DYNAMICS OF TWO DIFFUSIVELY COUPLED UNIMODAL MAPS WITH ZERO TOPOLOGICAL ENTROPY

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Abstract

The paper studies the dynamics of a coupled map lattice of two sites where the local map is a unimodal map for which the period of every periodic orbit is a power of two. With the help of the existence of nested subsets of the phase space and their forward invariance after some iterations, it is shown that every orbit passing such subsets is eventually periodic and the period is a power of two. A criterion for an orbit to visit a particular subset is given and a sufficient condition for the existence of periodic orbits satisfying this criterion is obtained.

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1. Coupled Unimodal Maps

The aim of this paper is to investigate the effects of a diffusive coupling on the dynamics of a simple dynamical system with discrete time, namely the direct product of two unimodal maps.

Physically, models with nonlinear forcing and diffusive coupling can be viewed as simple models for the time evolution of reaction-diffusion systems (see e.g. the chapter by Kaneko in [4]). Understanding the dynamics of these models is then useful to predict the asymptotic behaviour in these systems, in particular, to distinguish between chaotic and stable regimes and to describe the various patterns in the stable case.

Mathematically, we want to describe the topological changes in the non-wandering set of a class of two-dimensional non-hyperbolic systems. Just like the Hénon map, our dynamical systems are derived from a simple one-dimensional map through coupling and demonstrate rich dynamics.

The system we consider is a coupled map lattice of two sites. It means that the dynamics is given by the composition of the direct product of two identical one-dimensional maps with a convex coupling. The reason for considering only two sites is simplicity. Moreover, we believe this analysis will serve as a starting point for the investigation of the dynamics of lattices with more sites.

Although majority of works on coupled map lattices has considered more than two sites, analytic and numerical studies of two diffusively coupled maps can be found in the literature. These studies determine the parameter regions for the existence of some periodic orbits and study their bifurcations [2, 4, 5], or describe the structure of the attractor [3].

Our analysis emphasizes on the global topological aspects of the dynamics. Instead of considering systems with a specific formula such as the quadratic map, we study a class of maps with common topological features: unimodal maps with a family of kneading sequences. The study focuses on two topics: the asymptotically periodic behaviour of an orbit passing a sequence of nested squares and the emergence of non-synchronised orbits when the coupling gradually relaxes.

We combine specific properties of the kneading sequences with dynamical properties of the coupling to show the existence of a sequence of nested squares. Each of these squares is forward invariant by some iterates of the dynamics. An orbit visiting such a square is eventually periodic. Moreover, by considering the relative position of an orbit with respect to the diagonal, we give a criterion for an orbit to visit a specific square in the sequence. Finally, we give a condition for existence of periodic orbits satisfying this criterion.

We consider discrete dynamical systems generated by the composition of the direct product of two identical maps, the local map, with a diffusive coupling. The local map is a unimodal map on $[-1, 1]$. It is defined to have the following properties.

(1) $f(0) = 1$;

(2) f is increasing on $[-1,0]$ and decreasing on $[0,1]$;

(3) f is C^1 and $f'(x)=0$ only if $x=0$;

(4) $f([f(1),1]) \subset [f(1),1]$.

A well-known example is the quadratic map $f_a(x) = 1-ax^2$ where $0 < a \leq 2$.

The phase space $\mathcal{M}$ of our coupled system is the square $[f(1),1]^2$. In $\mathcal{M}$, the dynamics is given by the map F_ε defined by

$$F_\varepsilon(x,y) = ((1-\varepsilon)f(x)+\varepsilon f(y), (1-\varepsilon)f(y)+\varepsilon f(x)), \tag{1}$$

with $0 \leq \varepsilon \leq 1$.

The map F_ε can be viewed as the composition of two maps. $F_\varepsilon = L_\varepsilon \circ F_0$ where the coupling $L_\varepsilon(x,y) = ((1-\varepsilon)x+\varepsilon y, (1-\varepsilon)y+\varepsilon x)$ is the linear convex combination of coordinates. It follows that F_ε maps $\mathcal{M}$ into itself and the dynamics is well-defined.

The orbit with an initial point (x,y) is denoted by the sequence $\{(x^t,y^t)\}_{t\in\mathbb{N}}$ defined by $(x^0,y^0)=(x,y)$ and $(x^{t+1},y^{t+1}) = F_\varepsilon(x^t,y^t)$, $t\in\mathbb{N}$. For convenience, our index set $\mathbb{N}$ starts from 0.

The map F_ε commutes with the symmetry map $\theta : (x,y) \mapsto (y,x)$. An orbit with $x^t \neq y^t$, $t \in \mathbb{N}$ is said to be symmetric if there exists t_0 such that $(x^{t_0}, y^{t_0}) = \theta(x^0,y^0)$. Note that a symmetric orbit of F_ε is periodic.

The paper is organised as follows. The next section summarizes the main results of the paper. We begin with the description of the properties of the kneading sequences of the unimodal maps in Section 3. We then introduce fundamentals sets of F_ε and give their dynamical properties in Section 4. With these tools provided, we give details of proofs in Section 5.

2. Main Results

The dynamics of a unimodal map is largely determined by its *kneading sequence*, i.e., the location of the iterates of the point 1. Our study of coupled maps also relies on such information from the local map. We focus on maps with simple kneading sequences, either finite or 2^n-periodic, which implies zero topological entropy [6]. For the quadratic map $f_a(x) = 1-ax^2$, it means that the parameter $a>0$ can take any value smaller than the Feigenbaum's accumulation point of the period-doubling cascade.

We now recall definitions of itineraries and kneading sequences. The book by Collet and Eckmann [1] is a good source for more details.

The *itinerary* $I(x) = \{I_i(x)\}_{i\in\mathbb{N}}$ of a point $x \in [-1,1]$ is either an infinite sequence of L's and R's, or a finite sequence of L's and R's, followed by C. The i-th element of $I(x)$ is defined by

$$I_i(x) = \begin{cases} L & \text{if} \quad f^i(x) < 0 \\ C & \text{if} \quad f^i(x) = 0 \\ R & \text{if} \quad f^i(x) > 0 \end{cases}$$

The *kneading sequence* of a unimodal map is the itinerary of the point 1.

The simplest kneading sequences are $I^n := R^{*n} * R^\infty$, which is 2^{n+1}-periodic, and $J^n := R^{*n} * RC$, which is finite (see Sections 3.1 and 3.2 for definitions).

As in the case of unimodal maps where the study of the dynamics of intervals is important in the understanding the dynamics of the map, the dynamics of certain simple sets (we call them *fundamental sets*) under the coupled map F_ε plays a major role. We shall see that the dynamics of these fundamental sets is determined by the kneading sequence.

Among these fundamental sets, we consider squares invariant under the symmetry map θ. Given $a, b \in [-1,1]$, let

$$C_{\{a,b\}} = [\min\{a,b\}, \max\{a,b\}]^2.$$

To describe the dynamics of F_ε, we consider the squares $C_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$. Using properties of itineraries I^n and J^n, we first obtain a translation of the interval dynamics associated to f to the dynamics of the coupled system.

Theorem 2.1. *Assume that the kneading sequence of f is I^n or J^n for some $n \in \mathbb{N}$. Then, the following statements hold.*

(i) $\mathcal{M} = C_{\{f(0), f^2(0)\}} \supset C_{\{f^2(0), f^4(0)\}} \supset \cdots \supset C_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}} \supset \cdots \supset C_{\{f^{2^{n-1}}(0), f^{2^n}(0)\}} \supset C_{\{0, f^{2^n}(0)\}}$, *if* $n \geq 1$.

(ii) If $n \geq 1$ and $0 \leq k \leq n-1$, then the set $C_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$ is forward invariant under the map $F_\varepsilon^{2^k}$:

$$F_\varepsilon^{2^k}(C_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}) \subset C_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}},$$

For any $n \in \mathbb{N}$, the set $C_{\{0, f^{2^n}(0)\}}$ is forward invariant under the map $F_\varepsilon^{2^n}$:

$$F_\varepsilon^{2^n}(C_{\{0, f^{2^n}(0)\}}) \subset C_{\{0, f^{2^n}(0)\}}.$$

FIGURE 1. The squares $C_{\{f^{2^k}(0),f^{2^{k+1}}(0)\}}$ for k up to 2.

(iii) For any $1 \leq k \leq n$ *and any* $(x,y) \in C_{\{0,f^{2^k}(0)\}}$,

$$(x-y)(x^{2^k} - y^{2^k}) \leq 0.$$

If $(x,y) \in C_{\{0,f(0)\}}$, *then*

$$(x-y)(x^1 - y^1) \leq 0 \quad \textit{if} \quad 0 \leq \varepsilon \leq \frac{1}{2},$$

and

$$(x-y)(x^1 - y^1) \geq 0 \quad \textit{if} \quad \frac{1}{2} \leq \varepsilon \leq 1.$$

In other words, we have a nested sequence of squares (see Figure 1), and an orbit starting in one of these sets visits it periodically. Moreover, the position of a point in the square $C_{\{0,f^{2^k}(0)\}}$ with respect to the diagonal alternates after 2^k iterations. The case $k=0$ is different when $\varepsilon \leq \frac{1}{2}$ and $\varepsilon \geq \frac{1}{2}$ because we consider an odd number of iterations.

