Fractal Dimensions and Quantum Evolution Associated with Sparse Potential Jacobi Matrices

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Abstract

We study the quantum dynamics generated via Schrödinger equation by sparse-potential Jacobi matrices on $l_2(\mathbf{Z}_+)$. Exact bounds for the upper and lower intermittency functions governing the asymptotic growth of moments are derived in terms of the fractal dimensions of the spectral measure. Numerical experiments suggest that these bounds are sharp in the case of very sparse barriers.

1 Introduction

We consider an evolution equation in Hilbert space $\mathcal{H}$ with scalar product $(\cdot,\cdot)$:

$$i\frac{\partial \phi_t}{\partial t} = H\phi_t, \quad \phi_{t=0} = \phi \in \mathcal{H}. \tag{1}$$

H is here a self-adjoint operator with spectral resolution $H = \int \lambda dE_\lambda$. Much interest has been devoted in the last years to the following question: which information on the dynamical properties of $\{\phi_t, t \in \mathbf{R}\}$ are encoded in the spectral measure

$$d\mu_\phi(\lambda) := d(\phi, E_\lambda \phi), \tag{2}$$

and–of course–in which form they can be decoded.

Conventional wisdom about this problem rests on the so-called RAGE theorem [1] which provides a general qualitative answer: if the spectral measure is continuous, then time-averages of the projection of ϕ_t in a given finite dimensional subspace of $\mathcal{H}$ tend to zero as time goes to infinity. On the other hand, this theorem has a poor quantitative content and its interpretation is somewhat misleading, in particular in regards to the relation between dynamical localization and the pure point nature of the spectrum. Need for refinements arose from the discovery of many types of singular spectra in one-electron models solid state physics, like random or quasi-periodic potentials: the interest in the transport properties of such systems [2, 3], and in the dynamics of kicked systems [4, 5], required more complete and quantitative answers.

A first major step in this direction was accomplished by I. Guarneri [6, 7] in a pioneering work which lead, after some refinements [8, 9], to a general bound relating "growth exponents" as defined below to the Hausdorff and fractal dimensions of the spectral measure μ_ϕ. More precisely, let $\{e_n\}_{n\in\mathbf{N}}$ be an orthonormal basis in $\mathcal{H}$, and let the moment of order α be defined by

$$\nu_\alpha(t) := \frac{1}{T}\int_0^T \sum_n n^\alpha |(e_n, \phi_t)|^2 \, dt, \quad \alpha \in \mathbf{R}. \tag{3}$$

We obviously assume that this definition makes sense for a suitable ϕ. We the define the *growth exponents* $\beta^\pm(\alpha)$ via

$$\beta^\pm(\alpha) = \frac{1}{\alpha} \lim_{T\to\infty} \binom{sup}{inf} \frac{\log \nu_\alpha(T)}{\log T}, \quad \alpha > 0. \tag{4}$$

The variation of these exponents with the moment index α is highly non-trivial: this has been also denoted the *quantum intermittency* function [11]. The result quoted above is a general inequality for $\beta^-(\alpha)$:

$$\beta^-(\alpha) \geq \dim_H(\mu_\phi), \tag{5}$$

which calls in cause $\dim_H(\mu_\phi)$, the Hausdorff dimension of the spectral measure. Upper exponents are described by a more recent bound by I. Guarneri and H. Schulz-Baldez [12], to the extent that

$$\beta^+(\alpha) \geq \dim_p(\mu_\phi). \tag{6}$$

The new dimension appearing in this bound, $\dim_p(\mu_\phi)$, is the packing (or Tricot) fractal dimension. These dimensions can be defined in terms of the local exponents as follows.

Let μ be any probability measure. The local upper and lower dimensions $d^\pm_\mu(x)$ the measure μ at the point $x \in \mathbf{R}$ are defined by

$$d^\pm_\mu(x) := \lim_{T\to\infty} \binom{sup}{inf} \frac{\log \mu([x - \frac{1}{T}, x + \frac{1}{T}])}{-\log(T)}. \tag{7}$$

Usually, this equation is written in terms of $\varepsilon = \frac{1}{T}$: the above form is better in the present context, for it shows that as time progresses, a better resolution of the spectral measure is obtained.

The Hausdorff and Tricot dimensions follow from the local dimensions as essential superior limits:

$$\dim_H(\mu_\phi) := \mu - \text{ ess sup } d^-_\mu(x), \tag{8}$$

and

$$\dim_p(\mu_\phi) := \mu - \text{ ess sup } d^+_\mu(x). \tag{9}$$

A natural question is whether the bounds (5) and (6) are sharp, and in which sense. There is at least one reason why they cannot be so, for any choice of the basis in Hilbert space. This is the fact that the l.h.s. of (5) and (6) depend on the chosen basis while the r.h.s. do not. It is then clear that even within the same basis set, re-labelling the basis elements can radically alter growth exponents, and can even make them infinite. One of the crucial contents of the inequality (5) is that this action *cannot* render them zero, in any case. One should also mention that basis independent growth exponents can also be defined [8, 13, 14] for the "effective dimension" or "entropy" of ϕ_t and of the associated density matrices, but we shall not discuss them here.

The above leads to the idea that if these bounds have a chance to be sharp, this could be for some "natural" or "dynamical" basis adapted to the time evolution of the solution of eq. (1). It was suggested in [15] that the most convenient basis for eq. (1) is that of the *orthogonal polynomials* of μ, that is, the basis of the cyclic subspace generated by $\{H^n\phi, n \in \mathbf{N}\}$ in which H is a tridiagonal matrix. As a matter of fact, most models of quasi-periodic systems so far studied in the literature are expressed in such a basis from the very start [16], and so will be the model we consider here.

And yet, even in this basis, sharpness of the bounds (5) and (6) is rather exceptional. First of all, it must be also noted that the l.h.s. of (5) and (6) involve the moment index α, while the r.h.s. does not [17]. The functions $\beta^\pm(\alpha)$ are clearly non decreasing in α, and examples exist where they are strictly increasing [15, 11] . This is the effect of the so-called *quantum intermittency* phenomenon [19].

Evidently then, the bounds we considered are at their best in the limit $\alpha \to 0$, which defines the exponents $\beta^\pm(0)$. These are associated to the evolution of the logarithmic moment (see sect. 3). Even here, it has been observed [19] that the *logarithmic gap* $\gamma := \beta^-(0) - \dim_H(\mu_\phi)$ in general is *strictly* larger than zero. The sole non-trivial example where equality has been detected is the dynamics generated by Jacobi matrices of real Julia sets [15, 19, 20, 21]). We shall mention again this system, for it is exceptional in many respects. A new case with zero logarithmic gap will be presented in this paper.

The case of real Julia set matrices, and of Jacobi matrices associated with balanced measures of iterated function systems has unveiled the relation of the growth exponents with the *multifractal* *dimensions* $D(q)$, $q \in \mathbf{R}$, of the spectral measure μ_ϕ: this relation is exact only in the first case, and can be summed up in the relation $\beta(\alpha) = D(1-\alpha)$ [15, 19, 21]. The second case, to the contrary, has taught us that this relation is only approximate, and that discrepancies are brought about by subtle "renormalization" properties of the orthogonal polynomials of μ_ϕ [19]. This powerful tool is not available in the general case, and the quest for the *intermittency* function β is still open.

