SOBOLEV, HARDY AND CLR INEQUALITIES ASSOCIATED WITH PAULI OPERATORS IN $\mathbb{R}^3$

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ABSTRACT. In a previous article, the first two authors have proved that the existence of zero modes of Pauli operators is a rare phenomenon; inter alia, it is proved that for $|\vec{B}| \in L^{3/2}(\mathbb{R}^3)$, the set of magnetic fields $\vec{B}$ which do not yield zero modes contains an open dense subset of $[L^{3/2}(\mathbb{R}^3)]^3$. Here the analysis is taken further, and it is shown that Sobolev, Hardy and Cwikel-Lieb-Rosenbljum (CLR) inequalities hold for Pauli operators for all magnetic fields in the aforementioned open dense set.

1. INTRODUCTION

The Pauli operator is formally defined by

$$\mathbb{P}_A = \left\{ \boldsymbol{\sigma} \cdot \left(\frac{1}{i} \boldsymbol{\nabla} + \vec{A} \right) \right\}^2 \equiv \sum_{j=1}^{3} \left\{ \sigma_j \left(\frac{1}{i} \partial_j + A_j \right) \right\}^2 \tag{1}$$

where $\vec{A} = (A_1, A_2, A_3)$ is a vector potential which is such that $curl\, \vec{A} = \vec{B}$, the magnetic field, and $\boldsymbol{\sigma} = \vec{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$ is the triple of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

The expression (1) defines a non-negative self-adjoint operator in $[L^2(\mathbb{R}^3)]^2$; its precise definition will be given in §2.

It is well-known that there exist magnetic fields $\vec{B}$ for which $\mathbb{P}_A$ has zero modes, i.e. eigenvectors corresponding to an eigenvalue at 0 (see [2]). An important consequence is that there can't be Sobolev and Hardy-type inequalities associated with such magnetic fields. Another important implication is that for such magnetic fields there can't be an analogue of the Cwikel-Lieb-Rosenblum inequality for the number of negative eigenvalues of $\mathbb{P}_A + V$ in terms of some L^p norm of the scalar potential V; this follows since any small negative perterbation V would produce negative eigenvalues, irrespective of the size of this norm of V. However in [1] it is proved that the existence of zero modes is a rather rare phenomenon: specifically, it is proved that, with nul denoting nullity,

(i) for $|\vec{B}| \in L^{3/2}(\mathbb{R}^3)$, nul $\mathbb{P}_{tA} = 0$ except for a finite number of values of t in any compact subset of $[0, \infty)$,

(ii) $\left\{ \vec{B} \;:\; \text{nul}\ \mathbb{P}_A = 0,\ curl\, \vec{A} = \vec{B} \text{ and } |\vec{B}| \in L^{3/2}(\mathbb{R}^3) \right\}$ contains an open dense subset of $\left[L^{3/2}(\mathbb{R}^3) \right]^3$.

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In this note, we use the analysis in [1] to prove that

$$\mathbb{P}_A \ \geq \ \delta(B)\ S_A \tag{3}$$

where $\delta(B)$ is a measure of the distance from $\vec{B}$ to the set of fields which yield zero modes, and S_A is the magnetic Schrödinger operator. Consequences of (3) are Sobolev, Hardy and CLR inequalities for $\mathbb{P}_A$ for all magnetic fields $\vec{B}$ in $\left[L^{3/2}(\mathbb{R}^3)\right]^3$ which lie in the sets described in (i) and (ii) above.