This alternation of positions gives crucial information about the asymptotic behaviour of orbits of F_ε. An orbit $\{(x^t,y^t)\}_{t\in\mathbb{N}}$ is said to be eventually periodic if there exists $T \geq 1$ such that

$$\lim_{t\to\infty}(x^t - x^{t+T}) = \lim_{t\to\infty}(y^t - y^{t+T}) = 0.$$

An orbit is said to be synchronised if $x^t = y^t$ for all t and eventually synchronised if

$$\lim_{t\to\infty}(x^t - y^t) = 0.$$

Theorem 2.2. *Assume that the kneading sequence of f is I^n or J^n for some $n \geq 1$. Then, for any initial condition $(x,y) \in C_{\{f^i(0),f^{i+2^n}(0)\}}$ where $0 \leq i < 2^n$, the orbit is eventually periodic of period 2^p for some $1 \leq p \leq n+1$. Moreover, if the orbit is not eventually synchronised, then the period is 2^{n+1}.*

The proof of this theorem depends mainly on the alternation of the sign (Statement *(iii)* of Theorem 2.1) under the invariance given by Statement *(ii)*. In general, i.e. for $k < n$, if we assume that an orbit belongs to $C_{\{0,f^{2^k}(0)\}}$ after every 2^k iteration, then the result of the theorem also applies.

Corollary 2.3. *Assume that the kneading sequence of f is I^n or J^n for some $n \geq 1$. If an orbit is such that $(x^{t\cdot 2^k}, y^{t\cdot 2^k}) \in C_{\{f^i(0),f^{i+2^k}(0)\}}$ for all $t \in \mathbb{N}$ and for some $0 \leq i < 2^k$, $1 \leq k < n$, then the orbit is eventually periodic of period 2^p, $1 \leq p \leq k+1$. Moreover, if the orbit is not eventually synchronised, then the period is 2^{k+1}.*

Our conclusions on asymptotic periodicity depend crucially on the fact that we consider the squares $C_{\{f^i(0),f^{i+2^k}(0)\}}$ and not the bigger ones $C_{\{f^{i+2^k}(0),f^{i+2^{k+1}}(0)\}}$. Indeed, an orbit going through the other regions of the bigger squares may not be periodic. For instance, the system may have a limit cycle emerging from a Hopf bifurcation as was shown for the quadratic map in [2].

We see from Statement *(iii)* that the case $k = 0$ needs to be considered separately since the alternation of sign is no longer true when $\varepsilon \geq \frac{1}{2}$.

Corollary 2.4. *If the kneading sequence of f is I^0 or J^0, then every orbit of F_ε is eventually periodic of period 1 or 2.*

Moreover, if $0 \leq \varepsilon \leq \frac{1}{2}$, then any non-synchronised periodic orbit is of period 2.

On the other hand, for such kneading sequence and $\varepsilon > \frac{1}{2}$, a non-synchronised periodic orbit can have period 1 or 2. For instance, this is the case when $\varepsilon = 1$.

For S-unimodal maps, i.e. unimodal maps with negative Schwarzian derivative, if the kneading sequence is I^n, then f has a stable periodic orbit of period either

2^n or 2^{n+1} [1]. In both cases, the orbit of the point 1 is attracted to the stable periodic orbit. We use this information to show that if the latter has period 2^n, then any orbit of the coupled system initiated from the set $C_{\{0,f^{2^n}(0)\}}$ asymptotically approaches the diagonal.

Theorem 2.5. *Assume that the kneading sequence of a S-unimodal f is I^n for some $n \in \mathbb{N}$ and that the 2^n-periodic orbit in $[f(1),1]$ is stable. Then, for any initial condition $(x,y) \in C_{\{f^i(0),f^{i+2^n}(0)\}}$, $0 \leq i < 2^n$, the orbit is eventually synchronised and eventually periodic of period 2^n.*

When ε is close to $\frac{1}{2}$, any orbit is asymptotically synchronised because the distance $|x^t - y^t|$ exponentially decreases. To understand the mechanism leading the system out of synchrony when ε is moving away from $\frac{1}{2}$, we study the emergence of non-synchronised periodic orbits. A local analysis showed that the periodic orbits created from the period-doubling bifurcation of synchronised periodic orbits are symmetric [2]. This argument shows that such orbits exist provided the system is sufficiently close to the bifurcation. Here we use the properties of I^n and J^n to show that the existence of such orbits obtained from bifurcation arguments is indeed global.

Before proving their existence, we show that such orbits must visit the squares given in Theorem 2.1.

Theorem 2.6. *Assume that the kneading sequence of f is I^n or J^n for some $n \in \mathbb{N}$. Then, the following assertions hold.*

(i) Any symmetric 2^{k+1}-periodic orbit passes the square $C_{\{f^{2^k}(0),f^{2^{k+1}}(0)\}}$ if $k < n$, and passes the square $C_{\{0,f^{2^n}(0)\}}$ if $k = n$.

(ii) No symmetric 2^{k+1}-periodic orbit exists for $k > n$. No symmetric 2-periodic orbit exists for $\frac{1}{2} \leq \varepsilon \leq 1$.

We denote by $\{x_i\}_{i=1}^{2^k}$ the 2^k-periodic orbit ($k \in \mathbb{N}$) of f.

Theorem 2.7. *Assume that the kneading sequence of f is I^n or J^n for some $n \in \mathbb{N}$. If*

$$(1-2\varepsilon)^{2^k} \prod_{i=1}^{2^k} f'(x_i) < -1, \tag{2}$$

for some $0 \leq k \leq n$, then F_ε has a symmetric 2^{k+1}-periodic orbit.

For any $k \in \mathbb{N}$, we have (see the proof of Lemma 5.11)

$$\prod_{i=1}^{2^k} f'(x_i) < 0.$$

Therefore, if this periodic orbit of f is unstable, the condition (2) is satisfied provided that ε is sufficiently close to 0, or 1 if $k \geq 1$. However, for $\varepsilon > \frac{1}{2}$ and $k = 0$, this condition cannot hold as stated in the previous Theorem.

It is an open question if the inequality (2) is a necessary condition for the existence of a symmetric 2^{k+1} periodic orbit. In the special case of the quadratic map and $k = 0$, the answer is affirmative [2]. In particular, since the kneading sequence of this map is I^0 or J^0 when $a \leq 1$, this implies the following result.

Corollary 2.8. *For the map $f_a(x) = 1 - ax^2$ with $a \leq 1$, every orbit of F_ε is eventually synchronised if and only if $(1-2\varepsilon)f'(x_1) \geq -1$ where x_1 is the fixed point of f.*

Finally, the proof that (2) with $k = 0$ is a sufficient condition for the existence of a period-2 symmetric orbit holds for any kneading sequence starting with $R\cdots$, and hence holds for any unimodal map.

3. Properties of Unimodal Maps

Properties of itineraries and their operations were extensively discussed in [1]. We only collect the ones needed in this article. To make the article as much self-contained as possible, we provide brief proofs of properties that are not explicitly proved in [1].

We begin with a lemma on the relative position of iterates of two points when their itineraries are the same on the first symbols. Recall that a finite sequence of L's and R's is said to be *odd* if it has an odd number of R's, *even*, otherwise.

Lemma 3.1. *Let $a \in [-1, 1]$, $a \neq 0$ and $t \in \mathbb{N}$ be such that $I_i(a) = I_{i-1}(1) \neq C$, $1 \leq i \leq t$ if $t \geq 1$. Then we have*

$$a(f^{t+1}(a) - f^{t+1}(0)) > 0 \quad \textit{if} \quad \{I_i(a)\}_{0 \leq i \leq t} \quad \textit{is even},$$

and

$$a(f^{t+1}(a) - f^{t+1}(0)) < 0 \quad \textit{if} \quad \{I_i(a)\}_{0 \leq i \leq t} \quad \textit{is odd}.$$

Proof. We use induction. For $t = 0$, the result follows from the piecewise monotonicity of f. Now, given $t \in \mathbb{N}$, assume that $\{I_i(a)\}_{0\leq i\leq t+1}$ is even. Then either $\{I_i(a)\}_{0\leq i\leq t}$ is even and $I_{t+1}(a) = I_t(1) = L$, or $\{I_i(a)\}_{0\leq i\leq t}$ is odd and $I_{t+1}(a) = I_t(1) = R$. In the first case, we have

$$(f^{t+1}(a) - f^{t+1}(0))(f^{t+2}(a) - f^{t+2}(0)) > 0$$

and using the induction hypothesis, it follows that

$$a(f^{t+2}(a) - f^{t+2}(0)) > 0.$$

Similarly, in the second case, we have

$$(f^{t+1}(a) - f^{t+1}(0))(f^{t+2}(a) - f^{t+2}(0)) < 0$$

and then

$$a(f^{t+2}(a) - f^{t+2}(0)) > 0.$$

Consequently, the statement for $t + 1$ holds in the case of an even sequence. The odd case is proved similarly. □

3.1. **The sequences I^n.** We refer to [1] for the definition of the $*$-product. Now, let A^∞ denotes the infinite concatenation $AAA\cdots$ and let $R^{*n} * A$ denotes the n-fold $*$-product $R*(R*(\cdots(R*A)))$. According to these notations, the sequence $I^n = \{I^n_i\}_{i\in\mathbb{N}} = R^{*n} * R^\infty$ is a sequence of L's and R's defined inductively as follows.

$$I^0 = R^\infty \quad \text{and} \quad I^{n+1} = R * I^n,\ n \in \mathbb{N}.$$

It follows from the definition of the $*$-product that

$$I^{n+1}_{2i} = R,\ i \in \mathbb{N}, \tag{3}$$

and

$$I^{n+1}_{2i+1} = \check{I}^n_i,\ i \in \mathbb{N}, \tag{4}$$

where $\check{R} = L$ and $\check{L} = R$. For the convenience of the reader, we list the first four sequences

$$\begin{aligned} I^0 &= RR\cdots \\ I^1 &= RLRL\cdots \\ I^2 &= RLRRRLRR\cdots \\ I^3 &= RLRRRLRLRLRRRLRL\cdots \end{aligned}$$

According to expressions (3) and (4), one can easily verify that I^n is a 2^n-periodic sequence with the following properties.

