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Asymptotic Dynamics of Nonlinear Schrödinger Equations:

Resonance Dominating and Radiation Dominating Solutions

Tai-Peng Tsai*, Horng-Tzer Yau†

Courant Institute
New York University
New York, NY, 10012

Abstract

We consider a linear Schrödinger equation with a small nonlinear perturbation in $\mathbb{R}^3$. Assume that the linear Hamiltonian has exactly two bound states and its eigenvalues satisfy some resonance condition. We prove that if the initial data is near a nonlinear ground state, then the solution approaches to certain nonlinear ground state as the time tends to infinity. Furthermore, the difference between the wave function solving the nonlinear Schrödinger equation and its asymptotic profile can have two different types of decay: 1. The resonance dominating solutions decay as $t^{-1/2}$. 2. The radiation dominating solutions decay at least like $t^{-3/2}$.

*ttsai@cims.nyu.edu
†Work partially supported by NSF grant DMS-0072098, yau@cims.nyu.edu

1 Introduction

Let H_0 be the Hamiltonian $H_0 = -\Delta + V - e_0$ with V a smooth localized potential and $e_0 < 0$ the ground state energy to $-\Delta + V$. Consider the nonlinear Schrödinger equation

$$i\partial_t \psi = (-\Delta + V)\psi + \lambda |\psi|^2 \psi, \qquad \psi(t = 0) = \psi_0 \tag{1.1}$$

where λ is a small positive or negative parameter. Our goal is to understand its asymptotic evolution as $t \to \infty$. The nonlinear bound states to the Schrödinger equation (1.1) are solutions to the nonlinear equation

$$(-\Delta + V)Q + \lambda |Q|^2 Q = EQ, \tag{1.2}$$

For any nonlinear bound state, $\psi_t = Qe^{-iEt}$ is a solution to the nonlinear Schrödinger equation. We may obtain a family of such bound states from minimizing the corresponding nonlinear energy functional:

$$\mathcal{H}[\phi] = \int \frac{1}{2}|\nabla \phi|^2 + \frac{1}{2}V|\phi|^2 + \frac{1}{4}\lambda|\phi|^4 \, dx \ .$$

For each $N > 0$ and λ sufficiently small, subject to the constraint $\|\phi\|_{L^2}^2 = N$, there is a unique positive minimizer Q of $\mathcal{H}$, which solves (1.2) for some $E = E(N)$ and has exponential decay as $x \to \infty$. We call this family nonlinear ground states.

Instead of N, we can use E as the parameter. From now on we will refer to this continuous family as $\{Q_E\}_E$. Let

$$H_E = -\Delta + V - E + \lambda Q_E^2 \ . \tag{1.3}$$

We have $H_E Q_E = 0$. Since λ is small, the spectral properties of H_E are similar to those of H_0.

Suppose the initial data of the nonlinear Schrödinger equation ψ_0 is near some Q_E. If $-\Delta + V$ has only one bound state, it was proved in [7] that the evolution will eventually settle down to some ground state Q_{E_∞} with E_∞ close to E. Suppose now that $-\Delta + V$ has multiple bound states, say, two bound states: a ground state ϕ_0 with eigenvalue e_0 and an excited state ϕ_1 with eigenvalue e_1, i.e., $H_0\phi_1 = e_{01}\phi_1$ where $e_{01} = e_1 - e_0$. The question is whether the evolution with initial data ψ_0 near some Q_E will eventually settle down to some ground state Q_{E_∞} with E_∞ close to E? Furthermore, can we characterize the asymptotic behavior of the evolution?

The family of ground states is stable in the sense that if

$$\inf_{\Omega, E} \left\| \psi_0 - Q_E \, e^{i\Omega} \right\|_{L^2}$$

is small, it remains so for all t. Let $\|\cdot\|_{L^2,loc}$ denote local L^2 norm. One expects that this difference is actually approaching to zero, i.e.,

$$\lim_{t\to\infty} \inf_{\Omega,E} \left\|\psi_t - Q_E\, e^{i\Omega}\right\|_{L^2,loc} = 0 \tag{1.4}$$

In this paper, we shall answer this question positively in the case of two bound states. Furthermore, we estimate precisely the rate of relaxation for certain class of initial data.

We start with a simple question concerning the asymptotic dynamics around a fixed ground state profile at $t = \infty$. More precisely, we first fix an E and the corresponding nonlinear ground state Q_E. Our goal is to analyze the detailed asymptotic behavior of those solutions converging to this nonlinear ground state as $t \to \infty$. Although this problem seems to be simple, we found that there are two different types of behaviors for $\psi_t - Q_E e^{i\Omega}$: one is called resonance dominating solutions ; the other radiation dominating solutions. We first explain the Resonance condition on H_0 and the meanings of these solutions. Recall that the ground state and the excited state to H_0 are denoted by ϕ_0 and ϕ_1 respectively. The following two conditions are assumed for this paper.

Assumption A1: Resonance condition. Let $e_{01} = e_1 - e_0$ be the spectral gap of the ground state. We assume that $2e_{01} > |e_0|$ so that $2e_{01}$ is in the continuum spectrum of H_0. Furthermore,

$$\left(\phi_0\phi_1^2\,,\,\mathrm{Im}\,\frac{1}{H_0 - 0i - 2e_{01}}\,\mathbf{P}_{\mathrm{c}}^{H_0}\,\phi_0\phi_1^2\right) > 0\ . \tag{1.5}$$

Assumption A2 The bottom of continuum spectrum to $-\Delta + V$, 0, is not a generalized engenvalue, i.e., not a resonance.

Fixed an E and its ground state Q_E. Let $\mathcal{L}^{(\mathrm{or})}$ be the operator obtained from linearizing the Schrödinger equation (1.1) around the trivial evolution Qe^{-iEt}, i.e.,

$$\mathcal{L}^{(\mathrm{or})}k = -i\left\{H_E k + \lambda Q^2(k+\bar{k})\right\}\ ,$$

where H_E is defined in (1.3). It is more convenient to work with operators orthogonal to the ground state Q. Let Π be the projection which eliminates the Q-direction:

$$\Pi h = h - (c_0 Q, h)Q\ , \quad c_0 = (Q,Q)^{-1}$$

and let X denote its image,

$$X = \Pi(L^2)\ .$$

Define the operator $\mathcal{L}$ acting on the space $X = \Pi L^2$ by

$$\mathcal{L}h = -i\left\{Hh + \lambda\Pi Q^2\Pi(h+\bar{h})\right\}\ .$$

The operator can be written in matrix form

$$\mathcal{L} = \begin{bmatrix} 0 & L_- \\ L_+ & 0 \end{bmatrix}, \qquad L_- = H, \qquad L_+ = H + 2\lambda \Pi Q^2 \Pi.$$

With respect to $\mathcal{L}$, we define generalized "eigenspaces":

$$\mathtt{E}_\nu(\mathcal{L}) := \left\{ \psi : \mathcal{L}^2 \psi = -\nu^2 \psi \right\} = \left\{ u + iv : u, v \text{ real}, \, L_- L_+ u = \nu^2 u, \, L_+ L_- v = \nu^2 v \right\} \ .$$

Since the original Hamiltonian H_0 has only two bound states with the ground state projected out, there is exactly one value for ν, called κ, and $\dim_{\mathbb{R}} \mathtt{E}_\kappa = 2$. From simple perturbation theory, we know that $\kappa = e_{01} + O(\lambda)$. We can normalized u and v such that $(u, L_+ u) = \kappa$ and $v = \kappa^{-1} L_+ u$. Then $(u, v) = 1$ and the space $\mathtt{E}_\kappa(\mathcal{L})$ is just $\mathrm{span}_{\mathbb{R}} \{u, iv\}$.

Define the inner product

$$((\psi, \phi)) = (\mathrm{Re}\, \psi, \mathrm{Re}\, \phi) + (\mathrm{Im}\, \psi, \mathrm{Im}\, \phi) \ .$$

The space of continuum spectrum of $\mathcal{L}$ can be characterized by

$$\mathtt{H}_\mathrm{c}(\mathcal{L}) := \{ \psi : \psi \perp \mathtt{E}_\nu(\mathcal{L}^*) \text{ for all } \nu \} \ ,$$

where $\mathcal{L}^*$ is the adjoint w.r.t. the inner product just defined. It is also clear from the definition that

$$X = \mathtt{E}_\kappa(\mathcal{L}) \oplus \mathtt{H}_\mathrm{c}(\mathcal{L}) \ .$$

Notice that this is not an orthogonal decomposition. Explicitly, the eigenspace $\mathtt{E}_\kappa(\mathcal{L}^*)$ is simply $\mathrm{span}_{\mathbb{R}} \{v, iu\}$. Hence the continuum space is characterized by

$$\mathtt{H}_\mathrm{c}(\mathcal{L}) = \{ w_1 + i w_2 : w_1 \perp v, w_2 \perp u \} \ .$$

For any initial data ψ_0 near Q, we can solve γ_0 and ℓ_0 uniquely such that

$$\psi_0 = [Q + \gamma_0 Q + \ell_0] \, e^{i\theta_0}, \quad \ell_0 \perp Q \ .$$

It is more convenient to express ψ_0 in term of Q and $R = R_E$ defined by

$$R_E = \partial_E Q_E$$

We shall see in next section that R_E is of order λ^{-1}. We now rewrite the initial data as

$$\psi_0 = [Q + a_0 R + h_0] \, e^{i\theta_0}, \quad h_0 \perp Q \ .$$

(cf. Lemma 9.1) From the decomposition of X, we can write

$$h_0 = \alpha_0 u + \beta_0 i v + \eta_0, \quad \eta_0 \in \mathtt{H}_\mathrm{c}(\mathcal{L}) \ .$$

Denote the L^2 norm of ψ_0 by n. Since the evolution is nonlinear, we can always re-scale λ so that n, the L^2 norm of ψ_0, takes any fixed value. It is more convenient to allow n to be a parameter between, say, 1 and 10. Define $\mathcal{I}_\lambda$ to be the interval so that the L^2 norm of the ground state Q_E is between 1 and 10. Let $z_0 = \alpha_0 + i\beta_0$. Define the notations

$$\langle x \rangle = \sqrt{1+x^2}, \qquad \{t\}_\varepsilon = \varepsilon^{-2} + 2\Gamma t, \quad \{t\}_\varepsilon \sim \max\left\{\varepsilon^{-2}, t\right\}$$

where Γ is a constant to be specified later on. For the moment, we remark that Γ is of order $2\lambda^2$ times the quantity in (1.5). When the subscript ε is understood, we shall drop it.

Theorem 1.1 (1) **[Resonance dominating solutions]** *There exists small parameters λ_0 and ε_0 such that for any $E \in \mathcal{I}_\lambda$ and $|\lambda| \le \lambda_0$ the following hold. Suppose that*

$$0 < |z_0| = \epsilon \le \epsilon_0,$$

$$\|\eta_0\|_{H^2\cap W^{2,1}(\mathbb{R}^3)} \le C|z_0|^{2+\sigma}\ ,$$

for some small constant σ. Then we can find a small real number $a_0 = a_0(E, h_0)$ such that the solution $\psi(t)$ to the Schrödinger equation (1.1) *with the initial data $\psi_0 = Q + a_0 R + h_0$ can be decomposed as*

$$\psi(t) = [Q + a(t)R + h(t)]\, e^{i\Omega(t)} \tag{1.6}$$

with

$$a(0) = a_0, \quad h(0) = h_0, \quad \Omega(0) = 0,$$

and $a(t), h(t) \to 0$ as $t \to \infty$ in the following sense: Let $h(t) = \zeta(t) + \eta(t)$ with $\zeta(t)$ the component in $\mathtt{E}_\kappa(\mathcal{L})$ and $\eta(t)$ the component in $\mathtt{H}_c(\mathcal{L})$. Then we have

$$|a(t)| \le C\,\{t\}^{-1}$$

$$\zeta(t)\ldots\text{local, smooth, } \|\zeta(t)\|_{L^2} \sim \{t\}^{-1/2}\ ,$$

$$\eta(t)\ldots\text{dispersive wave, } \|\eta(t)\|_{L^4} \le C\,\{t\}^{-3/4+\sigma},\ \ \|\eta(t)\|_{L^2_{loc}} \le C\,\{t\}^{-1}\ ,$$

where $\{t\} = \varepsilon^{-2} + t$. Also, $\Omega(t)/t \to -E$.

(2) **[Radiation dominating solutions]** *For any $\chi_\infty \in H^2 \cap W^{2,1}(\mathbb{R}^3)$ small, there exist solutions of the form*

$$\psi(x,t) = [Q(x) + a(t)R(x)]\ e^{i\Omega(t)} + \chi(x,t)$$

such that

$$|a(t)| \le Ct^{-2}, \qquad \|\chi(\cdot,t)\|_{L^2_{loc}} \le Ct^{-3/2}$$

and $\chi = \chi_1 + \chi_2$, where $\chi_1 = e^{i\Delta t}\chi_\infty$ and $\|\chi_2(\cdot,t)\|_{L^2} \le o(t^{-3/2})$.

We have used the notation $f \sim g$ for $Cg \le f \le C^{-1} g$ for some constant C. Thus the decay of the excited state for solutions constructed in part (1) is given precisely. Consequently, the two types of solutions are completely different. Return to the general question concerning the asymptotic mass of the ground state profile at $t = \infty$ (1.4). Instead of minimizing the L^2 norm, we prefer to determine the ground state profile by the condition

$$\left(\psi_t - Q_{E(t)}\, e^{i\Theta(t)}\,,\, Q_{E(t)}\right) = 0 \tag{1.7}$$

We shall prove that there is a unique solution to this equation provided that the initial data is near some $Q_{\text{in}} = Q_{E_{\text{in}}}$ (Lemma 9.2). Furthermore, we can determine the change between $E(t)$ and E_{in}.

Theorem 1.2 *Let λ_0 and ε_0 be given as in the first theorem. There is a small constant σ such that for any $E_{\text{in}} \in \mathcal{I}_\lambda$ with $\lambda \le \lambda_0$ the following properties hold:*

(1) *Suppose the initial data*

$$\psi_0 = Q_{\text{in}} + a_{\text{in}} R_{\text{in}} + h_{\text{in}}$$

satisfies that

$$\|\psi_0 - Q_{\text{in}}\|_{H^2 \cap W^{2,1}(\mathbb{R}^3)} \sim \|h_{\text{in}}\|_{H^2 \cap W^{2,1}(\mathbb{R}^3)} + |a_{\text{in}}|\lambda^{-1} \le \epsilon_0^{2+2\sigma}\ .$$

Then there are solutions $E(t)$ and $\Theta(t)$ to (1.7) such that

$$\lim_{t\to\infty} E(t) = E_\infty, \qquad |E(t) - E_\infty| \le C/\langle t\rangle\ .$$

Furthermore, if we decompose ψ_t as in (1.6) with $Q = Q_\infty$ being the profile at time $t = \infty$ then we have

$$\begin{aligned}
&|a(t)| \le C\,\{t\}^{-1}, \qquad h(t) = \zeta(t) + \eta(t)\ ,\\
&\zeta(t)\ldots \textit{local, smooth, } \|\zeta(t)\|_{L^2} \le \{t\}^{-1/2}\ ,\\
&\eta(t)\ldots \textit{dispersive wave, } \|\eta(t)\|_{L^4} \le C\,\{t\}^{-3/4},\ \ \|\eta(t)\|_{L^2_{loc}} \le C\,\{t\}^{-1}\ .
\end{aligned}$$

where $\{t\} = \{t\}_{\varepsilon_0}$.

(2) *Suppose that*

$$0 < \varepsilon = |z_{\text{in}}| \le \epsilon_0,$$

$$\|\eta_{\text{in}}\|_{H^2 \cap W^{2,1}(\mathbb{R}^3)} \le C\varepsilon^{2+\sigma}, \qquad \lambda^{-1}|a_{\text{in}}| \le C\varepsilon^{2+2\sigma}\ ,$$

for some small constant σ. Then ψ_0 belongs to the class of resonance dominating solutions with respect to the final profile Q_{E_∞}.

We shall see in next section that R_E is of order λ^{-1}. This explains that $a_{\rm in}$ has to be order λ in order Theorem 1.2 to be correct. Notice that once we have proved the assertion (1) of Theorem 1.2, the second assertion follows from Theorem 1.1. The proof of Theorem 1.1 is complicated due to the construction of solutions to the Schrödinger equation with the boundary condition of h_0 at time $t = 0$ and that of a at the time $t = \infty$. If we are interested only in Theorem 1.2, we can omit this construction and the proof will be much simpler. We feel that this construction may be needed in other contents and so we keep it.

The resonance solutions for nonlinear Klein-Gordon equations were first proved in an important paper [8] by Soffer and Weinstein. They consider real solutions to the nonlinear Klein-Gordon equation

$$\partial_t^2 u + B^2 u = \lambda u^3, \qquad B^2 := (-\Delta + V + m^2), \tag{1.8}$$

with λ a small nonzero number. Assume that B^2 has only *one* eigenvector (the ground state) ϕ, $B^2\phi = \Omega^2\phi$, with the resonance condition $\Omega < m < 3\Omega$ (and some positivity assumption similar to that appears in assumption A1). Rewrite real solutions to equation (1.8) as $u = a\phi + \eta$ with

$$a(t) = \operatorname{Re} A(t) e^{i\Omega t}, \quad \operatorname{Re} A'(t) e^{i\Omega t} = 0.$$

Then A and η satisfy the equations

$$\dot{A} = \frac{1}{2i\Omega}\, e^{-i\Omega t}\left(\phi, \lambda(a\phi + \eta)^3\right) \tag{1.9}$$

$$(\partial_t^2 + B^2)\eta = P_c \lambda (a\phi + \eta)^3 \tag{1.10}$$

Theorem 1.1 in [8] states that *all* solutions decay as

$$A(t) \sim \langle t\rangle^{-1/4}, \quad \|\eta(t)\|_{L^\infty} \sim \langle t\rangle^{-3/4}.$$

In particular, the ground state is unstable and will decay as a resonance with rate $t^{-1/4}$.

We first remark that the proof in [8] has only established the upper bound $t^{-1/4}$. Furthermore, an universal lower bound of the form $t^{-1/4}$ is in fact incorrect. From the previous work of [1, 2], it is clear that radiation dominating solutions decaying much faster than $t^{-1/4}$ exist. Similar to Theorem 1.1, we have two cases:

1. $\eta(0) \ll A(0)$: the dominant term on the right side of (1.9) is $\lambda a^3 \phi^3$.
2. $\eta(0) \gg A(0)$: the dominant term is $\lambda \eta^3$.

In case 2, another type of solutions arises, namely, those with decay rate

$$A(t) \sim \langle t \rangle^{-2}, \quad \|\eta\|_{L^\infty} \sim \langle t \rangle^{-3/2}.$$

We shall sketch a construction of such solutions at the end of section 10.

Notice that all solutions in [8] decay as a function of t. Therefore, we can view [8] as a study of asymptotic dynamics around vacuum. Although most works concerning asymptotic dynamics of nonlinear evolution equations have been concentrated on cases with vacuum as the unique profile at $t = \infty$, more interesting and relevant cases are asymptotic dynamics around solitons (such as the Hartree equations [2]). The soliton dynamics have extra complications involving translational invariance. The current setting of nonlinear Schrödinger equations with local potentials eliminates the translational invariance and constitutes an useful intermediate step. This simplifies greatly the analysis but preserves a key difficulty which we now explain.

Recall that we need to approximate the wave function ψ_t by nonlinear ground states for all t. Since we aim to show that the error between them decay like $t^{-1/2}$, we have to track the nonlinear ground states with accuracy at least like $t^{-1/2}$. While the nonlinear ground states approximating the wave function ψ_t can in principle be defined, say, via equation (1.4) or (1.6), neither characterizations are useful unless we know the wave functions ψ_t precisely. Furthermore, even assuming we can track the approximate ground states reasonably well, the linearized evolution around these approximate ground states will be based on *time-dependent, non-self adjoint* operators $\mathcal{L}_t$. At this point we would like to mention the approach of [7] based on perturbation around the unitary evolution e^{itH_0}, where H_0 is the original self-adjoint Hamiltonian. While we do not know whether this approach can be extended to the current setting by combining with ideas of [8] (It is announced in [8] that its method can be extended to nonlinear Schrödinger equations as well. As the writing of this paper, there is no preprint available), such an approach will be difficult to extend to the Hartree or other equations with non-vanishing solitons. The main reason is that these dynamics are not perturbation of linear dynamics. We believe that perturbation around the profile at $t = \infty$ is a more nature setup. In this approach, at least we do not have to worry about the time dependence of approximate ground states in the beginning. But the linearized operator $\mathcal{L} = \mathcal{L}_\infty$ is still non-self adjoint and it does not commute with the multiplication by i. So calculations and estimates based on $\mathcal{L}$ are rather complicated. Our first idea is to map this operator to a self-adjoint operator by a bounded transformation in Sobolev space. This map simplifies many calculations and in a sense brings the problem back to the self-adjoint case at least as far as estimates are concerned, see section 3. We then identify and calculate carefully leading oscillatory terms. Once this is done, the estimate of error terms are rather straightforward.

The estimates obtained from integrating the equation from $t = \infty$ can be viewed as uniform bounds for all T, provided that the approximate nonlinear ground states for all T are known. On the other hand, to pin down the approximate nonlinear ground states, we need to have precise estimates on the wave functions. It is thus nature to consider continuity method by assuming that the approximate nonlinear ground states and various estimates on the wave function are known up to time T. We then show that these estimates continue to hold up to time $T + \delta T$, with δT small but fixed, provided that *all estimates are re-adjusted w.r.t. to the new nonlinear ground state at* $T + \delta T$. There is some similarity between this idea and the standard renormalization group argument. Borrowing standard terminology, we decompose the wave function ψ_t into "marginal" and "irrelavent" directions at any time T. From T to $T + \delta T$, we run the dynamics of the Schrödinger equation. Now at time $T + \delta T$, the estimates for the irrelavent directions can be passed easily to time $T + \delta T$. The estimates on marginal direction however cannot be easily passed from T to $T + \delta T$. But those are terms we can essentially calculated rather precisely and their estimates can be established independently. Because of this similarity, we call this method dynamical mass renormalization. The marginal terms here are those with high oscillations.

Although we restrict to the small coupling constant case, in future study we hope to replace them by spectral assumptions on the operators $\mathcal{L}$. This paper is divided into 10 sections:

Section 2. The set-up of the problem

Section 3. Properties of the linearized operator

Section 4. Main oscillation terms

Section 5. Estimates on dispersive wave

Section 6. Excited state equation and normal form

Section 7. Change of the mass of the ground state

Section 8. Contraction mapping

Section 9. Dynamical renormalization of mass

Section 10. Radiation dominating solution

It is a great pleasure to thanks M. Weinstein for explaining to us the beautiful idea in the work [8] and, in particular, to call our attention to the toy model in [8] which contains the basic idea of the resonance decaying in a very illuminating way.