2. Preliminaries

The formal operator (1) can be written as

$$\mathbb{P}_A = S_A + \vec{\sigma}\cdot\vec{B}, \quad \vec{B} = curl\,\vec{A}, \tag{4}$$

where S_A is the formal magnetic Schrödinger operator

$$S_A = \left(\frac{1}{i}\boldsymbol{\nabla} + \vec{A}\right)^2 \mathbb{I}_2 \ \equiv \ \sum_{j=1}^{3}\left(\frac{1}{i}\partial_j + A_j\right)^2 \mathbb{I}_2, \tag{5}$$

$\mathbb{I}_2$ being the 2×2 identity matrix and $\vec{\sigma}\cdot\vec{B}$ the Zeeman term. A gauge transformation $\vec{A}\mapsto\vec{A}+df$ does not affect nul $\mathbb{P}_A$ and so our results will be independent of gauge. We denote $[L^2(\mathbb{R}^3)]^2$ by $\mathcal{H}$, and its usual inner product and norm by

$$(f,g) \ = \ \int_{\mathbb{R}^3} f\cdot\bar{g}\ d\mathbf{x}, \quad \|f\| \ = \ (f,f)^{1/2}$$

where $f\cdot\bar{g}$ is the $\mathbb{C}^2-$inner product. We shall assume throughout that

$$A_j \ \in \ L^2_{loc}(\mathbb{R}^3), \quad j=1,2,3 \tag{6}$$

and

$$|\vec{B}| \ \in \ L^{3/2}(\mathbb{R}^3). \tag{7}$$

Also, the following facts from [1] will be needed:

- The operators $\mathbb{P}_A$, S_A are defined to be the Friedrichs extensions of (4), (5) respectively on $\left[C_0^\infty(\mathbb{R}^3)\right]^2$. They have the same form domain, namely $\mathcal{Q}(S_A)$, the completion of $[C_0^\infty(\mathbb{R}^3)]^2$ with respect to the norm

$$\|\varphi\|_{1,A} \ = \ \left\{\left\|\left(\frac{1}{i}\boldsymbol{\nabla}+\vec{A}\right)\varphi\right\|^2 + \|\varphi\|^2\right\}^{1/2}. \tag{8}$$

- S_A and the form sum $\mathbb{P} := \mathbb{P}_A + |\vec{B}|$ have no zero modes, and so have dense domain and range in $\mathcal{H}$. Also, $\mathcal{D}(\mathbb{P}^{1/2}) = \mathcal{D}(S_A^{1/2}) = \mathcal{Q}(S_A)$.
- Let H^1_A, $\mathbb{H}^1_B$ be the completion of $\mathcal{Q}(S_A)$ with respect to the norms

$$\|\varphi\|_{\mathrm{H}^1_A} \ := \ \|S_A^{1/2}\varphi\| \tag{9}$$

$$\|\varphi\|_{\mathbb{H}^1_B} \ := \ \|\mathbb{P}^{1/2}\varphi\| \tag{10}$$

respectively. We may choose an $\vec{A}$ to satisfy $div\,\vec{A} = 0$ and $|\vec{A}| \in L^3(\mathbb{R}^3)$ (see [1, Lemma 2.2]) and in this case we have the continuous embeddings

$$\mathbb{H}_B^1 \hookrightarrow \mathrm{H}_A^1 \hookrightarrow \mathrm{H}_0^1 \hookrightarrow \left[L^6(\mathbb{R}^3)\right]^2. \tag{11}$$

Here, H_0^1 is the space H_A^1 with $A_j = 0,\ j = 1,2,3$; it is not a subspace of $\mathcal{H}$ but, on account of the Hardy inequality, can be identified with the functional space

$$\left\{u \in \left[\mathrm{H}_{loc}^1(\mathbb{R}^3)\right]^2 \ : \ \|u\|_{\mathrm{H}_0^1}^2 + \left\|\frac{u}{|\cdot|}\right\|^2 < \infty\right\} \tag{12}$$

and $\|\cdot\|_{\mathrm{H}_0^1}$ is equivalent to the norm

$$\left\{\|u\|_{\mathrm{H}_0^1}^2 + \left\|\frac{u}{|\cdot|}\right\|^2\right\}^{1/2}. \tag{13}$$

