Exponentially accurate error estimates of quasiclassical eigenvalues

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Abstract.

We study the behaviour of truncated Rayleigh-Schrödinger series for the low-lying eigenvalues of the one-dimensional, time-independent Schrödinger equation, in the semiclassical limit $\hbar \to 0$. Under certain hypotheses on the potential $V(x)$, we prove that for any given small $\hbar > 0$ there is an optimal truncation of the series for the approximate eigenvalues, such that the difference between an approximate and exact eigenvalue is smaller than $\exp(-C/\hbar)$ for some positive constant C. We also prove the analogous results concerning the eigenfunctions.

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1. Introduction

Let us consider the time-independent Schrödinger equation

$$\left[-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V(x)\right]\tilde{\Psi}(x)=E\tilde{\Psi}(x) \tag{1}$$

where the potential energy $V(x)$ satisfies the following assumptions:

(H1) Let $V(x)$ be a C^∞ real function on $-\infty<x<\infty$, with a unique global minimum $V(0)=0$ at $x=0$, $V''(0)=1$, and $\liminf_{|x|\to\infty}V(x)>0$.

This hypothesis is sufficient to ensure the existence of discrete eigenvalues of (1) below the continuous spectrum for small $\hbar$, as several authors have proved using different approaches [1, 6, 5]. Also, the eigenvalues and eigenvectors have asymptotic expansions in powers of $\hbar$ given by formal Rayleigh-Schrödinger (R-S) series. The papers [1, 5] deal with the one dimensional case, whereas [6] extends this result to several dimensions.

As far as we know there are no results concerning the behaviour of truncations of these series, which should provide approximate solutions of equation (1). The main goal of this paper is to prove that for sufficiently small values of $\hbar$ there is an optimal truncation of approximate eigenvalues, such that the difference between an approximate and exact eigenvalue is smaller than $\exp(-\text{constant}/\hbar)$ for some positive constant. We also prove the analogous results concerning the eigenfunctions. A more precise statement of these assertions is the following: Suppose E_N and $\tilde{\Psi}_N$ are the N^{th} approximations (given by the R-S series) of E_k and $\tilde{\Psi}_k$, the exact k^{th} eigenvalue and eigenfunction of (1). Let $B=B(k)>0$ be a constant to be defined later. Then

Theorem. For each $0<g<B^{-2}$, there exists $0<\hbar_g$, such that for each $\hbar\le\hbar_g$, there is $N(\hbar)$, so that

$$\left|E_{N(\hbar)}(\hbar)-E_k(\hbar)\right| \;\le\; \Lambda\exp\left(-\frac{\Gamma}{\hbar}\right)$$

for some $\Gamma>0$ and $\Lambda>0$.

Theorem. If g, $\hbar_g$, $N(\hbar)$, Γ and Λ are defined as in the theorem above, then

$$\left\|\tilde{\Psi}_{N(\hbar)}(\hbar;x)-\tilde{\Psi}_k(\hbar;x)\right\| \;\le\; 8\,\Lambda\,\exp\left(-\frac{\Gamma}{\hbar}\right).$$

Our technique is closely related to one developed in [3]. Basically we calculate upper bounds for each term in the R-S series for both eigenvalues and eigenfuctions. Then we combine these to obtain a recursion relation that yields an estimate for the growth of these terms. From that we compute an estimate of the difference of the two sides of (1) after truncation at order N; this estimate behaves like $ab^N\hbar^{\frac{N}{2}}(N!)^{\frac{1}{2}}$. For each $\hbar$ we choose N to minimize this quantity. This and some standard results of functional analysis yield our results. A critical part of this procedure makes use of control on decay of the harmonic oscillator eigenfunctions and some assumptions about analiticity of the potential $V(x)$. Results about the spacing of eigenvalues for small $\hbar$ are also crucial for our results about the eigenfunctions.

Hypothesis (H1) contains the requirement of uniqueness of the global minimum of $V(x)$. If $V(x)$ has multiple wells with the same minimum, our technique approximate energies that are exponentially close to the spectrum, but the states we construct may not be good approximations to eigenfunctions. It is well known that when $V(x)$ has multiple wells, tunneling plays a central role. In these systems there are subsets of eigenvalues that differ from one another by an exponetially small function of $\hbar$, and the eigenfuctions can be large in more than one well [7, 4]. For that reason our technique, as presented here, only works if the potential energy function has a unique well. In fact, the assumption that there is a unique well is not required in [1, 6] (but is in [5]) in order to prove existence of eigenvalues as mentioned above.

This paper is organized as follows. In Section 2 we make a transformation of equation (1) that allows us to obtain a manageable recursion relation for the n^{th} term of the R-S series. In Section 3 we state and prove an estimate of the growth of these terms. We define a residual error function for equation (1) and prove an estimate for it. The main results are stated precisely in Section 4. The Appendix is devoted to a boring computation needed in Section 3.

We only analize the one dimensional problem. The multidimensional case will be discussed elsewhere.

2. Preliminaries

As was already mentioned, we need a further assumption about $V(x)$:

(H2) Suppose that $V(x)$ has an analytic extension to a neighborhood of the region $S_\delta = \{z : |\mathrm{Im} z| \leq \delta\}$ and satisfies $|V(z)| \leq M \exp(\tau |z^2|)$ when $z \in S_\delta$, for some positive constants $M > 0$ and $\frac{1}{4} > \tau > 0$.

This hypothesis allows us to control the behaviour of derivatives of $V(x)$ far from the potential well.

It is convenient to transform equation (1). We scale $x \to \hbar^{\frac{1}{2}} x$ and then divide by $\hbar$. This yields

$$\left[-\frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(\hbar; x) \right] \tilde{\Psi}(\hbar; x) = E(\hbar) \tilde{\Psi}(\hbar; x) \tag{2}$$

(H1) implies that we may write $V(\hbar; x) = \frac{1}{2} x^2 + W(\hbar; x)$ with obvious definition of $W(\hbar; x)$. Now let us write (2) as

$$[H_0 + W(\hbar; x)] \, \tilde{\Psi}(\hbar; x) = E(\hbar) \tilde{\Psi}(\hbar; x) \tag{3}$$

where $H_0 = -\dfrac{1}{2} \dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + \dfrac{1}{2} x^2$ is the harmonic oscillator hamiltonian with eigenvalues $e_k = k + 1/2$ and eigenstates ϕ_k, for integer $k \geq 0$. Because $V(x)$ satisfies (H1), a theorem [1, 6, 5] allows us to assert that for each nonnegative integer k, there exists $\hbar_0$ such that there are at least $k+1$ eigenvalues of (1) for all $\hbar \leq \hbar_0$. Moreover,

$$\lim_{\hbar \to 0} E_l(\hbar) = e_l$$

for $0 \leq l \leq k+1$. Also, each $E_l(\hbar)$ has an asymptotic expansion in powers of $\hbar^{1/2}$ around e_l. The analogous statement holds for eigenfunctions.

The function $W(\hbar; x)$ can be asymptotically approximated by its Taylor series at any order n

$$W(\hbar; x) = \sum_{l=3}^{n} \hbar^{\frac{l-2}{2}} c_l x^l + \mathrm{O}\left(\hbar^{\frac{n-1}{2}} x^{n+1}\right), \tag{4}$$

where $c_l = \frac{1}{l!} V^{[l]}(0)$.

