A degenerate bifurcation in the Hénon-Heiles family

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Abstract

The normalised Hénon-Heiles family exhibits a degenerate bifurcation when passing through the separable case "$\beta = 0$". We clarify the relation between this degeneracy and the integrability at $\beta = 0$. Furthermore we show that the degenerate bifurcation carries over to the Hénon-Heiles family itself.

§1 Introduction

Originally motivated by galactic dynamics, the so-called *Hénon-Heiles family* of Hamiltonian systems in two degrees of freedom has gained a paradigmatic status. It served as a model in further applications to *e.g.* molecular dynamics or the (restricted) three body problem and was also studied in its own right. This family is defined by

$$H(x_1, x_2, p_1, p_2) \;=\; \frac{1}{2}(p_1^2 + p_2^2) \;+\; \frac{\omega^2}{2}(x_1^2 + x_2^2) \;+\; \varepsilon\,(\alpha\frac{x_1^3}{3} \;+\; \beta x_1 x_2^2) \quad . \tag{1}$$

The pioneering work [Hénon,Heiles;64] contained numerical experiments on the case $\alpha = 1$, $\beta = -1$, showing how "the integrability of the system breaks down" as the perturbation is switched on. Later studies, *e.g.* [Churchill,Pecelli,Rod;79,80], were concerned with the behaviour of the system for "not so small" values of ε .

When the case $\alpha = 1$, $\beta = -1$ was included in the family (1), cf. [Braun;73], [Kummer;76], [Verhulst;79], [Cushman;82], [Miller;91], it turned out that some of its features were due to a degenerate normal form and not shared by other members of the family. In [Yanguas;98], [Ferrer,Lara,Palacián,San Juan,Viartola,Yanguas;98a,b] the original three-degrees-of-freedom system considered by [Hénon,Heiles;64] was studied for non-zero

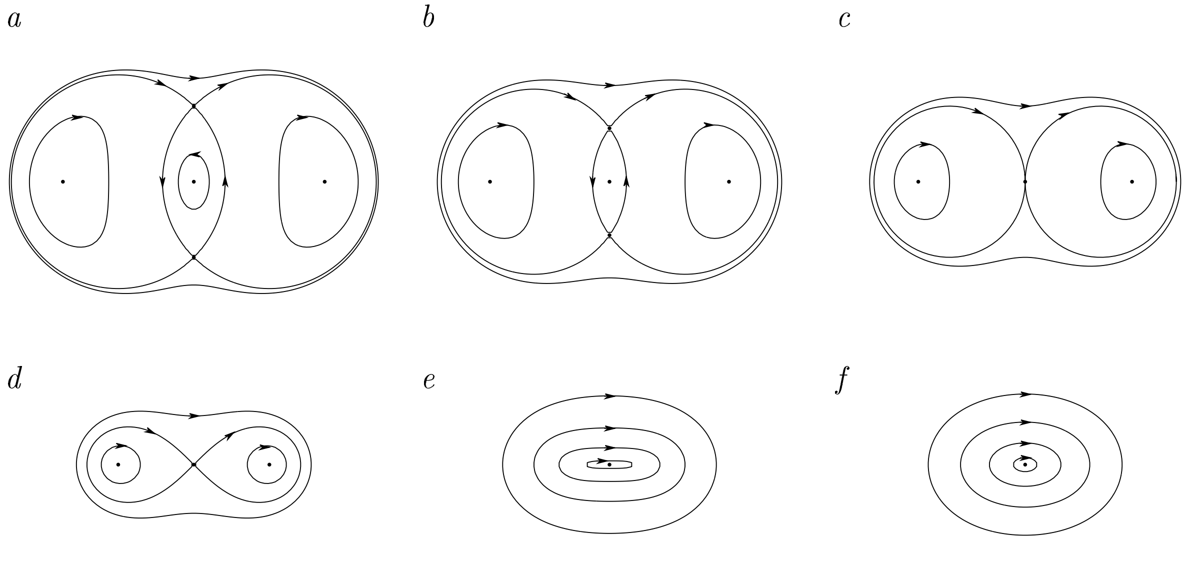

Figure 1.1: Two Hamiltonian pitchfork bifurcations governed by two different reversing symmetries.

values of the third component of the angular momentum, also exhibiting a “less chaotic” behaviour of the system.

