ON THE NUMBER OF NEGATIVE EIGENVALUES OF SCHRÖDINGER OPERATORS WITH AN AHARONOV-BOHM MAGNETIC FIELD

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Abstract. It is proved that for $V_+ = \max(V, 0)$ in the subspace $L^1(\mathbb{R}^+,\ L^\infty(\mathbb{S}^1),\ rdr)$ of $L^1(\mathbb{R}^2)$, there is a Cwikel-Lieb-Rosenblum type inequality for the number of negative eigenvalues of the operator $\left(\frac{1}{i}\vec{\nabla}+\vec{A}\right)^2 - V$ in $L^2(\mathbb{R}^2)$ when $\vec{A}$ is an Aharonov-Bohm magnetic potential with non-integer flux. It is shown that $L^1(\mathbb{R}^+,\ L^\infty(\mathbb{S}^1),\ rdr)$ can not be replaced by $L^1(\mathbb{R}^2)$ in the inequality.

1. Introduction

The Schrödinger operators $T_A(V)$ considered are self-adjoint realisations in $L^2(\mathbb{R}^2)$ of expressions of the form

$$T_A(V) \ := \ \left(\frac{1}{i}\vec{\nabla} + \vec{A}\right)^2 - V \ \equiv \ S_A - V, \tag{1.1}$$

where $curl\,\vec{A}$ is a magnetic field of Aharonov-Bohm type, namely, in polar coordinates

$$\vec{A} \ = \ \frac{\Psi(\theta)}{r}(\sin\theta, -\cos\theta) \tag{1.2}$$

for $\Psi \in L^\infty(\mathbb{S}^1)$; thus $curl\,\vec{A} = 0$ for $r \neq 0$ and $\vec{A}\cdot\vec{x} = 0$. For such $\vec{A}$ it was proved by Laptev and Weidl in [4] that the Hardy-type inequality

$$\int_{\mathbb{R}^2} \frac{|u(\mathbf{x})|^2}{|\mathbf{x}|^2}\, d\mathbf{x} \ \le \ C \int_{\mathbb{R}^2} \left|\left(\frac{1}{i}\vec{\nabla} + \vec{A}\right)u(\mathbf{x})\right|^2 d\mathbf{x}, \tag{1.3}$$

holds, with sharp constant $C = \left(\min_{k\in\mathbb{Z}} |k+\tilde{\Psi}|\right)^{-1}$, $\tilde{\Psi}$ being the magnetic flux $\tilde{\Psi} := \frac{1}{2\pi}\int_0^{2\pi} \Psi(\theta)\, d\theta$. It was predicted in [4] that a consequence of (1.3) should be an inequality of Cwikel-Lieb-Rosenblum (CLR) type for the number of negative eigenvalues (bound states) of $T_A(V)$ when V belongs to some appropriate function space. The confirmation of this prediction is the main result in this paper. Two vital components in the analysis were established in [2]. In [2, Theorem 1] the following

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embedding theorem was obtained: for the space H_A^1 defined as the completion of $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ with respect to the norm $t_A[\,\cdot\,]^{1/2}$, where

$$t_A[\varphi] = \int_{\mathbb{R}^2} \left(\left| \left(\frac{1}{i}\vec{\nabla} + \vec{A} \right) \varphi \right|^2 + |\varphi|^2 \right) d\mathbf{x}, \tag{1.4}$$

for $\vec{A}$ in (1.2) and $u \in H_A^1$,

$$\|u\|_X^2 \le C \left\| \left(\frac{1}{i}\vec{\nabla} + \vec{A} \right) u \right\|^2, \tag{1.5}$$

where $X = L^\infty(\mathbb{R}^+,\ L^2(\mathbb{S}^1),\ rdr) \equiv L^\infty(\mathbb{R}^+,\ rdr) \otimes L^2(\mathbb{S}^1)$ with

$$\|u\|_X = \operatorname*{ess\,sup}_{r\in\mathbb{R}^+} \left\{ \int_0^{2\pi} |u(r,\theta)|^2 \, d\theta \right\}$$

and $C = \left(\min_{k\in\mathbb{Z}} |k + \tilde{\Psi}| \right)^{-1}$ is sharp. When $V \in L^1(\mathbb{R}^+,\ L^\infty(\mathbb{S}^1),\ rdr) \equiv L^1(\mathbb{R}^+,\ rdr) \otimes L^\infty(\mathbb{S}^1)$, it was proved in [2, Theorem 2] that $S_A^{-1/2} V S_A^{-1/2}$ is compact in $L^2(\mathbb{R}^2)$, and hence that $T_A(V)$ has the same essential spectrum as S_A, namely $[0,\infty)$. Therefore the negative spectrum of $T_A(V)$ consists only of eigenvalues. We prove the following:

Theorem 1. *Let $\vec{A}$ be given by (1.2), $V \in L^1_{loc}(\mathbb{R}^2)$ and $V_+ \in L^1(\mathbb{R}^+,\ L^\infty(\mathbb{S}^1),\ rdr)$, where $V_+ = \max(V,0)$, and suppose that the magnetic flux $\tilde{\Psi}$ is not an integer. Then $N(T_A(V))$, the number of negative eigenvalues of $T_A(V)$, is finite and*

$$N(T_A(V)) \le e\|V_+\|_Y \sum_{|m+\tilde{\Psi}| \le e\|V_+\|_Y} \frac{1}{|m + \tilde{\Psi}|}, \tag{1.6}$$

where $Y = L^1(\mathbb{R}^+,\ L^\infty(\mathbb{S}^1),\ rdr)$.

Note that in (1.6)

$$\|V_+\|_Y = \int_0^\infty \left\{ \operatorname*{ess\,sup}_{\theta\in\mathbb{S}^1} |V_+(r,\theta)| \right\} rdr.$$

Theorem 2. *Suppose that the hypothesis of Theorem 1 is satisfied. Then*

$$N(T_A(V)) \le c(\tilde{\Psi}) \|V_+\|_{L^1(\mathbb{R}^+,\ L^\infty(\mathbb{S}^1),\ rdr)}, \tag{1.7}$$

where $c(\tilde{\Psi})$ is a constant depending only on $\tilde{\Psi}$.

To prove Theorem 1 we make use of the abstract CLR inequality established by Levin and Solomyak in [5, Theorem 1.2]. For Theorem 2 we apply Theorem 1.2 in Laptev and Netrusov [3] concerning an operator of the form

$$-\Delta + \frac{b}{|\mathbf{x}|^2} - V, \quad b > 0$$

in $L^2(\mathbb{R}^2)$. This and our operator $T_A(V)$ have similar characteristics, and indeed both satisfy (1.7), but are not in fact comparable. It is well-known that Hardy's inequality is not valid in $\mathbb{R}^2$ and there is no CLR inequality in terms of $L^1(\mathbb{R}^2)$ norm of V. A Hardy-type inequality is introduced in [3] by the presence of the term $\frac{b}{|\mathbf{x}|^2}$, while in our case (1.3) fulfills the role. The resulting CLR inequalities require V_+ to belong to a subspace of $L^1(\mathbb{R}^2)$, namely $L^1(\mathbb{R}^+,\ L^\infty(\mathbb{S}^1),\ rdr)$ in our case. In § 3 we show that the $L^1(\mathbb{R}^+,\ L^\infty(\mathbb{S}^1),\ rdr)$ norm of V_+ can not be replaced by the $L^1(\mathbb{R}^2)$ norm on the right-hand side of (1.7).

2. Proof of the theorems

We need some preliminary results and remarks. Note that we may assume, without loss of generality, that $\Psi \equiv \tilde{\Psi}$ in (1.2) since the corresponding magnetic potentials $\vec{A}$ generate the same magnetic field and hence the corresponding operators are gauge equivalent.

The operator $T_A(V)$ is defined as the (Friedrichs) operator associated with the quadratic form

$$Q[u] \ = \int\limits_{\mathbb{R}^2} \left(\left| \left(\frac{1}{i}\vec{\nabla} + \vec{A} \right) u \right|^2 - V|u|^2 \right) d\mathbf{x}, \quad u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}) \tag{2.1}$$

with

$$V \ \in \ L^1_{loc}(\mathbb{R}^2). \tag{2.2}$$

If $V_+ \in L^1(\mathbb{R}^+,\ L^\infty(\mathbb{S}^1),\ rdr)$, it follows from [2, Theorem 2] that Q is bounded below in $L^2(\mathbb{R}^2)$. It is also closable and $T_A(V)$ is defined as the self-adjoint operator in $L^2(\mathbb{R}^2)$ whose form domain is the domain of the closure of Q, which we still denote by Q.