Because we are interested in the semiclassical limit (i.e. $\hbar \to 0$), we may consider $W(\hbar, x)$ as a perturbation of H_0 and proceed with standard perturbation theory. That is, we write down formal R-S series for $\tilde{\psi}$ and E around the k^{th} eigenstate/eigenvalue of H_0

$$\tilde{\Psi}(x) = \phi_k(x) + \hbar^{\frac{1}{2}} \tilde{\psi}_1(x) + \hbar^{\frac{2}{2}} \tilde{\psi}_2(x) + \hbar^{\frac{3}{2}} \tilde{\psi}_3(x) + \hbar^{\frac{4}{2}} \tilde{\psi}_4(x) + \ldots \tag{5}$$

$$E(\hbar) = e_k + \hbar^{\frac{1}{2}} \mathcal{E}_1 + \hbar^{\frac{2}{2}} \mathcal{E}_2 + \hbar^{\frac{3}{2}} \mathcal{E}_3 + \hbar^{\frac{4}{2}} \mathcal{E}_4 + \ldots, \tag{6}$$

substitute these into (3), equate powers of $\hbar^{\frac{1}{2}}$, etc. Before doing so, to simplify some of the estimates, we transform (3) in the following way. We define a new operator A_k by $A_k \phi_k = \phi_k$, $A_k \phi_j = |k-j|^{-\frac{1}{2}} \phi_j$ for $j \neq k$, and extend it to all of the underlying Hilbert space $\mathcal{H}$. Then A_k maps $\mathcal{H}$ onto the quadratic form domain of H_0. Because $\mathcal{D}(H_0 + W(\hbar; x))$ is a subset of $\mathcal{D}(H_0)$, for each $\tilde{\varphi} \in \mathcal{D}(H_0 + W(\hbar; x))$ there is $\varphi \in \mathcal{H}$ such that $\tilde{\varphi} = A_k \varphi$. We define $H_k = A_k(H_0 - e_k)A_k$. This operator satisfies $H_k \phi_j = -\phi_j$ for $j < k$, $H_k \phi_k = 0$, and $H_k \phi_j = \phi_j$ for $j > k$. With these operators, (3) can be rewritten as

$$\left[H_k + A_k W(\hbar; x) A_k\right] \Psi(x) = \left(E(\hbar) - e_k\right) (A_k)^2 \Psi(x) \tag{7}$$

and instead of (5) we have

$$\Psi(x) = \phi_k(x) + \hbar^{\frac{1}{2}} \psi_1(x) + \hbar^{\frac{2}{2}} \psi_2(x) + \hbar^{\frac{3}{2}} \psi_3(x) + \hbar^{\frac{4}{2}} \psi_4(x) + \ldots \tag{8}$$

Now let us equate powers of $\hbar^{\frac{1}{2}}$. After replacing $W(\hbar; x)$ by its Taylor series, we obtain, for $n = 1, 2, \ldots$,

$$H_k \psi_n + \sum_{l=1}^{n} c_{l+2} A_k x^{l+2} A_k \psi_{n-l} = \sum_{l=1}^{n} \mathcal{E}_l (A_k)^2 \psi_{n-l} \tag{9}$$

where $\psi_0 = \phi_k$.

For the moment we do not require that $\Psi(x)$ be normalized. Instead we choose each ψ_j to be orthogonal to ϕ_k. This does not affect the eigenvalues. Using the selfadjointness of $A_k x^l A_k$, we see that (9) requires

$$\mathcal{E}_n = \sum_{l=1}^{n} c_{l+2} \left\langle A_k x^{l+2} A_k \phi_k, \psi_{n-l} \right\rangle \tag{10}$$

$$\psi_n = (H_k)_r^{-1} \left(\sum_{l=1}^{n} \mathcal{E}_l (A_k)^2 \psi_{n-l} - \sum_{l=1}^{n} c_{l+2} A_k x^{l+2} A_k \psi_{n-l} \right) \tag{11}$$

where $(H_k)_r^{-1}$ is the inverse of the restriction of H_k to the subspace orthogonal to ϕ_k. If we define $P_{j\le m}$ to be the projection onto the subspace spanned by $\{ \phi_j \,:\, j \le m \}$ then an easy induction argument applied to (11) shows that $\psi_l \in \mathrm{Ran}(P_{j\le k+3l})$. Then, from (10) and (11) we immediately obtain the following inequalities

$$|\mathcal{E}_n| \;\le\; \sum_{l=1}^{n} |c_{l+2}| \left\| A_k x^{l+2} A_k P_{j\le k} \right\| \|\psi_{n-l}\| \tag{12}$$

$$\|\psi_n\| \le \sum_{l=1}^{n} |\mathcal{E}_l| \|\psi_{n-l}\| + \sum_{l=1}^{n} |c_{l+2}| \left\| A_k x^{l+2} A_k P_{j\le k+3(n-l)} \right\| \|\psi_{n-l}\| \tag{13}$$

To obtain explicit bounds from these, we use the following result

Lemma 1. For $m \ge 1$ and $n \ge 0$,

$$\|A_k x^m A_k P_{j\le n}\| \le 2^{\frac{m}{2}} (2+k) \left[\frac{(n+m-1)!}{(n+1)!} \right]^{\frac{1}{2}}$$

Proof: Since $\mathrm{Ran}(x A_k P_{j\le n}) \subset \mathrm{Ran}(P_{j\le n+1})$ we have $x A_k P_{j\le n} = P_{j\le n+1} x A_k P_{j\le n}$. Thus,

$$\|A_k x^m P_{j\le n}\| \le \|A_k x\| \left\| x^{m-2} P_{j\le n+1} \right\| \|x A_k\|$$

Let $\{\phi_i\}_{i=0}^{\infty}$ be the basis of eigenfunctions of H_0. Then any $\varphi \in \mathcal{H}$ can be written as $\varphi = \sum_{i=0}^{\infty} d_i \phi_i$. An easy calculation shows that

$$\begin{aligned} A_k x \varphi = {} & \frac{1}{\sqrt{2}} \left(\sum_{i=0}^{k-1} d_{i+1} \sqrt{\frac{i+1}{k-i}} \phi_i + d_{k+1} \sqrt{k+1} \phi_k + \sum_{i=k+1}^{\infty} d_{i+1} \sqrt{\frac{i+1}{i-k}} \phi_i \right) \\ & + \frac{1}{\sqrt{2}} \left(\sum_{i=1}^{k-1} d_{i-1} \sqrt{\frac{i}{k-i}} \phi_i + d_{k-1} \sqrt{k} \phi_k + \sum_{i=k+1}^{\infty} d_{i-1} \sqrt{\frac{i}{i-k}} \phi_i \right) \end{aligned}$$

so we can write $A_k x \varphi = \varphi_1 + \varphi_2$ where $\varphi_{1,2}$ are defined in the obvious way. Clearly, $\|A_k x \varphi\| \le \|\varphi_1\| + \|\varphi_2\|$. Now

$$\begin{aligned} \|\varphi_1\|^2 &= \frac{1}{2} \left(\sum_{i=0}^{k-1} |d_{i+1}|^2 \frac{i+1}{k-i} + |d_{k+1}|^2 (k+1) + \sum_{i=k+1}^{\infty} |d_{i+1}|^2 \frac{i+1}{i-k} \right) \\ &\le \frac{1}{2} \left(k \sum_{i=0}^{k-1} |d_{i+1}|^2 + (k+1)|d_{k+1}|^2 + (k+2) \sum_{i=k+1}^{\infty} |d_{i+1}|^2 \right) \\ &\le \frac{1}{2} (k+2) \|\varphi\|^2 \end{aligned}$$

and a similar argument yields

$$\|\varphi_2\|^2 \le \frac{1}{2} (k+1) \|\varphi\|^2$$

Hence,

$$\|A_k x\| \leq \sqrt{\frac{k+1}{2}} + \sqrt{\frac{k+2}{2}} \leq \sqrt{2(2+k)}$$

By taking the adjoint of previous estimate we have

$$\|x A_k\| \leq \sqrt{2(2+k)}$$

Finally, we make use of Lemma 5.1 of [1] to complete the proof. □

In (H2) we assume that $V(x)$ has an analytic extension, so (4) has a non-zero radius of convergence. Thus, there is a constant $c > 0$ such that $|c_l| \leq c$. Therefore, we may obtain nicer bounds from both (12) and (13), namely,

$$|\mathcal{E}_n| \;\leq 2c(2+k) \sum_{l=1}^{n} 2^{\frac{l}{2}} \left[\frac{(1+k+l)!}{(1+k)!}\right]^{\frac{1}{2}} \|\psi_{n-l}\| \tag{14}$$

$$\|\psi_n\| \leq \sum_{l=1}^{n} |\mathcal{E}_l| \|\psi_{n-l}\| + 2c(2+k) \sum_{l=1}^{n} 2^{\frac{l}{2}} \left[\frac{(1+k+3n-2l)!}{(1+k+3n-3l)!}\right]^{\frac{1}{2}} \|\psi_{n-l}\| \tag{15}$$

Finally, substituting (14) into (15), we obtain

$$\begin{aligned} \|\psi_n\| \leq 2c(2+k) \sum_{l=1}^{n} \sum_{i=1}^{l} 2^{\frac{i}{2}} \left[\frac{(1+k+i)!}{(1+k)!}\right]^{\frac{1}{2}} \|\psi_{l-i}\| \|\psi_{n-l}\| \\ + 2c(2+k) \sum_{l=1}^{n} 2^{\frac{l}{2}} \left[\frac{(1+k+3n-2l)!}{(1+k+3n-3l)!}\right]^{\frac{1}{2}} \|\psi_{n-l}\| \end{aligned} \tag{16}$$

We prove below that the second term in (16) behaves roughly like $a\sqrt{1+k+n}\,\|\psi_{n-1}\|$, and that this is the dominant term in (16). So, we expect the growth of $\|\psi_n\|$ (and hence $|\mathcal{E}_n|$) to be dominated by something like $a^n\sqrt{n!}$.