(A) For all $n \in \mathbb{N}$, the sequence $\{I_i^n\}_{0 \le i \le 2^n - 1}$ is odd [1].

(B) For all $n \ge 1$, $I_{i+2^n}^{n+1} = I_i^{n+1},\ 0 \le i \le 2^n - 2$.

(C) For all $n \in \mathbb{N}$, $I_i^{n+1} = I_i^n,\ 0 \le i \le 2^{n+1} - 2$.

(D) For all $n \ge 1$, the sequence $\{I_i^{n+1}\}_{0 \le i \le 2^n - 2}$ is even if n is even and is odd, otherwise.

(E) For all $n \ge 1$, $I_{2^{n-1}-1}^n = L$ and $I_{2^n-1}^n = R$ if n is even and $I_{2^{n-1}-1}^n = R$ and $I_{2^n-1}^n = L$ if n is odd.

Proof. All the proofs are done by induction.

(B) We have $I^2 = (RLRR)^\infty$. Then $I_{0+2}^2 = I_0^2$. The statement holds for $n = 1$.

Now assume the statement holds for some $n \ge 2$. Then, $\check{I}_{i+2^n}^{n+1} = \check{I}_i^{n+1},\ 0 \le i \le 2^n - 2$. Using (4), it follows that $I_{2i+2^{n+1}+1}^{n+2} = I_{2i+1}^{n+2},\ 0 \le i \le 2^n - 2$. On the other hand, the expression (3) implies that $I_{2i+2^{n+1}}^{n+2} = R = I_{2i}^{n+2},\ 0 \le i \le 2^n - 1$. The statement then holds for $n + 1$.

The proof of (C) is similar.

(D) The sequence $\{I_i^2\}_{0 \le i \le 0} = R$ is odd and the statement holds for $n = 1$.

Now we assume that the statement holds for some $n \in \mathbb{N}$, n odd. We have

$$\{I_i^{n+2}\}_{0 \le i \le 2^{n+1}-2} = \{I_{2i}^{n+2}\}_{0 \le i \le 2^n - 1} \cup \{I_{2i+1}^{n+2}\}_{0 \le i \le 2^n - 2}.$$

By (3), the sequence $\{I_{2i}^{n+2}\}_{0 \le i \le 2^n-1}$ is even. By (4) and by the induction assumption, the sequence $\{I_{2i+1}^{n+2}\}_{0 \le i \le 2^n - 2}$ is also even. Therefore, the statement holds for $n+1$ that is even. A similar argument is done in the case where n is even and this completes the induction.

(E) We have $I_0^1 = R$ and $I_1^1 = L$ and then the statement holds for $n = 1$. Let $n \ge 1$ be odd and assume that $I_{2^{n-1}-1}^n = R$ and $I_{2^n-1}^n = L$. By relation (4), we have $I_{2^n-1}^{n+1} = \check{I}_{2^{n-1}-1}^n = L$ and $I_{2^{n+1}-1}^{n+1} = \check{I}_{2^n-1}^n = R$ and then the statement holds for $n + 1$ which is even. The case n is even follows similarly. □

3.2. **The sequences J^n.** The sequences $J^n = \{J_i^n\}_{i=0}^{2^{n+1}-1} = R^{*n} * RC$ satisfy the following induction

$$J^0 = RC \quad \text{and} \quad J^{n+1} = R * J^n,\ n \in \mathbb{N}.$$

Again the definition of the $*$-product implies that

$$J_{2i}^{n+1} = R,\ 0 \le i \le 2^{n+1} - 1$$

and

$$J^{n+1}_{2i+1} = \check{J}^n_i,\ 0 \le i \le 2^{n+1}-1,$$

where $\check{R} = L$, $\check{L} = R$ and $\check{C} = C$. Consequently, $J^n = \{J^n_i\}_{0\le i\le 2^{n+1}-1}$, $J^n_{2^{n+1}-1} = C$ and

$$J^n_i = I^n_i, \quad 0 \le i \le 2^{n+1}-2. \tag{5}$$

The proof is similar to that of (B).

4. Dynamics of Fundamental Sets

With the properties of the kneading sequences provided, we can now turn to the analysis of the coupled system. In this section, we give definitions of fundamental sets and their properties.

Given $a, b \in [-1, 1]$, $a < b$, let

$$\overline{T}_{a,b} = \left\{(x, y) \in [-1, 1]^2 \ : \ a \le x \le b \text{ and } x \le y \le b\right\}$$

denote the triangle above the diagonal and $\underline{T}_{a,b} = \theta(\overline{T}_{a,b})$, the triangle below the diagonal. Moreover, if $a < 0 < b$, let

$$\overline{R}_{a,b} = \left\{(x, y) \in [-1, 1]^2 \ : \ a \le x \le 0 \text{ and } 0 \le y \le b\right\} \quad \text{and} \quad \underline{R}_{a,b} = \theta(\overline{R}_{a,b}).$$

For convenience, we expand the notation a little. Let S be any of the symbols $\overline{T}, \underline{T}, \overline{R}$ or $\underline{R}$. The set $S_{\{a,b\}}$ is defined by the equation below.

$$S_{\{a,b\}} = S_{\min\{a,b\},\max\{a,b\}}. \tag{6}$$

Note that the square $C_{\{a,b\}} = [\min\{a, b\}, \max\{a, b\}]^2 = \overline{T}_{\{a,b\}} \cup \underline{T}_{\{a,b\}}$.

We call all these sets fundamental sets of the phase space. The following properties hold for their images.

Lemma 4.1. *(i) If $-1 \le a < b \le 0$, then*

$F_\varepsilon(\overline{T}_{a,b}) \subset \overline{T}_{f(a),f(b)}$ *when* $0 \le \varepsilon \le \dfrac{1}{2}$ *and* $F_\varepsilon(\overline{T}_{a,b}) \subset \underline{T}_{f(a),f(b)}$ *when* $\dfrac{1}{2} \le \varepsilon \le 1$.

(ii) If $0 \le a < b \le 1$, then

$F_\varepsilon(\overline{T}_{a,b}) \subset \underline{T}_{f(b),f(a)}$ *when* $0 \le \varepsilon \le \dfrac{1}{2}$ *and* $F_\varepsilon(\overline{T}_{a,b}) \subset \overline{T}_{f(a),f(b)}$ *when* $\dfrac{1}{2} \le \varepsilon \le 1$.

(iii) If $-1 \le a < 0 < b \le 1$, then

$F_\varepsilon(\overline{R}_{a,b}) \subset \overline{T}_{f(a),1} \cup \underline{T}_{f(b),1}$ *when* $0 \le \varepsilon \le \dfrac{1}{2}$ *and* $F_\varepsilon(\overline{R}_{a,b}) \subset \underline{T}_{f(a),1} \cup \overline{T}_{f(b),1}$ *when* $\dfrac{1}{2} \le \varepsilon \le 1$.

Similar properties hold for $F_\varepsilon(\underline{T}_{a,b})$ and $F_\varepsilon(\underline{R}_{a,b})$ by applying the symmetry map θ.

Proof. (i) If $-1 \leq a < b \leq 0$, then by monotonicity of f, we have

$$F_0(\overline{T}_{a,b}) \subset \overline{T}_{f(a),f(b)}.$$

Now for any $a < b$, we have

$$L_\varepsilon(\overline{T}_{a,b}) \subset \overline{T}_{a,b} \text{ when } 0 \leq \varepsilon \leq \frac{1}{2} \text{ and } L_\varepsilon(\overline{T}_{a,b}) \subset \underline{T}_{a,b} \text{ when } \frac{1}{2} \leq \varepsilon \leq 1. \tag{7}$$

The statement then follows when combining this expression and the previous inclusion. Statement *(ii)* is proved analogously.

(iii) By monotonicity, we have

$$F_0(\overline{R}_{a,b}) \subset \left\{(x,y) \in [-1,1]^2 \ : \ f(a) \leq x \leq 1 \text{ and } f(b) \leq y \leq 1\right\} \subset \overline{T}_{f(a),1} \cup \underline{T}_{f(b),1},$$

and the statement follows from (7). $\square$

It follows from this lemma that, for any triangle $\overline{T}_{\{a,b\}}$, the image $F_\varepsilon(\overline{T}_{\{a,b\}}) \subset F_0(\overline{T}_{\{a,b\}})$ when $\varepsilon \leq \frac{1}{2}$ and $F_\varepsilon(\overline{T}_{\{a,b\}}) \subset F_0(\underline{T}_{\{a,b\}})$ when $\varepsilon \geq \frac{1}{2}$. Lemma 3.1 then has an immediate translation to the iterates of triangles.

