THE SHAPE OF ANALYTICITY DOMAINS OF LINDSTEDT SERIES: THE STANDARD MAP

ALBERTO BERRETTI, CORRADO FALCOLINI, AND GUIDO GENTILE

Abstract. The analyticity domains of the Lindstedt series for the standard map are studied numerically using Padé approximants to model their natural boundaries. We show that if the rotation number is a Diophantine number close to a rational value p/q, then the radius of convergence of the Lindstedt series becomes smaller than the critical threshold for the corresponding KAM curve, and the natural boundary on the plane of the complexified perturbative parameter acquires a flower-like shape with $2q$ petals. We conjecture that the natural boundary has typically a fractal shape, which only in particular cases degenerates to an apparently regular curve.

The standard map is a paradigmatic model for the transition from regular to stochastic motion in classical mechanics introduced by Chirikov [1]. It has also been studied in relation to problems of quantum mechanics and quantum chaos [2, 3], and statistical mechanics [4]; it is also relevant to problems in plasma physics [5]. It is a discrete one-dimensional dynamical system generated by the iteration of the symplectic map of the cylinder into itself, $T_\varepsilon : \mathbb{T}\times\mathbb{R} \mapsto \mathbb{T}\times\mathbb{R}$, given by

$$T_\varepsilon : \begin{cases} x' = x + y + \varepsilon \sin x, \\ y' = y + \varepsilon \sin x. \end{cases}$$

For some background information, we refer the reader to the enormous literature on the topic, and in particular to [6] for a review.

For $\varepsilon = 0$, the circles $y = \text{const.}$ are invariant curves on which the dynamics is given by rotation with angular velocity – rotation number – $\omega = y/2\pi$. As the perturbation is turned on, we face the classical KAM problem of determining which invariant curves survive and up to which size of the perturbative parameter ε (see [7, 8] for the optimal arithmetic condition on the rotation number for the stability of an invariant curve). It is well known that such invariant curves are given parametrically by the equation

$$\mathcal{C}_{\varepsilon,\omega} : \begin{cases} x = \alpha + u(\alpha,\varepsilon,\omega), \\ y = 2\pi\omega + u(\alpha,\varepsilon,\omega) - u(\alpha - 2\pi\omega,\varepsilon,\omega), \end{cases}$$

where in the α variable the dynamics on the curve $\mathcal{C}_{\varepsilon,\omega}$ is given by rotations $\alpha_{n+1} = \alpha_n + 2\pi\omega$; the function $u(\alpha, \varepsilon, \omega)$, called the *conjugating function*, satisfies the functional equation

$$\begin{aligned} \left(D_\omega^2 u\right)(\alpha, \varepsilon, \omega) &= u(\alpha + 2\pi\omega, \varepsilon, \omega) - 2u(\alpha, \varepsilon, \omega) + u(\alpha - 2\pi\omega, \varepsilon, \omega) \\ &= \varepsilon \sin(\alpha + u(\alpha, \varepsilon, \omega)), \end{aligned} \tag{1}$$

whose solutions are formally unique if we impose that $u(\alpha, \varepsilon, \omega)$ has zero average in the α variable. The study of the invariant curves $\mathcal{C}_{\varepsilon,\omega}$ and of their smoothness properties may then be reduced to the study of the existence and smoothness of the solutions of the functional equation (1).

The solutions of (1) can be studied perturbatively by expanding u in Taylor series in ε and in Fourier series in α: the resulting series is called the *Lindstedt series*

$$u(\alpha, \varepsilon, \omega) = \sum_{k=1}^{\infty} \varepsilon^k \, u^{(k)}(\alpha, \omega) = \sum_{k=1}^{\infty} \varepsilon^k \sum_{|\nu| \le k} e^{i\nu\alpha} \, u_\nu^{(k)}(\omega);$$

by inserting the expansion in (1) we can derive recursion relations for the coefficients $u_\nu^{(k)}(\omega)$, which, besides imposing the restriction on the range of the sum over k, are useful for numerical calculations and form also the basis of the *trees expansions* pioneered in [9]. It is the inversion of the operator D_ω^2 when solving the functional equation (1) which gives rise to the so called *small denominators problem*, appearing as the presence in the expressions for the coefficients $u_\nu^{(k)}(\omega)$ of the kernel of the operator D_ω^{-2}, given by $1/4\sin^2 \nu\pi\omega$.