3. The main estimates

From (16) we can derive a bound for the growth of $\|\psi_n\|$ for large n. To do so, we may assume without loss that $c \geq 1$. In addition, we need two technical results that are summarized in the following lemma.

Lemma 2.

(i) For each $k \geq 0$ there is a constant γ_k so that, for all $m \geq 0$,

$$\sum_{l=0}^{m} \left[\frac{(1+k+m-l)!(1+k+l)!}{(1+k+m)!}\right]^{\frac{1}{2}} \leq \gamma_k$$

(ii) For all $k \geq -1$ there is a constant β so that, for all $m \geq 0$,

$$\sum_{l=0}^{m} 2^{-\frac{5l}{2}} \left[\frac{(1+k+3m-2l)!(1+k+m-l)!}{(1+k+3m-3l)!(1+k+m)!}\right]^{\frac{1}{2}} \leq \beta$$

Proof: (i).

$$\sum_{l=0}^{m}\left[\frac{(1+k+m-l)!(1+k+l)!}{(1+k+m)!}\right]^{\frac{1}{2}}$$

$$\begin{aligned}&= 2[(1+k)!]^{\frac{1}{2}} + 2\left[\frac{(2+k)!}{1+k+m}\right]^{\frac{1}{2}} + \sum_{l=2}^{m-2}\left[\frac{(1+k+m-l)!(1+k+l)!}{(1+k+m)!}\right]^{\frac{1}{2}} \\ &\le 2[(1+k)!]^{\frac{1}{2}} + 2\left[\frac{(2+k)!}{1+k}\right]^{\frac{1}{2}} \\ &\quad + (m-4)\max_{2\le l\le [\![\frac{m}{2}]\!]}\left[\frac{(1+k+m-l)!(1+k+l)!}{(1+k+m)!}\right]^{\frac{1}{2}},\end{aligned}$$

where $[\![\alpha]\!]$ denotes the greatest integer less than or equal to α. If $l \le [\![\frac{m}{2}]\!] - 1$, then $2l+1 \le m$. Thus, $(1+k+m-l)!(1+k+l)!$ is decreasing for $l \le [\![\frac{m}{2}]\!]$. Therefore,

$$\sum_{l=0}^{m}\left[\frac{(1+k+m-l)!(1+k+l)!}{(1+k+m)!}\right]^{\frac{1}{2}} \le 2[(1+k)!]^{\frac{1}{2}} + 2\left[\frac{(2+k)!}{1+k}\right]^{\frac{1}{2}} + [(3+k)!]^{\frac{1}{2}}\frac{m-4}{m+k}$$

The last term converges as $m \to \infty$, so existence of the constant γ_k is guaranteed.

(ii). By cancelling common factors, we have

$$\sum_{l=0}^{m} 2^{-\frac{5l}{2}}\prod_{s=1}^{l}\left(\frac{1+k+3m-3l+s}{1+k+m-l+s}\right)^{\frac{1}{2}}$$

For $k \ge -1$ and $s \ge 0$, we have $0 \le 2(1+k+s)$. This implies

$$\frac{1+k+3m-3l+s}{1+k+m-l+s} \le 3$$

Therefore,

$$\sum_{l=0}^{m} 2^{-\frac{5l}{2}}\left[\frac{(1+k+3m-2l)!(1+k+m-l)!}{(1+k+3m-3l)!(1+k+m)!}\right]^{\frac{1}{2}} \le \sum_{l=0}^{m} 2^{-\frac{5l}{2}}3^{\frac{l}{2}}$$

and the right hand side converges to $\beta = \left(1-\sqrt{3/2^5}\right)^{-1}$. □

Theorem 1. For each $k \ge 0$ and for $n \ge 1$,

$$\|\psi_n\| \le 2^{3n}\left[2c(2+k)\left(\gamma_k^2[(1+k)!]^{-\frac{1}{2}} + \beta\right)\right]^n [(1+k+n)!]^{\frac{1}{2}}.$$

Proof: Fix $k \ge 0$. For $n = 1$ the inequality (16) becomes

$$\|\psi_1\| \le 2^2 c(2+k)2^{\frac{1}{2}}(2+k)^{\frac{1}{2}}.$$

From the proof of Lemma 2 it follows that $\gamma_k^2[(1+k)!]^{-\frac{1}{2}} + \beta \ge 1$. Thus,

$$2^2c(2+k)2^{\frac{1}{2}}(2+k)^{\frac{1}{2}} \le 2^3 2c(2+k)(\gamma_k^2[(1+k)!]^{-\frac{1}{2}} + \beta)\,[(2+k)!]^{\frac{1}{2}},$$

so the assertion is true for $n = 1$ and for each $k \ge 0$.

Now assume the inequality is satisfied for all $i < n$. Let $b = 2c(2+k)\left(\gamma_k^2[(1+k)!]^{-\frac{1}{2}} + \beta\right)$. As we have pointed out above, we may assume that $c \ge 1$.

We already know that $\gamma_k^2[(1+k)!]^{-\frac{1}{2}}+\beta \geq 1$, so we conclude that $b \geq 1$. From (16) it follows that

$$
\begin{aligned}
\|\psi_n\| \leq\ & 2c(2+k) \sum_{l=1}^{n} \sum_{i=1}^{l} 2^{\frac{i}{2}} \left[\frac{(1+k+i)!}{(1+k)!}\right]^{\frac{1}{2}} 2^{3(n-i)} b^{n-i} [(1+k+n-l)!(1+k+l-i)!]^{\frac{1}{2}} \\
& + 2c(2+k) \sum_{l=1}^{n} 2^{\frac{l}{2}} \left[\frac{(1+k+3n-2l)!}{(1+k+3n-3l)!}\right]^{\frac{1}{2}} 2^{3(n-l)} b^{n-l} [(1+k+n-l)!]^{\frac{1}{2}} \\
\leq\ & 2c(2+k) 2^{3n} b^{n-1} \sum_{l=1}^{n} \sum_{i=1}^{l} 2^{-\frac{5i}{2}} \left[\frac{(1+k+i)!}{(1+k)!}\right]^{\frac{1}{2}} [(1+k+n-l)!(1+k+l-i)!]^{\frac{1}{2}} \\
& + 2c(2+k) 2^{3n} b^{n-1} \sum_{l=1}^{n} 2^{-\frac{5l}{2}} \left[\frac{(1+k+3n-2l)!}{(1+k+3n-3l)!}\right]^{\frac{1}{2}} [(1+k+n-l)!]^{\frac{1}{2}} \\
\leq\ & 2c(2+k) 2^{3n} b^{n-1} \left[\frac{(1+k+n)!}{(1+k)!}\right]^{\frac{1}{2}} \sum_{l=1}^{n} \left[\frac{(1+k+n-l)!(1+k+l)!}{(1+k+n)!}\right]^{\frac{1}{2}} \\
& \times \sum_{i=1}^{l} \left[\frac{(1+k+l-i)!(1+k+i)!}{(1+k+l)!}\right]^{\frac{1}{2}} + 2c(2+k) 2^{3n} b^{n-1} [(1+k+n)!]^{\frac{1}{2}} \\
& \times \sum_{l=1}^{n} 2^{-\frac{5l}{2}} \left[\frac{(1+k+3n-2l)!(1+k+n-l)!}{(1+k+3n-3l)!(1+k+n)!}\right]^{\frac{1}{2}}
\end{aligned}
$$

We now use Lemma 2 to obtain

$$
\begin{aligned}
\|\psi_n\| &\leq \left[2c(2+k)\left(\gamma_k^2\,[(1+k)!]^{-\frac{1}{2}}+\beta\right)\right] 2^{3n} b^{n-1}\,[(1+k+n)!]^{\frac{1}{2}} \\
&= 2^{3n} b^{n}\,[(1+k+n)!]^{\frac{1}{2}}
\end{aligned}
$$

□

The next step is to bound the error made by taking a finite number of terms in both series (8) and (6). We use the following approach, which follows the ideas of [3]. For $N \geq 1$ define

$$
E_N = e_k + \sum_{n=1}^{N-1} \hbar^{\frac{n}{2}} \mathcal{E}_n \qquad \Psi_N(x) = \phi_k(x) + \sum_{n=1}^{N-1} \hbar^{\frac{n}{2}} \psi_n(x) \tag{17}
$$

