Renormalizing the Renormalization Group Pathologies

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Abstract

We review the status of the "pathologies" of the Renormalization Group encountered when one tries to define rigorously the Renormalization Group transformation as a map between Hamiltonians. We explain their origin and clarify their status by relating them to the Griffiths' singularities appearing in disordered systems; moreover, we suggest that the best way to avoid those pathologies is to use the contour representation rather than the spin representation for lattice spin models at low temperatures. Finally, we outline how to implement the Renormalization Group in the contour representation.

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Contents

1 Introduction

The Renormalization Group (RG) has been one of the most useful tools of theoretical physics during the past decades. It has led to an understanding of universality in the theory of critical phenomena and of the divergences in quantum field theories. It has also provided a nonperturbative calculational framework as well as the basis of a rigorous mathematical understanding of these theories.

Here is a (partial) list of rigorous mathematical results obtained by a direct use of RG ideas:

- Proof that in the lattice field theory $\lambda\phi^4$ in $d = 4$, with λ small, the critical exponent η takes its mean field value 0 [41], [32].

- Construction of a renormalizable, asymtotically free, Quantum Field Theory, the Gross-Neveu model in two dimensions [42, 43], [33].

- Construction of a perturbatively non renormalizable Quantum Field Theory, the Gross-Neveu model in $2+\epsilon$ "dimensions" (i.e. the dimension of spacetime is two but the propagator is made more singular in the ultraviolet) [44] (see also [14]) and the lattice $\lambda\varphi^4$ model in $d = 4-\epsilon$, at the critical point [13].

- Construction of pure non Abelian gauge theories in $d = 4$ (in finite volume) [1], [80].

- Analysis of the Goldstone picture in $d > 2$ [2] .

Other mathematical results related to statistical mechanics and using the RG include first order phase transitions in regular [40] and disordered [7] spin systems, which we shall discuss in this paper, and diffusion in random media [8]. Finally, the application of RG ideas to the theory of dynamical systems initiated by Feigenbaum [30, 31] is well known; less well known is the application to the study of large time asymptotics of nonlinear PDE's pioneered by Goldenfeld and Oono [48], [9, 10][1].

The textbook explanation of the (Wilsonian) RG goes roughly as follows: consider a lattice system with spins σ and Hamiltonian H. Cover the lattice with disjoint boxes B_x and associate with each box a variable s_x giving a coarse-grained description of the spins in B_x, e.g. for the so-called block spin transformation, s_x is a suitably normalized average of the spins σ_i for $i \in B_x$. Now define (formally)

$$\exp(-\beta' H'(s)) = \sum_{\sigma}^{s} \exp(-\beta H(\sigma)) \tag{1.1}$$

where the sum runs over all configurations σ satisfying the constraints defined by s. The transformation (1.1) is called a RG transformation (RGT) and H' is the effective or renormalized Hamiltonian. Now it is usual to parametrize Hamiltonians in term of coupling constants $\mathbf{J}$, i.e. to write

$$H = \Sigma J_{ij}\sigma_i\sigma_j + \Sigma J_{ijk}\sigma_i\sigma_j\sigma_k + \cdots \tag{1.2}$$

where the collection of numbers $\mathbf{J} = (J_{ij}, J_{ijk}, \cdots)$ include the pair couplings, the three-body couplings, the n-body couplings etc. Using this description, the map $\beta H \to \beta' H'$ defined by (1.1) gives rise to a map $\beta\mathbf{J} \to \beta'\mathbf{J}'$. Now, by studying this map (or, in practice, some truncation of it), its iteration, its fixed points and its flow around the latter, one obtains useful information about the original spin system with Hamiltonian H, in particular about its phase diagram and its critical exponents.

The crucial feature that makes the RG method useful is that, even if βH happens to describe the system close to its critical point, the transformation (1.1) (and its iterations) amount to studying a non critical spin system and that analysis can be performed with rather standard tools such as high or low temperature expansions. The reason why that nice property holds is that critical properties of a spin system come from large scale fluctuations in the system while the sum (1.1) runs only over its small scale fluctuations. And this, in turn, is because fixing the s variables effectively freezes the large scale fluctuations of the σ variables.

At least, this is the scenario which is expected to hold and is usually assumed without proof in most applications. However, before coming to our main point, it should be stressed that the successful applications of the RG method mentioned above do not follow *literally* the "texbook" description, for reasons that will be discussed later.