For sufficiently small ε the family (1) can be treated as a small perturbation of two harmonic oscillators in $1:1$ resonance. Since all motions are periodic such an isotropic harmonic oscillator constitutes a Hamiltonian system that is not only integrable, but even *superintegrable*. This complicates the perturbation analysis. Indeed, we expect that only exceptional periodic orbits survive the perturbation and that most motions follow a quasi-periodic trajectory, cf. [Arnol’d;63]. Contrary to perturbations of non-degenerate integrable systems, here the perturbation itself determines which periodic orbits survive and hence how the perturbed flow is organized. Often these questions can be decided by the first non-trivial normal form : those features of the normalised flow that are sufficiently non-degenerate should be present in the “original perturbed system” as well.

The redundancy of parameters in (1) has usually been used to simplify the family by letting $\alpha =: \lambda$, $\beta = 1$ and thus scale the separable case $\beta = 0$ away to “$\lambda = \infty$”. In [Miller;91] the slightly different scaling $\alpha =: \cos\phi$, $\beta =: \sin\phi$ revealed that a degenerate bifurcation occurs in the normal form of the family (1) when β passes through zero. The aim of the present paper is to clarify the reason for this degeneracy and to explain how it is related to the integrability of the case $\beta = 0$ where the bifurcation takes place.

Being a family of “natural mechanical systems”, the family (1) is reversible with respect to the involution $(x_1, x_2, p_1, p_2) \mapsto (x_1, x_2, -p_1, -p_2)$. Furthermore, the special form of the potential permits a second reversing involution $(x_1, x_2, p_1, p_2) \mapsto (x_1, -x_2, -p_1, p_2)$. These reversing symmetries are respected by the normalisation procedure, cf. [Churchill, Kummer,Rod;83], and they also carry over to the one-degree-of-freedom system one obtains on reducing the normal form.

In "non-symmetric" Hamiltonian systems with one degree of freedom the sole generic co-dimension one bifurcation is the centre-saddle bifurcation. A reversing symmetry also makes Hamiltonian pitchfork bifurcations generic in one-parameter families. In particular, a one-parameter family of one-degree-of-freedom systems with two different reversing symmetries may display two subsequent Hamiltonian pitchfork bifurcations as shown in *figure 1.1*, see [Coffey,Deprit,Miller;86], [Cushman;87], [Broer,Vegter;92] or [Guckenheimer,Mahalov;92].

From the results in [Miller;91] we infer the following. On passing through $\beta = 0$, with *e.g.* $\alpha, \varepsilon > 0$ constant, the reduced second order normal form of (1) has a bifurcation where the phase portrait changes directly from the first of those shown in *figure 1.1* to the last one. None of the intermediate phase portraits occur. Indeed, all four "further" equilibria uniformly approach the "central" equilibrium and arrive there when $\beta = 0$, leading to a single centre after the bifurcation. This degenerate bifurcation cannot be explained by the discrete symmetries of the family (1). However, for $\beta = 0$ the system defined by (1) is separable and hence integrable. We will see how this governs the degeneracy of the above bifurcation.

This paper is organized as follows. In the next section we explain the physical meaning of the reduced phase space (of the normal form) using suited Lissajous variables. Section 3 contains a detailed study of the behaviour of the reduced normalised system, when β passes through zero. In the final section the results obtained are carried over to the "original" family (1).

§2 Setting of the problem

We are concerned with the isotropic harmonic oscillator in two degrees of freedom, perturbed by an external conservative force field. The Hamilton function of the former is $\frac{1}{2}(p_1^2 + p_2^2) + \frac{\omega^2}{2}(x_1^2 + x_2^2)$, where ω is the (common) frequency. The force field gives rise to the potential energy $\alpha\frac{x_1^3}{3} + \beta x_1 x_2^2$ which is the general cubic term invariant under $x_2 \mapsto -x_2$. A scaling $(x, p, H) \mapsto (\varepsilon x, \varepsilon p, \varepsilon^2 H)$ turns the perturbed Hamilton function into (1) ; close to the origin the influence of the perturbing force field is weak. Throughout this paper we assume $\varepsilon > 0$ fixed.

Discarding the trivial case $\alpha = \beta = 0$, the parameters of the external force field become redundant : as we are interested in the passage through $\beta = 0$, we might as well scale α away and use $\frac{\beta}{\alpha}$ as the single parameter. Equivalently, we put $\alpha = 1$ in (1) and work with β as sole parameter.