2 The set-up of the problem

2.1 Ground state family

We first review the construction of the ground state family $\{Q_E\}_E$ mentioned in Section 1. The results we reviewed in this subsection follow from simple perturbation theory and we shall not give details. Denote the standard L_2 inner product by

$$(f,g) \equiv \int \bar{f} g \, d^3x \ .$$

Let $Q = Q_E$ satisfy

$$(-\Delta + V - e_0)Q + \lambda Q^3 = E' Q \ .$$

where $E' = E - e_0$. Let $Q_E = w\phi_0 + h$ with h real and orthogonal to ϕ_0. Then h satisfies

$$(H_0 - E')h + \lambda(w\phi_0 + h)^3 = E' w\phi_0,$$

We can solve w and h so that

$$w^2 = O(\lambda^{-1}E'), \qquad \|h\|_2 = O(\lambda w^3) \tag{2.1}$$

where we have used that the spectral gap of the operator $-\Delta + V - e_0$ is of order one.

Let $R = \partial_E Q_E$. Recall

$$L_+^{(\mathrm{or})} = -\Delta + V - E + 3\lambda Q^2.$$

Then by differentiating the equation of Q_E w.r.t E, we have

$$L_+^{(\mathrm{or})} R_E = Q_E$$

Denote by $|0\rangle$ the ground state to $L_+^{(\mathrm{or})}$ of norm 1. We also have

$$Q_E = w|0\rangle + O(\lambda w)$$

Hence

$$R = (L_+^{(\mathrm{or})})^{-1} \left[w|0\rangle + O(\lambda w) \right] = O(\lambda^{-1} w^{-1}) \, |0\rangle + O(\lambda w) = O(\lambda^{-1} w^{-2}) \, Q + O(w) \ . \tag{2.2}$$

2.2 The set-up of the problem

We consider solutions $\psi(t)$ to the equation (1.1) The picture is that the solution $\psi(t)$ can be decomposed to two parts: one represents a soliton, the other one the radiation. The radiation part will disperse to infinity. The soliton part will converge to a soliton Q_{N_∞}, while its L^2-norm is changing in time. Hence it is natural to consider solutions of the form

$$\psi = (Q_{E(t)} + h(t)) e^{i\Omega(t)} \ , \tag{2.3}$$

and study its evolution. If we consider a minimization problem

$$\inf_{\Omega,E} \left\| \psi - Q_E\, e^{i\Omega} \right\|_{L^2} \ .$$

Then $\psi = (Q_E + h)\, e^{i\Omega}$ with $\mathrm{Im}\, h \perp Q_E$, $\mathrm{Re}\, h \perp R_E$. Hence we almost have $h(t) \perp Q_{E(t)}$ in our problem, with some small correction.

Since this setup would introduce a time-dependent linear operator, we try to find a good approximation with a fixed $Q = Q_E$. For this fixed Q, let

$$H = -\Delta + V - E + \lambda Q^2 \ ,$$

(we have $HQ = 0$), and let Π be the projection which eliminates the Q-direction:

$$\Pi h = h - (c_0 Q, h) Q \ , \quad c_0 = (Q, Q)^{-1}$$

Let X denote its image,

$$X = \Pi(L^2) \ .$$

If we assume

$$\psi = (Q + k(t)) e^{i[-Et + \theta(t)]} \ .$$

Then the equation for $k(t)$ is

$$\partial_t k = \mathcal{L}^{(\mathrm{or})} k - iF(k) - i\dot{\theta}(Q + k) \ , \tag{2.4}$$

where

$$\mathcal{L}^{(\mathrm{or})} k = -i \left\{ Hk + \lambda Q^2 (k + \bar{k}) \right\} \ ,$$

and

$$F(k) = \lambda Q (2|k|^2 + k^2) + \lambda |k|^2 k \ .$$

In view of (2.3), it is natural to assume

$$k = aR + h, \quad R = \partial_E Q_E \ .$$

Here $a(t)$ is small and compensates the change of L^2 norm of the soliton. Since we would like $h(t) \perp Q$ for all t. Hence we look for solutions of the form

$$\psi = (Q + aR + h)\, e^{i\Omega}, \quad (\Omega = -Et + \theta(t)) \ .$$

We note that ψ can be written in this form if ψ is sufficiently close to Q. Specifically, we let $e^{i\Omega}$ be the phase of $P_Q \psi$ and choose a so that $\|P_Q \psi\|_{L^2} = \|Q + a P_Q R\|_{L^2}$. Then h is obtained by $h = \psi\, e^{-i\Omega} - Q - aR \in X$.

We also note, by differentiating the equation of Q w.r.t. E,

$$\mathcal{L}^{(\mathrm{or})} R = -iQ \ .$$

We substitute $k = aR + h$ into (2.4) and obtain

$$\partial_t h = -\dot{a}R - aiQ + \mathcal{L}^{(\mathrm{or})} h - iF(k) - i\dot{\theta}(Q + aR + h) \ .$$

Since we would like $h \perp Q$, we choose $\dot{a}$ and $\dot{\theta}$ so that the right side of the above equation is perpendicular to Q. Using $h \perp Q$ and $Hh \perp Q$ we obtain the equations for a and θ:

$$\begin{aligned} \dot{a} &= (c_1 Q, \operatorname{Im} F(k)) \ , \qquad c_1 = (Q, R)^{-1} \ , \\ \dot{\theta} &= -(1 + c_0 c_1^{-1} a)^{-1} \ \left[a + (c_0 Q, \lambda Q^2 (h + \bar{h})) + (c_0 Q, \operatorname{Re} F(k)) \right] \ , \end{aligned}$$

where $c_0 = (Q, Q)^{-1}$. Then the equation of h is

$$\begin{aligned} \partial_t h &= \mathcal{L} h + \Pi F_{all} \ , \\ \mathcal{L} h &= -i \left\{ Hh + \Pi \lambda Q^2 \Pi (h + \bar{h}) \right\} \ , \\ F_{all} &= -i\dot{\theta} h - iF(k) - \left[(c_1 Q, \operatorname{Im} F) + ia\dot{\theta} \right] R_\Pi \ , \quad R_\Pi := \Pi R \ . \end{aligned}$$

Here we have used the equation for $\dot{a}$.

Next we compute $F(k) = F(h + aR)$:

$$F(k) = \lambda Q(2|h|^2 + h^2) + 2\lambda a Q R(2h + \bar{h}) + 3\lambda a^2 Q R^2 + \lambda (aR + h)^2 (aR + \bar{h}) \ .$$

With respect to $\mathcal{L}$, we can decompose X as

$$X = \mathtt{E}_\kappa \oplus \mathbf{P}_{\mathrm{c}}^{\mathcal{L}}(X) \ ,$$

where the direct sum is not orthogonal, and

$$\mathtt{E}_\kappa = \{ \alpha u + \beta iv : \alpha, \beta \in \mathbb{R} \} \ , \quad u, v \text{ real}, L_+ u = \kappa v, L_- v = \kappa u, (u, v) = 1.$$

We write

$$h(t) = \alpha(t) u + \beta(t) iv + \eta(t) \ .$$

Their equations are

$$\begin{aligned} \dot{\alpha} &= \kappa \beta + (v, \operatorname{Re} F_{all}) \ , \\ \dot{\beta} &= -\kappa \alpha + (u, \operatorname{Im} F_{all}) \ , \\ \partial_t \eta &= \mathcal{L} \eta + \mathbf{P}_{\mathrm{c}}{}^{\mathcal{L}} \Pi (F_{all}) \ . \end{aligned}$$

(Note $(v, \Pi f) = (v, f)$, and similarly for u.) The linear parts of the equations for $\alpha(t)$ and $\beta(t)$ form a rotation. To single out the rotation, we study

$$z(t) = \alpha(t) + i\beta(t) = p\mu^{-1}, \quad \mu(t) = e^{i\kappa t}.$$

Here μ^{-1} captures the rotation part and we expect a slower oscillation in p. Note that $z(t)$ can be obtained from $h(t)$ by

$$z(t) = (v, \operatorname{Re} h(t)) + i\,(u, \operatorname{Im} h(t))\ .$$

The function $p(t)$ satisfies

$$\begin{aligned}
\mu^{-1}\dot p &= (v, \operatorname{Re} F_{all}) + i(u, \operatorname{Im} F_{all}) \\
&= (v, \operatorname{Im}(F + \dot\theta h) - (c_1 Q, \operatorname{Im} F) R_\Pi) + i(u, -\operatorname{Re}(F + \dot\theta h) - a\dot\theta R_\Pi) \\
&= (v, \operatorname{Im} F) - (v, R_\Pi)\,(c_1 Q, \operatorname{Im} F) + i(u, -\operatorname{Re} F) + [(v, \operatorname{Im} h) + i(u, -\operatorname{Re} h)]\dot\theta - ic_3 a\dot\theta \\
&= (\tilde v, \operatorname{Im} F) + i(u, -\operatorname{Re} F) + [(v, \operatorname{Im} h) + i(u, -\operatorname{Re} h) - ic_3 a]\dot\theta \\
&= (\tilde v, (F - \bar F)/2i) + i(u, -(F + \bar F)/2) + [(v, (h - \bar h)/2i) + i(u, -(h + \bar h)/2) - ic_3 a]\dot\theta \\
&= -i\left[(\tilde{u_+}, F) + (\tilde{u_-}, \bar F) + \left\{(u_+, h) + (u_-, \bar h) + c_3 a\right\}\dot\theta\right]\ ,
\end{aligned}$$

where $c_3 = (u, R_\Pi)$, $\tilde v = v - (v, R_\Pi) c_1 Q \notin X$, and

$$\begin{aligned}
u_+ &= \tfrac12(u + v) = O(1), \qquad & u_- &= \tfrac12(u - v) = O(\lambda), \\
\tilde{u_+} &= \tfrac12(u + \tilde v) = O(1), \qquad & \tilde{u_-} &= \tfrac12(u - \tilde v) = O(\lambda).
\end{aligned}$$

Their orders are derive from the simple facts: $(v, R_\Pi) = O(1)$, $c_2 = O(\lambda)$ and $c_3 = (u, R_\Pi) = O(1)$.

Summarizing, we have

$$\begin{aligned}
\psi &= (Q + aR + h)\, e^{i\Omega}, \\
h &= \zeta + \eta = (zu_+ + \bar z u_-) + \eta\ , \\
\dot a &= (c_1 Q, \operatorname{Im} F(k))\ , \qquad c_1 = (Q, R)^{-1}\ , \\
\dot\theta &= -(1 + c_0 c_1^{-1} a)^{-1}\ \left[a + (c_0 Q, \lambda Q^2 (h + \bar h)) + (c_0 Q, \operatorname{Re} F(k))\right]\ , \\
\partial_t \eta &= \mathcal{L}\eta + \mathbf{P}_c^{\mathcal{L}}\Pi(F_{all}) \\
z(t) &= \alpha(t) + i\beta(t) = p\mu^{-1}, \quad \mu(t) = e^{i\kappa t}, \\
\mu^{-1}\dot p &= -i\left[(\tilde{u_+}, F) + (\tilde{u_-}, \bar F) + \left\{(u_+, h) + (u_-, \bar h) + c_3 a\right\}\dot\theta\right]\ ,
\end{aligned} \tag{2.5}$$

where

$$F_{all} = -i\dot{\theta}h - iF(h+aR) - \left[(c_1Q, \operatorname{Im} F) + ia\dot{\theta}\right] R_\Pi \ , \quad R_\Pi := \Pi R \ , \tag{2.6}$$

$$F(h+aR) = \lambda Q(2|h|^2 + h^2) + 2\lambda aQR(2h+\bar{h}) + 3\lambda a^2QR^2 + \lambda(aR+h)^2(aR+\bar{h}) \ . \tag{2.7}$$

This is the equation we shall solve for the rest of this paper.

3 Properties of the linearized operator

3.1 Spectral decomposition

Recall that, for given E,

$$H_E = -\Delta + V - E + \lambda Q_E^2$$

and $H_E Q_E = 0$. Recall Π be the projection which eliminates the Q-direction:

$$\Pi h = h - (h, Q)Q \ .$$

and the operator $\mathcal{L}$ acting on the space $X = \Pi L^2$ by

$$\mathcal{L}h = -i\left\{Hh + \Pi\lambda Q^2\Pi(h+\bar{h})\right\} \ .$$

We also have $L_- = H$, $L_+ = H + 2\Pi\lambda Q^2\Pi$. With respect to $\mathcal{L}$, we can define generalized "eigenspaces":

$$\mathtt{E}_\nu(\mathcal{L}) := \left\{\psi : \mathcal{L}^2\psi = -\nu^2\psi\right\} = \left\{u + iv : u, v \text{ real}, \ L_-L_+u = \nu^2 u, L_+L_-v = \nu^2 v\right\} \ .$$

In our case, the only value for ν is κ. It is natural to define

$$\mathtt{H}_\mathrm{c}(\mathcal{L}) := \{\psi : \psi \perp \mathtt{E}_\nu(\mathcal{L}^*) \text{ for all } \nu\} \ ,$$

since it is invariant under $\mathcal{L}$. The orthogonality is defined with respect to the inner product

$$((\psi, \phi)) = (\operatorname{Re}\psi, \operatorname{Re}\phi) + (\operatorname{Im}\psi, \operatorname{Im}\phi) \ ,$$

and we know

$$\mathtt{E}_\nu(\mathcal{L}^*) = \{v + iu : u, v \text{ real}, \ u + iv \in \mathtt{E}_\nu(\mathcal{L})\} \ .$$

It is also clear from the definition that

$$X = \bigoplus_\nu \mathtt{E}_\nu(\mathcal{L}) \oplus \mathtt{H}_\mathrm{c}(\mathcal{L}) \ .$$

Hence,

$$\mathtt{H}_\mathrm{c}(\mathcal{L}) = \left\{\phi_1 + i\phi_2 : \phi_1 \perp v, \phi_2 \perp u, \text{ for all } v, \, u \text{ such that } L_+L_-v = \nu^2 v, L_-L_+u = \nu^2 u\right\} \ .$$

In our situation, since we only have two simple eigenvalues for H_0 and the ground state is factored out, we can prove by perturbation argument that there are real u and v such that

$$\mathtt{E}_\kappa = \{au + biv : a, b \in \mathbb{R}\}, \quad L_+ u = \kappa v, \quad L_- v = \kappa u, \quad (u, v) = 1 .$$

Since $L_+ L_-$ and $L_- L_+$ are not self-adjoint, it is not convenient to use them to characterize $\mathtt{H}_c(\mathcal{L})$. We define two operators

$$B = \Pi(L_-)^{1/2}\Pi, \quad A = \sqrt{B L_+ B}.$$

Note that both A and B are self-adjoint and **positive** on X. With these operators, if we define $w = \kappa^{-1/2} B v$, we have w is the normalized eigenvector for the operator A and

$$u = \nu^{-1/2} B w, \quad v = \nu^{1/2} B^{-1} w, \quad A w = \kappa w, \quad (w, w) = 1.$$

Hence the continuum spectrum of $\mathtt{H}_c(\mathcal{L})$ can be characterized by

$$\begin{aligned}
\mathtt{H}_c(\mathcal{L}) &= \left\{\psi = \psi_1 + i\psi_2 : \psi_1, \psi_2 \text{ real }, \psi_1 \perp B^{-1} w_\nu, \psi_2 \perp B w_\nu \text{ for all } \nu\right\} \\
&= \left\{\psi = \psi_1 + i\psi_2 : \psi_1, \psi_2 \text{ real }, B^{-1}\psi_1, B\psi_2 \in \mathtt{H}_c(A)\right\} \\
&= \left\{\psi = B\phi_1 + iB^{-1}\phi_2 : \phi_1, \phi_2 \text{ real }, \phi_1, \phi_2 \in \mathtt{H}_c(A)\right\} .
\end{aligned}$$

The maps B and B^{-1} change the differentiability of the functions. A better way is to use $A^{-1/2} B$ and $A^{1/2} B^{-1}$ instead. Notice that both operators are bounded in weighted Sobolev spaces $H^{s,p} \cap X$ since A and B are strictly positive and the numbers of differentiations cancel exactly in both operators. We have established a one-to-one correspondence between the spectral decomposition of X with respect to $\mathcal{L}$ and that with respect to A:

$$\begin{aligned}
\psi \in \mathtt{E}_\kappa(\mathcal{L}) &\longleftrightarrow U\psi \in \mathtt{E}_\kappa(A), \\
\psi \in \mathtt{H}_c(\mathcal{L}) &\longleftrightarrow U\psi \in \mathtt{H}_c(A),
\end{aligned}$$

where U is the operator

$$U(f + ig) = A^{1/2} B^{-1} f + i A^{-1/2} B g, \quad \text{for real } f, g \in X. \tag{3.1}$$

Moreover, the operators U and U^{-1} are bounded in weighted Sobolev spaces $H^{s,p} \cap X$.

We have several ways to decompose $\mathcal{L}$ as products:

$$\begin{aligned}
\mathcal{L} &= V_1^{-1} \begin{bmatrix} 0 & 1 \\ -A^2 & 0 \end{bmatrix} V_1 & \qquad V_1 &= \begin{bmatrix} B^{-1} & 0 \\ 0 & B \end{bmatrix} \\
&= V_2^{-1} \begin{bmatrix} jA & 0 \\ 0 & -jA \end{bmatrix} V_2 & \qquad j = \sqrt{-1}, \quad V_2 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \frac{1}{jA} \\ -jA & 1 \end{bmatrix} V_1 \\
&= U^{-1} \begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix} U & \qquad U &= \begin{bmatrix} A^{1/2} B^{-1} & 0 \\ 0 & A^{-1/2} B \end{bmatrix}
\end{aligned}$$

We prefer using the last one since it has a simpler form:

$$\mathcal{L} = U^{-1}(-iA)\, U \ ,$$

with U given in (3.1).

This decomposition will be especially useful when we integrate integrals of the form

$$\int_0^t e^{(t-s)\mathcal{L}}\, \mathbf{P}_{\mathrm{c}}^{\mathcal{L}}\, e^{2i\kappa s} \phi \, ds \ .$$

For this purpose, it is convenient to decompose

$$U = U_+ + U_- \mathtt{C}, \qquad U_\pm = A^{1/2} B^{-1} \pm A^{-1/2} B \ . \tag{3.2}$$

Here $\mathtt{C}$ is the conjugation operator, and both U_+ and U_- are self adjoint. Hence $[U, i] \neq 0$ but $[U_\pm, i] = 0$.

3.2 Estimates

Here we collect some estimates on the operator A.

Lemma 3.1 *We have the following estimates for the operator A:*

$$\left\| \langle x \rangle^{-\beta}\ e^{-itA}\ \mathbf{P}_{\mathrm{c}}^{A}\ \langle x \rangle^{-\beta}\ \phi \right\|_{L^2} \le C \left\langle t \right\rangle^{-3/2} \|\phi\|_{L^2} \tag{3.3}$$

and, for $q = 4, 8$,

$$\left\| e^{-itA}\ \mathbf{P}_{\mathrm{c}}^{A}\, \phi \right\|_{L^q} \le C \left\langle t \right\rangle^{-3\left(\frac{1}{2} - \frac{1}{q}\right)} \|\phi\|_{L^{q'}} \ . \tag{3.4}$$

For smooth local functions ϕ, we have

$$\left\| \langle x \rangle^{-\beta}\ e^{-itA}\, \frac{1}{A - 0i - 2\kappa}\, \mathbf{P}_{\mathrm{c}}^{A}\ \langle x \rangle^{-\beta}\ \phi \right\|_{L^2} \le C \left\langle t \right\rangle^{-6/5} \tag{3.5}$$

Notice when the operator A is the Laplacian, these estimates are obvious and can be checked directly. In order to prove them for the general operator A, we have to show that the restriction of A onto its continuum spectrum can be mapped into the standard Laplacian. This is exactly the wave operator. The wave operator for the standard operator $-\Delta + V$ is proved to be bounded in Sobolev space $W^{k,p}$ by Yajima [11]. We notice the operator A can be written as a perturbation of $-\Delta + V$ in the following way: Let $K = H^{-1/2} 2\lambda Q^2 H^{-1/2}$. Then $A = H^{1/2}(1+K)^{1/2} H^{1/2}$. Since

$$(1+K)^{1/2} = 1 + \left((1+K)^{1/2} - 1 \right) = 1 + \frac{K}{(1+K)^{1/2} + 1} =: 1 + K K_1 \ .$$

Hence

$$A = H^{1/2}\left\{1 + H^{-1/2}2\lambda Q^2 H^{-1/2}K_1\right\}H^{1/2} = H + 2\lambda Q^2 H^{-1/2}K_1 H^{1/2}\ .$$

Similarly, since

$$(1+K)^{1/2} = 1 + \frac{1}{2}K + \frac{-\frac{1}{4}K^2}{(1+K)^{1/2} + 1 + \frac{1}{2}K} =: 1 + \frac{1}{2}K - \frac{1}{4}K^2 K_2\ ,$$

we have

$$A = H + \lambda Q^2 - \lambda Q^2 H^{-1}\lambda Q^2 H^{-1/2}K_2 H^{1/2}\ .$$

We shall adapt methods in [11] to prove the this Lemma in a separate paper.

Lemma 3.2

$$\left(Qu_+^2\ ,\ \mathrm{Im}\,\frac{1}{A - 0i - 2\kappa}\,\mathbf{P}_c^A\,\Pi Q u_+^2\right) > 0\ . \tag{3.6}$$

Define

$$\Gamma \equiv 2\lambda^2\left(Qu_+^2\ ,\ \mathrm{Im}\,\frac{1}{A - 0i - 2\kappa}\,\mathbf{P}_c^A\,\Pi Q u_+^2\right) > 0\ . \tag{3.7}$$

This Lemma is just a simple continuity argument. Notice that if $\lambda = 0$ then the statement of this lemma is just the assumption A.1. Since λ is small, by continuity it holds for small λ. A more detailed argument will be given in the paper proving Lemma 3.1.

We shall need the following estimate on the commutator of U and i:

Lemma 3.3

$$\|[U,i]\phi\|_{L^{8/7}} \le O(\lambda)\,\|\phi\|_{L^4}\ . \tag{3.8}$$

The commutator $[U,i]$ can be expressed in terms of $[A^{1/2}, B]$. It is clear that $A^{1/2} = B$ when $\lambda = 0$. The rest of the proof is just simple perturbation argument. The details will be given in the paper proving Lemma 3.1.

4 Main oscillation terms

We now identify the main oscillation terms in equation (2.5). We shall use the complex amplitude of the excited state, z, as the reference. Recall $z(t) = e^{-i\kappa t}p(t)$. Eventually we will have $p(t) \sim t^{-1/2}$ and its oscillation is much smaller than κ. The change of mass on the direction of the nonlinear ground state is given by a. We will also show that $a = O(z^2)$ and the order of the dispersive wave $\eta(t)$ is also of order $O(z^2)$. Assuming these orders, the second order term in F, $F^{(2)}$, is given explicitly by:

$$F^{(2)} = \lambda Q(2|\zeta|^2 + \zeta^2) = z^2\phi_{(20)} + z\bar{z}\phi_{(11)} + \bar{z}^2\phi_{(02)}$$

where $\zeta = zu_+ + \bar{z}u_-$ and

$$\begin{aligned}\phi_{(20)} &= \lambda Q(u_+^2 + 2u_+u_-) = \lambda Q u_+^2 + O(\lambda^2),\\ \phi_{(11)} &= 2\lambda Q(u_+^2 + u_-^2 + u_+u_-) = O(\lambda),\\ \phi_{(02)} &= \lambda Q(u_-^2 + 2u_+u_-) = O(\lambda^2).\end{aligned} \tag{4.1}$$

We shall write $F^{(2)} = z^\alpha \phi_\alpha$, where α is a double indices (ij) with $i + j = 2$ and $i, j \geq 0$. The repeated indices mean summation. We shall use β later on to denote double indices summing to 3 and γ for 4.