Be that as it may, it is a very natural mathematical question to ask whether the transformation (1.1) can be well defined on some space of Hamiltonians

[1] See e.g. [4, 36, 85] for yet other applications of the RG.

and, if so, to study its properties. However, this program has met some difficulties. Although it can be justified at high temperatures [58] and even, in some cases, at *any* temperature above the critical one [5], it has been observed in simulations [53] that the RG transformation seems, in some sense, "discontinuous" as a map between spin Hamiltonians at low temperatures. These observations led subsequently to a rather extensive discussion of the so-called "pathologies" of the Renormalization Group Transformations: van Enter, Fernandez and Sokal have shown [23, 24] that, first of all, the RG transformation is not really discontinuous. But they also show, using results of Griffiths and Pearce [50, 51] and of Israel [58], that, roughly speaking, there does not exist a renormalized Hamiltonian for many RGT applied to Ising-like models at low temperatures[2].

More precisely, van Enter, Fernandez and Sokal consider various real-space RGT (block spin, majority vote, decimation)[3] that can be easily and rigorously defined as maps acting on measures (i.e. on probability distributions of the infinite volume spin system): if we start with a Gibbs measure μ corresponding to a given Hamiltonian H, then one can easily define the renormalized measure μ'. The problem then is to reconstruct a renormalized Hamiltonian H' (i.e. a set of interactions, like $\mathbf{J}'$ above) for which μ' is a Gibbs measure. Although this is trivial in finite volume, it is not so in the thermodynamic limit, and it is shown in [24] that, in many cases at low temperatures, even if H contains only nearest-neighbour interactions, there is no (uniformly) absolutely summable interaction (defined in (2.2) below) giving rise to a Hamiltonian H' for which μ' is a Gibbs measure. It has to be emphasized that this not merely a problem arising from difficulties in *computing* H', but rather that H' is simply not defined, at least according to a standard and rather general definition (allowing for long range and many body interactions); therefore, if one devices an approximate scheme for "computing H'", it is not clear at all, in view of the results of van Enter, Fernandez and Sokal, what object this scheme is supposed to approximate.

One should also mention that this issue is related to another one, of independent interest: when is a measure Gibbsian for some Hamiltonian? For example, Schonmann showed [86] that, when one projects a Gibbs measure (at low temperatures) to the spins attached to a lattice of lower dimension, the resulting measure is not, in general, Gibbsian. This is also a question arising naturally, for example in the context of interacting particle system, where one would like to determine whether the stationary measure(s) are Gibbsian or not, see for example [73] for a discussion of this issue.

What should one think about those pathologies? Basically, the answer is that, by trying to implement (1.1) at low temperatures, one is following the letter rather than the spirit of the RG, because one is using the spin variables, which are the wrong variables in that region. The fact that the usefulness of the RG method depends crucially on choosing the right variables has been known for a long time. The "good" variables should be such that a single RG

[2]In some cases, but for rather special transformations, even at high temperatures in particular in a large external field, see [22, 25].

[3]For a discussion of problems arising in the definition of the RG in momentum-space, see [28]

transformation, which can be interpreted as solving the statistical mechanics of the small scale variables with the large ones kept fixed, should be "noncritical" i.e. should be away from the parameter regions where phase transitions occur. But, as we shall explain, all the pathologies occur because, even when the s variables are fixed, the σ variables can still undergo a phase transition for some values of the s variables, i.e. they still have large scale fluctuations; or, in other words, the sum (1.1) does not amount to summing *only* over small scale fluctuations of the system, keeping the large ones fixed, which is what the RG idea is all about. However, such a summation over only small scale fluctuations can be performed, also at low temperatures, and can yield useful results there, as we shall see; but for that, one needs to use a representation of the system in terms of contours (i.e. the domain walls that separate the different ground states), instead of the spin representation. To apply the RG method, one inductively sums over the small scale contours, producing an effective theory for the larger scale contours [40, 7].

In the next section, we briefly explain what is the most general, but standard, notion of Gibbs states. Then we define (Section 3) the RG transformations, and the renormalized measures that can be shown to be not Gibbsian in the sense of the Section 2. Then, after explaining intuitively why pathologies occur (Section 4) and why this phenomenon is actually similar to the occurence of Griffiths' singularities in disordered systems (Section 5), we introduce a weaker notion of Gibbs state such that one can show that the renormalized measures are Gibbsian in that weaker sense (Section 6). Next, we explain how the RG works in the contour language (Section 7) and we end up with some conclusions and open problems (Section 8).

Since detailed proofs of all the results mentioned in this paper exist in the litterature, we shall not give them here and simply refer the reader to the relevant literature; moreover, our style will be mostly heuristic and non-mathematical, with some remarks added for the mathematically inclined reader.