- The map $\mathbb{P}^{-1/2}$ extends to a unitary map
$$U \ : \ \mathcal{H} \longrightarrow \mathbb{H}_B^1, \quad U = \mathbb{P}^{-1/2} \ \text{ on } \ Range(\mathbb{P}^{1/2}) \tag{14}$$
and
$$\mathcal{S} \ := \ |\vec{B}|^{1/2} U \ : \ \mathcal{H} \longrightarrow \mathcal{H}$$
is continuous, where $f \mapsto |\vec{B}|^{1/2} f \ : \ \mathbb{H}_B^1 \longrightarrow \mathcal{H}$ is continuous. In fact $\mathcal{S}\mathcal{S}^*$ is compact.
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$$\mathbb{P}_A u = 0 \ \Leftrightarrow \ F(U^{-1}u) = 0, \tag{15}$$
where $F = 1 - \mathcal{S}\mathcal{S}^*$. Note that in (15) it is understood that
$$u \in \mathcal{D}(\mathbb{P}_A) \subset \mathbb{H}_B^1.$$
Thus $\mathrm{nul}\mathbb{P}_A \ \le \ \mathrm{nul}F$, with equality if and only of
$$F\varphi = 0 \ \Rightarrow \ U\varphi \in \mathcal{H} \cap \mathbb{H}_B^1. \tag{16}$$

3. The main result

Theorem. *Suppose that $\vec{B}$ is such that $F = 1 - \mathcal{S}\mathcal{S}^*$ has no zero mode, and set*

$$\delta(B) \ := \inf_{\|f\|=1, Uf \in \mathcal{H}\cap\mathbb{H}_B^1} \left\|[1 - \mathcal{S}^*\mathcal{S}]f\right\|^2. \tag{17}$$

Then $\delta(B) > 0$ and

$$\mathbb{P}_A \ \ge \ \delta(B) S_A. \tag{18}$$

It follows that:

(i) *(Sobolev embedding) for all $\varphi \in \mathcal{D}(\mathbb{P}_A^{1/2}) = \mathcal{Q}(S_A)$,*

$$\left\|\mathbb{P}_A^{1/2}\varphi\right\|^2 \ \ge \ \frac{\delta(B)}{\gamma}\left\|\varphi\right\|_{[L^6(\mathbb{R}^3)]^2}^2, \tag{19}$$

where γ is the norm of the embedding $\mathrm{H}_0^1 \hookrightarrow \left[L^6(\mathbb{R}^3)\right]^2$;

(ii) *(Hardy inequality) for all $\varphi \in \mathcal{Q}(S_A)$,*

$$\|\mathbb{P}_A^{1/2}\varphi\|^2 \ \geq \ \frac{\delta(B)}{4}\left\|\frac{\varphi}{|\cdot|}\right\|^2; \tag{20}$$

(iii) *(CLR inequality) for $V_- \in L^{3/2}(\mathbb{R}^3)$, the number $N(\mathbb{P}_A+V)$ of negative eigenvalues λ_n of $\mathbb{P}_A + V$ satisfies*

$$N(\mathbb{P}_A+V) \ \leq \ c[\delta(B)]^{-3/2}\int_{\mathbb{R}^3} V_-^{3/2}\, d\mathbf{x} \tag{21}$$

where c is the best constant in the CLR inequality for S_A. A consequence of (21) is that

$$\sum \lambda_n^\nu \ \leq \ c[\delta(B)]^{-3/2}\int_{\mathbb{R}^3} V_-^{\nu+3/2}\, d\mathbf{x} \tag{22}$$

for any $\nu \geq 0$.

Proof. If F has no zero mode, the compact operator $\mathcal{S}\mathcal{S}^*$ on $\mathcal{H}$ does not have eigenvalue 1 and hence neither does $\mathcal{S}^*\mathcal{S}$, since

$$\sigma(\mathcal{S}\mathcal{S}^*)\setminus\{0\} \ = \ \sigma(\mathcal{S}^*\mathcal{S})\setminus\{0\}.$$

Hence $\delta(B) > 0$, and for any $f \in \mathcal{H}$ with $Uf \in \mathcal{H}\cap\mathbb{H}^1_B$

$$\begin{aligned}\delta(B)\|f\|^2 \ &\leq \ \|(1-\mathcal{S}^*\mathcal{S})f\|^2\\ &= \ \|f\|^2 - 2(\mathcal{S}^*\mathcal{S}f,f) + \|\mathcal{S}^*\mathcal{S}f\|^2.\end{aligned}$$