In fact, considerable difficulties arise outside the realm of I.F.S. and Julia Jacobi Hamiltonians, if one wants to test (5) and (6), and more generally the relations between spectral and dynamical properties. In typical examples from Schrödinger operator theory, for instance, both terms in the inequalities (5) and (6) are difficult to control, and even in numerical simulations the very existence of the phenomenon of quantum intermittency has frequently escaped detection. In addition, very few models with non-trivial Hausdorff dimension (*i.e.* different from zero and one) were known until recently [22]. In fact, even in the case of pure point measures, as found in the Anderson model or in almost periodic potentials in some regimes, it has taken a lot of effort to show that dynamical localization ($\beta^-(\alpha) = \beta^+(\alpha) = 0$ for all $\alpha > 0$) takes place: see *e.g.* [23]. Notice that this result is not an obvious consequence of RAGE theorem: in [24] del Rio *et al.* give an example of a rank one perturbation of the almost-Mathieu Hamiltonian having pure point spectrum with exponentially decaying eigenfunctions and $\beta^+_\alpha = 1$ for all α !

Notice moreover that in these examples on $l_2(\mathbf{Z}_+)$ the basis $\{e_n\}_{n\in\mathbf{N}}$ is the canonical choice $e_n(k) = \delta_{n,k}$: except of course when dynamical localization takes place one cannot expect it to be adapted to the dynamics of waves in quasi-periodic potentials, and yet one has to stay with it, because of its physical meaning.

All together, these results seemed to suggest that (5) and (6) are *at best* rather rough estimates. Fortunately, a recent work of Jitomarkaya and Last [25] provided an example of a discrete one-dimensional Schrödinger operator with non-trivial exact Hausdorff dimension of the local density of states. At the same time, this example is simple enough for analysis: it consists of sparse potential barriers placed along a homogeneous lattice. Since quantum motion is here elementary, being a sequence of tunneling processes through barriers, followed by ballistic motion in between these latter, one can hope to treat it theoretically to a fair extent. Moreover, the upper and lower growth exponents $\beta^\pm_\alpha$ should take here different values, and should give a correct overall description of the dynamics of the system, from which, in particular, we might draw a good test of the bounds mentioned above.

This is indeed what we shall do in the following. Our first main conclusion is that the bound (5) is sharp, at least in the limit of small positive α. We shall also show that for sparse enough barriers $\dim_p(\mu_\phi)$ is one, so that (6) is also sharp for all positive α. Moreover, we shall show that the obvious inequality $\beta^- \leq \beta^+$ holds in the strict sense: for small positive α theoretically, and in the general case by numerical experiments. The spectral counterpart of these results, *i.e.* $\dim_H(\mu_\phi) < \dim_p(\mu_\phi)$ reveals a rich multifractal structure of the local density of states.

The second main result of the paper is an *upper* bound for the lower intermittency exponents $\beta^-(\alpha)$, which appears numerically to be an exact description of these exponents for all values of α, in a case of very sparse barriers. We shall see that control of the function $\beta^\pm(\alpha)$ fully characterizes the dynamics of the moments of the wave-function ϕ_t.

This paper is organized as follows: in the next section we introduce the sparse barrier model and we enunciate the main results. These results are illustrated and completed by numerical computations in section 3. Proofs of the main theorems 2 and 3 and the related theory are presented in sections 4 and 5, respectively. The Conclusions complete the paper.

2 The sparse barrier model and main results

Consider the following Jacobi matrices on $l_2(\mathbf{Z}^+)$ (with $\mathbf{Z}^+ = \{1,2,3,\ldots\}$):

$$(H_\theta\psi)(k) = \psi(k+1) + \psi(k-1) + V(k)\psi(k) + \theta\delta_{1,k}\psi(k), \tag{10}$$

with Dirichlet boundary condition $\psi(0) = 0$. These matrices are labeled by the real parameter $\theta \in [\theta_0, \theta_1]$, $0 < \theta_0 < \theta_1 < \infty$, which also plays the role of a boundary condition. The potential $V(k)$ in eq. (10) is chosen to be null, except on a set B of selected *barrier* locations: $B = \{L_n, n \in \mathbf{Z}^+\}$. Precisely,

$$V(k) = \begin{cases} 0 & \text{if } k \notin B \\ k^{\frac{1-\eta}{2\eta}} & \text{if } k \in B \end{cases} \tag{11}$$

In this equation, η is a parameter which ranges in the open interval $(0, 1)$: the smaller η, the larger the exponent in the law (11) which links location and height of the barrier. The limit $\eta = 0$ corresponds to a Dirichlet condition at each L_n, and therefore to pure point spectrum. On the other hand, $\eta = 1$ gives barriers of constant height, and generically absolutely continuous spectrum if these are sparse enough.

Sparseness of the potential must be independently imposed: Following [25] we assume that the locations $\{L_n\}$ fulfill the inequality

$$\prod_{j=1}^{n-1} L_j \leq L_n^{\delta_n}, \tag{12}$$

for some vanishing sequence $\{\delta_n\}_{n\in\mathbf{N}}$.

We also take as initial wave function ϕ the Kronecker delta at site one:

$$\phi(k) = \delta_{1,k}. \tag{13}$$

From now on, unless otherwise specified, the short-hand notation μ will denote the spectral measure μ_ϕ so constructed.

Under these conditions the Hausdorff dimension of the spectral measure features an exact interpolation between the two extreme cases, in the sense defined by the following Theorem:

Theorem 1 *(Jitomarskaya, Last). Let H_θ be defined according to (10-12), with $\eta \in (0, 1)$. Then, for Lebesgue almost every θ the restriction of the spectral measure μ_ϕ to $[-2, 2]$ has Hausdorff dimension η.*

More was indeed proven in [25]: the restriction of μ_ϕ to $[-2, 2]$ has *exact dimension* η in the sense that $d_\mu^-(x) = \eta$ for μ-a.e. $x \in [-2, 2]$. This measure however is *not* exactly scaling, in the sense that $d_\mu^-(x)$ is not equal to $d_\mu^+(x)$ for μ-a.e. $x \in [-2, 2]$. In fact, we shall prove the following

Theorem 2 *Under the assumptions of Theorem 1 one has $d_\mu^+(x) = 1$ for μ-a.e. $x \in (-2, 2)$.*

It is easy to see that outside $[-2, 2]$ the spectrum of H_θ is pure point for all θ. We need to mention that for general θ Ref. [25] only provides a lower bound η and an upper bound $\frac{2\eta}{1+\eta}$ on the Hausdorff dimension of the restriction of μ to $[-2, 2]$.