4.1 Main oscillation term in a and F

We start with identifying the main oscillation terms of $a(t)$. We have fixed the boundary condition of a at $t = \infty$ and set $a(\infty) = 0$. Thus we can rewrite the equation for $a(t)$ into the following equivalent integral equation:

$$a(t) = \int_\infty^t (c_1 Q, \operatorname{Im} F(k))\, ds \; .$$

As the oscillation term of order z^2 in F comes from $F^{(2)}$, we have up to second order in z^2

$$(c_1 Q, \operatorname{Im} F) \sim A^{(2)}$$

where

$$A^{(2)} = \big(c_1 Q,\, \lambda Q \operatorname{Im} \zeta^2\big)$$

Since $\zeta = zu_+ + \bar{z}u_-$, we have

$$\operatorname{Im} \zeta = \operatorname{Im} z\,(u_+ - u_-), \quad \operatorname{Im} \zeta^2 = (\operatorname{Im} z^2)\,(u_+^2 - u_-^2) \; .$$

Therefore, we have

$$A^{(2)} = C_1 \operatorname{Im} z^2, \qquad C_1 = \big(c_1 Q,\, \lambda Q(u_+^2 - u_-^2)\big) = O(\lambda^2) \; ,$$

We can integrate $A^{(2)}$ by parts to have:

$$\begin{aligned}\int_\infty^t A^{(2)}\, ds &= C_1 \operatorname{Im} \int_\infty^t z^2\, ds = C_1 \operatorname{Im} \int_\infty^t \mu^{-2} p^2\, ds\\ &= C_1 \operatorname{Im} \frac{1}{-2i\kappa} \left\{ \mu^{-2} p^2 - \int_\infty^t \mu^{-2} 2p\dot{p}\, ds \right\}\end{aligned}$$

As we shall prove later on, the last integral is higher order term. The first term on the right hand side can be written explicitly as

$$C_1 \operatorname{Im} \frac{1}{-2i\kappa} \mu^{-2} p^2 = a_{20}(z^2 + \bar{z}^2) \ ,$$

where

$$a_{20} = \frac{C_1}{4\kappa} = \frac{\lambda}{4\kappa} \left\{ c_1 Q,\ Q(u_+^2 - u_-^2) \right\} = O(\lambda^2) \ , \tag{4.2}$$

We shall prove later on that the last term $a_{20}(z^2 + \bar{z}^2)$ is the main oscillatory term in a. We denote the rest by b, i.e.,

$$a = a_{20}(z^2 + \bar{z}^2) + b \tag{4.3}$$

As shall prove $a, b, \dot{a} = O(z^2)$, but $\dot{b} = O(z^3)$. In other words, b is the part of a that has slower oscillation. From the equation of a, we have the following equation for b:

$$\dot{b} = \big(c_1 Q, \operatorname{Im}(F - F^{(2)})\big) - 4(?) \operatorname{Re} a_{20} z \mu^{-1} \dot{p} \ . \tag{4.4}$$

Assuming that b and η are of order z^2, we can decompose F into

$$F = F^{(2)} + F^{(3)} + \widetilde{F}^{(3)} + F^{(4)}$$

where $F^{(2)}$ and $F^{(3)}$ denote terms of order z^2 and z^3, respectively, and $F^{(4)}$ denotes higher order terms:

$$\begin{aligned}
F^{(2)} &= \lambda Q(2|\zeta|^2 + \zeta^2) = z^2 \phi_{(20)} + z\bar{z}\phi_{(11)} + \bar{z}^2 \phi_{(02)} \\
F^{(3)} &= 2\lambda Q[(\zeta + \bar{\zeta})\eta^{(2)} + \zeta \bar{\eta}^{(2)}] + \lambda |\zeta|^2 \zeta + 2\lambda a_{20}(z^2 + \bar{z}^2) Q R (2\zeta + \bar{\zeta}) \\
\widetilde{F}^{(3)} &= 2\lambda b Q R (2\zeta + \bar{\zeta}) \\
F^{(4)} &= 2\lambda Q[(\zeta + \bar{\zeta})\eta^{(3)} + \zeta \bar{\eta}^{(3)}] + \lambda Q \left[2|\eta|^2 + \eta^2 \right] + 2\lambda a Q R (2\eta + \bar{\eta}) \\
&\quad + 3\lambda a^2 Q R^2 + \lambda \left[|k|^2 k - |\zeta|^2 \zeta \right] \ ,
\end{aligned} \tag{4.5}$$

We can also rewrite the equation of θ into

$$\dot{\theta} = c_2(z + \bar{z}) + F_\theta \ , \tag{4.6}$$

where $c_2 = (c_0 Q, \lambda Q^2 u) = O(\lambda)$ and

$$F_\theta = -(1 + c_0 c_1^{-1} a)^{-1} \ \left[a + (c_0 Q, \lambda Q^2(\eta + \bar{\eta})) + (c_0 Q, \operatorname{Re} F(k)) \right] \ . \tag{4.7}$$

4.2 Main oscillation term in η

We now identify the main oscillation term in η. We first rewrite the equation of η using the operator A as

$$
\begin{aligned}
&\partial_t \eta = \mathcal{L}\eta - \mathbf{P}_c^{\mathcal{L}}\, i\dot\theta\eta + \mathbf{P}_c^{\mathcal{L}}\, \Pi F^\sharp\ ,\\
&F^\sharp = -i\dot\theta\zeta - iF(k) - [(c_1 Q, \operatorname{Im} F) + ia\dot\theta] R_\Pi\ .
\end{aligned}
\tag{4.8}
$$

Notice that $F^\sharp$ and F^{all} differs only by the term $i\dot\theta\eta$. Observe also that $-i\dot\theta\zeta$ is not killed by $\mathbf{P}_c^{\mathcal{L}}$ since $[\mathcal{L}, i] \neq 0$. Let

$$
\eta^\diamond = U\eta\ .
$$

Since $\mathcal{L} = U^{-1}(-iA)U$ and $U\,\mathbf{P}_c^{\mathcal{L}} = \mathbf{P}_c^A\, U$, we have

$$
\begin{aligned}
\partial_t \eta^\diamond &= -iA\eta^\diamond - \mathbf{P}_c^A\, U i\dot\theta U^{-1}\eta^\diamond + \mathbf{P}_c^A\, U\Pi F^\sharp\\
&= -iA\eta^\diamond - i\dot\theta\eta^\diamond - \mathbf{P}_c^A\, [U, i]\dot\theta U^{-1}\eta^\diamond + \mathbf{P}_c^A\, U\Pi F^\sharp\ .
\end{aligned}
$$

Let $\tilde\eta = e^{i\theta}\eta^\diamond$ and use $U^{-1}\eta^\diamond = \eta$, we get

$$
\partial_t \tilde\eta = -iA\tilde\eta + e^{i\theta}\,\mathbf{P}_c^A\, U\Pi F^\sharp - e^{i\theta}\,\mathbf{P}_c^A\, [U, i]\dot\theta\eta\ . \tag{4.9}
$$

Hence

$$
\tilde\eta(t) = e^{-iAt}\tilde\eta_0 + \int_0^t e^{-iA(t-s)}\,\mathbf{P}_c^A\, \left\{ e^{i\theta}U\Pi F^\sharp - e^{i\theta}[U, i]\dot\theta\eta \right\} ds\ . \tag{4.10}
$$

Since $\eta = U^{-1}e^{-i\theta}\tilde\eta$ and U is bounded in Sobolev spaces, for the purpose of estimation we can treat η and $\tilde\eta$ the same.

From the the definition of $\tilde\eta$ (4.19), the integrand is $e^{i\theta}U\Pi F^\sharp - e^{i\theta}[U, i]\dot\theta\eta$. We first identify the main term in $F^\sharp$:

$$
F^\sharp = -i\dot\theta\zeta - iF(k) - [(c_1 Q, \operatorname{Im} F) + ia\dot\theta] R_\Pi := i\, z^\alpha \phi_\alpha^\sharp + F^{\sharp\sharp}.
$$

We have already decomposed F into orders in z. To identify the main term of $F^\sharp$, it remains to decompose $-i\dot\theta\zeta$ and $(c_1 Q, \operatorname{Im} F)R_\Pi$. From the equation of θ (2.5), we have

$$
-i\dot\theta\zeta = -ic_2(z + \bar z)(zu_+ + \bar z u_-) - iF_\theta\zeta
$$

Also

$$
(c_1 Q, \operatorname{Im} F)R_\Pi = (c_1 Q, \phi_{20} - \phi_{02})R_\Pi \operatorname{Im} z^2 + O(z^3).
$$

Recall the decomposition of F in (4.5)and $F^{(2)} = z^\alpha \phi_\alpha$. We can now write

$$
F^\sharp = i\, z^\alpha \phi_\alpha^\sharp + F^{\sharp\sharp},
$$

where $F^{\sharp\sharp}$ denote the higher order terms and $\phi^\sharp$ is defined as follows:

$$\begin{aligned}
\phi^\sharp_{20} &= -\phi_{20} - c_2 u_+ + \tfrac{1}{2}(c_1 Q, \phi_{20} - \phi_{02}) R_\Pi = -\phi_{20} - c_2 u_+ + O(\lambda^2)\ ,\\
\phi^\sharp_{11} &= -\phi_{11} - c_2(u_+ + u_-) \qquad\qquad\qquad\quad = O(\lambda)\ ,\\
\phi^\sharp_{02} &= -\phi_{02} - c_2 u_- + \tfrac{1}{2}(c_1 Q, \phi_{20} - \phi_{02}) R_\Pi = O(\lambda^2)\ .
\end{aligned} \tag{4.11}$$

Here we have used $\phi_{20} = O(\lambda) = \phi_{11}$, $\phi_{02} = O(\lambda^2)$, $c_2 = O(\lambda)$ and $R_\Pi = O(1)$. Also note that they are all real.

Recall (3.2). Since $Ui = (U_+ + U_- C)i = i(U_+ - U_- C)$, We have

$$\mathbf{P}^A_c\, e^{i\theta} U i z^\alpha \Pi \phi^\sharp_\alpha = \mathbf{P}^A_c\, e^{i\theta} i(U_+ - U_- C) z^\alpha \Pi \phi^\sharp_\alpha = e^{i\theta} i z^\alpha \Phi_\alpha$$

where

$$\begin{aligned}
&\Phi_{20} = \mathbf{P}^A_c \left\{ U_+ \Pi \phi^\sharp_{20} - U_- \Pi \phi^\sharp_{02} \right\}, \quad \Phi_{11} = \mathbf{P}^A_c \left\{ (U_+ - U_-) \Pi \phi^\sharp_{11} \right\},\\
&\Phi_{02} = \mathbf{P}^A_c \left\{ -U_- \Pi \phi^\sharp_{20} + U_+ \Pi \phi^\sharp_{02} \right\}\ .
\end{aligned}$$

Hence we can rewrite the integrand in (4.19) as

$$e^{i\theta} i z^\alpha \Phi_\alpha + \mathbf{P}^A_c \left\{ e^{i\theta} U \Pi F^{\sharp\sharp} - e^{i\theta} [U, i] \dot\theta \eta \right\}\ .$$

The leading term in $\tilde\eta$ is

$$\begin{aligned}
(I) &\equiv \int_0^t e^{-iA(t-s)} e^{i\theta} i z^\alpha \Phi_\alpha \, ds\\
&= \int_0^t e^{-itA} e^{is(A-0i)} \left(\mu^{[\alpha]} (e^{i\theta} p^2)(s) i \Phi_\alpha \right) ds \qquad (\mu = e^{i\kappa s})\\
&= e^{-itA} \left[\frac{e^{is(A-0i)} [e^{i\theta} z^\alpha](s)}{i(A - 0i + [\alpha]\kappa)} i \Phi_\alpha \right]_{s=0}^t + (II)\\
&= \tilde\eta^{(2)} - e^{-itA} (e^{i\theta} z^\alpha)(0) \widetilde{\eta}_\alpha + (II)\ ,
\end{aligned} \tag{4.12}$$

where

$$\tilde\eta^{(2)} = e^{i\theta} z^\alpha \widetilde{\eta}_\alpha, \qquad \widetilde{\eta}_\alpha = \frac{1}{A - 0i + [\alpha]\kappa} \Phi_\alpha$$

and (II) is the error from integration by parts,

$$(II) = -\int_0^t e^{-i(t-s)A} \left\{ \mu^\alpha \frac{d}{ds} (e^{i\theta} p^\alpha) \widetilde{\eta}_\alpha \right\} ds\ .$$

Also note the sign of $0i$ is so that the last two terms in (4.12) decay as $t \to \infty$.

We have identify the main oscillation term in $\tilde\eta$ and we denote the remaining term in by $\eta^{(3)}$:

$$\tilde\eta = \tilde\eta^{(2)} + \tilde\eta^{(3)}. \tag{4.13}$$

Notice that $\tilde{\eta}^{(2)}$ is not in L^2 and this is not an L^2 decomposition. It is, however, very useful for local behavior as it identifies the local oscillation.

From (4.19) we obtain the equation for $\tilde{\eta}^{(3)}$:

$$\begin{aligned}
\tilde{\eta}^{(3)}(t) = {} & e^{-iAt}\tilde{\eta}_0 - e^{-iAt}(e^{i\theta}z^\alpha)(0)\widetilde{\eta}_\alpha - \int_0^t e^{-iA(t-s)} \left\{\mu^{[\alpha]}\frac{d}{ds}(e^{i\theta}p^\alpha)\widetilde{\eta}_\alpha\right\} ds \\
& + \int_0^t e^{-iA(t-s)}\,\mathbf{P}_c^A \left\{e^{i\theta}U\Pi\left[F^{\sharp\sharp} + i\eta^2\bar{\eta}\right] - e^{i\theta}[U,i]\dot{\theta}\eta\right\} ds \\
& + \int_0^t e^{-iA(t-s)}\,\mathbf{P}_c^A\, e^{i\theta}U\Pi(-i\eta^2\bar{\eta})\, ds \\
= {} & \tilde{\eta}_1^{(3)} + \tilde{\eta}_2^{(3)} + \tilde{\eta}_3^{(3)} + \tilde{\eta}_4^{(3)} + \tilde{\eta}_5^{(3)} \ .
\end{aligned} \tag{4.14}$$

We treat $\tilde{\eta}_5^{(3)}$ separately because $\eta^2\bar{\eta}$ is a non-local term. Notice that $\tilde{\eta}^{(2)} \notin L^2$, but is still "orthogonal" to the eigenvector of A.

From the decomposition of $\tilde{\eta}$ and the relation $\eta = U^{-1}e^{-i\theta}\tilde{\eta}$, we have the corresponding decomposition for η:

$$\eta(t) = \eta^{(2)}(t) + \eta^{(3)}(t) \ .$$

where

$$\eta^{(j)} = U^{-1}e^{-i\theta}\tilde{\eta}^{(j)} = U^{-1}z^\alpha\widetilde{\eta}_\alpha, \quad j = 2,3 \ .$$

Summarizing, we have decompose a, F and η into terms in order of z. The main oscillatory terms of order z^2 in a is $a_{20}(z^2+\bar{z}^2)$ and

$$a = a_{20}(z^2+\bar{z}^2) + b \tag{4.15}$$

with

$$\dot{b} = \left(c_1 Q, \operatorname{Im}(F - F^{(2)})\right) - 4\operatorname{Re} a_{20} z\mu^{-1}\dot{p} \ . \tag{4.16}$$

The nonlinear term F is decomposed into orders in (4.5) with the second order term $F^{(2)}$ explicitly given. We also rewrite the equation of θ into

$$\dot{\theta} = c_2(z+\bar{z}) + F_\theta \ , \tag{4.17}$$

where $c_2 = (c_0 Q, \lambda Q^2 u) = O(\lambda)$ and

$$F_\theta = -(1 + c_0 c_1^{-1} a)^{-1}\ \left[a + (c_0 Q, \lambda Q^2(\eta+\bar{\eta})) + (c_0 Q, \operatorname{Re} F(k))\right] \ . \tag{4.18}$$

The dispersive wave η is related to $\tilde{\eta}$ by the relation $\eta = U^{-1}e^{-i\theta}\tilde{\eta}$ with $\tilde{\eta}$ satisfies

$$\tilde{\eta}(t) = e^{-iAt}\tilde{\eta}_0 + \int_0^t e^{-iA(t-s)}\,\mathbf{P}_c^A\ \left\{e^{i\theta}U\Pi F^\sharp - e^{i\theta}[U,i]\dot{\theta}\eta\right\} ds \ . \tag{4.19}$$

Furthermore, it can be decomposed into $\tilde\eta^{(2)} + \tilde\eta^{(3)}$ with

$$\tilde\eta^{(2)} = e^{i\theta} z^\alpha \widetilde{\eta}_\alpha, \qquad \widetilde{\eta}_\alpha = \frac{1}{A - 0i + [\alpha]\kappa}\Phi_\alpha$$

and $\tilde\eta^{(3)}$ satisfies the equation (4.14). For Theorem 1.1 the boundary conditions are

$$\begin{aligned} a(\infty) = 0 &= b(\infty) \\ 0 < |z_0| &\le \epsilon_0 \\ \|\eta_0\|_{H^2\cap W^{2,1}(\mathbb{R}^3)} &\le |z_0|^2 \end{aligned} \tag{4.20}$$

We now collect a few properties of the operator U. We can expand $U_\pm$ in order of λ as

$$U_+ = 1 + O(\lambda), \quad U_- = O(\lambda).$$

Notice that $\mathbf{P}_c^A\, U c_2 u_+ = O(\lambda^2)$ since $U u_+$ is almost orthogonal to $\mathtt{H}_c(A)$. Hence we have

$$\Phi_{20} = -\,\mathbf{P}_c^A\, \Pi\phi_{20} + O(\lambda^2), \quad \Phi_{11} = O(\lambda), \quad \Phi_{02} = O(\lambda^2)\ .$$

We may decompose

$$U^{-1} = U_+^\diamond + U_-^\diamond \mathtt{C}, \qquad U_\pm^\diamond = \tfrac12 (BA^{-1/2} \pm B^{-1}A^{1/2})\ . \tag{4.21}$$

then

$$\eta^{(2)} = \left(U_+^\diamond + U_-^\diamond \mathtt{C}\right) z^\alpha \widetilde{\eta}_\alpha = z^2\eta_{20} + z\bar z\eta_{11} + \bar z^2 \eta_{02}\ ,$$

where

$$\eta_{20} = U_+^\diamond \widetilde{\eta}_{20} + U_-^\diamond \widetilde{\eta}_{02}, \quad \eta_{11} = U^{-1}\widetilde{\eta}_{11}, \quad \eta_{02} = U_+^\diamond \widetilde{\eta}_{02} + U_-^\diamond \widetilde{\eta}_{20}\ .$$

If we expand

$$U_+ = 1 + O(\lambda), \quad U_- = O(\lambda), \quad U_+^\diamond = 1 + O(\lambda), \quad U_-^\diamond = O(\lambda),$$

then we get

$$\eta_{20} = \widetilde{\eta}_{20} + O(\lambda^2) = \frac{-1}{A - 0i - 2\kappa}\,\mathbf{P}_c^A\, \Pi\phi_{20} + O(\lambda^2)\ , \quad \eta_{11} = O(\lambda)\ , \quad \eta_{02} = O(\lambda^2)\ .$$

5 Estimates of dispersive wave

We now estimate solutions to the equations (2.5) with the decompositions into main oscillatory and higher order terms in section 4. We first need to choose a suitable norm. Define

$$\{t\} = \varepsilon^{-2} + 2\Gamma t, \quad \{t\} \sim \max\left\{\varepsilon^{-2}, t\right\}.$$

and

$$M(T) := \sup_{0\le t\le T} \left\{ \{t\}^{1/2} |z(t)| + \{t\}^{3/4-\sigma} \|\eta(t)\|_{L^4} + \{t\}^{1+\sigma/4} \left\|\eta^{(3)}(t)\right\|_{L^2_{\rm loc}} \right\}$$

Recall that b, the slow oscillation part of a, is defined via (4.15) and satisfies the equation (4.16). We assume that the function b is in the following space $\mathcal{B}_T$:

$$\mathcal{B}_T = \left\{ b(t) : |b(t)| \le D\{t\}^{-1},\, 0 \le t \le T \right\} , \tag{5.1}$$

where $D = 2B_{22}/\Gamma$ is some constant. The precise form of the constant is not important, it merely has to be an order one constant bigger than the true behavior of b which we shall derive. The class $\mathcal{B}$ is nonempty since it contains the constant function 0.

Our setting is thus given at the end of section 4 except we now assuming the estimate (5.1) on b (and thus on a too) up to time T. From now on, we fix T.

Theorem 5.1 *Suppose η, h, a, θ, z are solutions to the equations* (2.5). *Assuming the estimate* (5.1) *on b, we have*

$$M(t) \le 2 \qquad \textit{for all} \quad t \le T\ .$$

Moreover, if we further assume $|z_0| = \varepsilon > 0$ and $\|\eta_0\| \le |z_0|^{2+\sigma}$, then $|z(t)| \ge c\{t\}^{-1/2}$.

Our strategy is to show that $M(0) \le 3/2$ and that $M(t) \le 3/2$ if $M(t) \le 2$. By continuity of $M(t)$, this would imply that $M(t) \le 3/2$ for all $t \le T$. So for the rest of this and next sections, we shall use freely that $M(t) \le 2$ to prove that $M(T) \le 3/2$. The proof of this theorem will be completed after Lemma 6.1. We now start the proof.

Due to the presence of the non-local term h^3 in F, we first need a global norm estimate on η, which we choose to be $\|\eta(t)\|_{L^4}$. Our goal is to prove that

$$\|\eta(t)\|_{L^4} \le C\{t\}^{-3/4} \log\{t\}$$

which agrees with that of free evolution.

We first recall some basic facts concerning the Schrödinger equation which provide some useful feeling on the size of various quantities. Since we shall proceed with iteration scheme, our a, z and η do not solve the Schrödinger equation and we will not use these facts.

The H^1 norm of ψ_t is uniformly bounded if λ is sufficiently small. (It can be proved by using the conservation of the Hamiltonian and the Gagliado-Nirenberg inequality.) Hence $\|h(t)\|_{H^1}$ is uniformly bounded since $h = \psi e^{i\theta} - Q$. Assuming $z(t)$ is bounded, then $\zeta(t)$ and $\eta(t)$ are both bounded in H^1, uniformly in t, since $\zeta = zu_+ + \bar{z}u_-$ and $\eta = h - \zeta$.

Return to a global estimate for η. Since $\eta = U^{-1}e^{-i\theta}\tilde{\eta}$ and U is bounded in Sobolev spaces, for the purpose of estimation we can treat η and $\tilde{\eta}$ the same. Recall $[U, i]$ is a local operator satisfying the estimate at Lemma 3.3.