These are the truncation at order N of the R-S series. We define

$$
\begin{aligned}
\xi_N(x) &= A_k(H_0 + W(\hbar;x) - E_N) A_k \Psi_N(x) \\
&= \left[H_k + A_k W(\hbar;x) A_k - \sum_{j=1}^{N-1} \hbar^{\frac{j}{2}} \mathcal{E}_j (A_k)^2\right] \sum_{m=0}^{N-1} \hbar^{\frac{m}{2}} \psi_m(x)
\end{aligned} \tag{18}
$$

This function is the error in the time-independent Schrödinger equation due to truncation, at order N, of both (8) and (6). There is plethora of cancellations in (18), that are described in the Appendix. After making these cancellations, we have

$$
\xi_N(x) = \sum_{n=0}^{N-1} \hbar^{\frac{n}{2}} A_k W^{[N+1-n]}(\hbar;x) A_k \psi_n(x) - \sum_{n=N}^{2N-2} \hbar^{\frac{n}{2}} \sum_{l=n-N+1}^{N-1} \mathcal{E}_l (A_k)^2 \psi_{n-l}(x)
$$

where $W^{[j]}(\hbar;x) = V(\hbar;x) - \sum_{l=0}^{j} \hbar^{\frac{l-2}{2}} c_l x^l$ is the Taylor series error (at order j) of $V(\hbar;x)$. The Taylor theorem says that

$$W^{[j]}(\hbar;x) = \hbar^{\frac{j-1}{2}} \frac{V^{[j+1]}(\eta)}{(j+1)!} x^{j+1}$$

for some $\eta \in (-|x|,|x|)$. From this, we obtain

$$\begin{aligned}\xi_N(x) = \hbar^{\frac{N}{2}} \sum_{n=0}^{N-1} A_k \frac{V^{[N+2-n]}(\eta_n)}{(N+2-n)!} x^{N+2-n} A_k \psi_n(x) \\ - \sum_{n=N}^{2N-2} \hbar^{\frac{n}{2}} \sum_{l=n-N+1}^{N-1} \mathcal{E}_l (A_k)^2 \psi_{n-l}(x) \qquad (19)\end{aligned}$$

with $\eta_n \in (-|x|,|x|)$, $n = 0,\ldots,N-1$.

Our main result relies on an estimate of $\|(H - E_N)A_k\Psi_N(x)\|$, so we need an upper bound of the L^2-norm of $A_k^{-1}\xi_N(x)$. Note that A_k^{-1} is an unbounded operator, but that $\xi_N(x)$ is clearly in its domain.

Proposition 1. For each $k \geq 0$, there exist positive constants A, B and N_0 such that

$$\left\| A_k^{-1}\xi_N(x) \right\| \leq \sum_{n=N}^{2N} AB^n \hbar^{\frac{n}{2}} \left[(2+k+n)!\right]^{\frac{1}{2}}$$

whenever $N \geq N_0$ and $\hbar \leq 1$.

Proof: We separately prove estimates inside and outside a closed interval $[-R,R]$, where $R = \sqrt{2k+6N+3}$. Let χ_R be the characteristic function defined by $\chi_R(x) = 1$ if $x \in [-R,R]$, and $\chi_R(x) = 0$ otherwise. From (19) it follows that

$$\begin{aligned}\left\| A_k^{-1}\xi_N(x) \right\| &\leq \hbar^{\frac{N}{2}} \sum_{n=0}^{N-1} \left\| \frac{V^{[N+2-n]}(\eta_n)}{(N+2-n)!} x^{N+2-n} A_k \psi_n(x) \right\| \\ &\quad + \sum_{n=N}^{2N-2} \hbar^{\frac{n}{2}} \sum_{l=n-N+1}^{N-1} |\mathcal{E}_l| \, \|\psi_{n-l}(x)\| \\ &\leq \hbar^{\frac{N}{2}} \sum_{n=0}^{N-1} \left\| \frac{V^{[N+2-n]}(\eta_n)}{(N+2-n)!} x^{N+2-n} \left(1 - \chi_R(x)\right) A_k \psi_n(x) \right\| \\ &\quad + \hbar^{\frac{N}{2}} \sum_{n=0}^{N-1} \left\| \frac{V^{[N+2-n]}(\eta_n)}{(N+2-n)!} x^{N+2-n} \chi_R(x) A_k \psi_n(x) \right\| \\ &\quad + \sum_{n=N}^{2N-2} \hbar^{\frac{n}{2}} \sum_{l=n-N+1}^{N-1} |\mathcal{E}_l| \, \|\psi_{n-l}(x)\| \qquad (20)\end{aligned}$$

where we have used that $\hbar \leq 1$ implies $\hbar^{\frac{n}{2}} \leq \hbar^{\frac{N}{2}}$ for $n \geq N$.

To estimate the first term of (20) we need to control the Taylor series error outside $[-R,R]$. An easy argument using the Cauchy integral formula (and the hypothesis on V; see [2, 3]) shows that for $\eta \in (-|x|,|x|)$ and for all $n \geq 0$,

$$\delta^n \frac{\left|V^{[n]}(\eta)\right|}{n!} \leq M \exp\left[\tau(|x|+\delta)^2\right] \leq M \exp(2\tau\delta^2)\exp(2\tau x^2) =: K \exp(2\tau x^2) \qquad (21)$$

Later we will assume that $\delta \leq 1$. Also, we recall that the eigenfunctions of the harmonic oscillator H_0 are $\phi_k(x) = \pi^{-\frac{1}{4}}(2^k k!)^{-\frac{1}{2}} \exp(-\frac{1}{2}x^2) H_n(x)$, where $H_k(x)$ are the Hermite polynomials.

We have the following estimate

Lemma 3. For $n = 0, 1, \ldots, N-1$ and for $R = \sqrt{2k + 6N + 3}$,

$$\left\| \delta^{N+2-n} \frac{V^{[N+2-n]}(\eta_n)}{(N+2-n)!} x^{N+1-n} \left[1 - \chi_R(x)\right] P_{j \leq k+3n+1} \right\| \leq \frac{K 2^{\frac{2+k+3n}{2}} \left[(2+k+N+2n)!\right]^{\frac{1}{2}}}{\pi^{\frac{1}{4}} (1-4\tau)^{\frac{3+k+N+2n}{2}} \left[(k+3n)!\right]^{\frac{1}{2}}}$$

Proof of Lemma 3: Consider any $\phi_i(x)$ with $i = 0, 1, \ldots, k+3n+1$. We have

$$\begin{aligned} &\left\| \delta^{N+2-n} \frac{V^{[N+2-n]}(\eta_n)}{(N+2-n)!} x^{N+1-n} \left[1 - \chi_R(x)\right] \phi_i(x) \right\|^2 \\ &\qquad \leq K^2 \int_{I\!R \setminus [-R,R]} e^{4\tau x^2} x^{2(N+1-n)} \left|\phi_i(x)\right|^2 dx \\ &\qquad \leq \frac{2K^2}{\pi^{\frac{1}{2}} 2^i i!} \int_R^\infty e^{-(1-4\tau)x^2} x^{2(N+1-n)} \left|H_i(x)\right|^2 dx \end{aligned}$$

Because $R \geq \sqrt{2i+1}$ we may make use of Lemma 3.1 of [2] to estimate the last integral. This lemma states that $|H_i(x)| \leq 2^i |x|^i$ whenever $x \geq \sqrt{2i+1}$. Thus,

$$\begin{aligned} &\left\| \delta^{N+2-n} \frac{V^{[N+2-n]}(\eta_n)}{(N+2-n)!} x^{N+1-n} \left[1 - \chi_R(x)\right] \phi_i(x) \right\|^2 \\ &\qquad \leq \frac{2^{i+1} K^2}{\pi^{\frac{1}{2}} i!} \int_R^\infty e^{-(1-4\tau)x^2} x^{2(N+1-n+i)} dx \\ &\qquad \leq \frac{2^{i+1} K^2}{\pi^{\frac{1}{2}} i!} \int_0^\infty e^{-(1-4\tau)x^2} x^{2(N+1-n+i)} dx \\ &\qquad = \frac{2^i K^2 \Gamma\left(N+1-n+i+\frac{1}{2}\right)}{\pi^{\frac{1}{2}} i! (1-4\tau)^{N+1-n+i+\frac{1}{2}}} \\ &\qquad \leq \frac{2^i K^2 (N+1-n+i)!}{\pi^{\frac{1}{2}} i! (1-4\tau)^{N+1-n+i+\frac{1}{2}}} \end{aligned}$$