We now consider an initial state ϕ with associated spectral measure necessarily enclosed in $(-2, 2)$: this can be obtained letting

$$\phi = f(H_\theta)e_1$$

where e_1 is the first basis state of $l_2(\mathbf{Z}^+)$, eq. (13), where $f \in C_0^\infty(\mathbf{R})$, with $\text{supp} f \subset (-2, 2)$, and $\|f\|_\infty = 1$. For simplicity of notation we omit to specify the dependence of ϕ on θ. For the same reason, the spectral measure shall be concisely expressed as μ, and we shall also leave as implicit the dependence of the exponents β on ϕ.

It follows from the above construction of ϕ and by Theorem 1 that under the sparseness condition (12)

$$\dim_H(\mu) = \eta \tag{14}$$

for almost every θ, whereas Theorem 2 implies that

$$\dim_p(\mu) = 1 \tag{15}$$

for all θ.

Notice that as a consequence of (6) and (15) one has $\beta^+(\alpha) = 1$ for all $\alpha > 0$ so that the inequality (6) is obviously sharp in this case! Indeed, this result can be obtained directly, without prior recourse to (6), along the lines of the proof of Theorem 2 and under much weaker conditions on the sequence of L_n's than those in eq. (12). At the same time it must be observed that in so doing we have no general estimate on $\dim_p(\mu)$, since the proof of Theorem 2 given here definitely requires condition (12).

In a sense, also inequality (5) is sharp. This is made precise by the following Theorem, whose proof occupies the main part of this paper.

Theorem 3 *Let H_θ be given by (10) and (11), with $\eta \in (0,1)$, and assume that for some $a > 1$ and all $n \in \mathbf{Z}$ large enough*

$$L_{n+1} \geq aL_n. \tag{16}$$

Then, for all α, $0 < \alpha \leq 2$, and almost all θ one has

$$\beta^-(\alpha) \leq \eta \frac{\alpha + 1}{\eta\alpha + 1} \tag{17}$$

An immediate consequence of Theorem 3 and of (5) is that $\dim_H(\mu) \leq \eta$ since $\lim_{\alpha\to 0} \beta^-(\alpha) \leq \eta$. Furthermore it immediately follows from (5) and (14) that (5) is sharp in the following sense:

Corollary 1 *Under the assumptions of Theorem 1 one has*

$$\lim_{\alpha\to 0^+} \beta^-(\alpha) = dim_H(\mu)$$

for almost every θ.

Sections 4 and 5 will be devoted to the proof of theorems 2 and 3. Let us now illustrate the content of these theorems by numerical experiments.

3 Pictures of quantum motion within sparse barriers

The factual content of the theorems presented in the previous section is best illustrated by the results of numerical experiments. While it is certainly true that numerical experiments provide only finite-time simulations, and, as such, cannot rigorously prove or disprove any of the Theorems presented, it is equally true that they can investigate much wider situations than those required by the hypotheses we were bound to formulate. Therefore, numerical experiments constitute at the same time a verification and a stimulus for rigorous analysis and a source of heuristic intuition.

In turning to the case at hand, it is significant to show before anything else how motion between barriers takes place. To better achieve this goal, a minor variation with respect to eq. (10) and (11) has been performed: the Hamilton operator has been divided throughout by the constant value 2. In so doing, the continuum spectrum is contained in the interval $[-1, 1]$, and the speed of "free" propagation becomes unity. It is then possible to compare directly space (*i.e.* basis index, or lattice number) and time.

A first instance of this comparison is seen in fig. 1, where the following dynamical situation (which shall be kept constant throughout this section) is considered: barriers are situated at $L_1 = 1, L_2 = 10$, and $L_3 = 1000$. The parameter η takes the value $\eta = \frac{2}{5}$, so that the exponent

$$\xi = \frac{1 - \eta}{2\eta} \tag{18}$$

appearing in the potential, eq. (11), takes the value $\xi = \frac{3}{4}$. We shall vary this parameter later on. Plotted are the first and second moment, ν_1 and ν_2, as function of time, together with the barrier heights, depicted as vertical bars, as a function of position (measured in the same units as of time). Motion of the wave packet along the lattice (*i.e.* the basis $\{e_n\}$) corresponds to the growth of these moments. It is then apparent that when the wave packet reaches a barrier, this slows down its propagation, effectively arresting moment growth for a time span which grows with the height of the barrier. The essence of the dynamics under investigation is then the balance between in-barrier propagation and tunneling through barriers.

This is also evident when observing, at fixed time t, the projection of the wave packet ϕ_t on the basis states e_n, as we do in fig. 2: the "occupation probabilities" $|(e_n, \phi_t)|^2$ are plotted versus n, once again with barrier locations super-imposed.

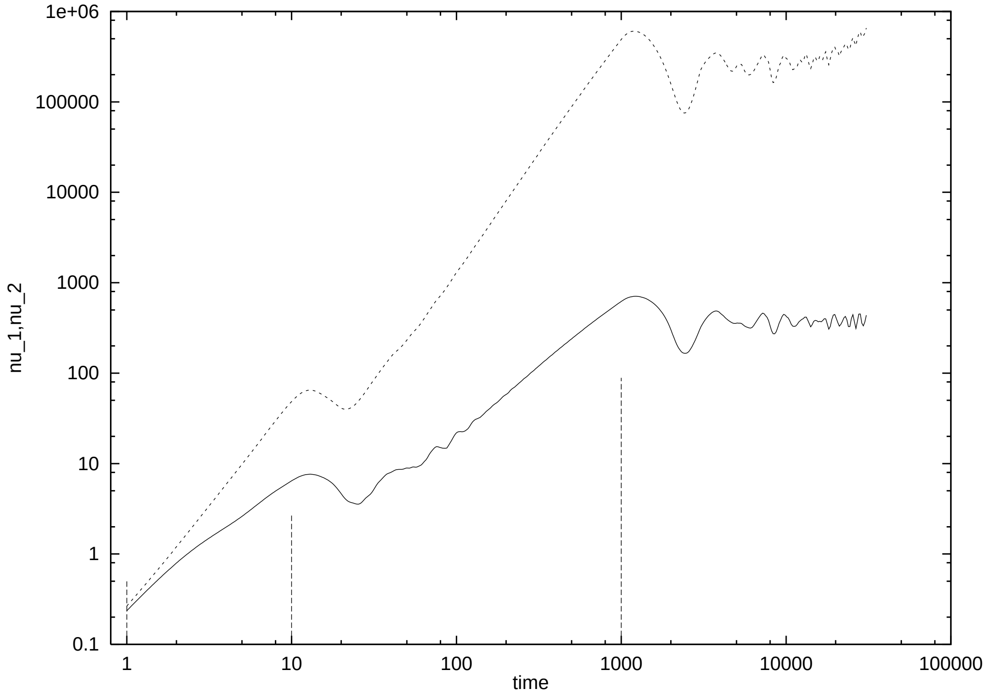

Figure 1: First and second moment ν_1 and ν_2 as a function of time. Also shown are barrier locations with their heights.

The crucial quantity in the analysis performed in this paper is the ratio

$$\rho_\alpha(t) = \frac{1}{\alpha} \frac{\log \nu_\alpha(t)}{\log t}, \tag{19}$$

whose upper and lower limits are $\beta^\pm(\alpha)$. Indeed, we find it interesting to investigate the behaviour of moments ν_α defined as in eq. (3) but *without* the Cesaro time averaging required by the theory. At the same time, we choose a single value of θ. In so doing, we stay closer to the actual dynamical behaviour of the system.