We will need the following calculus lemma:

Lemma 5.2 *Let $0 < d < 1 < m$. $\{t\} \equiv \varepsilon^{-2} + 2\Gamma t$.*

$$\int_0^t |t-s|^{-d} \{s\}^{-m} \, ds \le C\varepsilon^{2m-2} \{t\}^{-d} \ .$$

Also,

$$\int_0^t |t-s|^{-d} \{s\}^{-1} \, ds \le C \{t\}^{-d} \log(\varepsilon^2 \{t\}) \ .$$

If, instead, $d \ge m > 1$,

$$\int_0^t \langle t-s\rangle^{-d} \{s\}^{-m} \, ds \le C \{t\}^{-m} \ .$$

Proof. Denote the first integral by (I). If $t \le \varepsilon^{-2}$, then $\{t\} \sim \varepsilon^{-2}$ and

$$(I) \sim \int_0^t |t-s|^{-d}\varepsilon^{2m} \, ds \lesssim \varepsilon^{2m}\varepsilon^{-2(1-d)} = \varepsilon^{2m-2}\varepsilon^{2d} \sim \varepsilon^{2m-2} \{t\}^{-d} \ .$$

If $t \ge \varepsilon^{-2}$, then $\{t\} \sim 2\Gamma t$ and

$$\begin{aligned}(I) &\le \int_0^{t/2} Ct^{-d} \{s\}^{-m} \, ds + \int_{t/2}^t C|t-s|^{-d} C \{t\}^{-m} \, ds \\ &\le Ct^{-d}\varepsilon^{2(m-1)} + Ct^{1-d} \{t\}^{-m} \sim C \{t\}^{-d} \varepsilon^{2(m-1)} + C \{t\}^{1-d-m} \\ &\le C\varepsilon^{2m-2} \{t\}^{-d} \ .\end{aligned}$$

For the second case, denote the second integral by (II). If $t \le \varepsilon^{-2}$, then $\{t\} \sim \varepsilon^{-2}$ and

$$(II) \sim \int_0^t \langle t-s\rangle^{-d} \varepsilon^{2m} \, ds \le \varepsilon^{2m} \cdot C \sim C \{t\}^{-m} \ .$$

If $t \ge \varepsilon^{-2}$, then $\{t\} \sim \Gamma \langle t\rangle$ and

$$(II) \sim \int_0^t \langle t-s\rangle^{-d} \Gamma^{-m} \langle s\rangle^{-m} \, ds \le C\Gamma^{-m} \langle t\rangle^{-m} \sim C \{t\}^{-m} \ .$$

We conclude the lemma. **Q.E.D.**

5.1 Estimates in L^4 and L^2

Lemma 5.3 *Suppose that $\tilde{\eta}$ is given by equation* (4.19) *and recall $\eta = U^{-1}e^{-i\theta}\tilde{\eta}$. Assuming the estimate* (5.1) *on b and $M(T) \leq 2$, we have*

$$\|\eta(t)\|_{L^4} \leq C\,\{t\}^{-3/4}\log\{t\}\ .$$

Moreover, we have

$$\|\tilde{\eta}(t)\|_{L^2} \leq \varepsilon^{1/2}\ .$$

Proof. From the defining equation of $\tilde{\eta}$, we have

$$\|\widetilde{\eta}(t)\|_{L^4} \leq C\,\|\widetilde{\eta}_0\|_{H^1\cap L^{4/3}}\,\langle t\rangle^{-3/4} + \int_0^t C|t-s|^{-3/4}\left\{\left\|F^\sharp(s)\right\|_{L^{4/3}} + |\dot{\theta}|\,\|\eta(t)\|_4\right\}\,ds\ .$$

From the definitions of $F\sharp$ and ζ, we have

$$\left\|F^\sharp(s)\right\|_{L^{4/3}} + |\dot{\theta}|\,\|\eta(t)\|_4 \leq C|\dot{\theta}|\,\|\zeta(t)\|_{4/3} + \|F(s)\|_{L^{4/3}} + C|a\dot{\theta}|\ ,$$

From the Holder inequality and the definition of $\|F(s)\|_{L^{4/3}}$ we can bound $\|F(s)\|_{L^{4/3}}$ by

$$\|F(s)\|_{L^{4/3}} \leq C\left(\|h\|_{L^4}^2 + |a|\,\|h\|_{L^4} + |a|^2 + \left\|h^3\right\|_{L^{4/3}} + |a|^3\right)\ .$$

Since $\|h^3\|_{L^{4/3}} = \|h\|_{L^4}^3$ and

$$\|h(s)\|_{L^4} \leq \|\zeta(s)\|_{L^4} + \|\eta(s)\|_{L^4}\,,$$

we have from the assumption $M \leq 2$ that

$$\|h(s)\|_{L^4} \leq C\,\{s\}^{-1/2} + C\,\{s\}^{-3/4}\log\{s\} \leq C\,\{s\}^{-1/2}\ ,$$

Therefore we have $\|F(s)\|_{L^{4/3}} \leq C\,\{s\}^{-1}$ and thus

$$\left\|F^\sharp(s)\right\|_{L^{4/3}} \leq C\,\{s\}^{-1}\ .$$

Since $\langle t\rangle^{-1} \leq \varepsilon^{-2}\,\{t\}^{-1}$ and $\|\eta(t)\|_{L^4} \sim \|\widetilde{\eta}(t)\|_{L^4}$, we conclude

$$\|\eta(t)\|_{L^4} \leq C\varepsilon^{-3/2}\,\|\eta_0\|_{H^1\cap L^{4/3}}\,\{t\}^{-3/4} + \int_0^t |t-s|^{-3/4}C\,\{s\}^{-1}\,ds \leq C\,\{t\}^{-3/4}\log\{t\}\ .$$

Here we have used Lemma 5.2 in the last integration.

We now bound $\|\eta(t)\|_{L^2}$. Since

$$\frac{d}{2dt}(\tilde{\eta},\tilde{\eta}) = \mathrm{Re}(\tilde{\eta},\partial_t\tilde{\eta}) = \mathrm{Re}(\tilde{\eta}, e^{i\theta}\,\mathbf{P}_{\mathrm{c}}^{A}\,U\Pi F^\sharp - e^{i\theta}\,\mathbf{P}_{\mathrm{c}}^{A}\,[U,i]\dot{\theta}\eta)\ ,$$

we have

$$\frac{d}{dt}\|\tilde\eta\|_{L^2}^2 \le C\|\tilde\eta\|_{L^4}\cdot\left\{\left\|F^\sharp(s)\right\|_{L^{4/3}}+|\dot\theta|\,\|\eta(t)\|_4\right\}$$
$$\le C\{t\}^{-3/4}\log\{t\}\,C\{t\}^{-1}=C\{t\}^{-7/4}\log\{t\}\ .$$

In the second inequality we use our previous estimates. Hence

$$\|\tilde\eta\|_{L^2}^2\le C\varepsilon^4+\int_0^\infty C\{t\}^{-7/4}\log\{t\}\le\varepsilon\ ,$$

and we conclude $\|\tilde\eta(t)\|_{L^2}\le\varepsilon^{1/2}$. **Q.E.D.**

5.2 Local decay of $\eta^{(3)}$

Recall $\eta^{(3)}=U^{-1}e^{-i\theta}\tilde\eta^{(3)}$ and $\tilde\eta^{(3)}$ satisfies the equation (4.14). We want to show that $\tilde\eta^{(3)}$ is smaller than $\tilde\eta^{(2)}$ locally. Define the local L^2 norm by

$$\|f\|_{L^2_{\rm loc}}=\left\|\langle x\rangle^{-\beta_0}f\right\|_{L^2}$$

for a fixed sufficiently large $\beta_0>0$, which will become clear later on.

Lemma 5.4 *Assuming the estimate* (5.1) *on* b *and* $M(T)\le 2$ *for all* T*, we have*

$$\begin{aligned}\left\|\tilde\eta^{(3)}\right\|_{L^2_{loc}}&\le C\{t\}^{-1-\sigma/2}\le C\varepsilon^{\sigma/2}\{t\}^{-1-\sigma/4}\ ,\\ \left\|\eta^{(3)}\right\|_{L^2_{loc}}&\le C\varepsilon^{\sigma}\{t\}^{-1-\sigma/4}\end{aligned}\tag{5.2}$$

In particular, for a local function ϕ *we have*

$$\left|(\phi,\eta^{(3)})\right|\le C\{t\}^{-1-\sigma/4}\ ,\tag{5.3}$$

$$\left|(\phi,|\eta|^2+|\eta|^3)\right|\le C\|\eta\|_{L^2_{loc}}^2+C\|\eta\|_{L^2_{loc}}\|\eta\|_{L^4}^2\le C\{t\}^{-2}\ .\tag{5.4}$$

Proof. We first estimate $\tilde\eta_j^{(3)}$ appearing on the right side of the equation for $\tilde\eta^{(3)}$ (4.14). From the definition and the estimates Lemma 3.1 on A, we have

$$\left\|\tilde\eta_1^{(3)}\right\|_{L^2_{\rm loc}}\le C\langle t\rangle^{-9/8}\left(\|\eta_0\|_{H^1}+\|\eta_0\|_{L^{8/7}}\right)\le C\varepsilon^{\sigma/4}\{t\}^{-9/8}$$

$$\left\|\tilde\eta_2^{(3)}\right\|_{L^2_{\rm loc}}\le C\{t\}^{-9/8}\ .$$

$$\left\|\tilde\eta_3^{(3)}\right\|_{L^2_{\rm loc}}\le C\int_0^t\langle t-s\rangle^{-9/8}\{s\}^{-3/2}\,ds\le C\{t\}^{-9/8}\ .$$

To estimate $\tilde{\eta}_4^{(3)}$, we recall the definition of $F^{\sharp\sharp}$ and rewrite

$$\begin{aligned}F^{\sharp\sharp} + i\eta^2\bar{\eta} &= \left(F^{\sharp} - iz^{\alpha}\phi_{\alpha}^{\sharp}\right) + i\eta^2\bar{\eta} \\ &= -i[\dot{\theta} - c_2(z+\bar{z})]\zeta - i\left\{F - F^{(2)} - \eta^2\bar{\eta}\right\} \\ &\quad - \left[(c_1 Q, \mathrm{Im}(F - F^{(2)})) + ia\dot{\theta}\right] R_{\Pi} \ .\end{aligned}$$

Since $M \le 2$, we have $|\dot{\theta} - c_2(z+\bar{z})| \le C\{s\}^{-1}$, $|a| \le C\{s\}^{-1}$, $\left\|\langle x\rangle^{\beta_0}\left(F - F^{(2)} - \eta^2\bar{\eta}\right)\right\|_{L^2} \le C\{s\}^{-3/2}$, and $|(c_1 Q, \mathrm{Im}(F - F^{(2)}))| \le \{s\}^{-3/2}$. Hence, by Lemma 3.1 we have

$$\begin{aligned}\left\|\tilde{\eta}_4^{(3)}\right\|_{L^2_{\mathrm{loc}}} &\le \int_0^t \langle t-s\rangle^{-3/2}\left\|\langle x\rangle^{\beta_0}\left(F^{\sharp\sharp} + i\eta^2\bar{\eta}\right)(s)\right\|_{L^2} ds \\ &\le \int_0^t \langle t-s\rangle^{-3/2} C\{s\}^{-3/2}\, ds \\ &\le C\{t\}^{-3/2} \ .\end{aligned}$$

We now consider $\tilde{\eta}_5^{(3)}$. Denote the integrand $e^{-iA(t-s)}\,\mathbf{P}_{\mathrm{c}}^{A}\, e^{i\theta}U\Pi(-i\eta^2\bar{\eta})$ by $X(t,s)$ and decompose the time interval into $(0, t-\sigma)$ and $(t-\sigma, t)$. From the triangle inequality, we have

$$\left\|\tilde{\eta}_5^{(3)}\right\|_{L^2_{\mathrm{loc}}} \le \int_0^{t-\sigma} \|X(t,s)\,ds\|_{L^2_{\mathrm{loc}}} + \int_{t-\sigma}^t \|X(t,s)\,ds\|_{L^2_{\mathrm{loc}}}$$

We can bound the L^2_{loc} norm by either L^8 or L^4 norm. Thus

$$\left\|\tilde{\eta}_5^{(3)}\right\|_{L^2_{\mathrm{loc}}} \le C\int_0^{t-\sigma} \|X(t,s)\|_{L^8}\, ds + C\int_{t-\sigma}^t \|X(t,s)\|_{L^4}\, ds$$

From Lemma 3.1, we have

$$\|X(t,s)\|_{L^8} = \left\|e^{-iA(t-s)}\,\mathbf{P}_{\mathrm{c}}^{A}\, e^{i\theta}U\Pi(-i\eta^2\bar{\eta})\right\|_{L^8} \le \frac{1}{|t-s|^{9/8}}\left\|\eta^3\right\|_{L^{8/7}}$$

and

$$\|X(t,s)\|_{L^4} = \left\|e^{-iA(t-s)}\,\mathbf{P}_{\mathrm{c}}^{A}\, e^{i\theta}U\Pi(-i\eta^2\bar{\eta})\right\|_{L^8} \le \frac{1}{|t-s|^{9/8}}\left\|\eta^3\right\|_{L^{4/3}}$$

From the Holder inequality and the global estimate of η in Lemma 5.3 we have

$$\begin{aligned}\left\|\eta^3(s)\right\|_{L^{8/7}} &\le \|\eta\|_2^{1/2}\,\|\eta\|_4^{5/2} \le C\varepsilon^{1/4}\left(\{s\}^{-3/4}\ln\{s\}\right)^{5/2} \le C\varepsilon^{1/4}\{s\}^{-7/4} \ , \\ \left\|\eta^3(s)\right\|_{L^{4/3}} &\le \|\eta\|_{L^4}^3 \le \left(\{s\}^{-3/4}\ln(2+s)\right)^3 \le C\{s\}^{-2} \ .\end{aligned}$$

If $t \ge 2\varepsilon^{-2}$, then $\{t\} \sim t$. We choose $\sigma = t/2$ and thus

$$\begin{aligned}\left\|\tilde{\eta}_5^{(3)}\right\|_{L^2_{\mathrm{loc}}} &\le C\int_0^{t/2} \frac{1}{|t-s|^{9/8}} C\{s\}^{-7/4}\, ds + C\int_{t/2}^t \frac{1}{|t-s|^{3/4}} C\{s\}^{-2}\, ds \\ &\le C\int_0^{t/2} C\{t\}^{-9/8}\{s\}^{-7/4}\, ds + C\int_{t/2}^t \frac{1}{|t-s|^{3/4}} C\{t\}^{-2}\, ds \\ &\le C\{t\}^{-9/8}(\varepsilon^{-2})^{-3/4} + C\{t\}^{-2}t^{1/4} \le C\varepsilon^{5/4}\{t\}^{-9/8} \ .\end{aligned}$$

On the other hand, if $t < 2\varepsilon^{-2}$, then $\{t\} \sim \varepsilon^{-2}$ and we choose $\sigma = t$ to get

$$\begin{aligned}\left\|\tilde{\eta}_5^{(3)}\right\|_{L^2_{\mathrm{loc}}} &\le C\int_0^t \frac{1}{|t-s|^{3/4}} C(\varepsilon^{-2})^{-2}\,ds \\ &\le C\varepsilon^4 t^{1/4} \le C\varepsilon^{7/2} \sim C\varepsilon^{5/4}\{t\}^{-9/8} \ .\end{aligned}$$

Combining two cases, we conclude

$$\left\|\tilde{\eta}_5^{(3)}\right\|_{L^2_{\mathrm{loc}}} \le C\varepsilon^{5/4}\{t\}^{-9/8} \ .$$

The lemma follows from all the estimates on $\eta_j^{(3)}$, $j = 1, \cdots, 5$. **Q.E.D.**

6 Excited state equation and normal form

6.1 Excited state equation

Return to the basic equation (2.5) and recall the equation for $\dot{p}$:

$$\dot{p} = -i\mu\left[(\tilde{u_+}, F) + (\tilde{u_-}, \bar{F}) + \left\{(u_+, h) + (u_-, \bar{h}) + c_3 a\right\}\dot{\theta}\right] \ , \tag{6.1}$$

where $c_3 = (u, R_\Pi)$, $c_1 = (Q, R)^{-1}$. Recall that

$$z, p = O(z), \quad a, \eta = O(z^2), \quad \eta^{(3)} = O(z^3) \ .$$

We now expand the right hand side of the equation for $\dot{p}$ into terms in order of z:

$$\dot{p} = \mu\left\{c_\alpha z^\alpha + d_\beta z^\beta + d_1 bz + d_2 b\bar{z} + P^{(4)}\right\} \tag{6.2}$$

The coefficients will be computed later on and their properties are summarized in the following lemma. The proof of this lemma is just straightforward computation and the reader can check it rather easily.

Lemma 6.1 *We can rewrite the equation of p into the form* (6.2) *such that the coefficients d_1, d_2 and all c_α are purely imaginary. Moreover,* $\mathrm{Re}\, d_{21} = -\Gamma + O(\lambda^3)$, *with*

$$\Gamma \equiv 2\lambda^2\left(Qu_+^2, \mathrm{Im}\,\frac{1}{A - 0i - 2\kappa}\,\mathbf{P}_c^A\,\Pi Q u_+^2\right) \ge 0 \ . \tag{6.3}$$

Proof. There are two parts in p equation: $(\tilde{u_+}, F)+(\tilde{u_-}, \bar{F})$ and $\{(u_+, h)+(u_-, \bar{h})+c_3 a\}\,\dot{\theta}$. We first consider the second part:

$$
\begin{aligned}
&\{(u_+, h)+(u_-, \bar{h})+c_3 a\}\ \dot{\theta}\\
&=\{(u_+, \zeta+\eta^{(2)})+(u_-, \bar{\zeta}+\bar{\eta}^{(2)})+c_3 a\}\cdot\\
&\qquad\cdot\{(c_0 Q, \lambda Q^2(\zeta+\bar{\zeta}+\eta^{(2)}+\bar{\eta}^{(2)})+a+(c_0 Q, \operatorname{Re} F^{(2)})\}+\widetilde{P}_2^{(4)}\\
&=\big((u_+, \zeta)+(u_-, \bar{\zeta})\big)\cdot(c_0 Q, \lambda Q^2(\zeta+\bar{\zeta}))\\
&\quad+\big[(u_+, \eta^{(2)})+(u_-, \bar{\eta}^{(2)})+c_3 a\big]\cdot(c_0 Q, \lambda Q^2(\zeta+\bar{\zeta}))\\
&\qquad+\big((u_+, \zeta)+(u_-, \bar{\zeta})\big)\cdot\big[a+(c_0 Q, \lambda Q^2(\eta^{(2)}+\bar{\eta}^{(2)}))+(c_0 Q, \operatorname{Re} F^{(2)})\big]\\
&\qquad+P_2^{(4)}\ .
\end{aligned}
$$

Here $P_2^{(4)}$ denotes the remaining terms. We first observe that line 4, lines 5–6, and $P_2^{(4)}$ are of orders $O(z^2)$, $O(z^3)$ and $O(z^4)$, respectively. Hence line 4 contributes to $c_\alpha z^\alpha$, line 5–6 to $d_\beta z^\beta$ and $d_1 bz+d_2 b\bar{z}$, and $P_2^{(4)}$ to $P^{(4)}$. We also observe that the coefficients in line 4 are all real. Since there is a $-i\mu$ factor in front, their contribution to c_α are purely imaginary. Next we observe that the coefficients in lines 5–6 are real except those involving $\eta^{(2)}$. The terms with lowest λ-order are:

$$
(u_+, \eta^{(2)})\cdot(c_0 Q, \lambda Q^2(\zeta+\bar{\zeta}))\ +\ (u_+, \zeta)\cdot(c_0 Q, \lambda Q^2(\eta^{(2)}+\bar{\eta}^{(2)}))\ .
$$

The first part is of order λ^3 since $(u_+, \eta^{(2)})=(P_\kappa^A u_+ + O(\lambda),\, \mathbf{P}_{\mathrm{c}}^A\,\eta^{(2)}+O(\lambda^2))$, with $\mathbf{P}_{\mathrm{c}}^A\,\eta^{(2)}=O(\lambda)$. The second part has order λ^2 terms, but not on $z^2\bar{z}$. We conclude that the contribution to $\operatorname{Re} d_{21}$ from the second part of p-equation. is of order λ^3. Finally we observe that line 5–6 give a term $d_{1,2}bz+d_{2,2}b\bar{z}$, with real $d_{1,2}=O(1)$ and $d_{2,2}=O(\lambda)$. Since there is a $-i\mu$ factor in front, their contribution to d_1 and d_2 are purely imaginary.

Now we look at the first part of the p equation. The contribution to $c_\alpha z^\alpha$ is from $F^{(2)}$:

$$
-i\,\{(\tilde{u_+}, z^\alpha\phi_\alpha)+(\tilde{u_-}, \bar{z}^\alpha\phi_\alpha)\}\ ,
$$

Clearly all coefficients of z^α are purely imaginary. Together with the analysis of second part of p-equation., we know c_α are purely imaginary.

The contribution to $d_1 bz+d_2 b\bar{z}$ is from $\widetilde{F}^{(3)}$:

$$
-i\,\{(\tilde{u_+}, 2\lambda bQR(2\zeta+\bar{\zeta})+(\tilde{u_-}, 2\lambda bQR(\zeta+2\bar{\zeta})\}
$$

with $\zeta=zu_+ + \bar{z}u_-$. Hence all coefficients of bz and $b\bar{z}$ are purely imaginary. Together with the analysis of second part of p equation, we know d_1 and d_2 are purely imaginary.

The contribution to $P^{(4)}$ is from $F^{(4)}$:

$$P_1^{(4)} = -i\left[(\tilde{u_+}, F^{(4)}) + (\tilde{u_-}, \bar{F}^{(4)})\right] \ ,$$

and we have $P^{(4)} = P_1^{(4)} - iP_2^{(4)}$.

The contribution to $d_\beta z^\beta$ is from $F^{(3)}$:

$$-i[(\tilde{u_+}, F^{(3)}) + (\tilde{u_-}, \bar{F}^{(3)}] \ .$$

We only consider d_{21}. A coefficient in $F^{(3)}$ has to have imaginary part in order to have a real part contribution to d_{21}. Hence only the first group of terms in $F^{(3)}$, $2\lambda Q[(\zeta+\bar{\zeta})\eta^{(2)}$, has contribution to $\operatorname{Re} d_{21}$. Also, when we decompose $\eta^{(2)} = z^\alpha \eta_\alpha$, we can disregard $\eta_{(11)}$ since it is real. Now, $\eta_{(20)} = O(\lambda)$, $\eta_{(02)} = O(\lambda^2)$, $u_+ = O(1)$ and $u_- = O(\lambda)$, hence the main part of $(\operatorname{Re} d_{21}) z^2 \bar{z}$ is in

$$-i(\tilde{u_+},\, 2\lambda Q(\bar{z}u_+) z^2 \eta_{(20)})$$

Summarizing our efforts, we have

$$\begin{aligned}
\operatorname{Re} d_{21} &= \operatorname{Im}(u_+, 2\lambda Q u_+ \eta_{(20)}) + O(\lambda^3) = \operatorname{Im}(2\lambda Q u_+^2, \eta_{(20)}) + O(\lambda^3) \\
&= (2\lambda Q u_+^2, \operatorname{Im} \frac{-1}{A - 0i - 2\kappa} \mathbf{P}_{\mathrm{c}}^A \, \Pi \phi_{20}) + O(\lambda^3) \\
&= -(2\lambda Q u_+^2, \operatorname{Im} \frac{1}{A - 0i - 2\kappa} \mathbf{P}_{\mathrm{c}}^A \, \Pi \lambda Q u_+^2) + O(\lambda^3) \ .
\end{aligned}$$

Q.E.D.