To obtain last inequality here, we have used the fact that $\Gamma(x)$ is an increasing function when $x \geq 2$. Using the Schwarz inequality, we then conclude that

$$\begin{aligned} &\left\| \delta^{N+2-n} \frac{V^{[N+2-n]}(\eta_n)}{(N+2-n)!} x^{N+1-n} \left[1 - \chi_R(x)\right] P_{j \leq k+3n+1} \right\| \\ &\qquad \leq \left[\sum_{i=0}^{1+k+3n} \frac{2^i K^2 (N+1-n+i)!}{\pi^{\frac{1}{2}} i! (1-4\tau)^{N-n+i+1}} \right]^{\frac{1}{2}} \end{aligned}$$

Note that the terms on the right hand side are increasing in i. The sum is bounded by the number of terms times the $(2+k+3n)$ times the largest term. The lemma follows because $(2+k+3n)/(1+k+3n) \leq 2$. $\square$

From the proof of Lemma 1 we know that $\|xA_k\| \leq \sqrt{2(2+k)}$. Also, we are assuming that $\delta \leq 1$. Thus, the first term of (20) satisfies

$$\begin{aligned}
1^{\text{st}} \text{ term} &\leq \hbar^{\frac{N}{2}}\delta^{-N-2} \\
&\quad\times \sum_{n=0}^{N-1} \left\| \delta^{N+2-n}\frac{V^{[N+2-n]}(\eta_n)}{(N+2-n)!}x^{N+1-n}\left[1-\chi_R(x)\right]P_{j\leq k+3n+1}\right\| \|xA_k\|\,\|\psi_n\| \\
&\leq \sqrt{2(2+k)}\hbar^{\frac{N}{2}}\delta^{-N-2} \\
&\quad\times \sum_{n=0}^{N-1} \left\| \delta^{N+2-n}\frac{V^{[N+2-n]}(\eta_n)}{(N+2-n)!}x^{N+1-n}\left[1-\chi_R(x)\right]P_{j\leq k+3n+1}\right\| \|\psi_n\|
\end{aligned}$$

We use Lemma 3, the estimate of $\|\psi_n(x)\|$ of Theorem 1, and arrange factorials to obtain

$$\begin{aligned}
1^{\text{st}} \text{ term} &\leq \sqrt{2(2+k)}2^{\frac{2+k}{2}}\hbar^{\frac{N}{2}}\delta^{-N-2}K\pi^{-\frac{1}{4}}(1-4\tau)^{-\frac{k+3N-1}{2}}b^{N-1} \\
&\quad\times \max_{0\leq n\leq N-1}\left[\frac{(1+k+3n)(2+k+3n)}{2+k+n}\right]^{\frac{1}{2}} \\
&\quad\times \sum_{n=0}^{N-1} 2^{\frac{9n}{2}}\left[\frac{(2+k+N+2n)!(2+k+n)!}{(2+k+3n)!}\right]^{\frac{1}{2}}
\end{aligned}$$

Because $2+k+3n \leq 6+3k+3n$ for all $k\geq 0$ and $n\geq 0$, we have

$$\max_{0\leq n\leq N-1}\left[\frac{(1+k+3n)(2+k+3n)}{2+k+n}\right]^{\frac{1}{2}} \leq 3^{\frac{1}{2}}(k+3N-2)^{\frac{1}{2}}$$

We change the summation index to $l = N-n$ and then use the second inequality of Lemma 2 to conclude

$$\begin{aligned}
1^{\text{st}} \text{ term} &\leq \sqrt{2(2+k)}2^{\frac{2+k+9N}{2}}3^{\frac{1}{2}}\hbar^{\frac{N}{2}}\delta^{-N-2}K\pi^{-\frac{1}{4}}(1-4\tau)^{-\frac{k+3N-2}{2}}b^{N-1} \\
&\quad\times (k+3N-2)^{\frac{1}{2}}[(2+k+N)!]^{\frac{1}{2}} \\
&\quad\times \sum_{l=1}^{N} 2^{-\frac{5l}{2}}\left[\frac{(2+k+3N-2l)!(2+k+N-l)!}{(2+k+3N-3l)!(2+k+N)!}\right]^{\frac{1}{2}} \\
&\leq \sqrt{3(2+k)}2^{\frac{3+k+9N}{2}}\hbar^{\frac{N}{2}}\delta^{-N-2}K(1-4\tau)^{-\frac{k+3N}{2}}b^{N-1}\beta \\
&\quad\times (k+3N-2)^{\frac{1}{2}}[(2+k+N)!]^{\frac{1}{2}} \qquad (22)
\end{aligned}$$

We now estimate the second term in (20). A argument involving the Cauchy integral formula, similar to the one mentioned above, shows that

$$\sup_{n\geq 0,\, x\in[-R,R]} \delta^n\frac{\left|V^{[n]}(x)\right|}{n!} \leq\; M\exp\left[\tau(\sqrt{2k+6N+3})+\delta)^2\right] \leq D\exp(12\tau N)$$

where $D := M\exp(2\tau(2k+\delta^2+3))$. Thus, the second term in (20) satisfies

$$\begin{aligned}
2^{\text{nd}} \text{ term} &\leq \hbar^{\frac{N}{2}}\delta^{-N-2}\sum_{n=0}^{N-1}\left\|\delta^{N+2-n}\frac{V^{[N+2-n]}(\eta_n)}{(N+2-n)!}x^{N+2-n}\chi_R(x)A_kP_{j\leq k+3n}\right\| \|\psi_n\| \\
&\leq \hbar^{\frac{N}{2}}\delta^{-N-2} \\
&\quad\times \sum_{n=0}^{N-1}\left\|\delta^{N+2-n}\frac{V^{[N+2-n]}(\eta_n)}{(N+2-n)!}\chi_R(x)x^{N+1-n}P_{j\leq k+3n+1}\right\| \|xA_k\|\,\|\psi_n\|
\end{aligned}$$

$$\leq \sqrt{2(2+k)}\hbar^{\frac{N}{2}}\delta^{-N-2}De^{12\tau N}\sum_{n=0}^{N-1}\left\|x^{N+1-n}P_{j\leq k+3n+1}\right\|\,\|\psi_n\|$$

So using our bound from Theorem 1, Lemma 5.1 of [1], and switching indices by $n = N - l$, we obtain

$$\begin{aligned}
2^{\text{nd}}\text{ term} &\leq \sqrt{2(2+k)}2^{\frac{N+1}{2}}De^{12\tau N}\hbar^{\frac{N}{2}}\delta^{-N-2}b^{N-1} \\
&\quad\times\sum_{n=0}^{N-1}2^{\frac{5n}{2}}\left[\frac{(2+k+N+2n)!(1+k+n)!}{(1+k+3n)!}\right]^{\frac{1}{2}} \\
&\leq \sqrt{2(2+k)}2^{\frac{6N+1}{2}}De^{12\tau N}\hbar^{\frac{N}{2}}\delta^{-N-2}b^{N-1} \\
&\quad\times[(2+k+N)!]^{\frac{1}{2}}\max_{0\leq n\leq N-1}\left(\frac{2+k+3n}{2+k+n}\right)^{\frac{1}{2}} \\
&\quad\times\sum_{l=1}^{N}2^{-\frac{5l}{2}}\left[\frac{(2+k+3N-2l)!(2+k+N-l)!}{(2+k+3N-3l)!(2+k+N)!}\right]^{\frac{1}{2}} \\
&\leq \sqrt{3(2+k)}2^{3N+1}De^{12\tau N}\beta\hbar^{\frac{N}{2}}\delta^{-N-2}b^{N-1}[(2+k+N)!]^{\frac{1}{2}} \qquad (23)
\end{aligned}$$

Finally, to estimate the third term of (20) we need the inequality (14), the estimate of Theorem 1 and the first part of Lemma 2

$$\begin{aligned}
3^{\text{rd}}\text{ term} &\leq \sum_{n=N}^{2N-2}\hbar^{\frac{n}{2}}\sum_{l=1}^{N-1}|\mathcal{E}_l|\|\psi_{n-l}\| \\
&\leq 2c(2+k)\sum_{n=N}^{2N-2}\hbar^{\frac{n}{2}}\sum_{l=1}^{n}\sum_{i=1}^{l}2^{\frac{i}{2}}\left[\frac{(1+k+m)!}{(1+k)!}\right]^{\frac{1}{2}}\|\psi_{n-l}\|\|\psi_{l-i}\| \\
&\leq c(2+k)\left[(1+k)!\right]^{-\frac{1}{2}}\sum_{n=N}^{2N-2}2^{3n-2}b^{n-1}\hbar^{\frac{n}{2}}\left[(2+k+n)!\right]^{\frac{1}{2}} \\
&\quad\times\sum_{l=1}^{n}\left[\frac{(1+k+n-l)!(1+k+l)!}{(1+k+n)!}\right]^{\frac{1}{2}}\sum_{i=1}^{l}\left[\frac{(1+k+l-i)!(1+k+i)!}{(1+k+l)!}\right]^{\frac{1}{2}} \\
&\leq c(2+k)\left[(1+k)!\right]^{-\frac{1}{2}}\gamma_k^2\sum_{n=N}^{2N}2^{6n-2}b^{2n-3}\hbar^{\frac{n}{2}}[(2+k+n)!]^{\frac{1}{2}} \qquad (24)
\end{aligned}$$