Let $f = \mathbb{P}^{1/2}\varphi$. Then $Uf = \varphi$ and $\mathcal{S}f = |\vec{B}|^{1/2}\varphi$ from (14), and so

$$\begin{aligned}\delta(B)\|\mathbb{P}^{1/2}\varphi\|^2 \ &\leq \ \|\mathbb{P}^{1/2}\varphi\|^2 - 2\||\vec{B}|^{1/2}\varphi\|^2 + \|\mathcal{S}^*|\vec{B}|^{1/2}\varphi\|^2\\ &= \ \|\mathbb{P}_A^{1/2}\varphi\|^2 - \||\vec{B}|^{1/2}\varphi\|^2 + \|\mathcal{S}^*|\vec{B}|^{1/2}\varphi\|^2. \end{aligned} \tag{23}$$

since $\mathbb{P} = \mathbb{P}_A + |\vec{B}|$ in the form sense. We also have for any $h \in Range\,\mathbb{P}^{1/2}$

$$\|\mathcal{S}h\| \ = \ \||\vec{B}|^{1/2}\mathbb{P}^{-1/2}h\| \leq \|h\|$$

since $\mathbb{P} \geq |\vec{B}|$, and this implies $\|\mathcal{S}\| \leq 1$ in view of the range of $\mathbb{P}^{1/2}$ being dense in $\mathcal{H}$. Thus $\|\mathcal{S}^*\| = \|\mathcal{S}\| \leq 1$ and from (23)

$$\delta(B)\|\mathbb{P}^{1/2}\varphi\|^2 \ \leq \ \|\mathbb{P}_A^{1/2}\varphi\|^2,$$

whence $\mathbb{P}_A \geq \delta(B)\mathbb{P} \geq \delta(B)S_A$. The conclusions (i)-(iii) are consequences of these inequalities for S_A.

Remark. 1. If any one of the inequalities (19)-(21) is satisfied, $\mathbb{P}_A$ has no zero modes. Whether or not nul $\mathbb{P}_A = 0$ implies that $\delta(B) > 0$ is not clear. Note that the infimum in (17) is taken over the subspace of $\mathcal{H}$ in which $\mathbb{P}_A$ and F have common nullity.
2. Let $\mathcal{S}$ in (14) be denoted by $\mathcal{S}_B$ and $F_B = 1 - \mathcal{S}_B\mathcal{S}_B^*$. The results in [1] which yield (i) and (ii) in §1 are

(iii) for $|\vec{B}| \in L^{3/2}(\mathbb{R}^3)$, $nul\, F_{tB} = 0$ except for a finite number of values of t in any compact subset of $[0,\infty)$,

(iv) $\{\vec{B} \ : \ nul\, F_B = 0, \ |\vec{B}| \in L^{3/2}(\mathbb{R}^3)\}$ is an open dense subset of $[L^{3/2}(\mathbb{R}^3)]^3$.

Thus (18)-(22) represent the typical situation, and fail to hold only for exceptional magnetic fields $\vec{B}$.

It follows from (11) (see [1, Lemma 3.1]) that $\|\mathcal{S}\|^2 \leq \gamma_0^2 \||\vec{B}|\|_{L^{3/2}(\mathbb{R}^3)}$, where γ_0 is the norm of $\mathbb{H}_B^1 \hookrightarrow \left[L^6(\mathbb{R}^3)\right]^2$. Hence, if $\gamma_0 \|\vec{B}\|_{L^{3/2}(\mathbb{R}^3} < 1$,

$$\delta(B) \ \geq \ 1 - \gamma_0^2 \|\vec{B}\|_{L^{3/2}(\mathbb{R}^3)}$$

and from (21)

$$N(\mathbb{P}_A + V) \ \leq \ c[1 - \gamma_0^2 \|\vec{B}\|_{L^{3/2}(\mathbb{R}^3}]^{-3/2} \ \|V_-\|_{L^{3/2}(\mathbb{R}^3}^{3/2}. \tag{24}$$

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