6.2 Normal form

From Lemma 3.2, we have $\Gamma > 0$.

Lemma 6.2 *We can rewrite the equation* (6.2) *of p into a* **normal form***:*

$$\dot{q} = \delta_{21}|q|^2 q + d_1 b q + g \ . \tag{6.4}$$

where q is a perturbation of p given in the proof. The coefficient δ_{21} satisfying the relation

$$\operatorname{Re} \delta_{21} = \operatorname{Re} d_{21} \tag{6.5}$$

If we assume the estimate (5.1) *on b and $M(T) \le 2$, then the error term $g(t)$ given by* (6.11) *satisfies the bound*

$$|g(t)| \le C_1 \{t\}^{-3/2-\sigma} \tag{6.6}$$

for some constant C_1. Furthermore, there is a positive constant σ such that $|q(t)|$ is bounded by

$$(1-\sigma)\left\{t\right\}^{-1/2} \le |q(t)| \le (1+\sigma)\left\{t\right\}^{-1/2}\ . \tag{6.7}$$

and hence

$$(1-2\sigma)\left\{t\right\}^{-1/2} \le |z(t)| \le (1+2\sigma)\left\{t\right\}^{-1/2}\ .$$

Proof of Theorem 5.1 From Lemma 6.1, we have

$$\left\{t\right\}^{1/2}|z(t)| \le (1+2\sigma)$$

From Lemma 5.3 and Lemma 5.4 we can bound

$$\left\{t\right\}^{3/4-\sigma}\|\eta(t)\|_{L^4} + \left\{t\right\}^{1+\sigma/4}\left\|\eta^{(3)}(t)\right\|_{L^2_{\rm loc}}$$

by $C\varepsilon^{\sigma/4}$. Since ε is small, we have proved that $M(T) \le 3/2$ and this concludes Theorem 5.1. **Q.E.D.**

We now prove Lemma 6.1.

Proof. We have

$$\dot p = \mu\left[c_\alpha z^\alpha + d_\beta z^\beta + d_1 bz + d_2 b\bar z + P^{(4)}\right]$$

and we want to obtain the normal form (6.4). We will repeatedly use the following formula:

$$\mu^m p^\alpha = \frac{d}{dt}\left(\frac{\mu^m p^\alpha}{i\kappa m}\right) - \frac{\mu^m p^\alpha}{i\kappa m} f_\alpha(z) \tag{6.8}$$

where, if $\alpha = (\alpha_0\alpha_1)$, ($|\alpha| = \alpha_0 + \alpha_1 = 2, 3, 4\cdots$)

$$\begin{aligned} f_\alpha(z) &= (\alpha_0 + \alpha_1 \mathtt{C})(p^{-1}\dot p) \\ &= (\alpha_0 + \alpha_1 \mathtt{C}) z^{-1}\left[c_\alpha z^\alpha + d_\beta z^\beta + d_1 bz + d_2 b\bar z + P^{(4)}\right] \end{aligned} \tag{6.9}$$

and $\mathtt{C}$ denotes the conjugation operator. The formula is equivalent to integration by parts.

We first remove $c_\alpha z^\alpha$. Let

$$p_1 = p - \frac{c_\alpha}{i\kappa(1+[\alpha])}\mu z^\alpha.$$

Since $[\alpha]$ is even, $1+[\alpha] \ne 0$. By (6.8)

$$\dot p_1 = \dot p - c_\alpha \mu z^\alpha - \frac{c_\alpha}{i\kappa(1+[\alpha])}\mu z^\alpha f_\alpha(z)$$

Decomposing $f_\alpha(z)$ into two parts, we can write

$$\begin{aligned} d_\beta^+ z^\beta &= -\frac{c_\alpha}{i\kappa(1+[\alpha])}\mu z^\alpha(\alpha_0+\alpha_1\mathtt{C})z^{-1}c_{\tilde\alpha}z^{\tilde\alpha} \\ g_1 &= -\frac{c_\alpha}{i\kappa(1+[\alpha])}\mu z^\alpha(\alpha_0+\alpha_1\mathtt{C})z^{-1}\left[d_\beta z^\beta + d_1 bz + d_2 b\bar z + P^{(4)}\right] \end{aligned}$$

and we get

$$\dot p_1 = \delta_\beta \mu z^\beta + d_1\mu bz + d_2\mu b\bar z + \mu P^{(4)} + g_1$$

with $\delta_\beta = d_\beta + d_\beta^+$. Since d_β^+ are purely imaginary, due to that c_α are purely imaginary, we have the key relation

$$\operatorname{Re}\delta_\beta = \operatorname{Re} d_\beta \tag{6.10}$$

Next we remove $d_2\mu b\bar z$. Let

$$p_2 = p_1 - \frac{\mu d_2 b\bar z}{2i\kappa}\ .$$

We have

$$\begin{aligned} \dot p_2 &= \dot p_1 - \mu d_2 b\bar z - \frac{\mu^2 d_2}{2i\kappa}(\dot b\bar p + b\dot{\bar p}) \\ &= \mu\delta_\beta z^\beta + d_1\mu bz + (\mu P^{(4)} + g_1 + g_2)\ , \end{aligned}$$

where

$$g_2 = -\frac{\mu^2 d_2}{2i\kappa}(\dot b\bar p + b\dot{\bar p})\ .$$

Finally we deal with $\delta_\beta z^\beta$ terms. Let

$$q = p_2 - \sum_{\beta\neq(21)}\frac{\delta_\beta\mu z^\beta}{i\kappa(1+[\beta])}.$$

Note $1+[\beta]\neq 0$ for $\beta\neq(21)$. We have

$$\dot q = \dot p_2 - \mu\delta_\beta z^\beta + g_3\ ,$$

with

$$g_3 = -\sum_{\beta\neq(21)}\frac{\delta_\beta\mu^{1+[\beta]}}{i\kappa(1+[\beta])}\frac{d}{dt}(p^\beta).$$

Hence

$$\begin{aligned} \dot q &= \delta_{21}\mu z^2\bar z + \mu d_1 bz + (\mu P^{(4)} + g_1 + g_2 + g_3) \\ &= \delta_{21}|p|^2 p + d_1 bp + (\mu P^{(4)} + g_1 + g_2 + g_3) \\ &= \delta_{21}|q|^2 q + d_1 bq + g\ , \end{aligned}$$

where

$$g = (\mu P^{(4)} + g_1 + g_2 + g_3) + \delta_{21}(|p|^2 p - |q|^2 q) + d_1 b(p - q) \ . \tag{6.11}$$

Hence we have arrived at the normal form. (Note that, although we kept $\dot{p}$ and $\dot{b}$ in the definition of g_2, it should be replaced by the corresponding equations (6.2) and (4.16).) Also note we have

$$q = p - \frac{c_\alpha}{i\kappa(1+[\alpha])}\mu z^\alpha - \frac{\mu d_2 b \bar{z}}{2i\kappa} - \sum_{\beta \neq (21)} \frac{\delta_\beta \mu z^\beta}{i\kappa(1+[\beta])} \ . \tag{6.12}$$

Finally, we can check the size of g satisfying the bound (6.6). To conclude this lemma, it remains to prove the estimate on q. But this follows from the next lemma. **Q.E.D.**

6.3 Decay and continuity estimates

In this subsection we present some calculus lemmas which deal with the decay of $q(t)$ and its continuity on the error term $g(t)$. We will write

$$q(t) = \rho(t)\, e^{i\omega(t)} \ ,$$

where $\rho = |q|$ and ω is the phase of q. Recall

$$\{t\} = \varepsilon^{-2} + 2\Gamma t, \quad \{t\} \sim \max\left\{\varepsilon^{-2}, t\right\}.$$

Before we proceed with proof, we first explain some simple facts of an ordinary differential equation.

Example. We consider real functions $r(t) > 0$ which solve

$$\dot{r}(t) = -\Gamma r(t)^3 - \varepsilon\,(1+t)^{-3} \ .$$

We have the following facts.

a. All solutions $r(t)$ satisfy

$$r(t) \le (C + 2\Gamma t)^{-1/2} \quad \text{with } r(0) = C^{-1/2} \ .$$

b. There is a number r_1 such that if $r(0) > r_1$, then $r(t) \sim (C + 2\Gamma t)^{-1/2}$.

c. There is a unique global solution $r_0(t)$ such that

$$r_0(t) \sim t^{-2} \quad \text{as } t \to \infty.$$

d. If $r(0) < r_0(0)$, then $r(t) = 0$ in finite time.

e. If $r(0) > r_0(0)$, then

$$\int_0^\infty r(s)^2 ds = \infty.$$

Lemma 6.3 *Let $\varepsilon_0 > 0$ be small. Suppose $\rho(t)$ satisfy*

$$\dot{\rho} = -\rho^3 + g(t) \ ,$$

where $|g(t)| < \varepsilon_0 \langle t \rangle^{-3/2-\sigma}$, $\sigma > 0$. We can find $0 < \rho_1 \le \rho_2$, (depending on ε_0), such that
(a) If $\rho_1 < \rho(0) < \rho_2$, then $|\rho(t)| \sim C \langle t \rangle^{-1/2}$.
(b) If $0 < \rho(0) < \rho_2$, then $|\rho(t)| \le C \langle t \rangle^{-1/2}$.

The above example shows that the conclusion of (a) cannot be expected to hold if $\rho(0)$ is too small. This is the main reason that there are two type of asymptotic behavior: the resonance dominating solutions given by case (a) and the dispersion dominating ones by (b). The proof of this lemma is elementary and similar to the next lemma and thus we omit it.

Lemma 6.4 *Let $\Gamma > 0$, $\sigma > 0$, and $C_1 > 0$ be given constants, independent of ε_0. Suppose a positive function $\rho(t)$ satisfies*

$$\dot{\rho} = -\Gamma \rho^3 + g(t), \ , \tag{6.13}$$

(a) Suppose $\rho(0) = \varepsilon$ with $0 < \varepsilon \le \varepsilon_0$ and

$$|g(t)| \le C_1 \{t\}^{-3/2-\sigma} , \quad \{t\} \equiv \varepsilon^{-2} + 2\Gamma t \ .$$

Then there is a constant $m = m(\varepsilon_0) > 1$ such that

$$m^{-1} \{t\}^{-1/2} \le \rho(t) \le m \{t\}^{-1/2} \ .$$

Moreover, $m(\varepsilon_0) \to 1^+$ as $\varepsilon_0 \to 0^+$.
(b) Suppose that $\rho(0) \le \varepsilon_0$ and

$$|g(t)| \le C_1 \{t\}_0^{-3/2-\sigma} , \quad \{t\}_0 \equiv \varepsilon_0^{-2} + 2\Gamma t \ .$$

Then we have for some constant C

$$\rho(t) \le C \{t\}_0^{-1/2} \ .$$

Proof. We first prove part (a). Let $\rho_+ = m \{t\}^{-1/2}$ and $\rho_- = m^{-1} \{t\}^{-1/2}$, with $m > 1$ to be determined. We have $\rho_+(0) > \rho(0) > \rho_-(0)$. Moreover,

$$\dot{\rho}_+ = -\Gamma m^{-2} \rho_+^3 = -\Gamma \rho_+^3 + \Gamma(1 - m^{-2}) \rho_+^3 \ge -\Gamma \rho_+^3 + g \ ,$$

if

$$\Gamma(1 - m^{-2}) m^3 \{t\}^{-3/2} \ge C_1 \{t\}^{-3/2-\sigma} .$$

Also

$$\dot{\rho}_- = -\Gamma m^2 \rho_-^3 = -\Gamma \rho_-^3 - \Gamma(m^2-1)\rho_-^3 \le -\Gamma \rho_-^3 + g \ ,$$

if

$$\Gamma(m^2-1)m^{-3} \{t\}^{-3/2} \ge C_1 \{t\}^{-3/2-\sigma} \ .$$

Since $\{t\}^{-\sigma} \le \varepsilon^{2\sigma}$, both inequalities hold if

$$\Gamma(1-m^{-2})m^3, \, \Gamma(m^2-1)m^{-3} \ge C_1 \, \varepsilon_0^{2\sigma} \ .$$

This is true for $m > 1$ arbitrarily close to 1 by choosing ε_0 sufficiently small. By comparison, we have $\rho_-(t) < \rho(t) < \rho_+(t)$ for all t.

To prove part (b), note we can still define ρ_+ as a comparison function. Since $\rho_+(0) \ge \rho(0)$, the above argument still holds. **Q.E.D.**

The following lemma estimates the continuity of ρ in the error term $g(t)$. Here we consider the case $\rho(0) = \varepsilon > 0$ only.

Lemma 6.5 *Let $\Gamma > 0$, $\sigma > 0$, and $C_1 > 0$ be given constants, independent of ε_0. Suppose two positive function $\rho_1(t)$ and $\rho_2(t)$ satisfy*

$$\dot{\rho}_i = -\Gamma \rho_i^3 + g_i(t) \quad (i=1,2) \ , \qquad \rho_1(0) = \rho_2(0) = \varepsilon \ ,$$

with $0 < \varepsilon \le \varepsilon_0$ and

$$|g_i(t)| \le C_1 \{t\}^{-3/2-\sigma} \ , \quad |\boldsymbol{\delta} g(t)| \le \boldsymbol{\delta}_0 \{t\}^{-3/2-\sigma} \ ,$$

where $\boldsymbol{\delta} g = g_2 - g_1$ and $\boldsymbol{\delta}_0 \ge 0$ is a small number. Then we have

$$|\rho_2(t) - \rho_1(t)| \le \boldsymbol{\delta}_0 \varepsilon^\sigma (\Gamma\sigma)^{-1} \{t\}^{-(1+\sigma)/2} \ .$$

Note that the decay rate is improved.

Proof. Let $r = \boldsymbol{\delta}\rho = \rho_2 - \rho_1$. Then r satisfies

$$\dot{r} = -Gr + \boldsymbol{\delta} g, \quad r(0) = 0, \quad G = \Gamma(3\rho^2 + 3\rho r + r^2) \ .$$

We have

$$r(t) = \int_0^t e^{-\int_s^t G(\tau)d\tau} \boldsymbol{\delta} g(s) \, ds \ . \tag{6.14}$$

Note $|r(t)| < \rho_+(t) - \rho_-(t)$, hence

$$G(t) \ge \Gamma[3\rho_-^2 - 3\rho_+(\rho_+ - \rho_-)] = 3\Gamma[m^{-2} - m(m - m^{-1})] \{t\}^{-1} \ge 2\Gamma \{t\}^{-1}$$

if m is sufficiently close to 1. Hence

$$\int_s^t G(\tau)d\tau \geq \int_s^t 2\Gamma \{\tau\}^{-1} d\tau = 2\Gamma \int_s^t (2\Gamma)^{-1} [\ln\{\tau\}]_{\tau=s}^t \geq \frac{1+\sigma}{2} \ (\ln\{t\} - \ln\{s\})$$

and

$$e^{-\int_s^t G(\tau)d\tau} \leq \{t\}^{-(1+\sigma)/2} \cdot \{s\}^{((1+\sigma)/2} \ .$$

By (6.14) we have

$$\begin{aligned} |r(t)| &\leq \int_0^t \{t\}^{-(1+\sigma)/2} \{s\}^{(1+\sigma)/2} \boldsymbol{\delta}_0 \{s\}^{-3/2-\sigma} \ ds \\ &= \boldsymbol{\delta}_0 \{t\}^{-(1+\sigma)/2} \int_0^t \{s\}^{-1-\sigma/2} \ ds \leq \boldsymbol{\delta}_0 \varepsilon^\sigma (\Gamma\sigma)^{-1} \{t\}^{-(1+\sigma)/2} \ . \end{aligned}$$

Q.E.D.

7 Change of mass

For given Q_E and h_0, we want to find an $a(t) = a(E, h_0; t)$ which satisfies

$$a(\infty) = 0, \quad \dot{a} = (c_1 Q, \operatorname{Im} F(k)), \quad |a(t)| \leq C\lambda^2 \{t\}^{-1} \ .$$

7.1 Decomposition of $a(t)$

We use the following equivalent integral equation for $a(t)$:

$$a(t) = \int_\infty^t (c_1 Q, \operatorname{Im} F(k)) \, ds \ .$$

Note $k = aR + h = aR + \zeta + \eta$. We want to perform several integrations by parts so that we get the form

$$a(t) = O(t^{-1}) + \int_\infty^t O(t^{-2}) \, ds = O(t^{-1}) \ .$$

Using the decomposition of $F(k)$, we have

$$(c_1 Q, \operatorname{Im} F) = A^{(2)} + A^{(za)} + A^{(z\eta)} + A^{(z^3)} + A^{(4)} \ ,$$

where

$$\begin{aligned} A^{(2)} &= \left(c_1 Q, \ \lambda Q \operatorname{Im} \zeta^2\right) \\ A^{(za)} &= (c_1 Q, \ \operatorname{Im} 2\lambda a Q R \zeta) \\ A^{(z\eta)} &= (c_1 Q, \ \operatorname{Im} 2\lambda Q \zeta \eta) \\ A^{(z^3)} &= \left(c_1 Q, \ \operatorname{Im} \lambda |\zeta|^2 \zeta\right) \\ A^{(4)} &= \left(c_1 Q, \ \operatorname{Im} \left\{\lambda Q \eta^2 + 2\lambda Q R a \eta + \lambda \left[|k|^2 k - |\zeta|^2 \zeta\right]\right\}\right) \ , \end{aligned}$$

Since $\zeta = zu_+ + \bar{z}u_-$, hence

$$\operatorname{Im}\zeta = \operatorname{Im} z\,(u_+ - u_-), \quad \operatorname{Im}\zeta^2 = (\operatorname{Im} z^2)\,(u_+^2 - u_-^2)\ .$$

Therefore

$$\begin{aligned} A^{(2)} &= C_1 \operatorname{Im} z^2, & C_1 &= \left(c_1 Q,\ \lambda Q(u_+^2 - u_-^2)\right) = O(\lambda^2)\ , \\ A^{(za)} &= C_2\, a \operatorname{Im} z, & C_2 &= (c_1 Q, 2\lambda Q R(u_+ - u_-)) = O(\lambda)\ . \end{aligned}$$

First we integrate $A^{(2)}$:

$$\begin{aligned} \int_\infty^t A^{(2)}\,ds &= C_1 \operatorname{Im}\int_\infty^t z^2\,ds = C_1 \operatorname{Im}\int_\infty^t \mu^{-2}p^2\,ds \\ &= C_1 \operatorname{Im}\frac{1}{-2i\kappa}\left\{\mu^{-2}p^2 - \int_\infty^t \mu^{-2}2p\dot{p}\,ds\right\} \\ &= C_1 \operatorname{Re}\frac{1}{2\kappa}\left\{\mu^{-2}p^2 - \int_\infty^t \mu^{-1}2p\left[c_\alpha z^\alpha + d_\beta z^\beta + d_1 bz + d_2 b\bar{z} + P^{(4)}\right]\,ds\right\} \\ &= C_1 \operatorname{Re}\frac{1}{2\kappa}\left\{z^2 - \int_\infty^t 2z\left[c_\alpha z^\alpha + d_\beta z^\beta + d_1 bz + d_2 b\bar{z} + P^{(4)}\right]\,ds\right\} \\ &= a_{20}(z^2 + \bar{z}^2) + \int_\infty^t A_{2,3} + A_{2,4}\,ds\ , \end{aligned}$$

where

$$\begin{aligned} a_{20} &= \frac{C_1}{4\kappa} = \frac{\lambda}{4\kappa}\left(c_1 Q,\ Q(u_+^2 - u_-^2)\right) = O(\lambda^2)\ , \\ A_{2,3} &= -4a_{20}\operatorname{Re} z c_\alpha z^\alpha\ , \\ A_{2,4} &= -4a_{20}\operatorname{Re} z\left[d_\beta z^\beta + d_1 bz + d_2 b\bar{z} + P^{(4)}\right]\ . \end{aligned}$$

As it will become clear, we will have $a \le O(z^2)$ and it has two sources of contribution: one is from the term $a_{20}(z^2 + \bar{z}^2)$, which has strong oscillation; another part b (4.15) has slower oscillations. Before we can handle b, we first need to integrate $A^{(za)}$:

$$\begin{aligned} \int_\infty^t A^{(za)} &= \int_\infty^t C_2 a \operatorname{Im} z \\ &= C_2 \operatorname{Im}\frac{1}{-i\kappa}\left\{az - \int_\infty^t \mu^{-1}\frac{d}{dt}(ap)\right\} \\ &= C_2 \operatorname{Re}\frac{1}{\kappa}\left\{az - \int_\infty^t z(c_1 Q, \operatorname{Im} F) + a(c_\alpha z^\alpha + d_\beta z^\beta + d_1 bz + d_2 b\bar{z} + P^{(4)})\right\} \\ &= c_{za} a(z + \bar{z}) + \int_\infty^t A_{za,3} + A_{za,4}\,ds\ , \end{aligned}$$

where

$$\begin{aligned} c_{za} &= C_2\,/(2\kappa) = (c_1 Q, 2\lambda Q R(u_+ - u_-))\,/(2\kappa) = O(\lambda) \\ A_{za,3} &= c_{za}\operatorname{Re} z(c_1 Q, \operatorname{Im} F^{(2)}) \\ A_{za,4} &= c_{za}\operatorname{Re}\left\{z(c_1 Q, \operatorname{Im}(F - F^{(2)})\,) + a(c_\alpha z^\alpha + d_\beta z^\beta + d_1 bz + d_2 b\bar{z} + P^{(4)})\right\} \end{aligned}$$

We now integrate $A^{(z\eta)}$. Recall $\eta = U^{-1}\eta^\diamond = U^{-1}e^{-i\theta}\tilde\eta$. Let $U^{-1}(f_1+if_2) = U_1 f_1 + iU_2 f_2$, where $U_1 = BA^{-1/2}$ and $U_2 = B^{-1}A^{1/2}$ are self-adjoint. Then

$$
\begin{aligned}
A^{(z\eta)} &= (c_1 Q, \operatorname{Im} 2\lambda Q\zeta\eta) = \int dx\, 2c_1\lambda Q^2 \operatorname{Im}\zeta\eta \\
&= \int dx\, 2c_1\lambda Q^2 \operatorname{Im}\left\{(\operatorname{Re} z u + (\operatorname{Im} z) iv)U^{-1}\eta^\diamond\right\} \\
&= \int dx\, 2c_1\lambda Q^2 u(\operatorname{Re} z)U_2 \operatorname{Im}\eta^\diamond + 2c_1\lambda Q^2 v(\operatorname{Im} z)U_1 \operatorname{Re}\eta^\diamond \\
&= \int dx\, \phi_2(\operatorname{Re} z)\operatorname{Im}\eta^\diamond + \phi_1(\operatorname{Im} z)\operatorname{Re}\eta^\diamond
\end{aligned}
$$

where

$$
\phi_2 = U_2(2c_1\lambda Q^2 u), \quad \phi_1 = U_1(2c_1\lambda Q^2 v),
$$

and both are real. Hence

$$
\begin{aligned}
A^{(z\eta)} &= \int dx\, \phi_2 \frac{1}{2}(z+\bar z)\,\frac{1}{2i}(\eta^\diamond - \overline{\eta^\diamond}) + \phi_1 \frac{1}{2i}(z-\bar z)\,\frac{1}{2}(\eta^\diamond + \overline{\eta^\diamond}) \\
&= \operatorname{Im}\int dx(\phi_3 z + \phi_4 \bar z)\eta^\diamond\ , \quad \phi_3 = \frac{1}{2}(\phi_1+\phi_2), \quad \phi_4 = \frac{1}{2}(-\phi_1+\phi_2)\ .
\end{aligned}
$$