To characterize the breakdown of an invariant curve $\mathcal{C}_{\varepsilon,\omega}$ we introduce the *radius of convergence* of the Lindstedt series:

$$\rho(\omega) = \inf_{\alpha \in \mathbb{T}} \left(\limsup_{k \to \infty} \left| u^{(k)}(\alpha, \omega) \right|^{1/k} \right)^{-1},$$

and the *critical function*

$$\varepsilon_{\mathrm{c}}(\omega) = \sup\{\varepsilon' \ge 0 : \ \forall \, \varepsilon'' < \varepsilon' \ \mathcal{C}_{\varepsilon'',\omega} \text{ exists and is analytic}\};$$

clearly $\rho(\omega) \le \varepsilon_{\mathrm{c}}(\omega)$. In particular, the radius of convergence of the Lindstedt series is zero – so no KAM invariant curve exists – when ω is rational; when ω satisfies an irrationality condition known as the *Bryuno condition*, instead, it can be proved that $\rho(\omega) > 0$ – so that analytic invariant curves exist for ε small.

The analytic structure of u in ε is of particular interest, since it may explain, among other things, the relation between $\rho(\omega)$ and $\varepsilon_{\mathrm{c}}(\omega)$. It is believed, on the basis of [10, 11, 12], that u has a natural boundary on the complex ε plane, *i.e.* that its domain of analyticity is bounded by a continuous curve where singularities are dense, obstructing analytic continuation; this natural boundary appears to be independent on α. Its shape determines the relation between $\rho(\omega)$ and $\varepsilon_c(\omega)$: in fact, $\rho(\omega)$ is given by the distance of the natural boundary from the origin, while $\varepsilon_c(\omega)$ is given by its intersection with the real, positive axis (and for the standard map the natural boundary must be symmetric with respect to the imaginary axis).

For some rotation numbers, like the golden mean $\gamma = (\sqrt{5}-1)/2$, the natural boundary was found to be roughly circular [10, 11], with a very slight lenghtening along the imaginary axis for the golden mean [12], so that one would have $\varepsilon_c(\omega) = \rho(\omega)$. Qualitatively, we can note that such a situation practically arises for Diophantine numbers of the form $\omega = [a_1, a_2, a_3, \ldots]$, with small partial quotients a_k: in this sense the golden mean is the best possible case. Following hints from [15, 16], we can expect to see a sharp difference between $\rho(\omega)$ and $\varepsilon_c(\omega)$ when ω is close to a rational number (we shall come back later on what "close" really means).

Here we determine numerically, using Padé approximants, the natural boundary for the Lindstedt series of the standard map in the complex ε variable, when ω is a Diophantine number close to a rational one: see [10, 11] for the use of Padé approximants in this context and further references; it is enough here to recall that the poles of the Padé approximants model the shape of the natural boundary of the function being approximated.

In the following figures we plot the poles in the complex ε plane of some Padé approximants for $u(\alpha, \varepsilon, \omega)$ for $\alpha = 1$ and selected values of ω near rational values. We show only the data coming from high order Padé approximants ([240/240]), computed using high precision arithmetic (480 digits), though lower order Padé approximants and lower precision arithmetic show the same results. Spurious pole/zero pairs have been checked for and deleted. The independence of the analyticity domain on α has been tested by computing the Padé approximants for some selected values of α, and verifying that the shape of the natural boundary is essentially the same.

In figure 1 we see the shape of the natural boundary when $\omega = [10, 1^\infty] = 1/(10+\gamma)$, where $\gamma = (\sqrt{5}-1)/2$ is the golden mean; note that such an ω is close to 0. We clearly see that singularities close to the origin appear on the imaginary axis, so that $\rho(\omega) < \varepsilon_c(\omega)$ strictly in this case. Note also the two-lobed shape of the domain, generated by the two lines of singularities on the imaginary axis cutting deep into the analyticity domain.