Lemma 4.2. *Let $a \in [-1,1]$, $a \neq 0$ and $t \in \mathbb{N}$ be such that $I_i(a) = I_{i-1}(1) \neq C$, $1 \leq i \leq t$ if $t \geq 1$. For $0 \leq \varepsilon \leq \frac{1}{2}$, we have*

$$F_\varepsilon^{t+1}(\overline{T}_{\{0,a\}}) \subset \overline{T}_{\{f^{t+1}(0),f^{t+1}(a)\}} \quad \textit{if} \quad \{I_i(a)\}_{0 \leq i \leq t} \quad \textit{is even,}$$

and

$$F_\varepsilon^{t+1}(\overline{T}_{\{0,a\}}) \subset \underline{T}_{\{f^{t+1}(0),f^{t+1}(a)\}} \quad \textit{if} \quad \{I_i(a)\}_{0 \leq i \leq t} \quad \textit{is odd.}$$

For $\frac{1}{2} \leq \varepsilon \leq 1$, the same conclusion applies if t is odd. If t is even, we have

$$F_\varepsilon^{t+1}(\overline{T}_{\{0,a\}}) \subset \overline{T}_{\{f^{t+1}(0),f^{t+1}(a)\}} \quad \textit{if} \quad \{I_i(a)\}_{0 \leq i \leq t} \quad \textit{is odd,}$$

and

$$F_\varepsilon^{t+1}(\overline{T}_{\{0,a\}}) \subset \underline{T}_{\{f^{t+1}(0),f^{t+1}(a)\}} \quad \textit{if} \quad \{I_i(a)\}_{0 \leq i \leq t} \quad \textit{is even.}$$

Proof. The proof is similar to the one of Lemma 3.1. We only sketch the induction step for $\varepsilon \leq \frac{1}{2}$ and in the case $\{I_i(a)\}_{0 \leq i \leq t+1}$ is even. It means that either $\{I_i(a)\}_{0\le i\le t}$ is even and $I_{t+1}(a) = I_t(1) = L$, or $\{I_i(a)\}_{0\le i\le t}$ is odd and $I_{t+1}(a) = I_t(1) = R$. In the first case, using induction hypothesis, we have

$$F_\varepsilon^{t+2}(\overline{T}_{\{0,a\}}) \subset F_\varepsilon(\overline{T}_{\{f^{t+1}(0),f^{t+1}(a)\}}).$$

By Lemma 4.1 *(i)*, we conclude that

$$F_\varepsilon^{t+2}(\overline{T}_{\{0,a\}}) \subset \overline{T}_{\{f^{t+2}(0),f^{t+2}(a)\}}.$$

In the second case, a similar reasoning shows that

$$F_\varepsilon^{t+2}(\overline{T}_{\{0,a\}}) \subset F_\varepsilon(\underline{T}_{\{f^{t+1}(0),f^{t+1}(a)\}}) \subset \overline{T}_{\{f^{t+2}(0),f^{t+2}(a)\}},$$

and the statement holds for $t+1$. □

5. Proofs of Main Results

To prove the results, we start by considering the dynamics of fundamental sets when the kneading sequence of f is I^n or J^n. Then we use these properties to prove results on asymptotic behaviour of orbits, as well as the existence and the location of non-synchronised orbits.

5.1. **Proof of Theorem 2.1.** Instead of proving the results for a fixed itinerary I^n or J^n and various k, we focus on one square $C_{\{f^{2^k}(0),f^{2^{k+1}}(0)\}}$ for some fixed $k \ge 0$ and describe its dynamics for various values of n. The advantage of this approach is to use the following property of the sequences I^n and J^n: the first $2^{n+1}-1$ symbols of I^n are the same as those of I^{n+1}, or J^n. Thus, if a property of $f(x)$ depends only on the first $2^{n+1}-1$ symbols of the kneading sequence $I(1) = I^n$, the same property holds for any unimodal map with a kneading sequence $I(1) = I^m, m > n$, or $I(1) = J^m, m \ge n$.

The next statement shows that the points $f^{2^k}(0)$ and $f^{2^{k+1}}(0)$, $k < n$ are of different signs.

Proposition 5.1. *If $I(1) = I^n$ or J^n for some $n \ge k$ where $k \in \mathbb{N}$, then, $f^{2^k}(0) > 0$ when k is even and $f^{2^k}(0) < 0$ when k is odd.*

Proof. When $k = 0$, $f(0) = 1 > 0$ holds whatever the kneading sequence is. When $I(1) = I^k$ for some $k > 0$, the result follows from (E) and the relation $I_0(f^{2^k}(0)) = I^k_{2^k-1}$. If $I(1) = I^n$ with $n > k$, the result holds because of (C). In the case $I(1) = J^n$, $n \ge k$, the result can be obtained by using the expression (5). □

Statement *(i)* of Theorem 2.1 then follows immediately from the next Proposition.

Proposition 5.2. *If $I(1) = I^n$ or J^n for some $n \geq k+2$ where $k \in \mathbb{N}$, then*

$$C_{\{f^{2^{k+1}}(0), f^{2^{k+2}}(0)\}} \subset C_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}.$$

In order to prove this proposition and later results, we use the following lemma.

Lemma 5.3. *If $I(1) = I^n$ or J^n for some $n > k$ where $k \in \mathbb{N}$, then*

$$C_{\{f^{2^k}(0), f^{3\cdot 2^k}(0)\}} \subset C_{\{0, f^{2^k}(0)\}}.$$

Proof. We show that $0 < f^{3\cdot 2^k}(0) < f^{2^k}(0)$ when k is even. Assume that $k \geq 1$, the proof for $k = 0$ is similar.

Assume that $I(1) = I^{k+1}$. Then $f^{2^{k+1}}(0) \neq 0$ and by periodicity of I^{k+1}, we have

$$\{I_i(f^{2^{k+1}}(0))\}_{1 \leq i \leq 2^k - 1} = \{I_{i-1}^{k+1}\}_{1 \leq i \leq 2^k - 1} = \{I_{i-1}(1)\}_{1 \leq i \leq 2^k - 1}.$$

By a change of index, we obtain,

$$\{I_i(f^{2^{k+1}}(0))\}_{1 \leq i \leq 2^k - 1} = \{I_i^{k+1}\}_{0 \leq i \leq 2^k - 2},$$

and the latter is even by (D). Now, Proposition 5.1 says that $f^{2^{k+1}}(0) < 0$. Therefore the sequence $\{I_i(f^{2^{k+1}}(0))\}_{0 \leq i \leq 2^k - 1}$ is even.

Applying Lemma 3.1 with $a = f^{2^{k+1}}(0)$ and $t = 2^k - 1$, the inequality $f^{2^{k+1}}(0) < 0$ implies $f^{3\cdot 2^k}(0) < f^{2^k}(0)$. Using again the periodicity of I^{k+1}, we have

$$I_0(f^{3\cdot 2^k}(0)) = I_0(f^{2^k}(0)) = I_{2^k - 1}^{k+1}. \tag{8}$$

Property (E) then tells us that $f^{3\cdot 2^k}(0) > 0$ and the result is proved for $n = k+1$.

The arguments use only the first $3 \cdot 2^k - 1$ symbols of $I(1)$. Property (C) implies that the result holds for $I(1) = I^n$ with $n \geq k+2$. Due to the relation (5), the same is true in the case $I(1) = J^n$ with $n \geq k+1$. Similar arguments show that $f^{2^k}(0) < f^{3.2^k}(0) < 0$ when k is odd. $\square$

Proof of Proposition 5.2. According to Proposition 5.1, we only have to prove that $f^{2^{k+2}}(0) < f^{2^k}(0)$ when k is even and $f^{2^k}(0) < f^{2^{k+2}}(0)$ when k is odd.

Relation (8) showed that $f^{3.2^k}(0) \neq 0$. Assume that $I(1) = I^{k+2}$. By properties (B) and (C), we obtain

$$\{I_i(f^{3.2^k}(0))\}_{1 \leq i \leq 2^k - 1} = \{I_{i+3\cdot 2^k}^{k+2}\}_{0 \leq i \leq 2^k - 2} = \{I_i^{k+2}\}_{0 \leq i \leq 2^k - 2} = \{I_{i-1}(1)\}_{1 \leq i \leq 2^k - 1}.$$

The proof is now similar to that of the previous lemma. Assume that k is even. Then, $f^{3.2^k}(0) > 0$ and $\{I_{i-1}(1)\}_{1\le i\le 2^k-1}$ is even. Using the previous equality and Lemma 3.1, we conclude that $f^{2^{k+2}}(0) < f^{2^k}(0)$ and the result follows.

The arguments use only the first $2^{k+2}-1$ symbols of $I(1)$. Therefore, they are valid for I^n with $n > k+2$ and J^n with $n \ge k+2$. □

The proof of the remaining assertions of Theorem 2.1 depends on the following propositions.