Now let us collect what we have done so far. Estimate (22) may be written as

$$1^{\text{st}}\text{ term} \leq A_1B_1^N(k+3N-2)^{\frac{1}{2}}\hbar^{\frac{N}{2}}[(2+k+N)!]^{\frac{1}{2}}$$

for some constants A_1 and B_1. Choose $N_0 \geq 1$ so that $k + 3N_0 - 2 \leq 2^{N_0}$. Then

$$1^{\text{st}}\text{ term} \;\leq\; A_1\,(2B_1)^N\,\hbar^{\frac{N}{2}}\,[(2+k+N)!]^{\frac{1}{2}}$$

for all $N \geq N_0$. By the same reasoning, there are constants A_2, B_2, A_3 and B_3, such that estimates (23) and (24) are given by

$$2^{\text{nd}}\text{ term} \leq A_2\,B_2^N\,\hbar^{\frac{N}{2}}\,[(2+k+N)!]^{\frac{1}{2}}$$

$$3^{\text{rd}}\text{ term} \;\leq\; \sum_{n=N}^{2N}\;A_3\,B_3^n\,\hbar^{\frac{n}{2}}\,[(2+k+n)!]^{\frac{1}{2}}$$

Finally, define $A = 3\max\{A_1, A_2, A_3\}$ and $B = \max\{2B_1, B_2, B_3\}$ to complete the proof of Proposition 1. $\square$

4. Exponentially small truncation

Fix $k \geq 0$. Our goal is to obtain a nice estimate for $|E_N - E_k|$ and $\|\tilde{\Psi}_N - \tilde{\Psi}_k\|$, where E_k is the k^{th} eigenvalue of the actual Hamiltonian $H := H_0 + W(\hbar; x)$ and $\tilde{\Psi}_k$ is the corresponding *normalized* eigenfunction. $\tilde{\Psi}_N = \|A_k \Psi_N\|^{-1} A_k \Psi_N$; E_N and Ψ_N are the quantities obtained by truncation defined in (17).

Proposition 2. There exists $\hbar_k > 0$ so that for each $\hbar \leq \hbar_k$ there is an $N_k(\hbar) \geq N_0$ such that

$$|E_N(\hbar) - E_k(\hbar)| \ \leq \sum_{n=N}^{2N} A\, B^n\, \hbar^{\frac{n}{2}}\ [(2+k+n)!]^{\frac{1}{2}}$$

for all $\hbar \leq \hbar_k$ and $N_0 \leq N \leq N_k(\hbar)$.

Before starting the proof of this proposition let us state a general result concerning asymptotic series

Lemma 4. Suppose $\sum_{n=0} f_n \beta^n$ is asymptotic to $f(\beta)$ in the sense that given $N \geq N_0$, there exists C_N and $\beta(N)$ such that for all $\beta \leq \beta(N)$

$$\left| f(\beta) \ - \ \sum_{n=0}^{N-1} f_n\, \beta^n \right| \ < \ C_N\, \beta^N.$$

Then given $\epsilon > 0$, there exists $\beta(\epsilon) > 0$, such that for each $\beta \leq \beta(\epsilon)$, there is an $N(\beta) \geq N_0$ (maybe equal to ∞), so that

$$\left| f(\beta) \ - \ \sum_{n=0}^{N-1} f_n\, \beta^n \right| \ \leq \ \epsilon \tag{25}$$

whenever $N_0 \leq N < N(\beta)$.

Proof: Fix $\epsilon > 0$. Define $\beta_1(N_0) = (\epsilon\, C_{N_0}^{-1})^{1/N_0}$. Then for $N > N_0$, recursively choose positive numbers $\beta_1(N)$ that satisfy $\beta_1(N) < \min\{(\epsilon\, C_N^{-1})^{1/N},\ \beta_1(N-1)\}$. Then (25) will be satisfied whenever $\beta < \beta_1(N)$.

Define $\beta(\epsilon) = \beta_1(N_0)$, and define

$$N(\beta) \ = \ \begin{cases} N+1 & \text{if} \quad \beta_1(N+1) < \beta \leq \beta_1(N) \\ \infty & \text{if} \quad \beta < \beta_1(N) \text{ for all } N. \end{cases}$$

Then (25) holds whenever $N_0 \leq N \leq N(\beta)$. □

Proof of Proposition 2: Suppose $\hbar \leq \hbar_1 := \min\{1, \hbar_0\}$ and $N \geq N_0$. Then there exist at least $k+1$ eigenvalues of H and the estimate of Proposition 1 holds. Let $N_1(\hbar)$ be defined to be the largest $N \geq N_0$ which satisfies

$$\sum_{n=N_1(\hbar)}^{2N_1(\hbar)} A\, B^n\, \hbar^{\frac{n}{2}}\, [(2+k+n)!]^{\frac{1}{2}} \leq \frac{1}{4}$$

(this requirement can always be satisfied by reducing $\hbar_1$ if necessary). Then for all $N_0 \leq N \leq N_1(\hbar)$, we have

$$\sum_{n=N}^{2N} A\, B^n\, \hbar^{\frac{n}{2}}\, [(2+k+n)!]^{\frac{1}{2}} \leq \frac{1}{4} \tag{26}$$

On the other hand, note that $\Psi_N = \phi_k + \varphi_N$, where φ_N is orthogonal to ϕ_k because of the normalization we chose for the correction terms $\psi_n(x)$. Since $A_k\phi_k = \phi_k$, we conclude that $\|A_k\Psi_N(\hbar;x)\| \geq 1$. So Proposition 1 implies that

$$\|(H - E_N(\hbar))\, A_k\Psi_N(x)\| \ \leq \sum_{n=N}^{2N} AB^n\hbar^{\frac{n}{2}}\,[(2+k+n)!]^{\frac{1}{2}}\ \|A_k\Psi_N(\hbar;x)\| \tag{27}$$

We may assume that $E_N(\hbar) \notin \sigma(H)$, so $(H - E_N(\hbar))$ is invertible. It follows that

$$\left\{\sum_{n=N}^{2N} AB^n\hbar^{\frac{n}{2}}\,[(2+k+n)!]^{\frac{1}{2}}\right\}^{-1} \ \leq\ \left\|(H - E_N(\hbar))^{-1}\right\|.$$

Because H is selfadjoint, $\|(H-E)^{-1}\| = \mathrm{dist}\{E, \sigma(H)\}^{-1}$ by the Spectral Theorem. Thus,

$$\mathrm{dist}\,\{E_N(\hbar), \sigma(H)\} \ \leq \sum_{n=N}^{2N} AB^n\hbar^{\frac{n}{2}}\,[(2+k+n)!]^{\frac{1}{2}} \ \leq\ \frac{1}{4}, \tag{28}$$

for any given $\hbar \leq \hbar_1$, whenever $N_0 \leq N \leq N_1(\hbar)$.

Since $E_l(\hbar) \to e_l$ as $\hbar \to 0$ for $0 \leq l \leq k+1$, we can choose $\hbar_2$, so that $|E_l(\hbar) - e_l| < \frac{1}{4}$, for all $0 \leq l \leq k+1$ and $\hbar \leq \hbar_2$. Note that $|e_i - e_{i+1}| = 1$. As a consequence, we have

$$|E_k(\hbar) - E_\alpha(\hbar)| > \frac{1}{2} \tag{29}$$

for all $E_\alpha(\hbar) \in \sigma(H) \setminus \{E_k(\hbar)\}$, whenever $\hbar \leq \hbar_2$.

We know that $E_N(\hbar)$ is asymptotic to $E_k(\hbar)$. So, we may apply Lemma 4 (with $\beta = \hbar^{\frac{1}{2}}$) to see that there exists an $\hbar_3 > 0$, so that for each $\hbar \leq \hbar_3$, we have an $N_3(\hbar)$, such that

$$|E_k(\hbar) - E_N(\hbar)| \leq \frac{1}{4} \tag{30}$$

for all $N_0 \leq N \leq N_3(\hbar)$.