We rewrite

$$
\eta^\diamond(\tau) = e^{-i\theta}\tilde\eta = e^{-i\theta}e^{-iA\tau}f(\tau)
$$

where $f = e^{iA\tau}\tilde\eta$. The reason we work with f, instead of $\tilde\eta$, is that those terms in $\frac{d}{d\tau}f$ of same order (z^2) are explicit and do not involve differentiation. Now we have

$$
\begin{aligned}
\int_\infty^t A^{(z\eta)} &= \operatorname{Im}\int_\infty^t (\phi_3, z\eta^\diamond) + (\phi_4, \bar z\eta^\diamond)\, d\tau \\
&= \operatorname{Im}\int_\infty^t \left(\phi_3, e^{-i\tau(A+\kappa)}\left(pe^{-i\theta}f\right)\right) d\tau + \operatorname{Im}\int_\infty^t \left(\phi_4, e^{-i\tau(A-\kappa)}\left(\bar p e^{-i\theta}f\right)\right) d\tau
\end{aligned}
$$

We deal with the first integral, which is equal to (note $f \in \mathtt{H}_c(A)$)

$$
\begin{aligned}
&= \operatorname{Im}\left(\phi_3, \frac{1}{-i(A+\kappa)}e^{-i\tau(A+\kappa)}\left(pe^{-i\theta}f\right)\right) \\
&\quad - \operatorname{Im}\int_\infty^t \left(\phi_3, \frac{1}{-i(A+\kappa)}e^{-i\tau(A+\kappa)}\frac{d}{d\tau}\left(pe^{-i\theta}f\right)\right) d\tau \\
&= \operatorname{Re}\left(\frac{1}{A+\kappa}\mathbf{P}_c^A\, \Pi\phi_3, ze^{-i\theta}\tilde\eta\right) \\
&\quad - \operatorname{Re}\int_\infty^t \left(\frac{1}{A+\kappa}\mathbf{P}_c^A\, \Pi\phi_3, \left\{(\dot p/p - i\dot\theta)ze^{-i\theta}\tilde\eta + ze^{-i\theta}\,\mathbf{P}_c^A\left[e^{i\theta}U\Pi F^\sharp - e^{i\theta}[U,i]\dot\theta\eta\right]\right\}\right) d\tau \\
&= \operatorname{Re}(\phi_5, z\eta^\diamond) - \operatorname{Re}\int_\infty^t \left(\phi_5, \left\{(\dot p/p - i\dot\theta)z\eta^\diamond + z\,\mathbf{P}_c^A\left[U\Pi F^\sharp - [U,i]\dot\theta\eta\right]\right\}\right) d\tau
\end{aligned}
$$

where

$$\phi_5 = \frac{1}{A+\kappa}\,\mathbf{P}_c^A\,\Pi\phi_3\ .$$

We are careful in adding $\mathbf{P}_c^A\,\Pi$ so that ϕ_5 makes sense. We can do so since $f \in \mathtt{H}_c(A)$.

Similarly, the second integral is equal to

$$= \mathrm{Re}\,(\phi_6, \bar z\eta^\diamond) - \mathrm{Re}\int_\infty^t \Big(\phi_6, \Big\{(\overline{\dot p/p} - i\dot\theta)\bar z\eta^\diamond + \bar z\,\mathbf{P}_c^A\,\Big[U\Pi F^\sharp - [U,i]\dot\theta\eta\Big]\Big\}\Big)\,d\tau \tag{7.1}$$

with

$$\phi_6 = \frac{1}{A-\kappa}\,\mathbf{P}_c^A\,\Pi\phi_4\ .$$

We can rewrite their leading terms in the form

$$\mathrm{Re}(\phi_5, z\eta^\diamond) + \mathrm{Re}(\phi_6, \bar z\eta^\diamond) = \mathrm{Re}(z\phi_7 + \bar z\phi_8, \eta).$$

We get η from $\eta^\diamond$ by a similar argument as before.

We also need to extract terms of order z^3 from the integrand of the remainder. They are from $F^\sharp$:

$$A_{z\eta,3} := \mathrm{Re}(\phi_5, z\,\mathbf{P}_c^A\,U\Pi(-iz^\alpha\phi_\alpha^\sharp)) + \mathrm{Re}(\phi_6, \bar z\,\mathbf{P}_c^A\,U\Pi(-iz^\alpha\phi_\alpha^\sharp))$$

($\phi_\alpha^\sharp$ are defined in subsection 5.2.) Clearly they can be written as a sum of z^β, with $|\beta| = 3$. There is no a or η involved. We denote the others terms by $A_{z\eta,4}$. Note, although $e^{i\theta}$ appears in the computation, it does not show up in the last line.

Summarizing, we have

$$a = a_{20}\left(z^2 + \bar z^2\right) + c_{za}a(z+\bar z) + \mathrm{Re}(z\phi_7 + \bar z\phi_8, \eta) \tag{7.2}$$

$$+\int_\infty^t A^{(z^3)} + A_{2,3} + A_{(za,3)} + A_{(z\eta,3)} + A^{(4)} + A_{2,4} + A_{(za,4)} + A_{(z\eta,4)}\,ds \tag{7.3}$$

We now write

$$A^{(z^3)} + A_{2,3} + A_{(za,3)} + A_{(z\eta,3)} = A_\beta z^\beta\ .$$

where A_β is the coefficient for z^β and can be computed from the left side of the equation. Since this term is of lower, we will not precise estimates on A_β. From integration by parts, we have

$$\int_\infty^t A_\beta z^\beta\,ds = a_\beta z^\beta - \int_\infty^t a_\beta z^\beta f_\beta(z)\,ds, \quad a_\beta = \frac{A_\beta}{i[\beta]\kappa}\ .$$

where f_β is defined in (6.9). Note $[\beta] \neq 0$. Also note $\overline{a_\beta} = a_{\bar\beta}$.

Let $a^{(4)}$ denote the total of the remaining integrals. We have

$$a(t) = a_{20}(z^2 + \bar z^2) + a_\beta z^\beta + c_{za}a(z+\bar z) + \mathrm{Re}(z\phi_7 + \bar z\phi_8, \eta) + a^{(4)} \tag{7.4}$$

$$a^{(4)} = \int_\infty^t A^{(4)} + A_{2,4} + A_{(za,4)} + A_{(z\eta,4)} - a_\beta z^\beta f_\beta(z)\,ds\ . \tag{7.5}$$

It will be important to consider terms of order z^4, i.e., terms of the form

$$z^4,\ a^2,\ \eta^2,\ z^2a,\ z^2\eta,\ a\eta,\ z\eta^{(3)},$$

in the integrand of $a^{(4)}$. We are interested in those terms involving a.

We have the following Lemma.

Lemma 7.1 *In the integrand of $a^{(4)}$ (7.5), the only terms of order $O(z^4)$ and involving a can be sumed in the form $a(cz^2+\overline{cz^2})$. Terms without the oscillation factor μ, i.e., a^2 and $a|z|^2$, do not occur. Also, we do not have terms of the form $(z\phi+\bar{z}\phi,\eta^{(3)})$.*

Proof. Notice that the terms of order z^4 in the integrand of $a^{(4)}$ (7.5), the function $\eta^{(3)}$ does not appear explicitly. On the other hand, $\eta^{(3)}$ can occur if we replace η substitute η by its decomposition $z^\alpha\eta_\alpha+\eta^{(3)}$. But we can check the terms resulting from such substitutions are of the form $(\phi,\eta^{(3)})a$ for some function ϕ. This proves the last statement of the lemma.

To verify the first statement, we inspect the terms in its integrand in the order: $A^{(4)}$, $A_{2,4}$, $A_{za,4}$, $A_{z\eta,4}$, and $a_\beta z^\beta f_\beta(z)$.

1. In $A^{(4)}$, terms involving a are in

$$(c_1Q,2\lambda QRa\operatorname{Im}\eta)+(c_1Q,\lambda(aR+h)(aR+\bar{h})\operatorname{Im}h)\ .$$

those of order z^4 are

$$(c_1Q,2\lambda QRa\operatorname{Im}\eta^{(2)})=(c_1Q,2\lambda QRa\operatorname{Im}(z^2\eta_{20}+\bar{z}^2\eta_{02}))$$
$$(c_1Q,\lambda aR(\zeta+\bar{\zeta})\operatorname{Im}\zeta)=(c_1Q,\lambda aR\operatorname{Im}(z^2u_+^2+\bar{z}^2u_-^2))\ .$$

Terms with $|z|^2$ are killed by the operator Im, (η_{11} and u_+u_- are real), and there is no a^2 term.

2. In $A_{2,4}=-4a_{20}\operatorname{Re}z\left[d_\beta z^\beta+d_1bz+d_2b\bar{z}+P^{(4)}\right]$, $d_2b\bar{z}$ is killed by the Re since d_2 is purely imaginary. We do get a term $-4a_{20}\operatorname{Re}zd_1bz$, which does not give b^2 or $b|z|^2$ terms.

3. In $A_{za,4}=c_{za}\operatorname{Re}\left\{z(c_1Q,\operatorname{Im}(F-F^{(2)}))+a(c_\alpha z^\alpha+d_\beta z^\beta+d_1bz+d_2b\bar{z}+P^{(4)})\right\}$, the terms of order z^4 and involving a are

$$c_{za}\operatorname{Re}z(c_1Q,\operatorname{Im}2\lambda QRa\zeta)=c(z+\bar{z})a\operatorname{Im}z=ca\operatorname{Im}z^2$$
$$c_{za}\operatorname{Re}ac_\alpha z^\alpha=c_{za}a\operatorname{Re}(c_{20}z^2+c_{02}\bar{z}^2)$$

here we have removed $c_{11}|z|^2$ since it is purely imaginary.

4. In $A_{(z\eta,4)}$, from the first integral, the terms of order z^4 and involving a come from $F^\sharp$. They are

$$\operatorname{Re}\left(\phi_5,z\,\mathbf{P}_c^A\,U\Pi\left\{ia\zeta-i2\lambda aQR(2\zeta+\bar{\zeta})-\left[(c_1Q,2\lambda aQR\operatorname{Im}\zeta)-iac_2(z+\bar{z})\right]R_\Pi\right\}\right)$$

where $c_2 = (c_0 Q, \lambda Q^2(u_+ + u_-))$ is real. There is a similar copy from the second integral. We first observe that there is no a^2 term. Then we compute to see there is no $|z|^2 a$ term. (The key is that every z and $\bar{z}$ inside the $\{\}$ is accompanied by an i. When we expand it out, $iz\bar{z}a$ is killed by the Re operator in front.)

5. In $a_\beta z^\beta f_\beta(z)$, all terms of order z^4 are of the form z^γ, $|\gamma| = 4$. Hence no a^2 and $a|z|^2$ terms.

We conclude that all terms in the integrand of $a^{(4)}$ of order z^4 and involving a can be summed as $a \operatorname{Re} cz^2$, with a complex constant c. **Q.E.D.**

7.2 Equation for $b(t)$

It turns out that the function $a(t)$ is not ideal for the iteration. The main reason is its strong oscillation. As defined in (4.15), we write

$$a = a_{20}(z^2 + \bar{z}^2) + b$$

and work with b. The benefit is that, although $a(t), b(t), \dot{a}(t) \sim z^2$, but $\dot{b} \sim z^3$, see (4.16). In other words, $b(t)$ is the part of $a(t)$ that has no strong oscillation. Another related reason for using $b(t)$ is that

$$\boldsymbol{\delta} a(t) \sim \boldsymbol{\delta}_0 \{t\}^{-1} \log\{t\}, \quad \boldsymbol{\delta} b(t) \sim \boldsymbol{\delta}_0 \{t\}^{-1} .$$

To obtain the above result, we need to expand b further.

1. We substitute (4.15) into the equations of $a(t)$, (7.2)–(7.3). We find that the function $b(t)$ is of the form

$$b = cb(z+z) + \sum_{|\beta|=3} cz^\beta + \operatorname{Re}(z\phi_7 + \bar{z}\phi_8, \eta) + \int_\infty^t z^4 + \eta^2 + z^2\eta + z^2 b + b\eta + O(z^5)\, d\tau$$

We have ignored the complex constant factors in each term. Notice we do not have b^2, $|z|^2 b$ and $z\eta^{(3)}$ in the integrand, due to the discussion in last subsection.

2. Next, we substitute of $\eta = z^\alpha \eta_\alpha + \eta^{(3)}$ into the order z^4 terms, and z^4 terms only. In other words, we substitute $\eta = z^\alpha \eta_\alpha + \eta^{(3)}$ into terms of the form $z^2(\phi, \eta)$ and (ϕ, η^2). (We already did so for terms of the form $b(\phi, \eta)$.) Then in terms of order z^4 we no longer see η.

3. Finally, in the integrand, all terms of order z^4 are of the form z^γ, $|\gamma| = 4$, and $b \operatorname{Re} cz^2$. Now we integrate out $b \operatorname{Re} cz^2$,

$$\operatorname{Re} \int_\infty^t bcz^2 = \operatorname{Re} \frac{bcz^2}{-2i\kappa} - \operatorname{Re} \int_\infty^t \frac{c\mu^{-2}}{-2i\kappa} \frac{d}{ds}(bp^2)\, ds$$

where

$$\frac{d}{ds}(bp^2) = \dot{b}p^2 + 2bp\dot{p} = O(z^5) \ .$$

We also integrate out terms z^γ with $|\gamma| = 4$, $\gamma \neq (22)$:

$$\mathrm{Re} \int_\infty^t cz^\gamma = \mathrm{Re} \frac{cz^\gamma}{[\gamma]i\kappa} - \mathrm{Re} \int_\infty^t \frac{c\mu^{[\gamma]}}{[\gamma]i\kappa} \frac{d}{ds}(p^\gamma)\, ds = \mathrm{Re} \frac{cz^\gamma}{[\gamma]i\kappa} - \mathrm{Re} \int_\infty^t O(z^5)\, ds \ .$$

4. Now we arrive at the form

$$\begin{aligned} b = {} & \mathrm{Re}\left\{bz + bz^2 + z^3 + z^2\bar{z} + z^4 + z^3\bar{z}\right\} + \mathrm{Re}(z\phi_7 + \bar{z}\phi_8, \eta) \\ & + \int_\infty^t B_{22}|z|^4 + B_5(z, b, \eta^{(2)}, \eta^{(3)})\, d\tau \ . \end{aligned} \tag{7.6}$$

Here B_5 is a complicated polynomial in its arguments, which including U, U^{-1} and $((A - \kappa)^{-1}\phi, \cdot)$. It is of order $O(z^5)$. We have ignored the constant factors in each term in the last expression of b. Note that B_5 does not contain $e^{i\theta}$. (Its only appearance in this section is in the integration of $A^{(z\eta)}$, but we already observed that it disappears in the end.) We notice that the leading term in (7.6) is $\int_\infty^t B_{22}|z|^4$.

We have obtained the main estimate Theorem 5.1 assuming the estimate (5.1) on b. We now need to prove the existence of the solution and check the assumption on b. Define the mapping

$$\begin{aligned} S(b)(t) = {} & \mathrm{Re}\left\{bz + bz^2 + z^3 + z^2\bar{z} + z^4 + z^3\bar{z}\right\} + \mathrm{Re}(z\phi_7 + \bar{z}\phi_8, \eta) \\ & + \int_\infty^t B_{22}|z|^4 + B_5(z, b, \eta^{(2)}, \eta^{(3)})\, d\tau \ . \end{aligned} \tag{7.7}$$

Recall the class $\mathcal{B}_T$ of b

$$\mathcal{B}_T = \left\{b(t) : |b(t)| \le D\left\{t\right\}^{-1},\ 0 \le t \le T\right\} \ , \tag{7.8}$$

where $D = 2B_{22}/\Gamma$ is some constant. Our goal is to show that $S(b)$ maps $\mathcal{B}_T$ into itself. More precisely, we have the following Lemma.

Lemma 7.2 *Suppose that $M(t) \le 2$ and $|b(t)| \le D\left\{t\right\}^{-1}$, $0 \le t \le T$. Then we have*

$$|S(b)(t)| \le C_1(D)\left\{t\right\}^{-3/2} + \frac{B_{22}}{2\Gamma}\left\{t\right\}^{-1} \le D\left\{t\right\}^{-1} \ .$$

Hence S map $\mathcal{B}_T$ to itself.

The following bound on B_{22} will be essential in Section 9.

Lemma 7.3

$$B_{22} = O(\lambda^4) \ .$$

Note that, if this is true, then

$$a = O(a_{20}z^2) + O(B_{22}z^2/\Gamma) = O(\lambda^2 z^2), \quad aR = O(\lambda z^2).$$

Hence all coefficients of the r.s. of $\dot{a}$, $(c_1 Q, \operatorname{Im} F)$, are of order $O(\lambda^2)$. In fact, those terms involving $R = O(\lambda^{-1})$ are of order $O(\lambda^3)$, hence the estimates in the order of λ can be closed. Moreover, In F, although coefficients of terms in $F^{(2)}$ are of order $O(\lambda)$, the only term in $F^{(3)}$ which is not of order $O(\lambda^2)$ is from $\lambda|\zeta|^2\zeta$, namely the term $\lambda u_+^3 z^2 \bar{z}$. (Note $aR = O(\lambda)$ and $\eta_\alpha = O(\lambda)$.) Hence when we consider the coefficients from $F^{(3)}$, it is often sufficient to consider the term $\lambda u_+^3 z^2 \bar{z}$.

Proof. Recall B_{22} is the coefficient of $|z|^4$ in the remainders when we integrate out the $O(z^4)$ terms in $a^{(4)}$ (see (7.5)), i.e., in

$$A^{(4)} + A_{2,4} + A_{(za,4)} + A_{(z\eta,4)} - a_\beta z^\beta f_\beta(z) \ .$$

We inspect the term $|z|^4$ from each term above. Note that, since $\dot{a} = (c_1 Q, \operatorname{Im} F)$, the operator Im elliminates many $|z|^4$ terms, although the coefficients of other powers of z, say $z^3\bar{z}$, may be of a larger power.

From $A^{(4)}$, the contribution to the coefficient of $|z|^4$ is from $(c_1 Q, \lambda \operatorname{Im} \eta_{20}\eta_{02}) = O(\lambda^4)$ and $(c_1 Q, \lambda R a_{20}\eta_{20}) = O(\lambda^4)$.

From $A_{2,4}$ we have $-4a_{20} \operatorname{Re} z d_\beta z^\beta$. The coefficient of $|z|^4$ is $-4a_{20}(\operatorname{Re} d_{12}) = O(\lambda^4)$. (Note $d_{12} = O(\lambda^2)$ although $d_{21} = O(\lambda)$.)

From $A_{(za,4)}$, since $c_{za} = O(\lambda)$ and $c_1 = O(\lambda)$, the only term which is not of right order is from the term $\lambda u_+^3 z^2 \bar{z}$. We can check that it does not contribute to the coefficient of $|z|^4$.

From $A_{(z\eta,4)}$, the remainders are in the integrands of (7.1) and the other integral above it. Note that ϕ_5 and ϕ_6 are of order $O(\lambda^2)$. Since $\dot{p}/p$, $\dot{\theta}$, and η all have λ in the coefficients of their leading terms, the only possible trouble is from $F^{\sharp\sharp}$, where the only term which is of order z^3 but without a factor λ^2 in its coefficient is the term $\lambda|\zeta|^2\zeta$, with the leading term $\lambda u_+^3 z^2 \bar{z}$. Since all other terms' coefficients are of order λ^2 and this term does not contribute to $|z|^4$ in the end, the contribution from $A_{(z\eta,4)}$ to B_{22} is of order λ^4.

Finally, for $a_\beta z^\beta f_\beta(z)$, since the leading coefficients in f_β are $c_\alpha = O(\lambda)$, we would like to show that most a_β are of the order $O(\lambda^3)$. Recall that $a_\beta = A_\beta/(i[\beta]\kappa)$ and $A_\beta z^\beta = A^{(z^3)} + A_{2,3} + A_{za,3} + A_{z\eta,3}$. Now $A_{2,3} = O(a_{20}c_\alpha z^3) = O(\lambda^3 z^3)$, $A_{za,3} = O(c_{2a}c_1\lambda z^3) = O(\lambda^3 z^3)$, $A_{z\eta,3} = O(\phi_5, \phi_6) \cdot O(\phi_\alpha^\sharp z^3) = O(\lambda^3 z^3)$, and $A^{(z^3)} = (c_1 Q, \operatorname{Im} \lambda u_+^3 z^2 \bar{z}) + O(\lambda^3 z^3)$. Hence the only trouble term is $(c_1 Q, \operatorname{Im} \lambda u_+^3 z^2 \bar{z})$. Note that the integrand in the remainder, $(c_1 Q, \lambda u_+^3) \operatorname{Im} z^2 \bar{z} \frac{1}{i[21]\kappa} f_{21}(z)$, has no $|z|^4$ term due to the operator Im, hence the contribution from $a_\beta z^\beta f_\beta(z)$ to B_{22} is of order λ^4. The proof is complete. **Q.E.D.**

Assuming the previous bound on b, we can also estimate θ. Since θ is defined by

$$\theta(t) = S_3(b)(t) = [c_2/\kappa(z+\bar{z})]_0^t + \int_0^t F_\theta(s)\, ds$$

we have

$$|\theta(t)| \leq C \log\{t\} \ .$$

This estimate will not be used in the rigorous proof, but it provides an idea about its size.

8 Contraction map

We review what we have so far. Up to now, we have not shown the existence of the solution to the equations setup in section 4 with the boundary condition (4.20). Notice we cannot conclude this from the existence to the Schrödinger equation as the boundary condition of a is set at the time $t = \infty$ (4.20). We have obtained the main estimate Theorem 5.1 assuming the estimate (7.8) on b. We also proved a bound on the map S in Lemma 7.2, again, assuming an estimate on b. We now need to prove the existence of the solution and prove this bound on b. We shall achieve both goals simultaneously by proving the map S is a contraction map on the space $\mathcal{B} = \mathcal{B}_{T=\infty}$. Once we have proved this, we have constructed rigorously the solution needed in part (1) of Theorem 1.1 and established all the upper bounds in part (1). To conclude part (1) of Theorem 1.1, it remains to prove the lower bound in ζ. The size of ζ is given by z and thus by ρ. From part (s) of Lemma 6.4, we conclude the lower bound of ζ provided that we can check the bound on g. Since g is given explicitly in (6.12) and we can estimate all terms in g from the upper bounds in part (1) of Theorem 1.1. This concludes part (1) of Theorem 1.1 assuming that S is a contraction on the space $\mathcal{B}$. The rest of this section is a proof that S is a contraction.