The family (1) is an ε-small perturbation of the isotropic harmonic oscillator. The flow of the latter is periodic, moving along ellipses. This gives rise to a symmetry of the system which we call the *oscillator symmetry*. By means of a normalisation procedure one can push this symmetry through the Taylor series of the perturbation. Truncation yields normal forms of any order. As all uneven terms in normal forms of the isotropic harmonic oscillator vanish, the first nontrivial normal form is given by

$$\begin{aligned}\bar{H}(x,p) \;=\; & \tfrac{1}{2}(p_1^2+p_2^2) \;+\; \tfrac{\omega^2}{2}(x_1^2+x_2^2) \;+\; \tfrac{\varepsilon^2}{48\omega^6}\,[-5(p_1^2+\omega^2x_1^2)^2 \\ & -5\beta^2(p_2^2+\omega^2x_2^2)^2 - (12\beta+8\beta^2)(p_1^2+\omega^2x_1^2)(p_2^2+\omega^2x_2^2) \\ & +2\beta(1-6\beta)((p_1p_2+\omega^2x_1x_2)^2-\omega^2(x_1p_2-x_2p_1)^2)] \quad .\end{aligned} \tag{2}$$

It turns out that $\bar{H}$ is sufficient for our purposes, so we do not need to compute higher order normal forms.

By construction (2) is invariant under the oscillator symmetry, therefore we may reduce it to one degree of freedom. This is done best by means of the invariants of the oscillator symmetry. These can be written as functions of

$$\begin{aligned} I_0 &= \frac{1}{4\omega}(p_1^2+p_2^2+\omega^2(x_1^2+x_2^2)) \\ I_1 &= \frac{1}{2\omega}(p_1p_2+\omega^2x_1x_2) \\ I_2 &= \frac{1}{2}(x_1p_2-x_2p_1) \\ I_3 &= \frac{1}{4\omega}(p_1^2-p_2^2+\omega^2(x_1^2-x_2^2)) \quad . \end{aligned}$$

Accordingly, I_0, I_1, I_2, I_3 form a Hilbert basis of the ring of invariants and can be used as co-ordinates of the reduced phase space. These co-ordinates are constrained by the relation

$$I_0^2 \;=\; I_1^2 \;+\; I_2^2 \;+\; I_3^2 \quad . \tag{3}$$

In fact, both sides of (3) define a *Casimir element*, *i.e.* their Poisson brackets with the I_i vanish. The Poisson bracket on the reduced phase space is given by $\{I_i, I_j\} = \varepsilon_{ijk} I_k$, where the alternating Levi-Civita symbol ε_{ijk} denotes the sign of the permutation $\binom{1\,2\,3}{i\,j\,k}$ and vanishes if two of the i, j, k are equal or one of them is 0. Thus, denoting the common value in (3) by a^2 , we are led to the reduced Hamilton function

$$\begin{aligned}\bar{H}(I_1,I_2,I_3) \;=\; & 2a\omega \;+\; \tfrac{\varepsilon^2}{12\omega^4}\,[-a^2(5+12\beta+13\beta^2) \\ & -\;10a(1-\beta^2)I_3 \;-\; (5-12\beta-3\beta^2)I_3^2 \;+\; 2\beta(1-6\beta)(I_1^2-I_2^2)] \quad .\end{aligned} \tag{4}$$

As the sphere

$$\mathcal{S} \;=\; \left\{\; (I_1,I_2,I_3) \in \mathbb{R}^3 \;\middle|\; I_1^2+I_2^2+I_3^2 = a^2 \;\right\} \tag{5}$$

is invariant under the flow defined by (4), this performs the reduction to one degree of freedom.

For $\beta = 0$ the Hamilton function $\bar{H}$ inherits a further symmetry because in this case H itself is integrable. From (4) we infer that this symmetry amounts to rotation about the I_3-axis. Hence, cylindrical co-ordinates (Γ, γ) of the sphere given by

$$\begin{aligned} I_1 &= \sqrt{a^2 - \Gamma^2}\,\cos\gamma \\ I_2 &= \sqrt{a^2 - \Gamma^2}\,\sin\gamma \\ I_3 &= \Gamma \end{aligned}$$

may help to understand the passage through $\beta = 0$. There is a close relation of these co-ordinates to one of the three sets of Lissajous variables introduced in [Deprit;91]. Indeed, the second set is given by $L' = 2I_0$, $G' = 2\Gamma$, $g' = \frac{1}{2}\gamma$ and ℓ' conjugate to L'. These can also be obtained from

$$\begin{aligned} x_1 &= \sqrt{\frac{L' + G'}{\omega}}\,\sin(\ell' + g') \\ x_2 &= \sqrt{\frac{L' - G'}{\omega}}\,\sin(\ell' - g') \\ p_1 &= \sqrt{\omega(L' + G')}\,\cos(\ell' + g') \\ p_2 &= \sqrt{\omega(L' - G')}\,\cos(\ell' - g') \quad . \end{aligned}$$