Now (29) and (30) along with the second inequality of (28) imply that

$$\mathrm{dist}\,\{E_N(\hbar), \sigma(H)\} = |E_N(\hbar) - E_k(\hbar)|$$

for $\hbar \leq \hbar_k := \min\{\hbar_2, \hbar_3, \hbar_1\}$ and $N_0 \leq N \leq N_k(\hbar) := \min\{N_1(\hbar), N_3(\hbar)\}$. This completes the proof. $\square$

Remark: It is clear from inequality (26) that indeed we have $|E_N - E_k| \leq \frac{1}{4}$ for all $N_0 \leq N \leq N_1(\hbar)$. So without harm we can extend $N_k(\hbar)$ chosen in the proof to be $N_k(\hbar) = N_1(\hbar)$.

Theorem 3. For each $0 < g < B^{-2}$, there exists $0 < \hbar_g$, such that for each $\hbar \leq \hbar_g$, there is $N(\hbar)$, so that

$$\left|E_{N(\hbar)}(\hbar) - E_k(\hbar)\right| \ \leq\ \Lambda \exp\left(-\frac{\Gamma}{\hbar}\right) \tag{31}$$

for some $\Gamma > 0$ and $\Lambda > 0$.

Proof: We mimic a proof given in [2]. Fix $0 < g < B^{-2}$. Then $0 < B^2 g < 1$, so there is $\omega > 0$, such that $B^2 g = \exp(-\omega)$. Consider the function

$$f(\hbar) = Ag \exp\left[-\frac{\omega(1+k)}{4}\right] \hbar^{-\frac{4+k}{2}} \exp\left(-\frac{\omega g}{2\hbar}\right).$$

We know that $f(\hbar) > 0$ on $(0, \infty)$, and $f(\hbar) \to 0$ as $\hbar \to 0$ or $\hbar \to \infty$ so let us choose

$$\hbar_4 = \inf\left\{\hbar : f(\hbar) \geq \frac{1}{4}\right\}$$

and then set

$$\hat{\hbar}_g = \sup\left\{\hbar : \hbar \leq \min\{\hbar_k, \hbar_4\} \text{ and } \left[\!\left[\frac{g}{\hbar}\right]\!\right] \geq 2 + k + 2N_0\right\}$$

For $\hbar \leq \hat{\hbar}_g$ define $N(\hbar)$ by the identity $2+k+2N(\hbar) = [\![\frac{g}{\hbar}]\!]$. Then clearly $N(\hbar) \geq N_0$. On the other hand, since we can assume $B \geq 1$ and $2 + k + n \leq g/\hbar$ for $N(\hbar) \leq n \leq 2N(\hbar)$ we have

$$\begin{aligned}
\sum_{n=N(\hbar)}^{2N(\hbar)} AB^n \hbar^{\frac{n}{2}} \left[(2+k+n)!\right]^{\frac{1}{2}} &\leq \sum_{n=N(\hbar)}^{2N(\hbar)} AB^n \hbar^{\frac{n}{2}} (2+k+n)^{\frac{2+k+n}{2}} \\
&\leq A\hbar^{-\frac{2+k}{2}} \sum_{n=N(\hbar)}^{2N(\hbar)} \left[B^2 \hbar (2+k+n)\right]^{\frac{2+k+n}{2}} \\
&\leq A\hbar^{-\frac{2+k}{2}} \sum_{n=N(\hbar)}^{2N(\hbar)} \left(B^2 g\right)^{\frac{2+k+n}{2}}
\end{aligned}$$

Now use that $B^2 g = \exp(-\omega) < 1$ and the fact $a^n \geq a^{n+1}$ if $a < 1$ to obtain

$$\begin{aligned}
&\sum_{n=N(\hbar)}^{2N(\hbar)} AB^n \hbar^{\frac{n}{2}} \left[(2+k+n)!\right]^{\frac{1}{2}} \\
&\quad\leq A\hbar^{-\frac{2+k}{2}} \sum_{n=N(\hbar)}^{2N(\hbar)} \exp\left\{-\frac{\omega}{2}\left[2+k+N(\hbar)\right]\right\} \\
&\quad= A\hbar^{-\frac{2+k}{2}} e^{-\frac{\omega}{4}(2+k)} \left[1+N(\hbar)\right] \exp\left\{-\frac{\omega}{4}\left[2+k+2N(\hbar)\right]\right\} \\
&\quad\leq A\hbar^{-\frac{2+k}{2}} e^{-\frac{\omega}{4}(2+k)} \left[2+k+2N(\hbar)\right] \exp\left[-\frac{\omega}{4}\left(\frac{g}{h}-1\right)\right] \\
&\quad\leq Age^{-\frac{\omega}{4}(1+k)} \hbar^{-\frac{4+k}{2}} \exp\left(-\frac{\omega g}{4\hbar}\right) \qquad (32) \\
&\quad\leq Age^{-\frac{\omega}{4}(1+k)} \hbar_4^{-\frac{4+k}{2}} \exp\left(-\frac{\omega g}{4\hbar_4}\right) \\
&\quad\leq \frac{1}{4}.
\end{aligned}$$

Thus, $N(\hbar) \leq N_k(\hbar)$. Therefore, Proposition 2 holds for $\hbar < \hat{\hbar}_g$, which along with (32) implies

$$\left|E_{N(\hbar)}(\hbar) - E_k(\hbar)\right| \;\leq\; Age^{-\frac{\omega}{4}(1+k)} \hbar^{-\frac{4+k}{2}} \exp\left(-\frac{\omega g}{4\hbar}\right),$$

for all $\hbar \leq \hat{\hbar}_g$. Finally, define

$$\hbar_g = \max\left\{\hbar \leq \hat{\hbar}_g : \hbar^{-\frac{4+k}{2}} \exp\left(-\frac{\omega g}{8\hbar}\right) \leq 1\right\}$$

Then (31) is true for all $\hbar \leq \hbar_g$ with $\Gamma := \omega g/8$ and $\Lambda := Ag \exp\left(-\omega(1+k)/4\right)$. □

Proposition 3. Let h_k and $N_k(\hbar)$ be defined as in Proposition 2. Let $\tilde{\Psi}_N(\hbar; x)$ be the vector obtained by normalizing the vector obtained by our construction. Let $\tilde{\Psi}_k(\hbar; x)$ be the normalized exact eigenvector of H. Then

$$\left\|\tilde{\Psi}_N(\hbar; x) - \tilde{\Psi}_k(\hbar; x)\right\| \;\leq\; 8 \sum_{n=N}^{2N} A\, B^n\, \hbar^{\frac{n}{2}}\, \left[(2+k+n)!\right]^{\frac{1}{2}},$$

for all $\hbar \leq \hbar_k$ and $N_0 \leq N \leq N_k(\hbar)$.

Proof: Let us note that (27) means

$$\left\|(H - E_N(\hbar))\, \tilde{\Psi}_N(\hbar; x)\right\| \leq \sum_{n=N}^{2N} AB^n \hbar^{\frac{n}{2}} \left[(2+k+n)!\right]^{\frac{1}{2}} \tag{33}$$

We can write $\tilde{\Psi}_N(\hbar; x) = w_N \tilde{\Psi}_k(\hbar; x) + \Omega_N(\hbar; x)$, where Ω_N is orthogonal to $\tilde{\Psi}_k$ and $|w_N|^2 + \|\Omega_N(\hbar; x)\|^2 = 1$. Since these functions are defined up to a global phase, we can assume that indeed $0 < w_N \leq 1$. From the normalization condition, we obtain

$$\|\Omega_N(\hbar; x)\| \;\geq\; \|\Omega_N(\hbar; x)\|^2 \;=\; 1 - |w_N|^2 \;=\; (1+w_N)(1-w_N) \;\geq\; 1 - w_N.$$

So, we have

$$\begin{aligned}\left\|\tilde{\Psi}_N(\hbar; x) - \tilde{\Psi}_k(\hbar; x)\right\| &= \left\|(1-w_N)\tilde{\Psi}_k(\hbar; x) + \Omega_N(\hbar; x)\right\| \\ &\leq (1-w_N) + \|\Omega_N(\hbar; x)\| \leq 2\, \|\Omega_N(\hbar; x)\|\end{aligned} \tag{34}$$