Proposition 5.4. *If $I(1) = I^n$ or J^n for some $n \ge k, k \ge 1$, then*

$$F_\varepsilon^{2^k}(\overline{T}_{\{0,f^{2^k}(0)\}}) \subset \underline{T}_{\{f^{2^k}(0),f^{2^{k+1}}(0)\}}.$$

If $I(1) = I^n$ or J^n for some $n \ge 0$, then

$$F_\varepsilon(\overline{T}_{\{0,f(0)\}}) \subset \underline{T}_{\{f(0),f^2(0)\}} \quad \text{if} \quad 0 \le \varepsilon \le \frac{1}{2} \quad \text{and} \quad F_\varepsilon(\overline{T}_{\{0,f(0)\}}) \subset \overline{T}_{\{f(0),f^2(0)\}} \quad \text{if} \quad \frac{1}{2} \le \varepsilon \le 1.$$

Obviously, Statement *(iii)* follows from this proposition.

Proof. Assume that $k \ge 1$. We proceed similarly as in the proof of Lemma 5.3 to check the conditions of Lemma 4.2 with $a = f^{2^k}(0)$ and $t = 2^k - 1$.

We first consider the case $I(1) = I^k$.

We have

$$\{I_i(f^{2^k}(0))\}_{1\le i\le 2^k-1} = \{I_{i+2^k-1}(1)\}_{1\le i\le 2^k-1} = \{I_{i-1}(1)\}_{1\le i\le 2^k-1},$$

where the second equality is obtained by periodicity. Similarly, we have

$$\{I_i(f^{2^k}(0))\}_{0\le i\le 2^k-1} = \{I_{i+2^k-1}(1)\}_{0\le i\le 2^k-1}.$$

This sequence is odd by Property (A) and periodicity. One can now apply Lemma 4.2 to obtain the desired result.

The arguments use only the first $2^{k+1}-1$ symbols of $I(1)$. Property (C) implies that the result holds for $I(1) = I^n$ with $n \ge k+1$. Due to the relation (5), the same is true in the case $I(1) = J^n$ with $n \ge k$.

For $k = 0$, we have

$$F_0(\overline{T}_{\{0,f(0)\}}) \subset \underline{T}_{\{f(0),f^2(0)\}},$$

independently of the kneading sequence. The results then follows from Relation (7). □

Proposition 5.5. *For all $k \in \mathbb{N}$, we have*

$\underline{T}_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}} \subset \underline{T}_{\{0, f^{2^k}(0)\}}$ *if $I(1) = I^k$ or J^k. The inclusion is strict except when $I(1) = J^k$.*

When $I(1) = I^n$ or $I(1) = J^n, n > k$, we have the strict inclusion $\underline{T}_{\{0, f^{2^k}(0)\}} \subsetneq \underline{T}_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$.

Proof. We want to apply Lemma 3.1 with $a = f^{2^k}(0)$ and $t = 2^k - 1$. Assume that $I(1) = I^k$ and k is even. The case when k is odd can be proved similarly.

By periodicity, we have

$$\{I_i(f^{2^k}(0))\}_{1 \le i \le 2^k - 1} = \{I_{i-1}(1)\}_{1 \le i \le 2^k - 1}.$$

Similarly, by periodicity and (A), the sequence $\{I_i(f^{2^k}(0))\}_{0 \le i \le 2^k - 1}$ is odd. Lemma 3.1 then says that the inequality $0 < f^{2^k}(0)$ implies $f^{2^{k+1}}(0) < f^{2^k}(0)$. In addition, by periodicity and property (E), we have $I_0(f^{2^{k+1}}(0)) = I^k_{2^k - 1} = R$ (The case $k = 0$ can be checked directly). Therefore, $0 < f^{2^{k+1}}(0) < f^{2^k}(0)$ and the result holds.

In the case $I(1) = J^k$, the two sets are the same since $I_0(f^{2^{k+1}}(0)) = J^k_{2^{k+1}-1} = C$.

If $I(1) = I^{k+1}$, $k+1$ is odd when k is even. Thus, by property (E), $I_0(f^{2^{k+1}}(0)) = I^{k+1}_{2^{k+1}-1} = L$. Therefore, $f^{2^{k+1}}(0) < 0$ and $f^{2^{k+1}}(0) < 0 < f^{2^k}(0)$. Thus, $\underline{T}_{\{0, f^{2^k}(0)\}} \subsetneq \underline{T}_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$.

Finally, the previous arguments use only the properties of $\{I_i(1)\}_{0 \le i \le 2^{k+1} - 2}$. Therefore, they hold if $I(1) = I^n$ or J^{n-1} for some $n > k + 1$. ☐

It remains to prove Statement *(ii)* of the theorem. Combine Propositions 5.4 and 5.5 to obtain

$$F_\varepsilon^{2^k}(C_{\{0, f^{2^k}(0)\}}) \subset C_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}} \text{ if } k \le n - 1 \quad \text{and} \quad F_\varepsilon^{2^n}(C_{\{0, f^{2^n}(0)\}}) \subset C_{\{0, f^{2^n}(0)\}}.$$

Statement *(ii)* is then proved for $k = n$. Now if $k < n$, since $f^{2^k}(0)$ and $f^{2^{k+1}}(0)$ are of different sign, we have a partition of the set $C_{\{f^{2^{k+1}}(0), f^{2^k}(0)\}}$:

(9)
$$C_{\{f^{2^{k+1}}(0), f^{2^k}(0)\}} = C_{\{0, f^{2^k}(0)\}} \cup C_{\{0, f^{2^{k+1}}(0)\}} \cup \overline{R}_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}} \cup \underline{R}_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}.$$

Based on the same arguments used in the proof of Proposition 5.4, we apply Lemma 4.2 with $a = f^{2^{k+1}}(0)$ and $t = 2^k - 1$ to obtain

$$F_\varepsilon^{2^k}(C_{\{0, f^{2^{k+1}}(0)\}}) \subset C_{\{f^{2^k}(0), f^{3 \cdot 2^k}(0)\}} \subset C_{\{0, f^{2^k}(0)\}}. \tag{10}$$

The second inclusion is from Lemma 5.3.

Now, we only have to consider the images of the rectangles. Assume for simplicity, that k is even. Then, $\overline{R}_{\{f^{2^k}(0),f^{2^{k+1}}(0)\}} = \overline{R}_{f^{2^{k+1}}(0),f^{2^k}(0)}$. Applying Lemma 4.1 *(iii)*, we obtain

$$F_\varepsilon(\overline{R}_{f^{2^{k+1}}(0),f^{2^k}(0)}) \subset \overline{T}_{f^{2^{k+1}+1}(0),1} \cup \underline{T}_{f^{2^k+1}(0),1} = \overline{T}_{f^{2^{k+1}}(1),1} \cup \underline{T}_{f^{2^k}(1),1}.$$

Since $n > k$, the itineraries of 1, $f^{2^{k+1}}(1)$, and $f^{2^k}(1)$ are the same from 0 to $2^k - 1$. Thus, we have

$$F_\varepsilon^{2^k-1}(\underline{T}_{f^{2^k+1}(0),1}) \subset F_\varepsilon^{2^k-1}(C_{\{f^{2^k+1}(0),f(0)\}}) \subset C_{\{f^{2^{k+1}}(0),f^{2^k}(0)\}}$$

and

$$F_\varepsilon^{2^k-1}(\underline{T}_{f^{2^{k+1}+1}(0),1}) \subset F_\varepsilon^{2^k-1}(C_{\{f^{2^{k+1}+1}(0),f(0)\}}) \subset C_{\{f^{3\cdot 2^k}(0),f^{2^k}(0)\}} \subset C_{\{0,f^{2^k}(0)\}}.$$

The last inclusion is from Lemma 5.3. This completes the proof of Theorem 2.1.

Remark. Assume that k is even and $n > k$. Since L_ε is a convex combination of coordinates, we have

$$F_\varepsilon^{2^k-1}(\underline{T}_{f^{2^k}(1),1}) \subset F_0^{2^k-1}(\underline{T}_{f^{2^k}(1),1}) = F_0^{2^k}(\overline{T}_{0,f^{2^k}(0)}).$$

Proposition 5.4 then shows that

$$F_\varepsilon^{2^k-1}(\underline{T}_{f^{2^k}(1),1}) \subset \underline{T}_{f^{2^{k+1}}(0),f^{2^k}(0)}. \tag{11}$$

Therefore, the conclusion of Statement *(iii)* also applies to any point in $\overline{R}_{f^{2^{k+1}}(0),f^{2^k}(0)}$ if the first iterate of the point belongs to $\underline{T}_{f^{2^k}(1),1}$. On the other hand, if the first iterate of such a point belongs to $\overline{T}_{f^{2^{k+1}}(1),1}$, then, the inclusion (11) applied to $k+1$ shows that the 2^{k+1}-iterate belongs to $T_{\{f^{2^{k+2}}(0),f^{2^{k+1}}(0)\}}$.