The limit of ρ_α for vanishing α is of particular interest, as we have seen. Numerically, it is difficult to directly investigate this limit. A more reliable estimate is effected by considering instead the quantity

$$\rho_0(t) = \frac{\nu_{\log}(t)}{\log t}, \tag{20}$$

where the *logarithmic moment* $\nu_{\log}(t)$ is defined by

$$\nu_{\log}(t) := \sum_n \log(|n|) |(e_n, \phi_t)|^2. \tag{21}$$

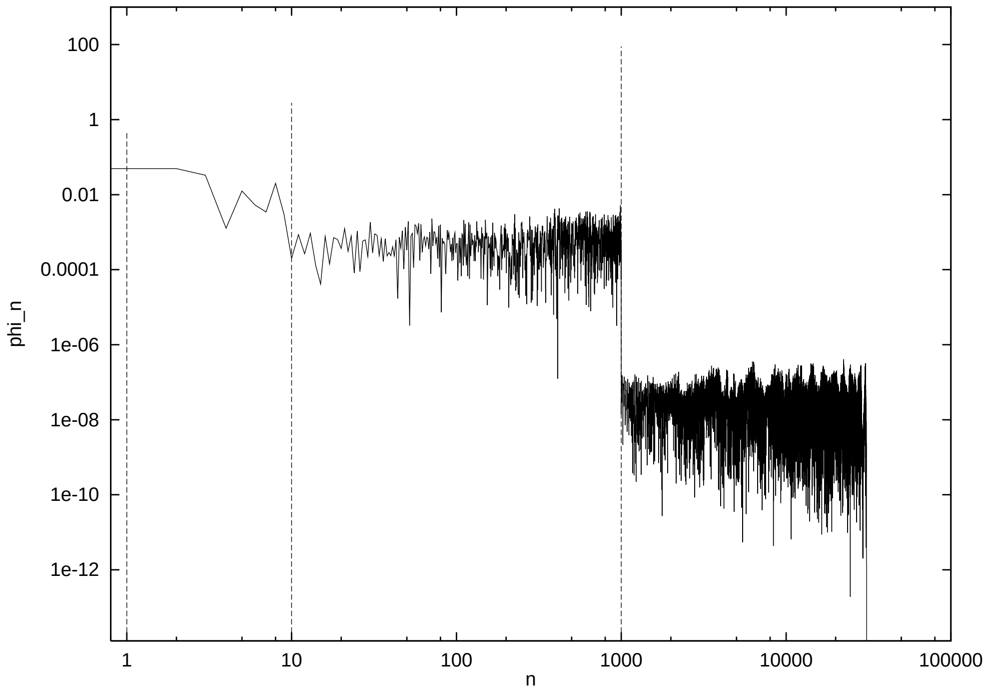

Figure 2: Quantum occupation probabilities $|(e_n, \phi_t)|^2$ versus n at fixed time $t = 30699$. Also shown are barrier locations with their heights.

Again, Cesaro averaging is not effected in the numerical computations of $\nu_{\log}(t)$.

These ratios are plotted in fig. 3 versus time, for various values of α, including zero: $\alpha = 0, 1, 2, 3, 4, 5$ (lowest five curves, in order, from bottom up). Barrier are here located as in the case of figures 1 and 2, and are again indicated by vertical lines. The height parameters are now different, and are $\eta = \frac{5}{9}$ and $\xi = \frac{2}{5}$.

This figure, as well as the preceding two, is limited to a time range which envelops three barriers only. And yet, this range is quite challenging: the basis size reached by the wave packet is about 200,000. Within these limitations, the figure brings evidence of most of the theoretical results of this paper: let us illustrate them in sequence.

Firstly, let us consider the local maxima occurring at the barrier "times" $t_n = L_n$. At these times, the head of the wave-packet arrives at a barrier. In passing from one maximum to the next, at fixed α, the value of the maximum grows. Theorem 2 assures that these values tend to the largest possible value, one. This is certainly more evident in figure for the largest value of α, $\alpha = 5$, due to the monotonicity of these quantities with α.

Indeed, the limit of $\alpha \to \infty$ can be investigated by another auxiliary quantity, the *effective basis size* of the wave packet: this is defined as the number of wave-packet projections $|(e_n, \phi_t)|^2$ larger than a fixed, very low threshold. This quantity has theoretical and computational relevance, as discussed in [28]. The related logarithmic ratio, ρ_∞, defined analogously with eq. (19), appears in the figure as the topmost curve, which regularly slides down on the horizontal line at value one.

To sum up, we expect the local maxima at barrier locations to converge to the value one for *any* α, the earlier the larger the value of α. This illustrates Theorem 2 and the consequent result $\beta^+(\alpha) = 1$ of the previous section.

Another crucial result evident in fig. 3 is the fact that β^- is strictly less than β^+. The former quantity can be inferred by the local *minima* of the curves $\rho_\alpha(t)$, occurring after the barrier times, and before tunneling has taken place: it is at these times that the blocking effect of these latter is maximal. We must discriminate here between the rapid oscillations which are due to "bouncing" of the wave packet, and which presumably fade away in the limit of infinite time (as they do with

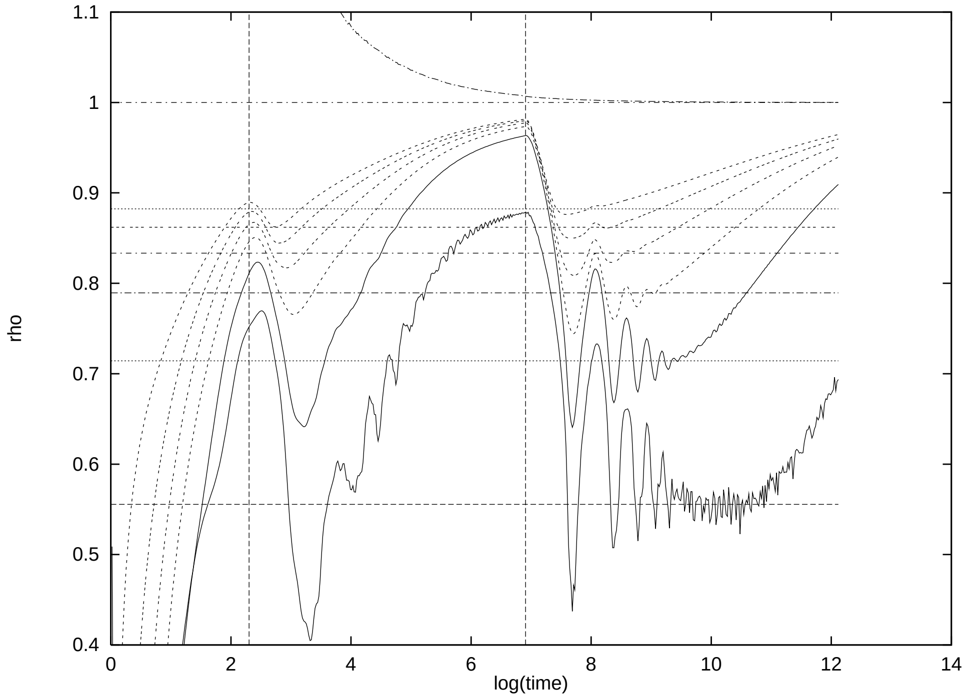

Figure 3: Moment growth ratio $\rho_\alpha(t)$ versus time, for the case described in the text, and $\alpha = 0, 1, 2, 3, 4, 5, \infty$ (bottom to top curves). The horizontal lines mark the values $\rho = \frac{5}{9}, \frac{5}{7}, \frac{15}{19}, \frac{5}{6}, \frac{25}{29}, \frac{15}{17}, 1$, which are to be orderly associated to the mentioned curves via eq. (22). The vertical lines mark the location of the barriers.