In figure 2 we see the shape of the natural boundary when $\omega = [2, 10, 1^\infty] = (10+\gamma)/(21+2\gamma)$, which is close to $1/2$. Again $\rho(\omega) < \varepsilon_c(\omega)$ strictly, and observe the four-lobed shape of the domain, with four lines of singularities, disposed radially as the 4-th roots of -1, cutting into the domain of analyticity.

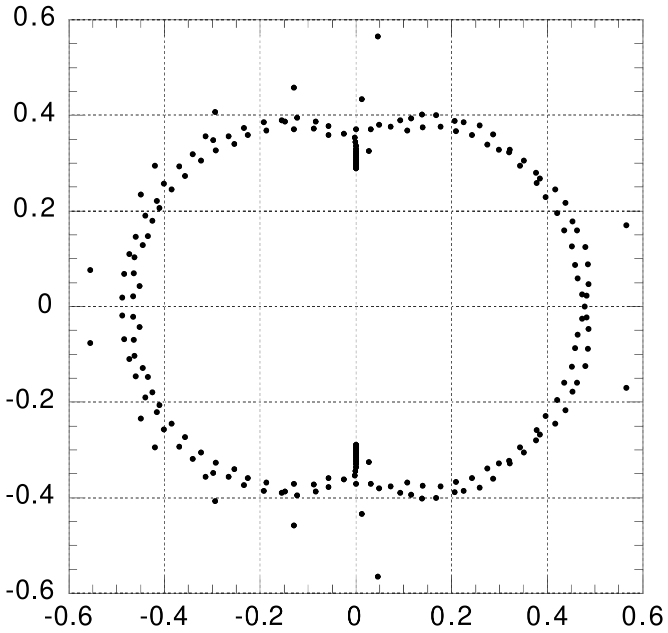

FIGURE 1. Poles of the Padé approximant [240/240], $\omega = [10, 1^\infty]$, $\alpha = 1$. Greene's method gives $\varepsilon_c(\omega) = 0.4768$.

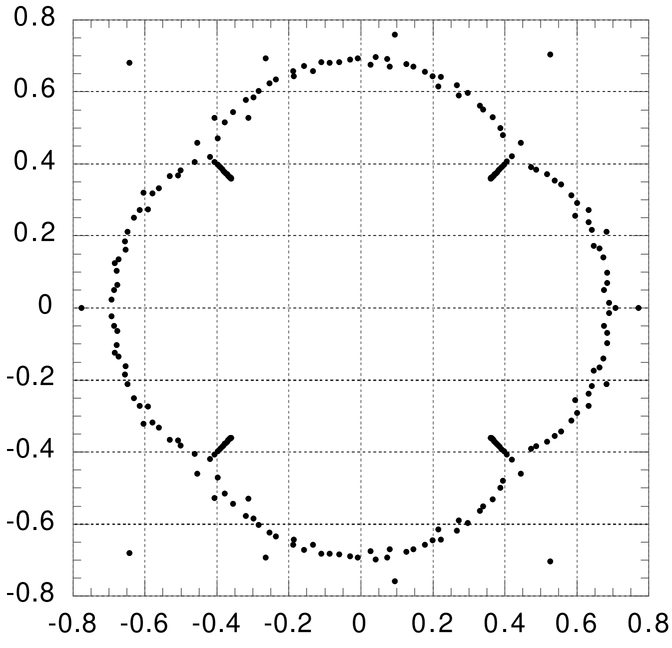

FIGURE 2. Poles of the Padé approximant [240/240], $\omega = [2, 10, 1^\infty]$, $\alpha = 1$. Greene's method gives $\varepsilon_c(\omega) = 0.6762$.