5.2. **Proof of Results on Asymptotic Behaviour.** *Proof of Theorem 2.2.* By Statement *(ii)* of Theorem 2.1, we may assume that $i = 0$. Otherwise, iterate the point (x, y) $2^n - i$ times to obtain

$$F_\varepsilon^{2^n-i}(x,y) \in C_{\{f^{2^n}(0),f^{2^{n+1}}(0)\}} \subset C_{\{0,f^{2^n}(0)\}}.$$

Let $\{(x^t, y^t)\}_{t\in\mathbb{N}}$ be an orbit with initial condition $(x, y) \in C_{\{0,f^{2^n}(0)\}}$. If $(x, y) = (x^{2^{n+1}}, y^{2^{n+1}})$, there is nothing to prove. Therefore, we assume that $(x, y) \neq (x^{2^{n+1}}, y^{2^{n+1}})$. We divide the situation into four cases:

(a) $(x \le x^{2^{n+1}})(y \le y^{2^{n+1}})$.

(b) $(x^{2^{n+1}} \le x)(y^{2^{n+1}} \le y)$.

(c) $(x \le x^{2^{n+1}})(y^{2^{n+1}} \le y)$.

(d) $(x^{2^{n+1}} \le x)(y \le y^{2^{n+1}})$.

The notation $(x_1 \le x_2)(y_1 \le y_2)$ means that both inequalities hold and at least one inequality is strict.

In the cases (a) and (b), we need the following lemma.

Lemma 5.6. *Let $a \in [-1,1]$, $a \neq 0$ and $t \in \mathbb{N}$ be such that $I_i(a) = I_{i-1}(1) \neq C$, $1 \le i \le t$ if $t \ge 1$. Then, for any $(x_1,y_1),(x_2,y_2) \in C_{\{0,a\}}$ such that $(x_1 \le x_2)(y_1 \le y_2)$, we have*

$$(x_1^{t+1} \le x_2^{t+1})(y_1^{t+1} \le y_2^{t+1}) \quad \textit{if} \quad \{I_i(a)\}_{0\le i\le t} \quad \textit{is even;}$$

$$(x_2^{t+1} \le x_1^{t+1})(y_2^{t+1} \le y_1^{t+1}) \quad \textit{if} \quad \{I_i(a)\}_{0\le i\le t} \quad \textit{is odd.}$$

Proof of the lemma: We prove it inductively. Using the piecewise monotonicity of f, we have

$$(x_1^1 \le x_2^1)(y_1^1 \le y_2^1) \quad \text{if} \quad I_0(a) = L \quad \text{and} \quad (x_2^1 \le x_1^1)(y_2^1 \le y_1^1) \quad \text{if} \quad I_0(a) = R.$$

Hence, the statement holds for t=0. Now, given $t \in \mathbb{N}$, let us assume that $\{I_i(a)\}_{0\le i\le t+1}$ is even. Then either $\{I_i(a)\}_{0\le i\le t}$ is even and $I_{t+1}(a) = I_t(1) = L$, or $\{I_i(a)\}_{0\le i\le t}$ is odd and $I_{t+1}(a) = I_t(1) = R$.

In the first case, by induction hypothesis, we have $(x_1^{t+1} \le x_2^{t+1})(y_1^{t+1} \le y_2^{t+1})$ and

$$(x_1^{t+1}, y_1^{t+1}), (x_2^{t+1}, y_2^{t+1}) \in C_{\{f^{t+1}(0), f^{t+1}(a)\}}$$

by Lemma 4.2. Using the same argument as the one in the case $t = 0$, we obtain

$$(x_1^{t+2} \le x_2^{t+2})(y_1^{t+2} \le y_2^{t+2}). \tag{12}$$

In the second case, the inequalities $(x_2^{t+1} \le x_1^{t+1})(y_2^{t+1} \le y_1^{t+1})$ also imply (12). Therefore, the statement holds for $t+1$ in the even case. The odd case follows similarly. □

In the case (a), we apply the previous lemma twice with $a = f^{2^n}(0)$ and $t = 2^n - 1$ (see proof of Proposition 5.4), to obtain by induction,

$$(x^{t\cdot 2^{n+1}} \le x^{(t+1)\cdot 2^{n+1}})(y^{t\cdot 2^{n+1}} \le y^{(t+1)\cdot 2^{n+1}}),\ t \ge 1,$$

and the orbit cannot be periodic. Since the sequences $\{x^{t.2^{n+1}}\}$ and $\{y^{t.2^{n+1}}\}$ are bounded and monotonic, they must converge. By continuity of f, the limit is a periodic orbit of F_ε whose period is a factor of 2^{n+1}.

The case (b) is proved analogously.

In the cases (c) and (d), we use the following lemma. Note that if $\varepsilon = \frac{1}{2}$, these two cases cannot occur. So we assume $\varepsilon \neq \frac{1}{2}$ in the following statements.

Lemma 5.7. *Let $a \in [-1, 1]$, $a \neq 0$ and $t \in \mathbb{N}$ be such that $I_i(a) = I_{i-1}(1) \neq C$, $1 \leq i \leq t$, if $t \geq 1$. Then, for any $(x_1, y_1), (x_2, y_2) \in C_{\{0,a\}}$ such that $(x_1 \leq x_2)(y_2 \leq y_1)$, At least one of the following properties holds.*

(i) *There exists $1 \leq i \leq t+1$ such that*

$$(x_1^i \leq x_2^i)(y_1^i \leq y_2^i) \quad \text{or} \quad (x_2^i \leq x_1^i)(y_2^i \leq y_1^i).$$

(ii) $(x_1^{t+1} \leq x_2^{t+1})(y_2^{t+1} \leq y_1^{t+1})$ *if $\{I_i(a)\}_{0 \leq i \leq t}$ is even and $0 \leq \varepsilon < \frac{1}{2}$, or $\{I_i(a)\}_{0 \leq i \leq t}$ is odd and $\frac{1}{2} < \varepsilon \leq 1$.*

(iii) $(x_2^{t+1} \leq x_1^{t+1})(y_1^{t+1} \leq y_2^{t+1})$ *if $\{I_i(a)\}_{0 \leq i \leq t}$ is odd and $0 \leq \varepsilon < \frac{1}{2}$, or $\{I_i(a)\}_{0 \leq i \leq t}$ is even and $\frac{1}{2} < \varepsilon \leq 1$.*

Proof of the lemma: We again use induction. Assume that $t = 0$. If $I_0(a) = L$, then, we have by assumption $(f(x_1) \leq f(x_2))(f(y_2) \leq f(y_1))$ and therefore for $0 \leq \varepsilon < \frac{1}{2}$,

$$x_1^1 - y_1^1 = (1 - 2\varepsilon)(f(x_1) - f(y_1)) < (1 - 2\varepsilon)(f(x_2) - f(y_2)) = x_2^1 - y_2^1.$$

There are three cases: $(x_1^1 \leq x_2^1)(y_1^1 \leq y_2^1)$, $(x_2^1 \leq x_1^1)(y_2^1 \leq y_1^1)$ or $(x_1^1 \leq x_2^1)(y_2^1 \leq y_1^1)$, and the conclusion holds for $t = 0$ in the case $\{I_i(a)\}_{0 \leq i \leq 0}$ is even. If $\varepsilon > \frac{1}{2}$, then the three cases are $(x_1^1 \leq x_2^1)(y_1^1 \leq y_2^1)$, $(x_2^1 \leq x_1^1)(y_2^1 \leq y_1^1)$ and $(x_2^1 \leq x_1^1)(y_2^1 \leq y_1^1)$ and the conclusion also holds. The odd case is proved similarly.

Now, given $t \in \mathbb{N}$, we assume that the statement holds. In case *(i)*, there is nothing to prove. In the cases *(ii)* or *(iii)*, the statements follow from arguments similar to $t = 0$. □

Now, we go back to the proof of the theorem. In the case (c), we apply Lemma 5.7 with $a = f^{2^n}(0)$ and $t = 2^n - 1$. Then, if we are in the case *(i)*, we conclude as in the cases (a) and (b). Otherwise, we are in the case *(iii)* and we repeat the argument. By induction, we are either in the case (a) or (b) after a finite number of steps or we have

$$(x^{t.2^{n+1}} \leq x^{(t+1).2^{n+1}})(y^{(t+1).2^{n+1}} \leq y^{t.2^{n+1}}), \quad t \in \mathbb{N}$$

In this case, the sequences $\{x^{t.2^{n+1}}\}$ and $\{y^{t.2^{n+1}}\}$ also converge and the orbit is asymptotically periodic with a period a factor of 2^{n+1}. The case (d) is proved similarly.

Moreover, by Statement *(iii)* of Theorem 2.1, if a periodic orbit in $C_{\{0,f^{2^n}(0)\}}$ is such that $x^t \neq y^t$, $t \in \mathbb{N}$, then its period must be a multiple of 2^{n+1}.

To see this, assume otherwise that the period is $2^t \cdot m$ where $0 \leq t \leq n$ and m is odd. Then after $2^n \cdot m$ iterations, the point lies on the other side of the diagonal according the statement *(iii)* of Theorem 2.1. But it should lie on the same side on the diagonal because $2^n \cdot m$ is a multiple of the period. We have a contradiction. □

Proof of Theorem 2.5. By assumption we have $I(f^t(0)) = I(f^{t+2^n}(0)) \neq C$, for any $t \geq 1$. It follows from applying Lemma 4.1 repeatedly that

$$F_\varepsilon^{m\cdot 2^n}(C_{\{f^t(0),f^{t+2^n}(0)\}}) \subset C_{\{f^{t+m\cdot 2^n}(0),f^{t+(m+1)\cdot 2^n}(0)\}},\ m \geq 1.$$

By Proposition 5.4, this inclusion also holds for $t = 0$.