Since we will not be concerned with the other two sets of Lissajous variables, we drop the primes from now on. The variables (ℓ, L, g, G) are generalized action-angle variables of the isotropic harmonic oscillator in the sense of [Nehorošev;72]. Indeed, the angle ℓ measures how that flow proceeds along the ellipses. In particular, the normalisation procedure turns ℓ into a cyclic angle. The physical meaning of the other three variables is instructive as well : ωL expresses the energy of the isotropic harmonic oscillator, ωG is the difference of energies of the two modes $\frac{1}{2}(p_1^2 + \omega^2 x_1^2)$ and $\frac{1}{2}(p_2^2 + \omega^2 x_2^2)$ and the angle g measures the phase difference between these two oscillators.

§3 The one-degree-of-freedom problem

After normalisation and reduction to one degree of freedom, the two reversing symmetries of (1) turn into the two reversing symmetries $I_2 \mapsto -I_2$ and $I_1 \mapsto -I_1$ of (4). Hence, the composite

$$(I_1, I_2, I_3) \mapsto (-I_1, -I_2, I_3) \tag{6}$$

yields a (non-reversing) discrete symmetry of (4). This explains why $(0,0,\pm a)$ are always equilibria of the flow defined by (4).

We now use a further reduction to explicitly divide out the $\mathbb{Z}_2$-symmetry generated by (6). To this end we again work with invariants. As the symmetry group in question is discrete, its reduction will not lead to a further decrease of the dimension of the reduced phase space. But the two points $(0,0,\pm a)$ have non-trivial isotropy groups, *i.e.* they remain fixed under (6), and therefore lead to singular points. The twice reduced phase space is the "lemon"

$$\mathcal{L} \;:=\; \left\{ \, (x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 = (a+z)^2 (a-z)^2 \, \right\}$$

with Poisson bracket

$$\{f,g\} \;=\; (\nabla f \times \nabla g \mid \nabla R) \quad ,$$

where $(..|..)$ is the inner product, R denotes the Casimir element $x^2+y^2-(a+z)^2(a-z)^2$, and the reduction is performed putting

$$x = I_1^2 - I_2^2 \qquad y = 2I_1I_2 \qquad z = I_3 \quad .$$

To simplify subsequent formulas we omit the constant term from $\bar{H}$, divide by $\frac{\varepsilon^2}{12\omega^4}$ and rescale time. In this way the Hamilton function (4) becomes

$$\mathcal{H}(x,y,z) \;=\; 2\beta(1-6\beta)x \;-\; 10a(1-\beta^2)z \;-\; (5-12\beta-3\beta^2)z^2 \tag{7}$$

and defines a Hamiltonian system that is reversible with respect to $y \mapsto -y$. Indeed, both reversing symmetries of (4) reduce to this reflection. We remark that all equilibria of the system are isolated and that the sole degenerate equilibrium occurs for $\beta = 0$ at the singular point $(0,0,-a)$.

Since y does not enter in (7), the level sets $\{\mathcal{H} = h\}$ are parabolic cylinders in $\mathbb{R}^3$ which contain the y-axis. As $\beta \to 0$ they degenerate to pairs of planes, and for $\beta = 0$ the "double plane" $\{\mathcal{H} = 5a^2\}$ passes exactly through the singular point $(0,0,-a)$ where the degenerate bifurcation takes place.

We dive a little bit into the normalisation procedure to understand the relation between the integrability of the original Hamilton function H (defined by (1)) when $\beta = 0$ and the especially simple level sets of $\mathcal{H}$ for $\beta = 0$. When pushing the oscillator symmetry through the Taylor series of the perturbation, one uses a sequence of co-ordinate transformations that turn the perturbed system into its normal form, modulo higher order terms. For $\beta = 0$ the Hamilton function H defines a separable system and the difference of the energies of the (x_1,p_1) and (x_2,p_2) subsystems is an integral as well. Transforming this integral to the new co-ordinates yields a second integral of $\bar{H}$. As shown in [Miller;91] this integral is a function that only depends on I_3 and not on I_1 or I_2, *i.e.* I_3 itself is invariant under the flow of the vector field $X_{\mathcal{H}}$. This means that level sets $\{\mathcal{H} = h\}$ are (pairs of) planes perpendicular to the z-axis.