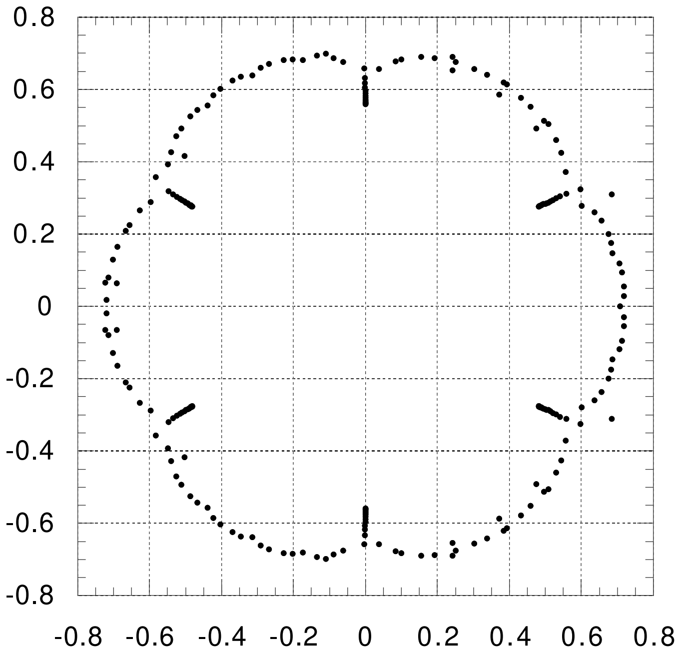

FIGURE 3. Poles of the Padé approximant [240/240], $\omega = [3, 12, 1^{\infty}]$, $\alpha = 1$. Greene's method gives $\varepsilon_c(\omega) = 0.7047$.

As we take rotation numbers close to other rational numbers p/q, we see $2q$-lobed analyticity domains generated by $2q$ lines of singularities, disposed as the $2q$-th roots of -1: we call *dominant* such singularities.

For example, in figure 3 we see the shape of the natural boundary when $\omega = [3, 12, 1^{\infty}] = (12+\gamma)/(37+3\gamma)$, close to $1/3$, and in figure 4 we see the shape of the natural boundary when $\omega = [4, 14, 1^{\infty}] = (15+\gamma)/(61+4\gamma)$, close to $1/4$: observe the shapes of the natural boundaries and their relations with q. It appears that there is no relevant dependence of the shape on the numerator p: for example in figure 5 we see the shape of the natural boundary when $\omega = [2, 2, 12, 1^{\infty}] = (25+2\gamma)/(62+5\gamma)$, which is close to $2/5$; a similar shape is obtained for a rotation number close to $1/5$.

In [13, 14] it was shown that there exists

$$\bar{u}_{p/q}(\alpha, \varepsilon) = \lim_{\eta \to 0} u(\alpha, (2\pi\eta)^{2/q}\varepsilon, p/q + \eta), \tag{2}$$

where the limit is taken along any path in the complex ω plane tending to p/q non-tangentially to the real axis. In particular this implies

$$u(\alpha, \varepsilon, p/q + \eta) = \bar{u}_{p/q}(\alpha, (2\pi\eta)^{-2/q}\varepsilon) + \text{corrections}, \tag{3}$$

where the corrections can be proved to be of order $(2\pi\eta)^{2/q}$. If we *assume* that the relation (2) holds also for *real* values of η, provided that the limit is replaced

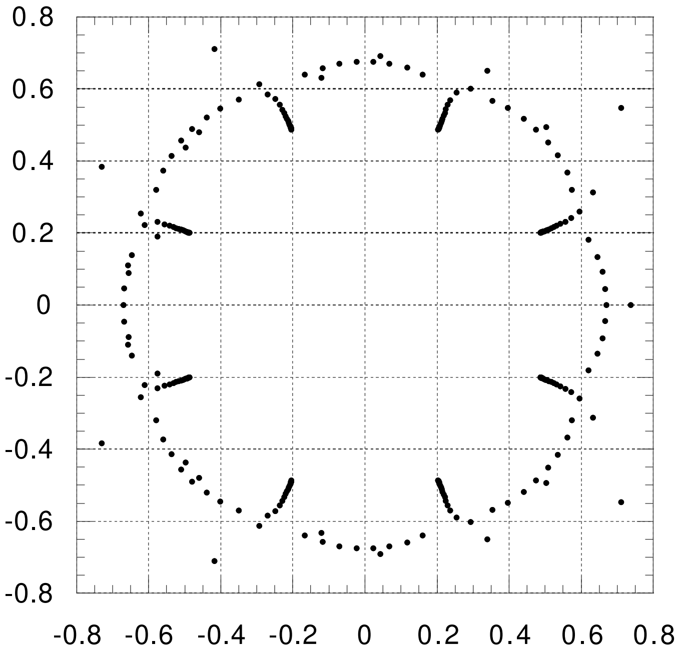

FIGURE 4. Poles of the Padé approximant [240/240], $\omega = [4, 15, 1^\infty]$, $\alpha = 1$. Greene's method gives $\varepsilon_c(\omega) = 0.6663$.