We first recall the setting. For each $b \in \mathcal{B}$, we can solve our system to get

$$z, p, q = S_1(b), \quad \eta = S_2(b), \quad \theta = S_3(b) \ .$$

More specifically, for fixed E and $h_0 \perp Q$, our system is

$$\begin{aligned}
\dot{q} &= \delta_{21}|q|^2 q + ic_4 bq + g(b,\eta,\theta;t) \ , \\
&\quad q = p + O(z^2), \ \text{given by (6.12)}, \ z = \mu^{-1} p \ , \\
&\quad q(0) \text{ given by } p(0) = z(0) = (v, \operatorname{Re} h_0) + i(u, \operatorname{Im} h_0) \ , \\
\tilde{\eta}(t) &= e^{-iAt}\tilde{\eta}_0 + \int_0^t e^{-iA(t-s)}\, \mathbf{P}_{\mathrm{c}}^{A} \left\{ e^{i\theta} U \Pi F^{\sharp} - e^{i\theta}[U,i]\dot{\theta}\eta \right\} ds \ , \\
\eta &= U^{-1} e^{-i\theta}\tilde{\eta}, \quad \eta(0) = \mathbf{P}_{\mathrm{c}}^{\mathcal{L}}\, h_0 \ , \\
\theta(t) &= \theta_0 + (c_2/\kappa)\,[z+\bar{z}]_0^t + \int_0^t F_\theta(b,\eta,\theta;s)\, ds \ .
\end{aligned}$$

Note $F^\sharp$ is given by (4.8), and that $q = p + c\mu z^\alpha + c\mu b\bar{z} + c\mu z^\beta$, for some constants c. In this system all a are already replaced by $a_{20}(z^2+\bar{z}^2)+b$. Hence we do not have a. We will also write

$$\rho = |q|, \quad q = \rho\, e^{i\omega} \ .$$

From the equation of q we can write ω explicitly as

$$\omega(t) = \int_0^t \left[\operatorname{Im} \delta_{21}\rho^2 + c_4 b + \operatorname{Im}(g/q) \right] ds \tag{8.1}$$

Assuming the estimate on b in Lemma 7.2, we can bound ω by

$$|\omega(t)| \le C \log\{t\} \ .$$

This provides an idea on the size of ω, but we shall not need it in the following rigorous proof for b.

Return to the proof that $S(b)$ is a contraction. Given $b, b' \in \mathcal{B}$, we can define z, η, θ and z', η', θ' correspondingly. We use $\boldsymbol{\delta}$ to denote the difference of quantities in these two sets. For example,

$$\boldsymbol{\delta} b = b - b', \quad \boldsymbol{\delta}\eta = \eta - \eta', \quad \boldsymbol{\delta}|z| = |z| - |z'| \ .$$

If

$$|\boldsymbol{\delta} b(t)| \le \boldsymbol{\delta}_0 \{t\}^{-1} \ ,$$

we want to show

$$|\boldsymbol{\delta} S(b)(t)| \le \tfrac{1}{2}\boldsymbol{\delta}_0 \{t\}^{-1} \ . \tag{8.2}$$

Let us define

$$\begin{aligned}
N(T) = \sup_{0\le t\le T} \Big\{ &\{t\}^{1/2+\sigma} |\boldsymbol{\delta}|z|(t)| + \{t\}^{1/2} (\log\{t\})^{-1} |\boldsymbol{\delta} z(t)| \\
&+ \{t\}^{3/4-\sigma} \|\boldsymbol{\delta}\eta(t)\|_{L^4} + \{t\}^{1+\sigma/4} \left\|\boldsymbol{\delta}\eta^{(3)}(t)\right\|_{L^2_{\mathrm{loc}}} \\
&+ (\log\{t\})^{-1} |\boldsymbol{\delta}\theta(t)| \Big\}
\end{aligned}$$

The key here is that $\boldsymbol{\delta}|z|$ has a faster decay than $|z|$. We first want to show that $N(T) \leq C\boldsymbol{\delta}_0$ uniformly for all T. Our strategy is to show that

$$N(T) \leq C\varepsilon^\sigma N(T) + C\boldsymbol{\delta}_0 \tag{8.3}$$

for all T. After this is proved, by choosing ε_0 sufficiently small we have $N(T) \leq C\boldsymbol{\delta}_0$. We also have

$$\begin{aligned}
|\boldsymbol{\delta} S(b)(t)| &\leq \{t\}^{-1} N \{t\}^{-1/2} \log\{t\} + \{t\}^{-1/2} \boldsymbol{\delta}_0 \{t\}^{-1} \\
&\quad + \int_\infty^t \{s\}^{-3/2} \; N \{s\}^{-1/2-\sigma} + \{s\}^{-2-\sigma} (N + \boldsymbol{\delta}_0) \, ds \\
&\leq C\boldsymbol{\delta}_0 \{t\}^{-1-\sigma} \leq \tfrac{1}{2}\boldsymbol{\delta}_0 \{t\}^{-1}
\end{aligned}$$

which shows (8.2). Now we focus on proving (8.3).

Let $t \leq T$. We will write $N = N(T)$. Also, when we say things like $|\boldsymbol{\delta}(\rho^2)| \leq C|\rho\boldsymbol{\delta}\rho|$, what we really means is $|\boldsymbol{\delta}(\rho^2)| \leq C|\rho\boldsymbol{\delta}\rho| + C|\rho'\boldsymbol{\delta}\rho|$.

1. We first estimate $\boldsymbol{\delta} g$. By (6.11) and the definitions of $P^{(4)}$, g_1, g_2, and g_3, we have

$$|\boldsymbol{\delta} g(t)| \leq C(N + \boldsymbol{\delta}_0) \{t\}^{-3/2-\sigma}$$

Note $C(N + \boldsymbol{\delta}_0) \leq C_1$ if $\boldsymbol{\delta}_0$ is sufficiently small. Hence

$$m^{-1} \{t\}^{-1/2} \leq \rho(t), \rho'(t) \leq m \{t\}^{-1/2} \; .$$

By Lemma 6.5,

$$|\boldsymbol{\delta}\rho(t)| \leq C(N + \boldsymbol{\delta}_0)\varepsilon^\sigma \{t\}^{-(1+\sigma)/2}$$

2. From the equation for ω (8.1), we can bound the variation of ω by

$$\begin{aligned}
|\boldsymbol{\delta}\omega(t)| &\leq \int_0^t C|\boldsymbol{\delta}(\rho^2)| + C|\boldsymbol{\delta} b| + C|\boldsymbol{\delta} g| \{s\}^{1/2} + C|g/q^2||\boldsymbol{\delta} q| \, ds \\
&\leq \int_0^t C \{s\}^{-1-\sigma} \varepsilon^\sigma N + C \{s\}^{-1} \boldsymbol{\delta}_0 + C(N + \boldsymbol{\delta}_0) \{s\}^{-3/2-\sigma+1/2} \\
&\qquad + C(N + \boldsymbol{\delta}_0) \{s\}^{-3/2-\sigma+1} N \{s\}^{-1/2} \log\{s\} \; ds \\
&\leq [C\varepsilon^\sigma N + C\boldsymbol{\delta}_0] \log\{t\}
\end{aligned}$$

Hence the variation of z is bounded by

$$|\boldsymbol{\delta} z(t)| \leq |\rho\boldsymbol{\delta}\omega| + |\boldsymbol{\delta}\rho| \leq (C\varepsilon^\sigma N + C\boldsymbol{\delta}_0) \{t\}^{-1/2} \log\{t\} \; .$$

3. Since $\theta(t) = c_2[(z + \bar{z})]_0^t + \int_0^t F_\theta \, ds$, we have

$$|\boldsymbol{\delta}\theta(t)| \leq C|\boldsymbol{\delta} z| + \int_0^t \boldsymbol{\delta} F_\theta \, ds \leq (C\varepsilon^\sigma N + C\boldsymbol{\delta}_0) \log\{t\} \; .$$

4. To estimate $\boldsymbol{\delta}\eta$,

$$\boldsymbol{\delta}\eta = \int_0^t e^{-iA(t-s)}\,\mathbf{P}_{\mathrm{c}}^A\,\boldsymbol{\delta}\left\{e^{i\theta}U\Pi\left[F^{\sharp} + i\eta^2\bar{\eta}\right] - e^{i\theta}[U,i]\dot{\theta}\eta\right\}\,ds$$

we note

$$\left\|\boldsymbol{\delta}\left\{e^{i\theta}U\Pi F^{\sharp} - e^{i\theta}[U,i]\dot{\theta}\eta\right\}(s)\right\|_{L^{4/3}} \le C|\boldsymbol{\delta}\theta|\,\{s\}^{-1} + (C\varepsilon^{\sigma}N + C\boldsymbol{\delta}_0)\,\{s\}^{-1}$$

hence

$$\|\boldsymbol{\delta}\eta(t)\|_{L^4} \le \int_0^t \frac{C}{|t-s|^{3/4}}\,(C\varepsilon^{\sigma}N + C\boldsymbol{\delta}_0)\,\{s\}^{-1}\log\{s\}\;ds \le C\varepsilon^{\sigma}(N+\boldsymbol{\delta}_0)\,\{s\}^{-3/4+\sigma}$$

5. To estimate $\boldsymbol{\delta}\eta^{(3)}$,

$$\begin{aligned}
\boldsymbol{\delta}\tilde{\eta}^{(3)}(t) = &-\int_0^t e^{-iA(t-s)}\left\{\mu^{[\alpha]}\boldsymbol{\delta}\left\{\frac{d}{ds}(e^{i\theta}p^{\alpha})\right\}\widetilde{\eta}_{\alpha}\right\}\,ds \\
&+\int_0^t e^{-iA(t-s)}\,\mathbf{P}_{\mathrm{c}}^A\,\boldsymbol{\delta}\left\{e^{i\theta}U\Pi\left[F^{\sharp\sharp} + i\eta^2\bar{\eta}\right] - e^{i\theta}[U,i]\dot{\theta}\eta\right\}\,ds \\
&+\int_0^t e^{-iA(t-s)}\,\mathbf{P}_{\mathrm{c}}^A\,\boldsymbol{\delta}\left\{e^{i\theta}U\Pi(-i\eta^2\bar{\eta})\right\}\,ds \\
= &\;\boldsymbol{\delta}\tilde{\eta}_3^{(3)} + \boldsymbol{\delta}\tilde{\eta}_4^{(3)} + \boldsymbol{\delta}\tilde{\eta}_5^{(3)}\;.
\end{aligned}$$

Hence

$$\left\|\boldsymbol{\delta}\tilde{\eta}^{(3)}(t)\right\|_{L^2_{\mathrm{loc}}} \le (C\varepsilon^{\sigma}N + C\boldsymbol{\delta}_0)\,\{s\}^{-1-3\sigma/4}$$

Note that we estimated $\|\eta\|_{L^2}$ in section 5 in order to estimate $\||\eta|^2\eta\|_{L^{8/7}}$, which is used in estimating $\left\|\tilde{\eta}_5^{(3)}\right\|_{L^2_{\mathrm{loc}}}$. However, we do not need $\|\boldsymbol{\delta}\eta\|_{L^2}$ here since

$$\left\|\boldsymbol{\delta}(|\eta|^2\eta)\right\|_{L^{8/7}} \le C\,\|\eta\|_{L^2}^{1/2}\,\|\eta\|_{L^4}^{3/2}\,\|\boldsymbol{\delta}\eta\|_{L^4}\;.$$

Compare with Subsection 5.3. Other estimates are similar to those in that subsection.

Since $\eta^{(3)} = U^{-1}e^{-i\theta}\tilde{\eta}^{(3)}(t)$,

$$\left\|\boldsymbol{\delta}\eta^{(3)}(t)\right\|_{L^2_{\mathrm{loc}}} \le |\boldsymbol{\delta}\theta|\,\left\|\tilde{\eta}^{(3)}(t)\right\|_{L^2_{\mathrm{loc}}} + \left\|\boldsymbol{\delta}\tilde{\eta}^{(3)}(t)\right\|_{L^2_{\mathrm{loc}}} \le C\varepsilon^{\sigma/4}(N+\boldsymbol{\delta}_0)\,\{s\}^{-1-\sigma/2}$$

5. Summarizing, we have shown (8.3). Choosing ε_0 sufficiently small we have $N \le C\boldsymbol{\delta}_0$.

6. The above shows that the map $S(b)$ is a contraction mapping. hence there is a fixed point $b = S(b)$, which gives us a solution.

9 Dynamical renormalization of mass

As explained in the introduction, we expect that the solution to be of the form

$$\psi(t) = [Q_{E(t)} + h(t)]e^{i\Theta(t)}$$

with a changing $E(t)$, and we use $[Q_E + aR_E + h]e^{i\Theta(t)}$ as an approximation. In Theorem 5.1 we conclude that $M(T) \le 2$ assuming $|b(t)| \le D\{t\}^{-1}$ for $t \in [0,T]$. We cannot extend this result beyond T. The reason is that, since $E(t)$ is changing and we expect that $E(t)$ will converge to some E_∞ which is likely to be different from $E(0)$, after certain time $[Q_E + aR_E + h]e^{i\Theta(t)}$ is no longer a good approximation and $|a(t)|$ will no longer decay. To overcome this difficulty, for each time step $T = k\Delta T$, we propose to re-choose $E = E_k$ so that the new $a(t)$ satisfies $a(T) = 0$. Then we prove that the new $a(t)$ has the same estimate as the old $a(t)$ if our time increment ΔT is sufficiently small. Then we estimate the change of $E(t)$, especially taking into account of its oscillation, by studying $a(t)$, and prove that $E(t)$ does converge. The rest of the proof essentially follows that of Theorem 1.1.

9.1 Renormalization lemma

Lemma 9.1 *Suppose $\|\psi - Q_E\|_2 \ll 1$, then we can rewrite*

$$\psi = ((1+a)Q_E + k)e^{i\theta} = (Q_E + a'R_E + h)e^{i\theta}$$

uniquely with $h, k \perp Q$. Moreover, a, a', θ, k and h are small.

Proof. First we can write $\psi = (1+\alpha)Q_E + k_1$, with $k_1 \perp Q_E$, and α a complex number. Since $\psi - Q_E$ is small, α and k_1 are both small. Hence we can find small real a and θ such that $(1+\alpha) = (1+a)e^{i\theta}$. Let $k = k_1 e^{-i\theta}$. Then we have $\psi = ((1+a)Q_E + k)e^{i\theta}$. We have $R_E = c_1^{-1}c_0 Q_E + \Pi_E R_E$, and hence $aQ_E = c_1 c_0^{-1} a(R_E - \Pi_E R_E)$. Let $a' = c_1 c_0^{-1} a$ and $h = k - a'\,\Pi_E R_E$. We have $h \perp Q_E$ and $aQ_E + k = a'R_E + h$. **Q.E.D.**

Let

$$W = H^2 \cap W^{2,1}(\mathbb{R}^3)\ .$$

Lemma 9.2 *Let τ_0 and δ_0 be sufficiently small and suppose $E \in \mathcal{I}_\lambda$ with $dist(E, \partial\mathcal{I}_\lambda) > C\tau_0\lambda$. Suppose that $\psi = [Q_E + aR_E + h]\,e^{i\Theta}$, with $|\lambda^{-1}a| = \tau < \tau_0$, $\|h\|_W = \delta < \delta_0$. Then we can find $\widetilde{E}$ with $|\widetilde{E} - (E+a)| \le \tau\lambda/2$ such that, if we write uniquely*

$$\psi = [Q_{\widetilde{E}} + \widetilde{a}R_{\widetilde{E}} + \widetilde{h}]\,e^{i\widetilde{\Theta}}, \quad \widetilde{h} \perp Q_{\widetilde{E}} \tag{9.1}$$

using Lemma 9.1, then $\widetilde{a} = 0$ *and we have* $\left\|h - \widetilde{h}\right\|_W < \tau/2$*, and* $|\Theta - \widetilde{\Theta}| < \tau/2$*. In fact,* $(\widetilde{E}, \widetilde{\Theta})$ *is the unique solution of*

$$\left(\psi - Q_{E'}\, e^{i\Theta'}\,,\, Q_{E'}\right) = 0\,, \qquad (|E - E'| \le 2\lambda\tau_0), \tag{9.2}$$

for $\|\psi - Q_E\|_W$ *sufficiently small.*

Proof. We will define a sequence $\{E_k\}_{k=1,2,3\cdots}$ which converges to $\widetilde{E}$. Let $E_1 = E + a$. We can write uniquely

$$\psi = [Q_{E_1} + a_1 R_{E_1} + h_1]\, e^{i\Theta_1}, \quad h_1 \perp Q_{E_1}$$

using Lemma 9.1. Denote $\delta Q = Q_E + a R_E - Q_{E_1}$ and $\delta\Theta = \Theta - \Theta_1$. We have $\|Q_E - Q_{E_1}\| \le O(aR) \le C|a|\lambda^{-1} \le C\tau$, and $\|\delta Q\| \le C a^2 (\partial_E^2 Q) \le C a^2 \lambda^{-2} \le C\tau^2$. Note we have

$$a_1 R_{E_1} + h_1 = Q_{E_1}(e^{i\delta\Theta} - 1) + [\delta Q + h]\, e^{i\delta\Theta}\ .$$

Hence $0 = \mathrm{Im}(Q_{E_1}, a_1 R_{E_1} + h_1) = O(\delta\Theta) + O((Q_{E_1}, \delta Q + h))$, and hence

$$|\delta\Theta| \le O(\delta Q) + C|(Q_E, h)| + C|(Q_{E_1} - Q_E, h)| \le C\tau^2 + 0 + C\tau\delta\ .$$

Also,

$$O(\lambda^{-1})\, a_1 = (Q_{E_1}, a_1 R_{E_1} + h_1) = c_0^{-1}(e^{i\delta\Theta} - 1) + O((Q_{E_1}, \delta Q + h))\ ,$$

hence

$$\lambda^{-1}|a_1| \le C|e^{i\delta\Theta} - 1| + C\tau^2 + C\tau\delta \le C\tau^2 + C\tau\delta\ .$$

Finally,

$$\begin{aligned}
\|h_1 - h\|_W &\le \left\|h_1 - h\, e^{i\delta\Theta}\right\|_W + \left\|h\, e^{i\delta\Theta} - h\right\|_W \\
&\le \left\|(Q_{E_1}(e^{i\delta\Theta} - 1) + \delta Q e^{i\delta\Theta} - a_1 R_{E_1}\right\|_W + C|\delta\Theta| \\
&\le C\tau^2 + C\tau\delta\ .
\end{aligned}$$

If we choose $C\tau + C\delta \le 1/3$, then we have

$$\lambda^{-1}|a_1|,\ |\delta\Theta|,\ \|h_1 - h\|_W \le \tau/3\ .$$

Now for $k \ge 2$ we define $E_k = E_{k-1} + a_{k-1}$, and define a_k, h_k and Θ_k correspondingly by Lemma 9.1. We follow the previous estimates to get

$$\lambda^{-1}|a_k|,\ |\Theta_k - \Theta_{k-1}|,\ \|h_k - h_{k-1}\|_W \le 3^{-k}\tau$$

Note that the size of h_k may increase in the process, but since

$$C|a_k| + C\,\|h_k\|_W \le C\frac{|a_{k-1}|}{3} + C(\|h_{k-1}\|_W + \frac{|a_{k-1}|}{3}) \le C|a_{k-1}| + C\,\|h_{k-1}\|_W \le \cdots \le 1/3$$

our condition for estimates is always satisfied. Hence $E+\sum_k a_k$ converges to a limit $\widetilde{E}$ with $|\widetilde{E}-E| \le 3\lambda\tau/2$ and $|\widetilde{E}-E-a| \le \lambda\tau/2$. Writing ψ in the form (9.1) with respect to $\widetilde{E}$, we have $\widetilde{a} = \lim_k a_k = 0$ and $|\widetilde{\Theta}-\Theta| \le \sum_k |\Theta_{k+1}-\Theta_k| \le \tau/2$. We conclude $(\widetilde{E},\widetilde{\Theta})$ is a solution of (9.2).

To show the uniqueness of (9.2), we first note that it is locally unique by inspecting the equations obtained by taking derivatives of (9.2) with respect to E' and Θ'. Now suppose we may write

$$\psi = [Q_1 + h_1]e^{i\Theta_1} = [Q_2 + h_2]e^{i\Theta_2}$$

with $h_1 \perp Q_1$ and $h_2 \perp Q_2$, . Then we have $[Q_1 + h_1] = [Q_2 + h_2]e^{i\delta\Theta}$. Since both h_1 and h_2 are small, taking projection on Q_1 we get $\delta\Theta$ is small. The local uniqueness implies the claim.

Q.E.D.

9.2 Proof of Theorem 1.2

First we proceed an induction argument to find the desired renormalization at each time step and the corresponding estimates. Recall

$$W = H^2 \cap W^{2,1}(\mathbb{R}^3).$$

Our initial datum is $\psi_0 = Q_{E_{\rm in}} + a_{\rm in}R_{E_{\rm in}} + h_{\rm in}$, with $\|\psi_0 - Q_{E_{\rm in}}\|_W \le C^{-1}\varepsilon$. Applying Lemma 9.1 to ψ_0, we can find E_0, h_0, and Θ_0 such that $\psi_0 = [Q_{E_0} + h_0]\, e^{i\Theta_0}$. Moreover, we have $|E_0 - E_{\rm in}| \le C^{-1}|a_{\rm in}| \le C^{-1}\lambda\varepsilon$, and hence $\left\|Q_{\rm in} - Q_0 e^{i\Theta_0(0)}\right\|_W \le \frac{1}{2}\varepsilon$. This is the beginning of our induction argument. We will choose $\Delta T > 0$ sufficiently small. Then we define $T_k = k\Delta T$ and will define E_k, h_k, etc. at time T_k. Also, we abbreviate $Q_k = Q_{E_k}$, $R_k = R_{E_k}$, etc..

Note, Theorem 5.1 remains valid if we replace the assumption $|b(t)| \le D\{t\}^{-1}$ by $|a(t)| \le D\{t\}^{-1}$. Also recall that $D = O(\lambda^2)$ by Lemmas 7.2 and 7.3.