If the level sets of (7) remained parabolic cylinders throughout the passage through $\beta = 0$ with, say, the maximum of the parabola reaching $z = -a$ exactly at $\beta = 0$, the

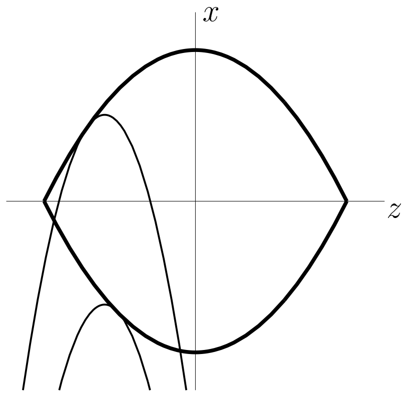

Figure 3.1: Possible tangencies within $\{y = 0\}$ of the reduced phase space and two different energy levels, for fixed value $\beta < 0$ of the parameter.

bifurcation scenario would be as displayed in *figure 1.1*. The first bifurcation would occur for some $\beta_* < 0$ and the second for some $\beta^* > 0$. Indeed, equilibria on $\mathcal{L}$ only occur at $\{y = 0\}$, and the parabola $\{\mathcal{H} = h\} \cap \{y = 0\}$ can touch the upper arc left to its maximum and the lower arc right to its maximum, see *figure 3.1*.

The reason is that $\mathcal{L}$ is a surface of revolution, such that a tangent plane to $\mathcal{L}$ can only contain the y-axis (and, hence, be able to coincide with a tangent plane of the parabolic cylinder $\{\mathcal{H} = h\}$) at points where y vanishes. The contour $\mathcal{L} \cap \{y = 0\}$ in the (x, z)-plane is given by

$$x \;=\; \pm\,(a^2 \;-\; z^2) \quad , \quad -1 \leq z \leq 1 \quad ,$$

and the parabola $\{\mathcal{H} = h\} \cap \{y = 0\}$ has the equation

$$x \;=\; \frac{5 - 12\beta - 3\beta^2}{2\beta(1 - 6\beta)}\, z^2 \;+\; \frac{10a(1 - \beta^2)}{2\beta(1 - 6\beta)}\, z \;+\; \frac{h}{2\beta(1 - 6\beta)} \quad .$$

When reconstructing the flow on the sphere $\mathcal{S}$ given by (5), all equilibria on $\mathcal{L}$ give rise to two equilibria on $\mathcal{S}$, except for the equilibria at the singular points $(0, 0, \pm a)$. It is exactly the combined effect of the regular equilibria reaching the singular point and the degeneration to pairs of planes that leads to the 'omission of the intermediate phase portraits', *i.e.* to the degeneracy of the bifurcation at $\beta = 0$.

To explain the mechanism that leads to the degenerate bifurcation, we can fix for every parameter value $\beta < 0$ the parabola $q(\beta)$ that passes through the singular point, see *figure 3.2*. This parabola intersects the contour of $\mathcal{L}$ in two further points, and obviously the one with the larger z co-ordinate $z(\beta)$ yields an upper bound of the z co-ordinates of the two regular equilibria. As $\beta \nearrow 0$ the parabola $q(\beta)$ gets more and more pointed to approach the double line $\{\, (x, z) \in \mathbb{R}^2 \mid z = -a \,\}$. Thus, the upper bound $z(\beta)$ tends to $-a$ as $\beta \nearrow 0$ and forces both equilibria to simultaneously reach the singular point for $\beta = 0$.

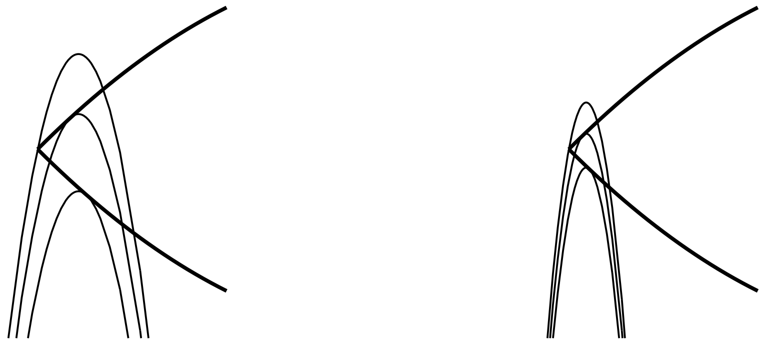

Figure 3.2: The z co-ordinate $z(\beta)$ of the intersection of the parabola $q(\beta)$ through the singular point with the lower arc of the contour $\mathcal{L} \cap \{y = 0\}$ is an upper bound for the z co-ordinates of the two tangent points.