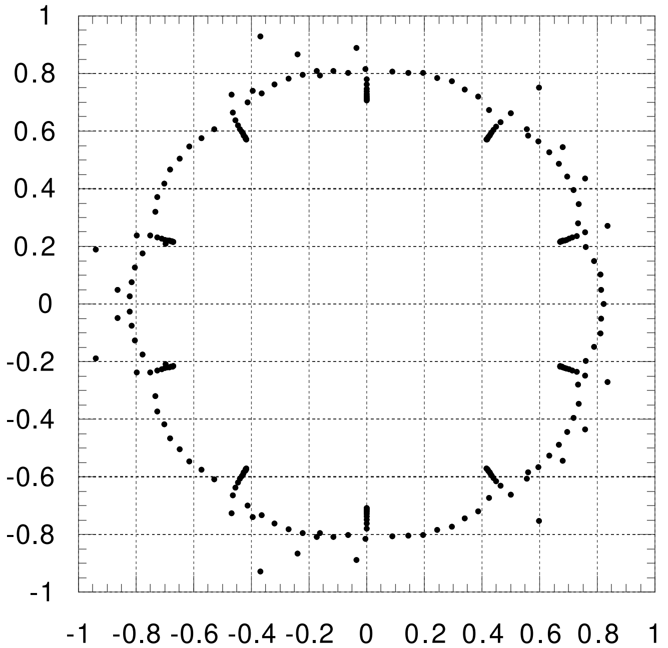

FIGURE 5. Poles of the Padé approximant [240/240], $\omega = [2, 2, 12, 1^\infty]$, $\alpha = 1$. Greene's method gives $\varepsilon_c(\omega) = 0.8160$.

TABLE 1. Radius of convergence for some values of the rotation number ω: $\rho_1(\omega)$ is the value given by formula (4), while $\rho_P(\omega)$ is the value obtained numerically by using Padé approximants.

ω	p/q	$C_{p/q}$	$\rho_1(\omega)$	$\rho_P(\omega)$
$[2,10,1^\infty]$	1/2	0.125000	0.514	0.510
$[3,12,1^\infty]$	1/3	0.041667	0.557	0.555
$[4,15,1^\infty]$	1/4	0.026042	0.528	0.526
$[10^\infty]$	0/1	1.000000	0.320	0.313
$[10,1^\infty]$	0/1	1.000000	0.290	0.284
$[2,2,12,1^\infty]$	2/5	0.008713	0.707	0.704

with the limit superior [8], so that (3) can be written also for the values of ω we are considering, then we obtain that $\rho(\omega)$ can be approximated as

$$\rho(\omega) \approx \eta^{2/q} \left(q C_{p/q}^{-1} \lambda_c \right)^{1/q}, \tag{4}$$

where $C_{p/q}$ is the same constant as defined in [14], and $\lambda_c \approx 4\pi^2 \cdot 0.828 \approx 32.669$ (see [13]).

Then for the radius of convergence $\rho(\omega)$, by comparing the values $\rho_1(\omega)$ obtained through formula (4) and the values $\rho_P(\omega)$ obtained by Padé approximants, we find the results listed in table 1.

In all cases $\rho(\omega) < \varepsilon_c(\omega)$ strictly; furthermore the discrepancy between the two values increases as ω gets close to p/q. In a forthcoming publication we shall turn this remark into a quantitative statement and compare the results from Padé approximants and from Greene's method for $\varepsilon_c(\omega)$.