But, as said before, 0 is attracted to the periodic orbit of f. It follows that the orbit of any initial condition in $C_{\{f^i(0),f^{i+2^n}(0)\}}$ must be eventually synchronised.

5.3. **Proof of Theorem 2.6.** The main property of a symmetric 2^{k+1}-periodic orbit we use in the proof is that the difference $x^t - y^t$ changes its sign every 2^k iterations. Hence, we start by collecting results from the proof of Theorem 2.1 on the changes of signs of $x^t - y^t$.

Lemma 5.8. *Assume that $I(1) = I^n$ or J^n for some $n \geq 1$. Let $\{(x^t, y^t)\}_{t\in\mathbb{N}}$ be an orbit of F_ε. Assume there exists $0 < j \leq n$ such that*

(i) $(x^{t\cdot 2^j}, y^{t\cdot 2^j}) \in C_{\{f^{2^j}(0),f^{2^j+1}(0)\}} \setminus C_{\{f^{2^j+1}(0),f^{2^j+2}(0)\}}$ for all $t \in \mathbb{N}$, if $j < n-1$,

(ii) $(x^{t\cdot 2^j}, y^{t\cdot 2^j}) \in C_{\{f^{2^j}(0),f^{2^j+1}(0)\}} \setminus C_{\{0,f^{2^j+1}(0)\}}$ for all $t \in \mathbb{N}$, if $j = n-1$,

(iii) $(x, y) \in C_{\{0,f^{2^j}(0)\}}$ if $j = n$.

Then

$$(x^{t\cdot 2^j} - y^{t\cdot 2^j})(x^{(t+1)\cdot 2^j} - y^{(t+1)\cdot 2^j}) \leq 0, \quad t \in \mathbb{N}.$$

In the case where one of the previous conditions is satisfied for $j = 0$ and $n \in \mathbb{N}$, the conclusion still holds if $\varepsilon \leq \frac{1}{2}$. If $\varepsilon \geq \frac{1}{2}$, the conclusion becomes

$$(x^t - y^t)(x^{t+1} - y^{t+1}) \geq 0, \quad t \in \mathbb{N}$$

Indeed, when $0 < j < n-1$, choose a point in $C_{\{f^{2^j}(0), f^{2^{j+1}}(0)\}} \backslash C_{\{f^{2^{j+1}}(0), f^{2^{j+2}}(0)\}}$. To compute its iterates, we consider the partition (9). If the point belongs to $C_{\{0, f^{2^j}(0)\}}$, then $(x-y)(x^{2^j} - y^{2^j}) \leq 0$ by Proposition 5.4. If the point is in one of the rectangles, using Relation (11), we deduce that either $(x-y)(x^{2^j} - y^{2^j}) \leq 0$ or the point belongs to $C_{\{f^{2^{j+1}}(0), f^{2^{j+2}}(0)\}}$ after 2^{j+1} iterations. The assumption of the case *(i)* ensures that the latter does not happen and hence, the sign changes after every 2^j iterations. For $j = n-1$, the arguments are identical. For all other cases, the conclusion follows immediately from Statements *(ii)* and *(iii)* of Theorem 2.1.

We now prove the theorem. Assume that we have a symmetric 2^{k+1}-periodic orbit for some $0 \leq k \leq n-2$. The proof is the same when $k = n-1$ or n. We have

$$(x-y)(x^{2^k} - y^{2^k}) < 0. \tag{13}$$

Consider the sequence of inclusions in Statement *(i)* of Theorem 2.1. We show that for all $0 \leq j \leq k$, there exists $t_j \in \mathbb{N}$ such that $(x^{t_j}, y^{t_j}) \in C_{\{f^{2^j}(0), f^{2^{j+1}}(0)\}}$.

When $j = 0$, it is obvious since the square $C_{\{f(0), f^2(0)\}} = \mathcal{M}$. For $j = 1$, assume otherwise that $(x^t, y^t) \in C_{\{f(0), f^2(0)\}} \setminus C_{\{f^2(0), f^4(0)\}}$ for all $t \in N$. By the lemma, $x^t - y^t$ changes sign after every iterate if $\varepsilon \leq \frac{1}{2}$, or keeps the same sign for all $t \in \mathbb{N}$ if $\varepsilon \geq \frac{1}{2}$. This implies

$$(x-y)(x^{2^k} - y^{2^k}) \geq 0$$

which contradicts to the assumption (13). Thus, there exists t_1 such that $(x^{t_1}, y^{t_1}) \in C_{\{f^2(0), f^4(0)\}}$. By Statement *(ii)* of Theorem 2.1, $(x^{t_1 + t\cdot 2}, y^{t_1 + t\cdot 2}) \in C_{\{f^2(0), f^4(0)\}}$, $t \in \mathbb{N}$. Apply the lemma again, we conclude that there exists $t_2 = t_1 + t \cdot 2$ for some $t \in \mathbb{N}$ such that $(x^{t_2}, y^{t_2}) \in C_{\{f^4(0), f^8(0)\}}$. The argument can be repeated until $j = k$.

Finally, all the orbits in $\mathcal{M}$ for the kneading sequences we are considering have the property that

$$(x-y)(x^{2^{n+1}} - y^{2^{n+1}}) \geq 0,$$

so none can be symmetric 2^{k+1}-periodic for $k > n$. This completes the proof of the theorem. □

Remark. The same arguments can be applied to show that a symmetric 2^{k+1}-periodic orbit cannot pass the square at the next level $C_{\{f^{2^{k+1}}(0), f^{2^{k+2}}(0)\}}$ when

$k < n-1$, or $C_{\{0,f^{2^n}(0)\}}$ when $k = n-1$. Moreover, this property and the statements of Theorem 2.6 hold for any orbit where the sign of $x^t - y^t$ changes for every 2^k iterates.

5.4. **Proof of Theorem 2.7.** Given $(x,y) \in \mathcal{M}$, let $P_1(x,y) = x$ and $P_2(x,y) = y$ denote the projections on each coordinate. Given $k \in \mathbb{N}$, we define the functions

$$h_k(x,y) = P_1 F_\varepsilon^{2^k}(x,y) - y \quad \text{and} \quad g_k(x,y) = P_2 F_\varepsilon^{2^k}(x,y) - x,$$

and we have to prove the existence of a point (x,y), $x \neq y$ such that

$$h_k(x,y) = g_k(x,y) = 0. \tag{14}$$

If $k = n$, then we look for this point in $C_{\{0,f^{2^k}(0)\}}$. When $k < n$, we prove the existence of such point in $C_{\{f^{2^k}(0),f^{2^{k+1}}(0)\}}$. Since $g_k(x,y) = h_k(y,x)$, we concentrate on the analysis of h_k.

For convenience, we assume that k is even. Hence $f^{2^k}(0) > 0$ and $f^{2^{k+1}}(0) < 0$ if $k < n$.

Our first goal is to show that the equations (14) define nice curves. To that purpose, we check the conditions of Implicit Function Theorem.

The forward invariance given by Statement *(ii)* of Theorem 2.1 and the continuity imply the following lemma.

Lemma 5.9. *If $k = n$, then for any $x \in [0, f^{2^k}(0)]$ there exists $y \in [0, f^{2^k}(0)]$ such that $h_k(x,y) = 0$.*

If $k < n$, then for any $x \in [f^{2^{k+1}}(0), f^{2^k}(0)]$ there exists $y \in [f^{2^{k+1}}(0), f^{2^k}(0)]$ such that $h_k(x,y) = 0$.

Lemma 5.10. *(i) If $k = n$, then $\frac{\partial h_k}{\partial y}(x,y) \neq 0$ for $(x,y) \in C_{\{0,f^{2^k}(0)\}}$.*

(ii) If $k < n$, then the partial derivatives of h_k are not simultaneously zero when $h_k(x,y) = 0$ in $C_{\{f^{2^k}(0),f^{2^{k+1}}(0)\}}$.

Proof. *(i)* We have $\frac{\partial h_0}{\partial y}(x,y) = \varepsilon f'(y) - 1 < 0$ on $C_{0,f(0)}$ and the result holds for $k = 0$. Now by the chain rule, we have for $k \geq 1$

$$\frac{\partial h_k}{\partial y}(x,y) = ((1-\varepsilon)f'(x^{2^k-1}), \varepsilon f'(y^{2^k-1})) \prod_{i=1}^{2^k-2} DF_\varepsilon(x^i,y^i) \begin{pmatrix} \varepsilon f'(y) \\ (1-\varepsilon)f'(y) \end{pmatrix} - 1, \tag{15}$$

where $DF_\varepsilon(x,y)$ denotes the Jacobian matrix computed at (x,y). Property (A) tells that the sequence $\{I_i^n\}_{0\leq i\leq 2^k-1}$ is odd. Lemma 4.2 applied with $\{I_i^n\}_{0\leq i\leq 2^k-1}$

shows that the coordinates x^i and y^i, $0 < i < 2^k$ have the same sign for any point in $C_{\{0,f^{2^k}(0)\}}$. Therefore, all the products

$$f'(y)\prod_{i=1}^{2^k-1} f'(z^i), \tag{16}$$

where z^i can be x^i or y^i, are non-positive. Since the product in the expression of $\frac{\partial h_k}{\partial y}(x,y)$ is a linear combination of terms (16) with non-negative coefficients, we conclude that $\frac{\partial h_k}{\partial y}(x,y) \le -1$ on $C_{\{0,f^{2^k}(0)\}}$ and *(i)* is proved.