Let

$$W(r) \ := \ \|V_+(r,\cdot)\|_{L^\infty(\mathbb{S}^1)}\ , \tag{2.3}$$

so that

$$\begin{aligned} \|W\|_{L^1(\mathbb{R}^+,\ rdr)} \ &= \ \|W\|_{L^1(\mathbb{R}^+,\ L^\infty(\mathbb{S}^1)\ rdr)} \\ &= \ \|V_+\|_{L^1(\mathbb{R}^+,\ L^\infty(\mathbb{S}^1)\ rdr)} \ < \ \infty\ . \end{aligned} \tag{2.4}$$

Then, from [2, Theorem 2] again, $T_A(W)$ is a lower semi-bounded, self-adjoint operator with essential spectrum $[0,\infty)$. Since $T_A(V) \geq T_A(W)$ we have

$$N(T_A(V)) \ \leq \ N(T_A(W)), \tag{2.5}$$

and so the theorems will follow if we prove them for $T_A(W)$. Hereafter in the proof we take V to be replaced by W in (2.1).

We need the following facts from [1] and [4]:

$$\begin{aligned} L^2(\mathbb{R}^2) &= L^2(\mathbb{R}^+; rdr) \otimes L^2(\mathbb{S}^1) \\ &= \underset{m\in\mathbb{Z}}{\oplus} \left\{ L^2(\mathbb{R}^+; rdr) \otimes \left[\frac{e^{im\phi}}{\sqrt{2\pi}} \right] \right\}, \end{aligned} \tag{2.6}$$

where $[\ \cdots\ ]$ denotes the linear span;

$$S_A = \underset{m\in\mathbb{Z}}{\oplus} \{ D_m \otimes 1_m \}, \tag{2.7}$$

where D_m is the Friedrichs operator in $L^2(\mathbb{R}^+; rdr)$ associated with the quadratic form

$$h_m[u] = \int_0^\infty \left(|u'(r)|^2 + \frac{(m+\alpha)^2}{r^2} |u(r)|^2 \right) rdr, \quad \alpha = \tilde{\Psi} \ ; \tag{2.8}$$

for any u in the form domain of $T_A(W)$ we have

$$u(r,\theta) = \sum_{m\in\mathbb{Z}} u_m(r) \left\{ \frac{1}{\sqrt{2\pi}} e^{im\theta} \right\} \tag{2.9}$$

and

$$Q[u] = \sum_{m\in\mathbb{Z}} \left\{ h_m[u_m] - \int_0^\infty W(r)|u_m(r)|^2 \ rdr \right\}, \tag{2.10}$$

where Q is given by (2.1), but with V replaced by W. It follows that

$$T_A(W) = \underset{m\in\mathbb{Z}}{\oplus} \{ (D_m - W) \otimes 1_m \}, \tag{2.11}$$

and the negative spectrum of $T_A(W)$ is the aggregate of the eigenvalues of the $D_m - W$.

To prove Theorem 1 we apply the result of Levin and Solomyak in [5, Theorem 1.2] to the operators $D_m - W$ in $L^2(\mathbb{R}^+;\ rdr)$. The form h_m in (2.8) is easily seen to satisfy the Beurling-Deny conditions and hence the operator D_m generates a Markov semigroup in $L^2(\mathbb{R}^+;\ rdr)$; indeed the part

$$\int_0^\infty |u'(r)|^2 \ rdr$$

in h_m corresponds to the restriction of the Laplacian in $L^2(\mathbb{R}^2)$ to spherically symmetric functions. We also know from the proof of [2, Theorem 1] that, for u in the domain of h_m

$$\|u\|^2_{L^\infty(\mathbb{R}^+)} \le \frac{1}{|m+\alpha|} \ h_m[u]. \tag{2.12}$$

Hence the Sobolev inequality in [5, (1.7)] is also satisfied with $q = \infty$. We therefore have

$$N(D_m - W) \ \leq \ \frac{e}{|m+\alpha|} \int_0^\infty W(r)\ rdr,$$

which yields

$$N(T_A(W)) \ \leq \sum_{|m+\alpha| \leq e \int_0^\infty W(r)\ rdr} \frac{e}{|m+\alpha|} \int_0^\infty W(r)\ rdr$$

from (2.11), whence (1.6).