A few words are necessary to explain what we mean by "close": strictly speaking, in fact, an irrational number is close to infinitely many rational values. For example, $[2,10,1^\infty]$ is close to 1/2 because its continued fraction starts as the expansion of $1/2 = [2]$ and then is followed by a larger integer (10, in this case); the next best approximant of $[2,10,1^\infty]$ is $[2,10] = 10/21$, but $[2,10,1^\infty]$ is *not* close to 10/21 as the following partial quotient in its continued fraction is a small integer (1, in this case). Instead, consider $\omega_1 = [10,1^\infty]$ and $\omega_2 = [10^\infty]$; the sequence of rational approximants obtained by truncation of their continued fractions is 0/1, 1/10, 1/11, ... for ω_1, and 0/1, 1/10, 10/101, ... for ω_2: so we can say that both are close to 0, but ω_2 is also close to 1/10, while ω_1 is not. It follows that the natural boundary corresponding to the rotation number

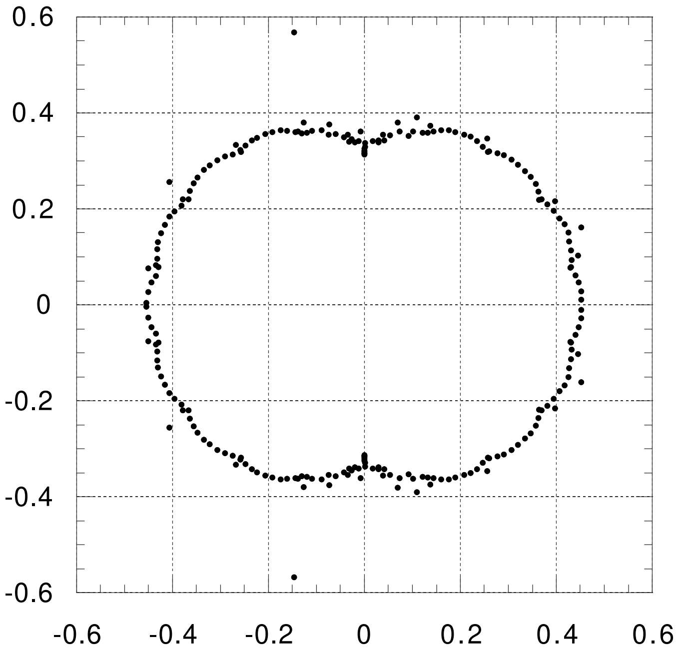

FIGURE 6. Poles of the Padé approximant [240/240], $\omega = [10^\infty]$, $\alpha = 1$.

ω_2 should be visibly influenced both by the singularities corresponding to 0/1 and by those corresponding to 1/10, which we could call *subdominant*. This is indeed the case, as we can see from figure 6; note that only the subdominant singularities near the real axis are really detectable as those near the imaginary axis give a negligeable effect with respect to the dominant ones.

Following ideas first introduced in [16], we believe that the natural boundary is created by accumulation of lines of singularities due to resonances, so *all* the rational approximants p_k/q_k of the rotation number should contribute, with $2q_k$ of lines of singularities to the build-up of the natural boundary; therefore it acquires a fractal shape, as each lobe is actually decomposed in a larger number of lobes on a smaller scale and so on; these smaller lobes are eventually not detectable with Padé approximants of feasible order with the computer technology available to us, *except when the nearness to two distinct rational values is strong*, like in the case of ω_2 above.

What occurs practically by increasing the values of a partial quotient a_k, for some k, in the continued fraction expansion for the rotation number ω is that the latter becomes closer to the convergent p_{k-1}/q_{k-1}, so that for $k = 2$, the effect of the singularity corresponding to such a rational value is sort of amplified: then the singularities detected by the Padé approximants tend to accumulate near such a dominant singularity, and the rest of the boundary of the analyticity domain appears as a set of a few scattered points without much visible structure: to obtain a more meaningful picture one should increase by far the order of the Padé approximants, which is not so easy from the numerical point of view.

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Alberto Berretti, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy
E-mail address: `berretti@mat.uniroma2.it`

Corrado Falcolini, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy
E-mail address: `falcolin@mat.uniroma2.it`

Guido Gentile, Dipartimento di Matematica, Università di Roma Tre, Largo San Leonardo Murialdo 1, 00146 Roma, Italy
E-mail address: `gentile@matrm3.mat.uniroma3.it`