi)\big)\big\|_\gamma\, ds + \bigg\|\int_0^t e^{-A(t-s)}Q\big(\Phi_\varrho^s(\xi)\big)\, dW(s)\bigg\|_\gamma .$$

The first term can be bounded by combining (5.1b) and **P3**. The second term is bounded by Proposition 8.4.

The proof Proposition 5.1 is complete.

8.5 Bounds on the Off-Diagonal Terms

Here, we prove Lemma 5.4. This is very similar to the proof of (D) of Proposition 5.1.

Proof. We start with (5.5b). Recall that here we do not write the cutoff ϱ. We choose $h \in \mathcal{H}_\mathrm{H}$ and $\xi \in \mathcal{H}$. The equation for $\Psi^t = \big(D_\mathrm{H}\Phi_\mathrm{L}^t(\xi)\big)h$ is :

$$\begin{aligned}\Psi^t &= \int_0^t e^{-A(t-s)}\Big(\big(DF_\mathrm{L}\circ\Phi_\varrho^s\big)(\xi)\big(D_\mathrm{H}\Phi^s(\xi)\big)h\Big)\, ds \\ &\quad + \int_0^t e^{-A(t-s)}\Big(\big(DQ_\mathrm{L}\circ\Phi_\varrho^s\big)(\xi)\big(D_\mathrm{H}\Phi^s(\xi)\big)h\Big)\, dW(s) \\ &\equiv R_1^t + R_2^t .\end{aligned}$$

Since $DF = DF_\varrho$ is bounded we get

$$\big\|R_1^t\big\| \;\le\; C\int_0^t \big\|\big(D_\mathrm{H}\Phi^s(\xi)\big)h\big\|\, ds \;\le\; Ct \sup_{s\in[0,t]}\big\|\big(D_\mathrm{H}\Phi^s(\xi)\big)h\big\| .$$

Using (5.1c), this leads to

$$\mathbf{E}\sup_{t\in[0,T]}\big\|R_1^t\big\|^p \;\le\; C^p t^p \mathbf{E}\sup_{s\in[0,t]}\big\|\big(D_\mathrm{H}\Phi^s(\xi)\big)h\big\|^p \;\le\; Ct^p\|h\|^p .$$

The term R_2^t is bounded exactly as in (8.27). Combining the bounds, (5.5b) follows.

Since Q_H is constant, see (4.1), we get for $\Psi^t = \big(D_\mathrm{L}\Phi_\mathrm{H}^t(\xi)\big)h$ and $h \in \mathcal{H}_\mathrm{L}$:

$$\Psi^t = \int_0^t e^{-A(t-s)}\Big(\big(DF_\mathrm{H}\circ\Phi_\varrho^s\big)(\xi)\big(D_\mathrm{L}\Phi^s(\xi)\big)h\Big)\, ds .$$

This is bounded like R_1^t and leads to (5.5a). This completes the proof of Lemma 5.4. □

8.6 Proof of Proposition 2.3

Here we point out where to find the general results on (1.7) which we stated in Proposition 2.3. Note that these are bounds on the flow *without* cutoff ϱ.

Proof of Proposition 2.3. There are many ways to prove this. To make things simple, without getting the best estimate possible, we note that a bound in L^∞ can be found in [Cer99, Prop. 3.2]. To get from L^∞ to $\mathcal{H}$, we note that $\xi \in \mathcal{H}$ and we use (1.7) in its integral form. The term $e^{-At}\xi$ is bounded in $\mathcal{H}$, while the non-linear term $\int_0^t e^{-A(t-s)} F\big(\Phi^s(\xi)\big)\, ds$ can be bounded by using a version of Lemma 8.8. Finally, the noise term is bounded by Proposition 8.4.

Furthermore, because of the compactness of the semigroup generated by A, it is possible to show [DPZ96, Thm. 6.3.5] that an invariant measure exists. □

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References

[BKL00] J. Bricmont, A. Kupiainen, and R. Lefevere, *Probabilistic Estimates for the Two Dimensional Stochastic Navier-Stokes Equations*, Preprint, 2000.

[Cer99] S. Cerrai, *Smoothing Properties of Transition Semigroups Relative to SDEs with Values in Banach Spaces*, Probab. Theory Relat. Fields **113** (1999), 85–114.

[Col94] P. Collet, *Non Linear Parabolic Evolutions in Unbounded Domains*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci **437** (1994), 97–104.

[DPZ92] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, University Press, Cambridge, 1992.

[DPZ96] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, London Mathematical Society Lecture Note Series, vol. 229, University Press, Cambridge, 1996.

[EH00] J.-P. Eckmann and M. Hairer, *Non-Equilibrium Statistical Mechanics of Strongly Anharmonic Chains of Oscillators*, Comm. Math. Phys. **212** (2000), 105–164.

[EPR99a] J.-P. Eckmann, C.-A. Pillet, and L. Rey-Bellet, *Non-Equilibrium Statistical Mechanics of Anharmonic Chains Coupled to Two Heat Baths at Different Temperatures*, Comm. Math. Phys. **201** (1999), 657–697.

[EPR99b] J.-P. Eckmann, C.-A. Pillet, and L. Rey-Bellet, *Entropy Production in Non-Linear, Thermally Driven Hamiltonian Systems*, J. Stat. Phys. **95** (1999), 305–331.

[ESM00] W. E, Y. G. Sinai, and J. C. Mattingly, *Gibbsian Dynamics and Ergodicity for the Stochastically Forced Navier-Stokes Equation*, Preprint, 2000.

[FM95] F. Flandoli and B. Maslowski, *Ergodicity of the* 2*-D Navier-Stokes Equation Under Random Perturbations*, Comm. Math. Phys. **172** (1995), 119–141.

[Hör67] L. Hörmander, *Hypoelliptic Second Order Differential Equations*, Acta Math. **119** (1967), 147–171.

[Hör85] L. Hörmander, *The Analysis of Linear Partial Differential Operators I–IV*, Springer, New York, 1985.

[KA00] S. B. Kuksin and A. Shirikyan, *Stochastic Dissipative PDE's and Gibbs Measures*, Preprint, 2000.

[Lun95] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, Basel, 1995.

[MS95b] H. M. Möller and H. J. Stetter, *Multivariate Polynomial Equations with Multiple Zeros Solved by Matrix Eigenproblems*, Numerische Mathematik **70** (1995), 311–329.

[Nor86] J. Norris, *Simplified Malliavin Calculus*, Lecture Notes in Mathematics **1204** (1986), 101–130.