Of course, these two equilibria are reduced from the four equilibria $E_2 - E_5$ computed in [Miller;91] and the explicit formulae given there immediately yield the coinciding collapse at $\beta = 0$. But the mechanism explained above allows one to draw the same conclusion for all higher order normal forms as well.

The argument goes as follows. For $\beta = 0$ the normalisation of (1) only affects the (x_1, p_1)-subsystem since the harmonic oscillator $\frac{1}{2}(p_2^2 + \omega^2 x_2^2)$ cannot be further simplified. Therefore, the higher order terms become functions in $\frac{1}{2}(p_1^2 + \omega^2 x_1^2) \stackrel{!}{=} a + z$. In particular, in a reduced normal form the energy level through $(0, 0, -a)$ is always a degenerate double plane. The reader is referred to [Sommer;97] for more details.

This suggests that the degenerate bifurcation of equilibria in the reduced normal form triggers a similar bifurcation of periodic orbits in the original family (1).

Remark : It is interesting to discuss the co-dimension of our degenerate bifurcation. As the mechanism involves integrability at the bifurcation parameter, the co-dimension within the 'universe' of all Hamiltonian systems with two degrees of freedom in infinite. In [Broer,Vegter;92] a 'universe' is studied where 1-parameter families of Hamiltonian systems are forced to be integrable at a certain parameter value, leading to a *Birkhoff bifurcation* where a whole circle of equilibria occurs. In our situation, this would correspond to the double plane at $\beta = 0$ intersecting $\mathcal{L}$ in a whole circle. Thus, in a generic family of such a 'universe' one would need an additional parameter to shift the double plane to pass through the singular point.

Remark : The system studied originally in [Hénon,Heiles;64] is a rotationally symmetric perturbation of the isotropic harmonic oscillator in three dimensions and reduced to two degrees of freedom by neglecting the (cyclic) angle in cylindrical co-ordinates. The $\mathbb{Z}_2$-symmetry in (1) is reminiscent of that original rotational symmetry. Reducing the rotational symmetry by means of invariants directly leads to the 'lemon-shaped' phase space we encountered after reduction of the 'left-over' $\mathbb{Z}_2$-symmetry, cf. [Cushman,Ferrer,Hanßmann;99].

§4 Implications for the original system

To reconstruct from the one-degree-of-freedom systems the flow in two degrees of freedom, we have to attach an S^1 to every point of the reduced phase space $\mathcal{S}$. This is best done in the Lissajous variables of Section 2, where the angle ℓ parametrising the attached S^1 has the (fast) frequency $\dot{\ell} = \frac{\partial H}{\partial L}$. In this way equilibria on $\mathcal{S}$ give rise to periodic orbits, while periodic orbits on $\mathcal{S}$ yield invariant 2-tori of the flow of $X_{\bar{H}}$.

To obtain information on the "original" flow, defined by X_H , we use that (in appropriate co-ordinates, given by the normalisation procedure) H is an $(\varepsilon^2)^2$-small perturbation of $\bar{H}$. Therefore, iso-energetic KAM-theory, cf. [Arnol'd;63], [Broer,Huitema;91], allows to conclude that most invariant 2-tori survive the perturbation if ε^2 is sufficiently small ; on each energy shell the complement of persisting 2-tori has a measure that tends to 0 as $\varepsilon \to 0$. The crucial point is the behaviour of the (slow) frequency under variation of the (reduced) periodic orbits on $\mathcal{S}$. Now this frequency is an analytic function and depends on ε only through a global multiplication. Hence, it suffices to show that this function is not constant. For $\beta < 0$ there are separatrices, where the frequency converges to zero, and for $\beta \geq 0$ the non-zero second Birkhoff coefficient of the equilibrium at $(0, 0, a)$ allows to draw this conclusion as well.

Concerning the periodic orbits of $X_{\bar{H}}$, the implicit mapping theorem yields their persistence as long as these are elliptic or hyperbolic. While the ones reconstructed from $(0, 0, \pm a)$ are elliptic (except for the one reconstructed from $(0, 0, -a)$ when $\beta = 0$), we get hyperbolic periodic orbits for $\beta < 0$ when the synchronisation angle g turns the 'ellipse' into a straight line. The other pair of periodic orbits for $\beta < 0$ is again elliptic and corresponds to a synchronisation g that maximises the area of the ellipse.