increasing α) and the underlying behaviour. In any case, it clearly appears that the difference $\beta^+(\alpha) - \beta^-(\alpha)$ is strictly larger than zero. We shall come back on this point momentarily, in figure 5.

Theoretically, the last result is implied in force of Theorem 3. Our numerical results lead us to the stronger statement, which asserts that

$$\beta^-(\alpha) = \eta \frac{\alpha + 1}{\eta\alpha + 1}. \tag{22}$$

Therefore, it seems that the estimate (17) is more than a bound: we interpolate rapid fluctuations, minima lie on the horizontal lines in the figure, which are obtained by eq. (22).

The fact that (17) is exact is trivially true for $\alpha = \infty$, and it was rigorously proven for $\alpha = 0$ in sect. 2. The numerical results show that it is plausible that this result holds for any α, and without Cesaro time averaging.

A further verification of eq. (22) can be effected by varying η, which corresponds, as we know, to varying the barrier heights, via the exponent ξ. This is done in fig. 4. The comments made for fig. (3) still apply. Indeed, as we are here varying the Jacobi Hamiltonian, this test is even more stringent than the previous.

The dynamical implications of this fact are of paramount importance, and will be discussed in the Conclusions. Here, we can now go back to the time evolution of the first moment, $\nu_1(t)$, for $\eta = \frac{5}{9}$, $\xi = \frac{2}{5}$, which is shown in fig. 5. It asymptotically oscillates between two power law lines, of the form $t^{\beta^\pm(1)}$, with unit prefactor: our bounds are then a quite satisfactory description of the dynamics. Similar conclusions can be drawn in all other cases, for various α and η. We are now ready to go back to exact analysis, for the proofs of the theorems of section 2.

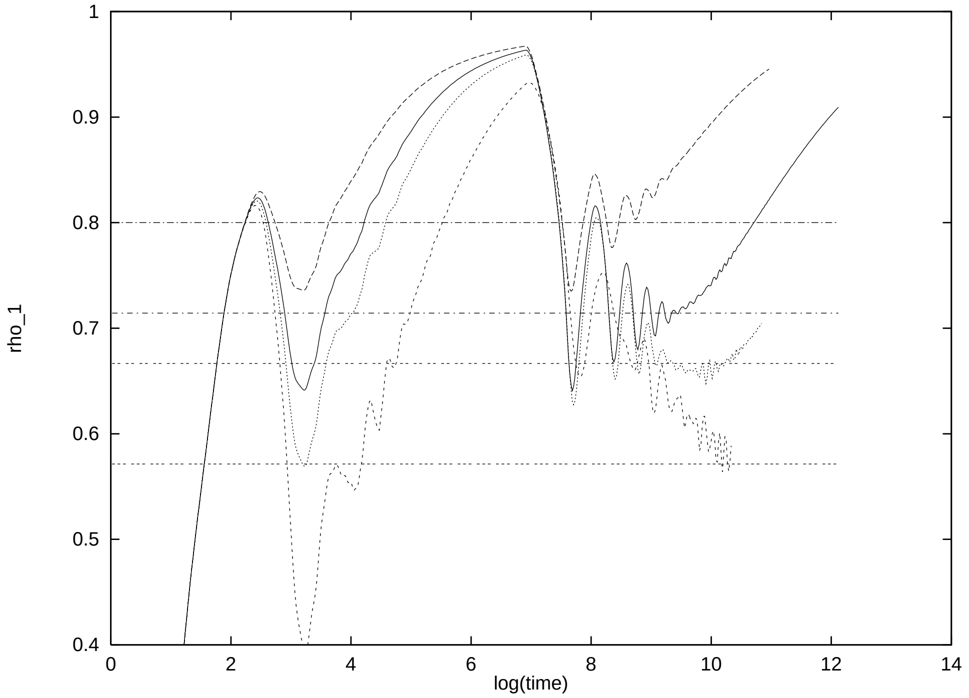

Figure 4: Moment growth ratio $\rho_1(t)$ versus time, for various values of η: $\eta = \frac{2}{5}, \frac{1}{2}, \frac{5}{9}, \frac{2}{3}$ (bottom to top curves). The horizontal lines mark the values $\beta^-(1) = \frac{4}{7}, \frac{2}{3}, \frac{5}{7}, \frac{4}{5}$, which follow orderly from eq. (22).

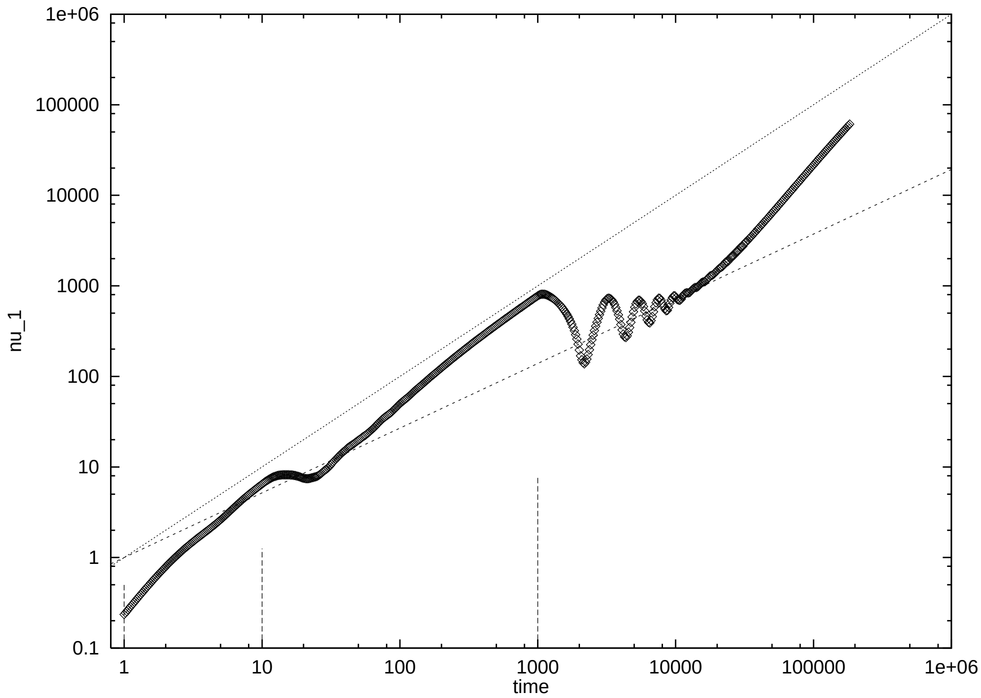

Figure 5: First moment $\nu_1(t)$ versus time, for the case described in the text, with $\eta = \frac{5}{9}$. The dotted lines have equation $t^{\beta^{\pm}(1)}$, with $\beta^+(1) = 1$, and $\beta^-(1) = \frac{5}{7}$. The vertical lines mark the location and height of the barriers.