(ii) Similarly, we have $\frac{\partial h_0}{\partial x}(x,y) = (1-\varepsilon)f'(x)$ and for $k \ge 1$

$$\frac{\partial h_k}{\partial x}(x,y) = ((1-\varepsilon)f'(x^{2^k-1}), \varepsilon f'(y^{2^k-1}))\prod_{i=1}^{2^k-2} DF_\varepsilon(x^i,y^i)\begin{pmatrix}(1-\varepsilon)f'(x)\\ \varepsilon f'(x)\end{pmatrix}. \tag{17}$$

By Property (C), we can repeat the arguments in the proof of *(i)* to obtain that $\frac{\partial h_k}{\partial x}(x,y) = 0$ only if $f'(x) = 0$, i.e. if $x = 0$.

If $x = 0$ and $y \in [0, f^{2^k}(0)]$, then repeating the arguments of *(i)* shows that $\frac{\partial h_k}{\partial y}(x,y) \le -1$.

On the other hand, we have shown in the proof of Statement *(ii)* of Theorem 2.1 that

$$F_\varepsilon^{2^k}(C_{\{0,f^{2^{k+1}}(0)\}}) \subset C_{\{f^{2^k}(0),f^{3\cdot 2^k}(0)\}},$$

and that $0 < f^{3\cdot 2^k}(0) < f^{2^k}(0)$ in the proof of Lemma 5.3. Hence if $x = 0$ and $y \in [f^{2^{k+1}}(0), 0]$, we have $P_1 F_\varepsilon^{2^k}(0,y) > 0$ and $h_k(0,y) > 0$. This completes the proof. □

Remark. The last part of the proof shows no symmetric 2^{k+1}-periodic orbit can pass $C_{\{0,f^{2^{k+1}}(0)\}}$ since the equation $h_k(x,y) = 0$ has no solution in this square.

By Implicit Function Theorem, for any point (x_0,y_0) not on the boundary of the square $C_{\{0,f^{2^k}(0)\}}$ or $C_{\{f^{2^k}(0),f^{2^{k+1}}(0)\}}$ satisfying $h_k(x_0,y_0) = 0$, there is an open neighbourhood of (x_0,y_0) such that its intersection with the set $\{(x,y) : h_k(x,y) = 0\}$ is diffeomorphic to an open interval. If (x_0,y_0) is on the boundary of these squares, the intersection is diffeomorphic to a semi-closed interval.

Lemma 5.11. *Assume the kneading sequence of f is I^n or J^n for some $n \in \mathbb{N}$. Then for any $0 \le k \le n$, f has a unique periodic point x_k^* of period 2^k in the interior of* $[0, f^{2^k}(0)]$.

Proof. Assume for simplicity, that k is even, the odd case can be proved analogously. We have $f^{2^k}(0) > 0$ and $f^{2^{k+1}}(0) < f^{2^k}(0)$. The continuity implies that f has a periodic point in the interior of $[0, f^{2^k}(0)]$ whose period is a factor of 2^k.

Assume that $k \geq 1$. Lemma 3.1 (see also the proof of Relation (10) applied to $k-1$) shows that

$$f^{2^{k-1}}([0, f^{2^k}(0)]) \subset [f^{2^{k-1}}(0), f^{3\cdot 2^{k-1}}(0)].$$

Lemma 5.3 implies that

$$[f^{2^{k-1}}(0), f^{3\cdot 2^{k-1}}(0)] \cap [0, f^{2^k}(0)] = \emptyset$$

Hence, the period of every periodic orbit in $[0, f^{2^k}(0)]$ is a multiple of 2^k. This proves that our periodic point has period 2^k.

Finally the map $f^{2^k}\Big|_{[0,f^{2^k}(0)]}$ is decreasing because the sequence $\{I_i^n\}_{0\leq i\leq 2^k-1}$ is odd (this property still holds when $k < n$ by Property (C)). Consequently, the periodic point is unique. □

Let H_k be the connected component of the curve $h_k(x,y) = 0$ going through (x_k^*, x_k^*).

Lemma 5.12. *The curve H_k joins the line segments $x = 0$ and $x = f^{2^k}(0)$ if $k = n$ and joins the line segments $x = f^{2^{k+1}}(0)$ and $x = f^{2^k}(0)$ if $k < n$.*

Proof. The case $n = k$ follows directly from Implicit Function Theorem since $\frac{\partial h_k}{\partial y} \neq 0$ on $C_{\{0,f^{2^k}(0)\}}$.

For $k < n$, we may assume that $0 < \varepsilon < 1$, the cases when $\varepsilon = 0$ or 1 are trivial.

If $0 < \varepsilon < 1$, then for any $a \leq x, y \leq b$, we have $a \leq P_1 L_\varepsilon(x,y), P_2 L_\varepsilon(x,y) \leq b$ with strict inequality unless $x = y = a$ or $x = y = b$.

Now, by Implicit Function Theorem, the curve H_k extends to the boundary of the square. We know that $f^{2^{k+1}}(0) \leq f^{2^k}(x), f^{2^k}(y) \leq f^{2^k}(0)$ in this square. The previous property implies that

$$f^{2^{k+1}}(0) < P_1 F_\varepsilon^{2^k}(x,y) < f^{2^k}(0),$$

unless $x = y = f^{2^{k+1}}(0)$ or $x = y = f^{2^k}(0)$. It means that the curve can only meet the boundary on the line segments $x = f^{2^{k+1}}(0)$ and $x = f^{2^k}(0)$. By Lemma 5.11 and the remark preceding this lemma, the curves crosses the diagonal of $C_{f^{2^{k+1}}(0), f^{2^k}(0)}$ once at the point (x_k^*, x_k^*). Therefore, H_k joins the two line segments. □

FIGURE 2. The curves H_0 and G_0 when $n \geq 1$ and the condition (2) is satisfied.

By symmetry, G_k, the connected component of the curve $g_k(x,y) = 0$ passing the point (x_k^*, x_k^*) joins the line segments $y = 0$ and $y = f^{2^k}(0)$ if $k = n$ and joins the line segments $y = f^{2^{k+1}}(0)$ and $y = f^{2^k}(0)$ if $k < n$.

The last lemma shows that, under the condition (2), the curves H_k and G_k intersect at (x_k^*, x_k^*) in a way that guarantees there is an intersection off-diagonal (see Figure 2).

Lemma 5.13. *If $(1-2\varepsilon)^{2^k}(f^{2^k})'(x_k^*) < -1$, then, at the point (x_k^*, x_k^*), the slope of the tangent line to H_k is smaller than -1 and the slope of the tangent line to G_k is larger than -1.*

Proof. Formulae (15) and (17) evaluated at (x_k^*, x_k^*) give us

$$\frac{\partial h_k}{\partial x}(x_k^*, x_k^*) = (f^{2^k})'(x_k^*)(1-\varepsilon, \varepsilon)\begin{pmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{pmatrix}^{2^k-2}\begin{pmatrix} 1-\varepsilon \\ \varepsilon \end{pmatrix},$$

and

$$\frac{\partial h_k}{\partial y}(x_k^*, x_k^*) = (f^{2^k})'(x_k^*)(1-\varepsilon, \varepsilon)\begin{pmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{pmatrix}^{2^k-2}\begin{pmatrix} \varepsilon \\ 1-\varepsilon \end{pmatrix} - 1.$$

We have

$$\begin{pmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{pmatrix} = (1-\varepsilon)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and then

$$\begin{pmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{pmatrix}^{2^k} = \sum_{i=0}^{2^{k-1}} C_{2^k}^{2i}(1-\varepsilon)^{2i}\varepsilon^{2^k-2i}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{i=1}^{2^{k-1}} C_{2^k}^{2i-1}(1-\varepsilon)^{2i-1}\varepsilon^{2^k-2i+1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The slope of the tangent lines follow directly from this expression. The slope for H_k is

$$S_k := -\frac{\partial h_k}{\partial x}(x_k^*, x_k^*)/\frac{\partial h_k}{\partial y}(x_k^*, x_k^*)$$

$$= -(f^{2^k})'(x_k^*)\sum_{i=0}^{2^{k-1}} C_{2^k}^{2i}(1-\varepsilon)^{2i}\varepsilon^{2^k-2i}/(f^{2^k})'(x_k^*)\sum_{i=1}^{2^{k-1}} C_{2^k}^{2i-1}(1-\varepsilon)^{2i-1}\varepsilon^{2^k-2i+1} - 1.$$

The slope for G_k is the inverse $1/S_k$. The denominator of S_k being negative, the condition $S_k < -1$ is equivalent to $(1-2\varepsilon)^{2^k}(f^{2^k})'(x_k^*) < -1$. □

The proof of the theorem is complete.

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