For the proof of Theorem 2 we apply Theorem 1.2 in Laptev and Netrusov [3]. We may assume, without loss of generality, that $\alpha \in (0,1)$, since the gauge transformation $f(r,\theta) \mapsto e^{in\theta} f(r,\theta)$, for $n \in \mathbb{Z}$, takes $\tilde{\Psi}$ into $\tilde{\Psi} + n$ and gives rise to unitary equivalent operators. For $m \geq 0$ we have

$$D_m \ \geq \ -\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) + \frac{\alpha^2}{r^2} + \frac{m^2}{r^2}\ , \tag{2.13}$$

and for $m < 0$,

$$D_m \ \geq \ -\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) + \frac{(1-\alpha)^2}{r^2} + \frac{(m+1)^2}{r^2}\ . \tag{2.14}$$

The two dimensional case of the operator considered in [3] has the form

$$L(b; W) \ \equiv \ -\Delta + \frac{b}{|\mathbf{x}|} - W \ = \ \bigoplus_{m \in \mathbb{Z}} \{(L_m(b) - W) \otimes 1_m\}, \quad b > 0,$$

in (2.6), where

$$L_m(b) \ = \ -\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) + \frac{b}{r^2} + \frac{m^2}{r^2}\ ,$$

and $L_m(b) - W$ is defined by the associated quadratic form in $L^2(\mathbb{R}^+;\ rdr)$. It follows from (2.13) and (2.14) that

$$\begin{aligned} N(\bigoplus_{m \geq 0} \{(D_m - W) \otimes 1_m\}) \ &\leq \ N(\bigoplus_{m \geq 0} \{(L_m(\alpha^2) - W) \otimes 1_m\}) \\ &\leq \ N(L(\alpha^2; W)) \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} N(\bigoplus_{m < 0} \{(D_m - W) \otimes 1_m\}) \ &\leq \ N(\bigoplus_{m < 0} \{(L_m((1-\alpha)^2) - W) \otimes 1_m\}) \\ &\leq \ N(L((1-\alpha)^2; W))\ . \end{aligned} \tag{2.16}$$

Therefore, from [3, Theorem 1.2] and (2.11)

$$N(T_A(W)) \ \leq \ c(\alpha)\ \|W\|_{L^1(\mathbb{R}^+,\ L^\infty(\mathbb{S}^1),\ rdr)}$$

and Theorem 2 is proved.

3. Remark

We shall prove the following: given $\varepsilon > 0$ there exists $V \in L^1(\mathbb{R}^2)$, $V \geq 0$, such that $\|V\|_{L^1(\mathbb{R}^2)} \leq \varepsilon$ but $T_A(V)$ has a negative eigenvalue. This has the consequence that in (1.6) and (1.7) the $L^1(\mathbb{R}^+,\ L^\infty(\mathbb{S}^1),\ rdr)$ norm of V_+ can not be replaced by the $L^1(\mathbb{R}^2)$.

Our starting point is the well-known fact that the CLR inequality is not valid for $-\Delta - V$ in $L^1(\mathbb{R}^2)$, i.e. there is no inequality

$$N(T_0(V)) \ \leq \ C\|V\|_{L^1(\mathbb{R}^2)}.$$

It follows that given $\varepsilon > 0$, there exist a compactly supported $V \in L^1(\mathbb{R}^2)$ such that $\|V\|_{L^1(\mathbb{R}^2)} < \varepsilon$ and $N(T_0(V)) \geq 1$. Hence, there exists $f \in C_0^\infty(\mathbb{R}^2)$ such that for some $\delta = \delta(\varepsilon) > 0$

$$(T_0(V)f, f) < -\delta.$$

We may suppose that V and f are supported in $B(0, R)$, the ball centre the origin and radius $R = R(\varepsilon)$. Define the translation

$$\tau : \ \vec{x} \ \mapsto \ \vec{x} + (2R, 0)$$

and set $\tilde{V} = V \circ \tau$, $\tilde{f} = f \circ \tau$, so that both $\tilde{V}$ and $\tilde{f}$ are supported in $B(-2R, R)$ which lies in the half plane $P = \{(x, y) : \ x < -R\}$. Since the magnetic field $B = curl\, \vec{A}$ is zero in the simply connected set P we have by Poincare's Theorem that there exist a gauge transformation

$$\phi(\mathbf{x}) \ \mapsto \ e^{ik(\mathbf{x})}\phi(\mathbf{x}), \quad k : \ P \to \mathbb{R}$$

which is such that $-\Delta$ and S_A are equivalent in P. Thus, with $g(\mathbf{x}) = e^{ik(\mathbf{x})}\tilde{f}(\mathbf{x})$, we have

$$\begin{aligned}(T_A(\tilde{V})g, g) \ &= \ (T_0(\tilde{V})\tilde{f}, \tilde{f}) \\ &= \ (T_0(V)f, f) \ < \ \delta\ .\end{aligned}$$

Our claim is therefore established.

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