4 Proof of Theorem 2.

The idea of the proof of Theorem 2 is the following. Because of the sparseness of the potential barriers the time evolution of a wave packet can be well approximated, on large time-scales, by a finite rank reduction of the full potential V. More precisely, let

$$H^{(N)} = H_0 + V^{(N)},$$

with

$$V^{(N)}(k) = \begin{cases} V(k) & \text{for } k < L_{N+1}, \\ 0 & \text{for } k \geq L_{N+1}, \end{cases}$$

where H_0 is the free discrete Laplacian on $\mathbf{Z}^+$, and where V is given by eq. (11) plus the θ dependent boundary condition. Accordingly, $H^{(N)}$ has absolutely continuous spectrum; in addition we shall see momentarily a good estimate of its local density of states which, by comparison, will yield estimates on the upper local exponents for H at certain energy scales.

Let therefore $\phi_t^{(N)}$, $t \in \mathbf{R}$, be the solution of Schrödinger's equation (1) with $H^{(N)}$ in place of H and ϕ given by eq. (13). Let also $\mu^{(N)}$ be the associated spectral measure. An elementary time estimate follows:

Lemma 1 *Assume $L_{n+1} \geq aL_n$ for some $a > 1$ and all integer n; let $\Delta_n := L_n - L_N$ for $n > N$. Then if N is large enough one has*

$$\|\phi_t - \phi_t^{(N)}\| \leq C 2^{-\Delta_{N+1}} \quad \text{if} \quad |t| < \frac{\Delta_{N+1}}{8e} \tag{23}$$

for some constant C independent of N.

Proof. Let $b_n(k) = \delta_{L_n,k}$. Expanding $\phi_s^{(N)} = \exp\{-iH^{(N)}s\}\phi$ in powers of s to order Δ_n gives:

$$|(b_n, \phi_s^{(N)})| \leq 2\frac{s^{\Delta_n}}{\Delta_n!}\|(H^{(N)})^{\Delta_n} b_n\| \leq 2\frac{s^{\Delta_n}}{\Delta_n!}\|H_0^{\Delta_n} b_n\|,$$

according to our definition of Δ_n, and thanks to the fact that $H^{(N)}$ couples only next neighbors sites. Stirling's formula, and $\|H_0\| \leq 2$ then imply that

$$|(b_n, \phi_s^{(N)})| \leq \sqrt{8\pi}\Delta_n^{-\frac{1}{2}} 4^{-\Delta_n}, \tag{24}$$

if $0 \leq s \leq \frac{\Delta_n}{8e}$. Now from Duhamel's formula one has that for $t > 0$

$$\phi_t - \phi_t^{(N)} = \sum_{n \geq N+1} L_n^{\frac{1-\eta}{2\eta}} \int_0^t b_{n,t-s}\,(b_n, \phi_s^{(N)})\,ds,$$

where obviously $b_{n,t-s}$ is the Schrödinger evoluted of the initial state b_n for time $t-s$. Accordingly,

$$\|\phi_t - \phi_t^{(N)}\| \leq \sum_{n \geq N+1} L_n^{\frac{1-\eta}{2\eta}} \int_0^t |(b_n, \phi_s^{(N)})| ds \leq \sqrt{8\pi} \sum_{n \geq N+1} L_n^{\frac{1-2\eta}{2\eta}} 4^{-\Delta_n} \leq \sqrt{8\pi} A^{\frac{1-2\eta}{2\eta}} \sum_{n \geq N+1} \Delta_n^{\frac{1-2\eta}{2\eta}} 4^{-\Delta_n},$$

with $A = \frac{a}{a-1}$ and where we have used (24) in the second inequality. Terms in the last sum in the equation above are bounded by $2^{-\Delta_n}$ for N large enough. The estimate $\Delta_n - \Delta_{N+1} \geq (a^{n-N-1} - 1)\Delta_{N+1}$ valid for $n \geq N+1$ then completes the proof. ∎

By duality on the previous Lemma, we expect that the spectral measures μ and $\mu^{(N)}$ should have comparable weights on an energy scale inversely proportional to Δ_{N+1}. A form of this duality is expressed by the next Lemma: Let $B_{\frac{1}{T}}(x) := (x - \frac{1}{T}, x + \frac{1}{T})$. Then,

Lemma 2 *Under the conditions of Lemma 1 one has, for all $\varepsilon > 0$ and all $x \in \mathbf{R}$*

$$|\mu(B_{\frac{1}{T}}(x)) - \mu^{(N)}(B_{\frac{2}{T}}(x))| \leq \frac{C_\varepsilon}{T^2}, \ \ for \ \ \frac{T_N}{2} \leq T \leq T_N, \tag{25}$$

where

$$T_N = (\frac{\Delta_{N+1}}{8e})^{\frac{1}{1+\varepsilon}},$$

and C_ε depends only on ε.

Proof. Let $f \in C_0^\infty(\mathbf{R})$ be positive, equal to 1 on $(-1,1)$, and with support in $(-2,2)$. For $T > 0$ let $f_T(\lambda) := f(\lambda T)$. Then,

$$\mu(B_{\frac{1}{T}}(x)) \leq \int f_T(\lambda - x) d\mu(\lambda) \leq \mu(B_{\frac{2}{T}}(x)),$$

and by Fourier transform

$$\mu(B_{\frac{1}{T}}(x)) \leq \frac{1}{T} \int e^{ixt} \hat{f}(\frac{t}{T})(\phi, e^{-iHt}\phi)\ dt \leq \mu(B_{\frac{2}{T}}(x)). \tag{26}$$

Of course, the same inequalities (26) also hold when $\mu^{(N)}$ and $H^{(N)}$ take the place of μ and H. Then,

$$|\mu(B_{\frac{1}{T}}(x)) - \mu^{(N)}(B_{\frac{2}{T}}(x))| \leq \int_{|t|<T^{1+\varepsilon}} \|\phi_t - \phi_t^{(N)}\|\ dt + \frac{2}{T}\int_{|t|>T^{1+\varepsilon}} |\hat{f}(\frac{t}{T})|\ dt.$$

The first integral above is certainly smaller than $\frac{1}{T^2}$ on account of (23), if N is large enough. The second integral is easily estimated by writing $|\hat{f}(\frac{t}{T})| \leq C_n(\frac{T}{t})^n$ for some constant C_n depending only on n. When n is large enough (depending on ε) the integral is then apparently smaller than $\frac{C_{n(\varepsilon)}}{T^2}$, which completes the proof. ∎

The next Lemma states that $\mu^{(N)}$ has absolutely continuous restriction to $(-2,2)$. More importantly, it also provides bounds on the weight of intervals under $\mu^{(N)}$, through a suitable control of the Radon-Nykodin derivative due to B. Simon [29].