Assume for $T_k = k\Delta T$ we have found E_k, $a_k(t)$, $h_k(t)$, $\Theta_k(t)$ so that

$$\begin{cases} \psi(t) = [Q_k + a_k(t)R_k + h_k]\, e^{i\Theta_k}, \quad E_k \in \mathcal{I}_\lambda\ , \\[2ex] h_k(t) \perp Q_k \quad \text{for } t \in [0,T_k]\ ; \quad \left\|\psi_0 - Q_k e^{i\Theta_k(0)}\right\|_W \le \varepsilon\ , \\[2ex] |a_k(t)| \le D\{t\}^{-1}, \quad a_k(T_k) = 0\ . \end{cases} \tag{9.3}$$

This is our induction hypothesis. By Theorem 5.1 we have $M_k(T_k) \le 2$. Since $a_k(T_k) = 0$, we have

$$a_k(t) = \left[a_{20}(E_k)(z_k^2 + \bar{z}_k^2) + \cdots\right]_T^t + \int_T^t B_{22}(E_k)|z_k|^4 + \cdots \tag{9.4}$$

Direct estimates gives us (cf. Lemma 7.2)

$$|a_k(t)| \le \frac{1}{2} D \{t\}^{-1} \qquad \text{for } t \in [0, T_k]$$

Moreover, since $|a_k(0)| \le \frac{1}{2} D\varepsilon$, we have

$$\begin{aligned}\left\|\psi_0 - Q_k e^{i\Theta_k(0)}\right\|_W &\le \left\|\psi_0 - Q_0 e^{i\Theta_0(0)}\right\|_W + \left\|Q_0 e^{i\Theta_0(0)} - Q_k e^{i\Theta_k(0)}\right\|_W \\ &\le \frac{1}{2}\varepsilon + \lambda^{-1}|a_k(0)| \le \frac{3}{4}\varepsilon \ .\end{aligned}$$

By continuity, there is a $\sigma > 0$ such that

$$|a_k(t)| \le D\{t\}^{-1} \qquad \text{for } t \in [0, T_k + \sigma]$$

Theorem 5.1 ensures that

$$M_k(T_k + \sigma) \le 2 \ .$$

We now return to (9.4). Since the derivatives of the terms outside the integral (say, $z_k^2 + \bar{z}_k^2$) are bounded by $\{t\}^{-1}$, for $t \in [T_k, T_k + \sigma]$ we have

$$\begin{aligned}|a_k(t)| &\le CT^{-1}(t - T) + CT^{-2}(t - T) \\ &\le C_3 T^{-1}(t - T) \le \frac{1}{2} D \{T\}^{-1} \le \frac{5}{8} D\{t\}^{-1}\end{aligned} \tag{9.5}$$

if $(t - T) \le D/(2C_3)$ and $(t - T) \le \{T\}/4 \le \varepsilon^{-2}/4$. Now we define

$$\Delta T = \min\left\{D/(2C_3)\,,\, \varepsilon^{-2}/4\right\} \ .$$

Note that both C_3 and D are of order λ^2, hence ΔT is of order 1. Recall $T_{k+1} = T_k + \Delta T$. We have thus proved that $\sigma \ge \Delta T$ and

$$|a_k(t)| \le \frac{5}{8} D\{t\}^{-1}, \qquad \text{for } t \in [0, T_{k+1}]$$

and $M_k(T_{k+1}) \le 2$. By Lemma 9.2, we can find E_{k+1} with

$$|E_{k+1} - E_k| \le \tfrac{3}{2}|a_k(T_{k+1})| \ , \tag{9.6}$$

such that we can rewrite

$$\psi(t) = [Q_{k+1} + a_{k+1}(t)R_{k+1} + h_{k+1}(t)]\, e^{i\Theta_{k+1}(t)}, \quad h_{k+1}(t) \perp Q_{k+1}$$

with $a_{k+1}(T_{k+1}) = 0$. Now we compare these two sets of data with respect to E_k and E_{k+1}, for $t \in [0, T_{k+1}]$. As in the proof of lemma 9.2, we have

$$\begin{aligned}|\Theta_k(t) - \Theta_{k+1}(t)| &\le C|(Q_{k+1}, h_k(t))| = C|(Q_k + O(\lambda^{-1}a_k(T_{k+1})), h_k(t))| \\ &\le 0 + C\lambda\{T\}^{-1}\{t\}^{-1/2} \le C\lambda\{t\}^{-3/2}\end{aligned} \tag{9.7}$$

and since

$$a_{k+1}(t)R_{k+1} + h_{k+1}(t) = [Q_k + a_k(t)R_k + h_k(t)]\, e^{i\Theta_k(t) - i\Theta_{k+1}(t)} - Q_{k+1}$$

Taking inner product with Q_{k+1} we get

$$\lambda^{-1}|a_{k+1}(t) - a_k(t)| \le C|\Theta_k(t) - \Theta_{k+1}| + C|a_k(t)|^2 + C\lambda \{T\}^{-1} \|h_k(t)\| \le C\lambda \{t\}^{-3/2}$$

Hence

$$|a_{k+1}(t)| \le |a_k(t)| + |a_{k+1}(t) - a_k(t)| \le \frac{5}{8} D \{t\}^{-1} + \frac{1}{8} D \{t\}^{-1} = \frac{3}{4} D \{t\}^{-1} \ ,$$

if ε is sufficiently small. By (9.6) and (9.7), we have

$$\left\|\psi_0 - Q_{k+1} e^{i\Theta_{k+1}(0)}\right\|_W \le \left\|\psi_0 - Q_k e^{i\Theta_k(0)}\right\|_W + C\lambda \{t\}^{-1} \le \varepsilon \ .$$

Also note

$$|E_0 - E_{k+1}| \le \frac{3}{2} |a_{k+1}(0)| \le 2D\varepsilon^2 \le C\lambda^2\varepsilon^2 \ .$$

Hence $E_{k+1} \in \mathcal{I}_\lambda$. We have thus proved the induction hypothesis (9.3) for $t = T_{k+1}$. The induction argument is completed.

Next we proceed to show that E_k has a limit. Let $n = 2^k$ and $T = T_n$. Since $a_n(T) = 0$, we have

$$a_n(t) = \left[a_{20}(E_n)(z_n^2 + \bar{z}_n^2) + \cdots\right]_T^t + \int_T^t B_{22}(E_n)|z_n|^4 + \cdots$$

for $t \in [0, T_n]$. In particular,

$$|a_n(T/2)| \le C \{T\}^{-1} + \int_{T/2}^T |B_{22}| \ \{s\}^{-2} \ ds \le 2D \{T\}^{-1}$$

(Note that the argument here differs from (9.5).) Hence, by considering $\psi = \psi(T/2)$, $E = E_n$ and $\widetilde{E} = E_{n/2}$ in Lemma 9.2, we have

$$|E_{n/2} - E_n| \le 2|a_n(T/2)| \le 4D \{T\}^{-1} \ .$$

This estimate shows that the sequence E_{2^k} converges to a limit E_∞. Moreover, choose k_1 so that $2\Gamma 2^{k_1}\Delta T > \varepsilon^{-2}$ and we have

$$|E_0 - E_\infty| \le |E_0 - E_{k_1}| + |E_{k_1} - E_\infty| \le C\lambda^2\varepsilon^2 + CD \sum_{k > k_1} \left\{2^k \Delta T\right\} \le C\lambda^2\varepsilon^2 \ .$$

Write $Q_\infty = Q_{E_\infty}$, $R_\infty = R_{E_\infty}$, and

$$\psi(t) = [Q_\infty + a_\infty(t)R_\infty + h_\infty(t)]\ e^{i\Theta_\infty(t)}$$

and compare this set of data with data at T_k, $t \in [0, T_k]$. As before we get $|a_\infty(t) - a_k(t)| \le CD\{t\}^{-3/2}$ and hence

$$|a_\infty(t)| \le \frac{3}{4} D\{t\}^{-1} \ .$$

Since k is arbitrary, this estimate is true for all $t \in [0, \infty)$. Since $|E_\infty - E_0| \le O(\lambda^2 \varepsilon^2)$, we have $\|\psi_0 - Q_\infty\|_W \le \varepsilon$. Theorem 5.1 shows that

$$M_\infty(T) \le 2 \qquad \text{for all } T \ .$$

The first part of Theorem 1.2 is thus proved.

Suppose the assumption of part (2) of Theorem 1.2 holds, that is,

$$0 < |z_{\rm in}| = \varepsilon \le \epsilon_0, \quad \|\eta_{\rm in}\|_W \le C\varepsilon^{2+\sigma}, \quad \lambda^{-1}|a_{\rm in}| \le C\varepsilon^{2+2\sigma}.$$

Let $a_\infty = a_\infty(0)$, $z_\infty = z_\infty(0)$, and $\eta_\infty = \eta_\infty(0)$. Apply Lemma 9.2 to $\psi = \psi_0$ and $E = E_{\rm in}$, $E = E_\infty$ respectively. In both cases, $\widetilde{E} = E_0$. We have

$$|a_{\rm in}| \le C\lambda\varepsilon^{2+2\sigma}, \quad |a_\infty| \le C\lambda^2\varepsilon^2$$
$$|E_0 - E_{\rm in}| \le C|a_{\rm in}|, \quad |E_0 - E_\infty| \le C|a_\infty|$$

Therefore $Q_{\rm in} - Q_\infty$ is of order $\lambda^{-1}|a_{\rm in} - a_\infty| = o(\varepsilon^2)$, and hence so is $h_{\rm in} - h_\infty$. Denote $P_\infty = P_c(E_\infty)\Pi_\infty$ and $P_{\rm in} = P_c(E_{\rm in})\Pi_{\rm in}$, we have

$$\begin{aligned} \|P_\infty h_\infty\|_W &\le \|P_{\rm in} h_{\rm in}\|_W + \|P_{\rm in}(h_\infty - h_{\rm in})\|_W + \|(P_\infty - P_{\rm in})h_\infty\|_W \\ &\le \varepsilon^{2+\sigma} + o(\varepsilon^2) + |E_\infty - E_{\rm in}|\varepsilon \\ &\le 2\varepsilon^{2+\sigma} \end{aligned}$$

By the last statement of Theorem 5.1, we have $|z_\infty(t)| \ge c\{t\}^{-1/2}$ for all time t. The last statement of Theorem 1.2 is proved. **Q.E.D.**

10 Radiation dominating solutions

Let W denote the wave operator for $A + E$, i.e.,

$$e^{-i(A+E)t}\xi_\infty \to e^{i\Delta t} W \xi_\infty \quad \text{as } t \to \infty \ .$$

We let $\xi_\infty = W^* \chi_\infty$, where χ_∞ is the profile of radiation in part (2) of Theorem 1.1. We will construct a radiation dominating solution with $e^{iAt}\xi_\infty$ as the leading term in h, thus proving part (2) of Theorem 1.1.

We recall that we set

$$\psi = (Q + aR + h)\, e^{i[-Et+\theta(t)]} \ ,$$

and h satisfies

$$\partial_t h = -\dot a R - aiQ + \mathcal{L}^{(\mathrm{or})} h - iF(aR+h) - i\dot\theta(Q + aR + h) \ .$$

To ensure $h(t) \perp Q$ all the time, we set

$$\begin{aligned} \dot a &= (c_1 Q, \operatorname{Im} F(k)) \ , \qquad c_1 = (Q,R)^{-1} \ , \\ \dot\theta &= -(1 + c_0 c_1^{-1} a)^{-1} \ \left[a + (c_0 Q, \lambda Q^2(h+\bar h)) + (c_0 Q, \operatorname{Re} F(k))\right] \ . \end{aligned}$$

(Recall $c_0 = (Q,Q)^{-1}$.) Then the equation of h is

$$\begin{aligned} \partial_t h &= \mathcal{L} h + \Pi F_{all} \ , \\ \mathcal{L} h &= -i \left\{ Hh + \Pi \lambda Q^2 \Pi (h + \bar h) \right\} \ , \\ F_{all} &= -i\dot\theta h - iF - \left[(c_1 Q, \operatorname{Im} F) + ia\dot\theta \right] R_\Pi \ , \quad R_\Pi := \Pi R \ , \\ F &= F(aR+h) \\ &= \lambda Q (2|h|^2 + h^2) + 2\lambda a QR(2h + \bar h) + 3\lambda a^2 Q R^2 + \lambda (aR+h)^2 (aR + \bar h) \ . \end{aligned}$$

We first rewrite the equation of h. The equation for h is

$$\begin{aligned} \partial_t h &= \mathcal{L} h - i\dot\theta h + \Pi F^\sharp \ , \\ F^\sharp &= -iF(k) - [(c_1 Q, \operatorname{Im} F) + ia\dot\theta] R_\Pi \ . \end{aligned}$$

Let

$$h^\diamond = Uh \ .$$

Since $\mathcal{L} = U^{-1}(-iA)U$, we have

$$\begin{aligned} \partial_t h^\diamond &= -iAh^\diamond - Ui\dot\theta U^{-1} h^\diamond + U\Pi F^\sharp \\ &= -iAh^\diamond - i\dot\theta h^\diamond - [U,i]\dot\theta U^{-1} h^\diamond + U\Pi F^\sharp \ . \end{aligned}$$

Let $\widetilde h = e^{i\theta} h^\diamond$ and use $U^{-1} h^\diamond = h$, we get

$$\partial_t \widetilde h = -iA\widetilde h + e^{i\theta} U \Pi F^\sharp - e^{i\theta}[U,i]\dot\theta h \ . \tag{10.1}$$

For a specificed datum ξ_∞ at infinity, we define a solution by the equations

$$\begin{aligned} \widetilde h(t) &= \xi(t) + g(t) \\ \xi(t) &= e^{-iAt} \xi_\infty \\ g(t) &= \int_\infty^t e^{-iA(t-s)} \left\{ e^{i\theta} U \Pi F^\sharp - e^{i\theta} [U,i] \dot\theta h \right\} \, ds \ . \end{aligned}$$

We also let $h = U^{-1} e^{-i\theta} \widetilde{h}$.

Suppose we have a solution, then the main term in $F^\sharp$ is of order t^{-3}, hence $g \sim t^{-2}$, $a \sim t^{-2}$, $\dot\theta \sim t^{-3/2}$, $\theta \sim t^{-1/2}$.

Note that $\xi(t)$ is fixed. We consider ξ_∞ small in the following norm

$$\|\xi_\infty\|_{H^2 \cap W^{2,1}} \le \varepsilon$$

with ε sufficiently small. Then

$$\|\xi(t)\|_{H^2} \le C_1 \varepsilon, \qquad \|\xi(t)\|_{W^{2,\infty}} \le C_1 \varepsilon |t|^{-3/2}. \tag{10.2}$$

Also, for the leading term $\lambda|\xi|^2\xi$ in F, we have, for $t > 1$,

$$\begin{aligned}
\left\| |\xi|^2 \xi \right\|_{L^2} &\le C \|\xi(t)\|_\infty^2 \|\xi(t)\|_2 \le C\varepsilon^3 \langle t \rangle^{-3}, \\
\left\| \nabla^2 (|\xi|^2 \xi) \right\|_{L^2} &\le C \|\xi(t)\|_\infty^2 \left\|\nabla^2 \xi(t)\right\|_2 + C \|\xi(t)\|_\infty \|\nabla \xi(t)\|_4^2 \\
&\le C(\langle t \rangle^{-3/2})^2 \cdot 1 + C \langle t \rangle^{-3/2} \cdot (\langle t \rangle^{-3/4})^2 \le C\varepsilon^3 \langle t \rangle^{-3}.
\end{aligned}$$

It $t \le 1$, we simply have $\||\xi|^2\xi\|_{H^2} \le C \|\xi\|_{H^2}^3 \le C\varepsilon^3$. We conclude

$$\left\| |\xi|^2 \xi \right\|_{H^2} \le C\varepsilon^3 \langle t \rangle^{-3}. \tag{10.3}$$

Similarly, if $\|g(t)\|_{H^2} \le C\varepsilon \langle t \rangle^{-2}$, one can prove, for example,

$$\left\| |\xi + g|^2 (\xi + g) \right\|_{H^2} \le C\varepsilon^3 \langle t \rangle^{-3}, \quad \|Q\xi g\|_{H^2} \le C\varepsilon^2 \langle t \rangle^{-7/2}, \quad \text{etc.}$$

Now we proceed to construct a solution. For convenience, we introduce a new variable $\omega = \dot\theta$. We will define a Cauchy sequence on the space

$$\begin{aligned}
\mathcal{A} = \Big\{ &(\omega, \theta, a, g) : [0, \infty) \to \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times H^2, \\
&|\omega(t)| \le \varepsilon^{1/2} \langle t \rangle^{-3/2}, \ |\theta(t)| \le \varepsilon^{1/2} \langle t \rangle^{-1/2}, \ |a(t)| \le \varepsilon \langle t \rangle^{-2}, \ \|g(t)\|_{H^2} \le \varepsilon \langle t \rangle^{-2} \Big\}
\end{aligned}$$

Here $H^2 = W^{2,2}(\mathbb{R}^3)$. We define our map by

$$\begin{aligned}
\omega^\triangle(t) &:= -(1 + c_0 c_1^{-1} a)^{-1} \left[a + (c_0 Q, \lambda Q^2 (h + \bar h)) + (c_0 Q, \operatorname{Re} F) \right] \\
\theta^\triangle(t) &:= \int_\infty^t \omega \, ds \\
a^\triangle(t) &:= \int_\infty^t (c_1 Q, \operatorname{Im} F) \, ds \\
g^\triangle(t) &:= \int_\infty^t e^{-iA(t-s)} \left\{ e^{i\theta} U \Pi \left\{ -iF - [(c_1 Q, \operatorname{Im} F) + ia\omega] R_\Pi \right\} - e^{i\theta} [U, i] \omega h \right\} ds,
\end{aligned}$$

where

$$F = F(aR + h), \quad h(t) := U^{-1} e^{-i\theta} \left(e^{-iAt} \xi_\infty + g(t) \right) .$$

Our initial data are

$$\omega(t) \equiv 0, \quad \theta(t) \equiv 0, \quad a(t) \equiv 0, \quad g(t) \equiv 0 .$$

Given $(\omega, \theta, a, g) \in \mathcal{A}$, using this assumption and (10.2)–(10.3), we have

$$\begin{aligned}
\left\| |g|^2 g \right\|_{H^2} &\le \|g\|_{H^2}^3 \le C\varepsilon^3 \langle t \rangle^{-6} \\
\|F\|_{H^2} &\le C\varepsilon^2 \langle t \rangle^{-3} \\
|\omega^\triangle(t)| &\le C\varepsilon \langle t \rangle^{-3/2} \le \varepsilon^{1/2} \langle t \rangle^{-3/2} \\
|\theta^\triangle(t)| &\le \int_\infty^t \varepsilon^{1/2} \langle s \rangle^{-3/2} \, ds \le \varepsilon^{1/2} \langle t \rangle^{-1/2} \\
|a^\triangle(t)| &\le \int_\infty^t C\varepsilon^2 \langle s \rangle^{-3} \, ds \le \varepsilon \langle t \rangle^{-2} \\
\left\| g^\triangle(t) \right\|_{H^2} &\le \int_\infty^t C\varepsilon^2 \langle s \rangle^{-3} + C\varepsilon^{3/2} \langle s \rangle^{-3} \, ds \le \varepsilon \langle t \rangle^{-2}
\end{aligned}$$

We have shown that $(\omega^\triangle, \theta^\triangle, a^\triangle, g^\triangle) \in \mathcal{A}$, that is, our mapping maps $\mathcal{A}$ into itself.

Next we show that it is a contraction. Given $(\omega_1, \theta_1, a_1, g_1), (\omega_2, \theta_2, a_2, g_2) \in \mathcal{A}$, we denote

$$\boldsymbol{\delta}_0 = \sup_t \left\{ \langle t \rangle^3 |\boldsymbol{\delta}\omega(t)|^2 + \langle t \rangle |\boldsymbol{\delta}\theta(t)|^2 + \langle t \rangle^2 |\boldsymbol{\delta}a(t)| + \langle t \rangle^2 \|\boldsymbol{\delta}g(t)\|_{H^2} \right\}$$

we know $\boldsymbol{\delta}_0 \le 8\varepsilon$. Notice that $-i|\xi|^2\xi$ is cancelled in $\boldsymbol{\delta}F$.

$$\begin{aligned}
\left\| \boldsymbol{\delta}(|g|^2 g) \right\|_{H^2} &\le \|g\|_{H^2}^2 \|\boldsymbol{\delta}g\|_{H^2} \le C\varepsilon^2 \boldsymbol{\delta}_0 \langle t \rangle^{-6} \\
\|\boldsymbol{\delta}F\|_{H^2} &\le C\varepsilon \, \boldsymbol{\delta}_0 \langle t \rangle^{-7/2} \le C\varepsilon \, \boldsymbol{\delta}_0 \langle t \rangle^{-3} \\
|\boldsymbol{\delta}\omega^\triangle(t)| &\le C\boldsymbol{\delta}_0 \langle t \rangle^{-3/2} \le \frac{1}{8} \boldsymbol{\delta}_0^{1/2} \langle t \rangle^{-3/2} \\
|\boldsymbol{\delta}\theta^\triangle(t)| &\le \int_\infty^t \boldsymbol{\delta}_0^{1/2} \langle s \rangle^{-3/2} \, ds \le \frac{1}{8} \boldsymbol{\delta}_0^{1/2} \langle t \rangle^{-1/2} \\
|\boldsymbol{\delta}a^\triangle(t)| &\le \int_\infty^t C\varepsilon \boldsymbol{\delta}_0 \langle s \rangle^{-3} \, ds \le \frac{1}{8} \boldsymbol{\delta}_0 \langle t \rangle^{-2} \\
\left\| \boldsymbol{\delta}g^\triangle(t) \right\|_{H^2} &\le \int_\infty^t C\varepsilon \, \boldsymbol{\delta}_0 \langle s \rangle^{-3} + C\varepsilon^{1/2} \boldsymbol{\delta}_0 \langle s \rangle^{-3} \, ds \le \frac{1}{8} \boldsymbol{\delta}_0 \langle t \rangle^{-2}
\end{aligned}$$

Therefore we have

$$\sup_t \left\{ \langle t \rangle^3 |\boldsymbol{\delta}\omega^\triangle(t)|^2 + \langle t \rangle |\boldsymbol{\delta}\theta^\triangle(t)|^2 + \langle t \rangle^2 |\boldsymbol{\delta}a^\triangle(t)| + \langle t \rangle^2 \left\| \boldsymbol{\delta}g^\triangle(t) \right\|_{H^2} \right\} \le \frac{1}{2} \boldsymbol{\delta}_0 .$$

These show that our map is a contraction mapping.

We conclude that we do have solutions with the main profile $e^{iAt}\xi_\infty$.

$$\psi(t) = [(1+a)Q + e^{iAt}\xi_\infty + g]\, e^{i\theta}$$

Since

$$e^{iAt}\xi_\infty \to e^{i(-\Delta+E)t} W\xi_\infty$$

for $W\xi_\infty = \chi_\infty$, and $(\theta(t)+Et)/t \to 0$ as $t \to \infty$, we have

$$[aQ + e^{iAt}\xi_\infty + g]\, e^{i\theta} - e^{-i\Delta t}\chi_\infty \to 0$$

as $t \to \infty$. Part (b) of Theorem 1.1 is thus proved.

Remark on radiation dominating solutions to Klein-Gordon equations

We now sketch a construction for radiation dominating solutions to Klein-Gordon equations. We follow the notation in the introduction. For a specified profile $\eta_\pm$, let

$$u = \xi + g$$

where

$$\xi(t) = e^{iBt}\eta_+ + e^{-iBt}\eta_-$$

and g denotes the rest. Then we have

$$\begin{aligned}
&(\partial_t^2 + B^2)\xi = 0, \\
&(\partial_t^2 + B^2) g = \lambda(\xi+g)^3 \ .
\end{aligned}$$

Hence $g(t)$ satisfies

$$g(t) = \int_\infty^t \left\{ e^{iB(t-s)} - e^{-iB(t-s)} \right\} \frac{1}{2iB} \lambda(\xi+g)^3 ds \ ,$$

Since the main source term is

$$\left\|\xi^3\right\|_2 \le \|\xi\|_\infty^2 \, \|\xi\|_2 \le Ct^{-3}$$

we get $\|g\|_2 \le \int_\infty^t Cs^{-3}\, ds = Ct^{-2}$. Then we proceed as in section 10 to construct one such solution by a contraction mapping argument.

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