The bifurcating orbit has the (normal) linearization $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and since the bifurcation is degenerate it is not possible to use the implicit mapping theorem for persistence of the whole bifurcation scenario. On the other hand the family defined by (1) degenerates (to a separable system) at $\beta = 0$. Therefore, we work directly with the equations of motion

$$\begin{aligned} \dot{x}_1 &= p_1 \\ \dot{p}_1 &= -\omega^2 x_1 - \varepsilon(x_1^2 + \beta x_2^2) \\ \dot{x}_2 &= p_2 \\ \dot{p}_2 &= -\omega^2 x_2 - 2\varepsilon\beta x_1 x_2 \end{aligned} \tag{8}$$

defined by H .

From the two equilibria $(0, 0, \pm a)$ one reconstructs two kinds of periodic motion, with all energy contained in the (x_1, p_1) or the (x_2, p_2) subsystem, respectively. The former is triggered by the symmetry $(x_1, p_1, x_2, p_2) \mapsto (x_1, p_1, -x_2, -p_2)$ of the system and as well present in the flow of X_H . The latter is triggered by a symmetry $(x_1, p_1, x_2, p_2) \mapsto (-x_1, -p_1, x_2, p_2)$, which is introduced by the normalisation procedure. However, for $\beta = 0$ the two subsystems decouple and there is a family of periodic orbits

$$x_1(t) \equiv 0 \equiv p_1(t) \qquad x_2(t) = \sqrt{\tfrac{2a}{\omega}} \sin \omega t \qquad p_2(t) = \sqrt{2a\omega} \cos \omega t$$

parametrised by their energy $H = 2a\omega$. These are the bifurcating periodic orbits, one on each energy shell.

Before proving that this family of periodic orbits undergoes a degenerate bifurcation, we give some heuristic remarks why this should be true. It is instructive to understand why, still for $\beta = 0$, the periodic orbits with all energy concentrated in the (x_1, p_1) subsystem are elliptic, while the ones with all energy concentrated in the (x_2, p_2) subsystem have trivial Floquet matrices. The (x_1, p_1) subsystem is an anharmonic oscillator, the frequency ν of the periodic orbits decreases monotonously when the energy is increased, cf. [Chow,Sanders;86]. Therefore, $\nu < \omega$ and the 'ellipse' generated by superposition with the harmonic oscillator in the (x_2, p_2) subsystem slowly changes its shape, $\dot{g} \neq 0$. In particular, the normal frequency of the periodic orbit concentrated in the (x_1, p_1) subsystem is given by $\omega - \nu$ and hence nonzero. However, $\omega - \nu$ does vanish at $x_1 \equiv 0 \equiv p_1$, whence the periodic orbit concentrated in the (x_2, p_2) subsystem has vanishing Floquet exponents.

When $\beta > 0$, the coupling $\varepsilon\beta x_1 x_2^2$ would transfer energy to the (x_1, p_1) subsystem if all energy were concentrated in the (x_2, p_2) subsystem. This indicates that there exists some 'equilibrium state' with small $\frac{1}{2}(p_1^2 + \omega^2 x_1^2)$ — a periodic orbit. Here the anharmonicity of the (x_1, p_1) oscillator is able to provide (normal) ellipticity of the periodic orbit. This is the elliptic periodic orbit reconstructed from $(0, 0, -a)$. As in the case $\beta = 0$, on each energy shell there is one family of invariant 2-tori between the two periodic orbits (which are, in fact, the ones predicted by Lyapunov's theorem on small oscillations, cf. [Moser;76], [Weinstein;73,78]) and there are no further isolated periodic orbits 'with short period'.

The situation for $\beta < 0$ is slightly different. Again the coupling $\varepsilon\beta x_1 x_2^2$ leads to an elliptic periodic orbit, but this time the 'frequency in the (x_1, p_1) subsystem' is increased rather than decreased, cf. (8). Consequently, the normal frequency of the periodic orbit now has opposite sign and on each energy shell there are perturbed 2-tori in $1:1$ resonance. This leads to further isolated periodic orbits, and these are exactly the additional four periodic orbits we encountered in the flow of $X_{\bar{H}}$. In particular, both the elliptic periodic orbits and the hyperbolic periodic orbits have 'to wait until $\beta = 0$' to coalesce with the 'central' elliptic periodic orbit.

To prove these assertions we work with β as the "small parameter", keeping $\varepsilon > 0$ fixed. When $\beta = 0$ both the series K obtained by indefinite normalisation of (1) and the sequence of normalising co-ordinate transformations ψ_n coverge, see [Rüssmann;64]. We denote the limit of the latter by $\phi := \lim_{n \to \infty} \psi_n$, *i.e.* we have $H \circ \phi = K$.