Lemma 3 *Let the sparseness condition (12) hold, and let ε, T_N be defined as in Lemma 1. There exist a sequence $\{\varepsilon_N\}_N$ with $\lim_{N\to\infty} \varepsilon_N = 0$ such that for all $x \in (-2,2)$:*

$$\mu^{(N)}(B_{\frac{1}{T}}(x)) \leq C_\varepsilon T^{\varepsilon_N - 1}, \ \ for \ \ \frac{T_N}{2} \leq T \leq T_N, \tag{27}$$

for some constant C_ε depending only on ε.

Proof. We refer to [29] (theorem 2J) for the proof that

$$\frac{d\mu^{(N)}}{d\lambda} \leq \frac{4}{\pi} \sup_n \|T^{(N)}(\lambda; n, 0)\|^6 \tag{28}$$

where $T^{(N)}$ is the fundamental matrix for $H^{(N)}$ at energy λ, defined by

$$T^{(N)}(\lambda; n, 0) = \prod_{k=1}^{N} A(\lambda, k),$$

and

$$A(\lambda, k) = \begin{pmatrix} \lambda - V(k) & -1 \\ 1 & 0 \end{pmatrix}.$$

Accordingly, for $\lambda \in [-2,2]$, one has

$$\|T^{(N)}(\lambda;n,0)\| \le C\prod_{k=1}^{n} V(L_k) \le C\prod_{k=1}^{N} V(L_k), \tag{29}$$

since $\|A(\lambda,k)\| = 1$ when $V(k)=0$. Now, the sparseness condition (12) implies that

$$\prod_{k=1}^{N} V(L_k) = \prod_{k=1}^{N} L_k^{\frac{1-\eta}{2\eta}} \le L_{N+1}^{\frac{1-\eta}{2\eta}\delta_{N+1}}, \tag{30}$$

where δ_{N+1} is the vanishing sequence element present in (12). The same condition also implies that $L_{N+1} > 2\Delta_{N+1}$, for N large enough. Therefore, eqs. (28-30) imply that

$$\frac{d\mu^{(N)}}{d\lambda} \le T_N^{\varepsilon_N}, \tag{31}$$

with $\varepsilon_N = 12\delta_{N+1}(1+\varepsilon)\frac{1-\eta}{2\eta}$. Then (27) follows. ∎

We can now complete the

Proof of Theorem 2. It follows from (25) and (27) that

$$\mu((x-\frac{1}{T_N}, x+\frac{1}{T_N})) \le CT_N^{\varepsilon_N - 1}$$

for all $x \in (-2,2)$. Since $\lim \varepsilon_N = 0$ it follows that $d_\mu^+(x) \le 1$. On the other hand Theorem 2J of [29] also provides a lower bound,

$$\frac{d\mu^{(N)}}{d\lambda} \ge \frac{1}{4\pi}\sup_n \|T^{(N)}(\lambda;n,0)\|^{-6},$$

which along the same lines leads to

$$\frac{d\mu^{(N)}}{d\lambda} \ge T_N^{-\varepsilon_N}.$$

This in turn implies $d_\mu^+(x) \ge 1$ and completes the proof. ∎

5 Proof of Theorem 3.

The main ingredients of the proof presented in this section are energy-space decoupling methods, and spectral averaging over the parameter θ. For notational convenience in this section, we shall adopt Dirac notation $\langle n|\phi\rangle$ for the scalar product (e_n, ϕ). Also, all constants appearing in the estimates will be denoted by the letter C.

Of particular importance is the resolvent $R_\theta(z) := (H_\theta - z)^{-1}$, with $\mathrm{Im}\, z \ne 0$, $\theta \in [\theta_0, \theta_1]$.

Lemma 4 *Let H_θ be given by (10), (11), and assume that (16) holds. If $I \subset (-2,2)$ is a closed interval, then for N large enough one has:*

$$|\langle L_N|R_\theta(E+\frac{i}{T})|1\rangle|^2 \le CTL_N^{-\frac{1}{\eta}}\, Im\ \langle 1|R_\theta(E+\frac{i}{T})|1\rangle, \tag{32}$$

for a constant C depending only on I.

Proof. Let $\chi_N(k) = \frac{k-L_{N-1}}{L_N-L_{N-1}}$ for $L_{N-1} < k \le L_N$ and zero elsewhere. The geometric resolvent equation (see *e.g.* [30]) gives

$$\chi_N R_\theta = R^{(N)}\chi_N - R^{(N)}[\chi, H_\theta]R_\theta, \tag{33}$$

where $R^{(N)}$ is the resolvent of any self-adjoint operator acting as H on the support of χ_N; we choose it to be $H_0 + V_N$, with $V_N(k) = V(L_N)$ if $k = L_N$ and zero elsewhere. It is then well known (see *e.g.* [31]) that

$$\langle L_N | R^{(N)} | k \rangle = \frac{\langle L_N | R_0 | k \rangle}{V(L_N) \langle L_N | R_0 | L_N \rangle + 1}.$$

An elementary computation then shows that

$$\sup_{E \in I} |\langle L_N | R^{(N)} (E + \frac{i}{T}) | p \rangle| \leq C(I) V(L_N)^{-1}, \tag{34}$$

where $C(I)$ only depends on I. Since $\chi_N |1\rangle = 0$ one obtains from eq. (33), (34), and Cauchy-Schwartz inequality

$$|\langle L_N | R_\theta | 1 \rangle|^2 \leq \frac{2}{(L_N - L_{N-1})^2} \sum_{k=L_{N-1}}^{L_N} |\langle L_N | R^{(N)} | k \rangle|^2 \sum_{k'=L_{N-1}}^{L_N} |\langle k' | R_\theta | 1 \rangle|^2 \leq$$

$$\leq \frac{4T}{(L_N - L_{N-1})^2} \text{Im } \langle 1 | R_\theta | 1 \rangle \sum_{k=L_{N-1}}^{L_N} |\langle L_N | R^{(N)} | k \rangle|^2 \leq \frac{4C^2(I) T}{L_N - L_{N-1}} L_N^{-\frac{1-\eta}{\eta}} \text{Im } \langle 1 | R_\theta | 1 \rangle \leq$$

$$\leq 4AC^2(I) T L_N^{-\frac{1}{\eta}} \langle 1 | R_\theta | 1 \rangle$$

with $A = \frac{a-1}{a}$. ∎

The next Lemma is a well known spectral averaging result under rank one perturbations with variable strength (see *e.g.* [32]):

Lemma 5

$$\int_{\theta_0}^{\theta_1} |\langle 1 | R_\theta (E + \frac{I}{T}) | 1 \rangle| \, d\theta < C \tag{35}$$

for some finite constant C independent of $E \in I$ and $T \neq 0$.