Since

$$(H - E_N(\hbar))\, \Omega_N(\hbar; x) = (H - E_N(\hbar))\, \tilde{\Psi}_N(\hbar; x) - w_N\, (E_k(\hbar) - E_N(\hbar))\, \tilde{\Psi}_k(\hbar; x),$$

it follows from (33) and Proposition 2 that

$$\|(H - E_N(\hbar))\, \Omega_N(\hbar; x)\| \;\leq\; 2 \sum_{n=N}^{2N} A\, B^n\, \hbar^{\frac{n}{2}}\, \left[(2+k+n)!\right]^{\frac{1}{2}}, \tag{35}$$

whenever $\hbar \leq \hbar_k$ and $N_0 \leq N \leq N(\hbar)$. On the other hand, recall that $E_N \notin \sigma(H)$ and also note that $(H - E_N)\, \Omega_N$ is orthogonal to $\tilde{\Psi}_k$. Thus, we have

$$\begin{aligned}\|\Omega_N(\hbar; x)\| &= \left\|(H - E_N(\hbar))_\perp^{-1}\, (H - E_N(\hbar))\, \Omega_N(\hbar; x)\right\| \\ &\leq \left\|(H - E_N(\hbar))_\perp^{-1}\right\|\, \|(H - E_N(\hbar))\, \Omega_N(\hbar; x)\|,\end{aligned} \tag{36}$$

where $(H - E_N)_\perp$ is the restriction of $(H - E_N)$ to the subspace orthogonal to $\tilde{\Psi}_k$. Now make use of the Spectral Theorem and (29) to see that

$$\left\|(H - E_N(\hbar))_\perp^{-1}\right\| \;=\; \inf_{x \in [0, E_{k-1}] \cup [E_{k+1}, \infty)} |x - E_N(\hbar)|^{-1} \;\leq\; 2. \tag{37}$$

Hence (35), (36) and (37) imply

$$\|\Omega_N(\hbar; x)\| \;\leq\; 4 \sum_{n=N}^{2N} A\, B^n\, \hbar^{\frac{n}{2}}\, \left[(2+k+n)!\right]^{\frac{1}{2}}, \tag{38}$$

for all $\hbar \leq \hbar_k$ and $N_0 \leq N \leq N_k(\hbar)$. Now combine (34) and (38). □

The next result follows from Proposition 3 in the same way that Theorem 3 does from Proposition 2.

Theorem 4. If g, $\hbar_g$, $N(\hbar)$, Γ and Λ are defined as in Theorem 3, then

$$\left\| \tilde{\Psi}_{N(\hbar)}(\hbar;x) - \tilde{\Psi}_k(\hbar;x) \right\| \;\leq\; 8\,\Lambda\,\exp\left(-\frac{\Gamma}{\hbar}\right).$$

Appendix

Here we simplify the formula (18) by using the identity (9)

$$\begin{aligned}
\xi_N(x) =&\; \left[H_k + A_k W(\hbar;x) A_k - \sum_{j=1}^{N-1} \hbar^{\frac{j}{2}} \mathcal{E}_j (A_k)^2 \right] \sum_{m=0}^{N-1} \hbar^{\frac{m}{2}} \psi_m(x) \\
=&\; \sum_{m=0}^{N-1} \hbar^{\frac{m}{2}} H_k \psi_m(x) + \sum_{m=0}^{N-1} \hbar^{\frac{m}{2}} A_k W(\hbar;x) A_k \psi_m(x) \\
&- \sum_{j=1}^{N-1} \sum_{m=0}^{N-1} \hbar^{\frac{j+m}{2}} \mathcal{E}_j (A_k)^2 \psi_m(x)
\end{aligned}$$

We use $W(\hbar;x) = \sum_{j=3}^{N+2} \hbar^{\frac{j-2}{2}} c_j x^j + W^{[N+2]}(\hbar;x)$ and change the index by $j \to j-2$. Using $H_k \phi_k = 0$, we then obtain

$$\begin{aligned}
\xi_N(x) =&\; \sum_{m=1}^{N-1} \hbar^{\frac{m}{2}} H_k \psi_m(x) + \sum_{m=0}^{N-1} \sum_{j=1}^{N} \hbar^{\frac{m+j}{2}} c_{j+2} A_k x^{j+2} A_k \psi_m(x) \\
&+ \sum_{m=0}^{N-1} \hbar^{\frac{m}{2}} A_k W^{[N+2]}(\hbar;x) A_k \psi_m(x) - \sum_{m=0}^{N-1} \sum_{j=1}^{N-1} \hbar^{\frac{j+m}{2}} \mathcal{E}_j (A_k)^2 \psi_m(x) \\
=&\; \sum_{n=1}^{N-1} \hbar^{\frac{n}{2}} H_k \psi_n(x) + \sum_{n=1}^{N-1} \hbar^{\frac{n}{2}} \sum_{j=1}^{n} c_{j+2} A_k x^{j+2} A_k \psi_{n-j}(x) \\
&+ \sum_{n=N}^{2N-1} \hbar^{\frac{n}{2}} \sum_{j=n-N+1}^{N} c_{j+2} A_k x^{j+2} A_k \psi_{n-j}(x) \\
&+ \sum_{m=0}^{N-1} \hbar^{\frac{m}{2}} A_k W^{[N+2]}(\hbar;x) A_k \psi_m(x) - \sum_{n=1}^{N-1} \hbar^{\frac{n}{2}} \sum_{j=1}^{n} \mathcal{E}_j (A_k)^2 \psi_{n-j}(x) \\
&- \sum_{n=N}^{2N-2} \hbar^{\frac{n}{2}} \sum_{j=n-N+1}^{N-1} \mathcal{E}_j (A_k)^2 \psi_{n-j}(x)
\end{aligned}$$

The first, second and fifth terms of last equation cancel because of (9). In the third term define $m = n - j$ and then $p = n - N$. This yields

$$\begin{aligned}
\xi_N(x) =&\; \sum_{n=N}^{2N-1} \hbar^{\frac{n}{2}} \sum_{m=n-N}^{N-1} c_{n-m+2} A_k x^{n-m+2} A_k \psi_m(x) \\
&+ \sum_{m=0}^{N-1} \hbar^{\frac{m}{2}} A_k W^{[N+2]}(\hbar;x) A_k \psi_m(x) - \sum_{n=N}^{2N-2} \hbar^{\frac{n}{2}} \sum_{j=n-N+1}^{N-1} \mathcal{E}_j (A_k)^2 \psi_{n-j}(x)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{p=0}^{N-1}\sum_{m=p}^{N-1} \hbar^{\frac{p+N}{2}} c_{p+N-m+2} A_k x^{p+N-m+2} A_k \psi_m(x) \\
&\quad + \sum_{m=0}^{N-1} \hbar^{\frac{m}{2}} A_k W^{[N+2]}(\hbar;x) A_k \psi_m(x) - \sum_{n=N}^{2N-2} \hbar^{\frac{n}{2}} \sum_{j=n-N+1}^{N-1} \mathcal{E}_j (A_k)^2 \psi_{n-j}(x) \\
&= \sum_{m=0}^{N-1} \hbar^{\frac{m}{2}} \sum_{p=0}^{m} \hbar^{\frac{p+N-m}{2}} c_{p+N-m+2} A_k x^{p+N-m+2} A_k \psi_m(x) \\
&\quad + \sum_{m=0}^{N-1} \hbar^{\frac{m}{2}} A_k W^{[N+2]}(\hbar;x) A_k \psi_m(x) - \sum_{n=N}^{2N-2} \hbar^{\frac{n}{2}} \sum_{j=n-N+1}^{N-1} \mathcal{E}_j (A_k)^2 \psi_{n-j}(x) \\
&= \sum_{m=0}^{N-1} \hbar^{\frac{m}{2}} A_k \left[\sum_{i=2}^{m+2} \hbar^{\frac{i+N-m-2}{2}} c_{i+N-m} x^{i+N-m} + W^{[N+2]}(\hbar;x) \right] A_k \psi_m(x) \\
&\quad - \sum_{n=N}^{2N-2} \hbar^{\frac{n}{2}} \sum_{j=n-N+1}^{N-1} \mathcal{E}_j (A_k)^2 \psi_{n-j}(x)
\end{aligned}
$$

Finally, note that $\hbar^{\frac{j-2}{2}} c_j x^j + W^{[j+1]}(\hbar;x) = W^{[j]}(\hbar;x)$. It then follows that

$$
\xi_N(x) = \sum_{m=0}^{N-1} \hbar^{\frac{m}{2}} A_k W^{[N-m+1]}(\hbar;x) A_k \psi_m(x) - \sum_{n=N}^{2N-2} \hbar^{\frac{n}{2}} \sum_{j=n-N+1}^{N-1} \mathcal{E}_j (A_k)^2 \psi_{m-j}(x)
$$

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