We now re-incorporate the parameter α in (1) and write the parameter dependence of the Hénon-Heiles family explicitly as $H_{\alpha\beta}$. To similarly obtain $\bar{H}_{\alpha\beta}$ in (2) one simply has to include factors α and α^2 in such a way that the powers of α and β always add up to the exponent of ε. For finite order n the normalisation works for $\beta \neq 0$ as well, yielding co-ordinate transformations $\psi_{n\beta}$. We have already noted that $H_{\alpha\beta} \circ \psi_{3\beta} - \bar{H}_{\alpha\beta} \sim \mathcal{O}(\varepsilon^4)$, and our present aim is to obtain a similar smallness in β.

The series K obviously starts with the first line of (2). Let $N_{\alpha\beta}$ denote the sum of K and the terms in the second and third lines of (2). Furthermore we write $\phi_\beta := \phi \circ \psi_3^{-1} \circ \psi_{3\beta}$. Thus ϕ_β transforms $H_{\alpha\beta}$ for $\beta = 0$ into K, and for $\beta \neq 0$ into $N_{\alpha\beta}$ + *higher order terms*. These terms are of orders $\alpha^3\beta$, $\alpha^2\beta^2$, $\alpha\beta^3$ or β^4, respectively (since terms of order α^4 are included in K). Putting $\alpha = \beta^{2/3}$ yields the desired estimate

$$\|H_{\beta^{2/3}\beta} \circ \phi_\beta \; - \; N_{\beta^{2/3}\beta}\|_{A(h)} \;\; \leq \;\; \Gamma(h) \cdot \beta^3 \tag{9}$$

for sufficiently small β. Here $\|\,..\,\|_{A(h)}$ denotes the supremum norm on the union $A(h)$ of the energy shells $\{\, H_{\beta^{2/3}\beta} = h' \,\}$ with $h' \leq h$, and $\Gamma(h)$ is a constant that only depends on h (and not on β).

As in Section 3 we reduce $N_{\beta^{2/3}\beta}$ to $\mathcal{L}$, omit the constant part, divide by $\frac{\varepsilon^2\beta^{4/3}}{12\omega^4}$ and rescale time. In this way we obtain

$$\mathcal{N} \;\; = \;\; 2\beta^{1/3}(1 - 6\beta^{1/3})x \; - \; 10a(1 - \beta^{2/3})z \; - \; (5 - 12\beta^{1/3} - 3\beta^{2/3})z^2 \; + \; \mathcal{O}(\varepsilon^2) \quad .$$

We have seen in Section 3 that, for sufficiently small $\varepsilon > 0$, the qualitative properties of $\mathcal{N}$ do not differ from those of $\mathcal{H}$.

Bifurcation Theorem

Let $H_{\alpha\beta}$ be given by (1) with α positive and ε sufficiently small. Consider an energy shell $\{H_{\alpha\beta} = h\}$ with h sufficiently small as well. Then the following bifurcation takes place as β passes through zero.

For $\beta \geq 0$ there are exactly two periodic orbits on $\{H_{\alpha\beta} = h\}$. At $\beta = 0$ one of those loses its ellipticity and bifurcates into two hyperbolic and three elliptic periodic orbits. These and the further elliptic periodic orbit are present for $\beta < 0$.

Proof :

The flows of $X_{H_{\alpha\beta}}$ and $X_{H_{\beta^{2/3}\beta}}$ are equivalent to $X_{H_{1\frac{\beta}{\alpha}}}$ and $X_{H_{1\beta^{1/3}}}$, respectively. The situation for $\beta = 0$ has been described above, so we only need to prove the assertions concerning $\beta \neq 0$. Therefore, we work with $H_{\beta^{2/3}\beta}$ using the reparametrisation $\beta \to \frac{\beta^3}{\alpha^3}$. Now these assertions are only about elliptic and hyperbolic periodic orbits, and we know them to hold true for $N_{\beta^{2/3}\beta}$. Using the estimate (9) we invoke the implicit mapping theorem to conclude the proof **;** **q.e.d.**

The smallness conditions on h and ε are also needed to guarantee the convergence of K and ϕ . For instance, the equilibrium at $(x,p) = (-\frac{1}{\varepsilon}, 0, 0, 0)$ with energy $h = \frac{\omega^2}{2\varepsilon^2} - \frac{1}{3\varepsilon^2}$ cannot be reached by our perturbation analysis.

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