From this lemma we can obtain a bound on the moments ν_α over the canonical basis (as defined in eq. (3)) and averaged over the boundary conditions. The initial state of the evolution, ϕ, is chosen to be $\phi = f(H_\theta) e_1$ where with $f \in C_0^\infty(\mathbf{R})$, and $\text{supp} f \subset I$, as in lemma 4.

Lemma 6 *Let $\gamma(\alpha) = \eta \frac{\alpha+1}{\eta\alpha+1}$, and $\{T_N\}_N$ a sequence of times defined by $T_N^{\gamma(\alpha)} = L_N$. Then, if $0 < \alpha \leq 2$ one has*

$$\int_{\theta_0}^{\theta_1} \nu_\alpha (T_N) d\theta \leq C T^{\alpha \gamma(\alpha)} \tag{36}$$

for some constant C independent of N and α.

Proof. Let $J_N(k) = 1$ if $k > L_N$ and zero otherwise. Then

$$\nu_\alpha(T) \leq L_N^\alpha + \frac{1}{T} \int_0^T \|X^{\frac{\alpha}{2}} J_N \phi_t\|^2 dt, \tag{37}$$

where the operator X used above has the obvious meaning of position operator: $(X\phi)(k) = k\phi(k)$. On the other hand (see *e.g.* [33] vol. III)

$$\frac{1}{T} \int_0^T \|X^{\frac{\alpha}{2}} J_N \phi_t\|^2 dt \leq \frac{e}{T} \int_{\mathbf{R}} \|X^{\frac{\alpha}{2}} J_N R_\theta (E + \frac{i}{T}) \phi\|^2 dE \tag{38}$$

from Parseval equality. We now refer to [34] for the derivation of the bound

$$\|[X^r, g(H_\theta)]\| \leq C \||\hat{g}|^{\frac{1}{2}}\|_{L_1} \|t|\hat{g}|^{\frac{1}{2}}\|_{L_1}, \quad 0 < r \leq 1, \tag{39}$$

where $\hat{g}(t)$ is the Fourier transform of g and C is some constant independent of g, r, and θ. Consider then the integral at r.h.s. in eq. (38): for $E \notin I$ we can apply eq. (39) to estimate the commutator of $X^{\frac{\alpha}{2}} J_N$ with $g(H_\theta) = R_\theta(E + \frac{i}{T}) f(H_\theta)$. Since $J_N e_1 = 0$ one easily obtains that

$$\int_{\mathbf{R}\setminus I} \| X^{\frac{\alpha}{2}} J_N R_\theta(E + \frac{i}{T}) f(H_\theta) e_1 \|^2 dE \leq C L_N^\alpha \tag{40}$$

if $0 < \alpha \leq 2$, for some constant C independent of N and α. It remains to evaluate the integral over I. One simply commutes $f(H_\theta)$ and $X^{\frac{\alpha}{2}} J_N$ to obtain

$$\int_I \| X^{\frac{\alpha}{2}} J_N R_\theta(E + \frac{i}{T}) \phi \|^2 \leq C \left(\int_I \| X^{\frac{\alpha}{2}} J_N R_\theta(E + \frac{i}{T}) e_1 \|^2 dE + L_N^\alpha \right). \tag{41}$$

Combining eqs. (37)-(41) gives

$$\nu_\alpha(T) \leq C \left(L_N^\alpha + \frac{1}{T} \int_I \| X^{\frac{\alpha}{2}} J_N R_\theta(E + \frac{i}{T}) e_1 \|^2 dE \right), \tag{42}$$

with C independent of N and α. We can employ again the geometric resolvent equation to get

$$J_N R_\theta e_1 = R_{\text{ext}}^N | L_N + 1 \rangle \langle L_N | R_\theta | 1 \rangle \tag{43}$$

where R_{ext}^N is the resolvent of $H_0 + V J_N$ (and is therefore independent of θ!). Then, eqs. (42) and (43) imply that

$$\int_{\theta_0}^{\theta_1} \nu_\alpha(T) d\theta \leq C \left(L_N^\alpha + \frac{1}{T} \int_I \| X^{\frac{\alpha}{2}} R_{\text{ext}}^N (E + \frac{i}{T}) | L_N + 1 \rangle \|^2 dE \times \sup_{E \in I} \int_{\theta_0}^{\theta_1} | \langle L_N | R_\theta(E + \frac{i}{T}) | 1 \rangle |^2 d\theta \right). \tag{44}$$

We can now estimate the last factor at r.h.s. of eq. (44) by Lemmas 4 and 5 as follows:

$$\sup_{E \in I} \int_{\theta_0}^{\theta_1} | \langle L_N | R_\theta(E + \frac{i}{T}) | 1 \rangle |^2 d\theta \leq C T L_N^{-\frac{1}{\eta}}. \tag{45}$$

As to the integral over I in eq. (44) we can use a rough ballistic estimate to get:

$$\frac{1}{T} \int_I \| X^{\frac{\alpha}{2}} R_{\text{ext}}^N (E + \frac{i}{T}) | L_N + 1 \rangle \|^2 dE \leq C (L_N + T)^\alpha. \tag{46}$$

Finally, eqs. (44)-(46) imply that

$$\int_{\theta_0}^{\theta_1} \nu_\alpha(T) d\theta \leq C \left(L_N^\alpha + L_N^{-\frac{1}{\eta}} (L_N + T)^{\alpha+1} \right). \tag{47}$$

Choosing now T_N as stated gives eq. (36). ∎

We are now ready for the

Proof of Theorem 3. Let

$$\Omega_N = \{ \theta \in [\theta_0, \theta_1] \text{ s.t. } \sup_{0 < \alpha \leq 2} \frac{\nu_\alpha(T_N)}{T_N^{\alpha\gamma(\alpha)}} > N^2 \}.$$

Then obviously $\sum_N |\Omega_N| < +\infty$ so that by Borel-Cantelli Lemma one has $\nu_\alpha(T_N) \leq N^2 T_N^{\alpha\gamma(\alpha)}$ for N large enough, and for almost every $\theta \in [\theta_0, \theta_1]$. Estimate (17) follows since $\log T_N$ grows at least like N. ∎

6 Conclusions

We have presented in this paper a detailed examination of the quantum motion taking place in sparse potential Jacobi matrices on $l_2(\mathbf{Z}_+)$. For this system, we have been able to derive exact bounds to the moment growth exponents $\beta^{\pm}(\alpha)$. These bounds have been examined numerically, and it is plausible to assume that they are indeed exact results, for very sparse barriers.

This has shown a strong form of intermittency: not only the function $\beta^-(\alpha)$ is increasing with α, but it is also different from $\beta^+(\alpha)$. By focusing attention also on the fractal properties of the associated spectral measure, we have shown that, in this example, the inequalities recently proposed for the growth exponents are sharp.

Sparseness of the barriers is at the same time a key for our derivation, and a limitation to its generality. A direction of future research will precisely involve releasing the sparseness request adopted herein.

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