2 Gibbs States

Since there exist many good references on the theory of Gibbs measures (also called Gibbs states), (see e.g. [24, 47, 57, 88, 87]) we shall only state the main definition and the basic properties.

To start with a concrete example, consider the nearest-neighbour Ising model on $\mathbf{Z}^d$. To each $i \in \mathbf{Z}^d$, we associate a variable $\sigma_i \in \{-1, +1\}$, and the (formal) Hamiltonian is

$$-\beta H = \beta J \sum_{\langle ij \rangle} (\sigma_i \sigma_j - 1) \tag{2.1}$$

where $\langle ij \rangle$ denotes a nearest-neighbour pair and β is the inverse temperature.

Obviously, the sum (2.1) makes sense only when it is restricted to a finite subset of the lattice. So, one would like to define Gibbs measures through the usual factor $Z^{-1} \exp(-\beta H)$ but using only in that formula restrictions of H to finite subsets of the lattice. One possibility is to first define Gibbs states in finite volume (with appropriate boundary conditions, and given by the RHS of (2.7) below) and then take all possible limits of such measures as the volume grows to infinity; however, there is a more intrinsic way to introduce Gibbs states directly in infinite volume, which we shall explain now. But, instead of defining the Gibbs measures only for the Ising Hamiltonian, we shall first introduce a more general framework, which will be needed later and which defines precisely what it means for a Hamiltonian to contain n-body potentials for all n (while the Hamiltonian (2.1) clearly includes only a two-body potential).

Let us consider spin variables σ_i taking values in a discrete set Ω (equal to $\{-1,+1\}$ above; everything generalizes to spins taking values in compact spaces which, in applications, are usually spheres). For a subset X of the lattice, denote the set of spin configurations on that set by Ω_X. Define an *interaction* $\Phi = (\Phi_X)$, as a family of functions

$$\Phi_X : \Omega_X \to \mathbf{R},$$

given for each finite subset X of $\mathbf{Z}^d$. Assume that Φ is

a) translation invariant.

b) uniformly absolutely summable:

$$\|\Phi\| \equiv \sum_{X \ni 0} \|\Phi_X\| < \infty \tag{2.2}$$

where $\|\Phi_X\| = \sup_{\sigma\in\Omega_X} |\Phi_X(\sigma)|$.

Φ_X should be thought of as an n-body interaction[4] between the spins in X with $n = |X|$. For the example of the Ising model, we have

$$\begin{aligned} \Phi_X(\sigma) &= \beta J(\sigma_i\sigma_j - 1) \text{ if } X = \{i,j\} \text{ and } i,j \text{ are nearest-neighbours.} && (2.3)\\ \Phi_X(\sigma) &= 0 \qquad\qquad\qquad \text{otherwise.} && (2.4)\end{aligned}$$

Note that, for convenience, we absorb the inverse temperature β into Φ.

Given an interaction Φ, one may define the Hamiltonian in any finite volume V, i.e. the energy of a spin configuration $\sigma \in \Omega_V$, provided boundary conditions are specified. Since we are allowing arbitarily long range interactions, boundary conditions mean specifying a spin configuration $\bar\sigma$ in the complement of V, i.e. $\bar\sigma \in \Omega_{V^c}$. The Hamiltonian is then given by

$$H(\sigma|\bar\sigma) = -\sum_{X\cap V\neq\emptyset} \Phi_X(\sigma\vee\bar\sigma) \tag{2.5}$$

where $\sigma\vee\bar\sigma$ denotes the total spin configuration. The sum (2.5) is a precise version of the formal sum (1.2) or (2.1).

The quantity $H(\sigma|\bar\sigma)$ is bounded by :

$$\begin{aligned} |H(\sigma|\bar\sigma)| &\le \sum_{x\in V}\sum_{X\ni x} \|\Phi_X\| \\ &= |V|\|\Phi\| && (2.6)\end{aligned}$$

[4]This set of interactions obviously forms a Banach space equipped with the norm (2.2) (note that our terminology differs slightly from the one of [24]: we add the word "uniformly" to underline the difference with respect to condition (6.1) below) .

i.e. is finite for all V finite under condition (2.2).

Definition. A probability measure μ on (the Borel sigma-algebra of) $\Omega_{\mathbf{Z}^d}$ is a *Gibbs measure* for Φ if for all finite subsets $V \in \mathbf{Z}^d$ its conditional probabilities satisfy, $\forall \sigma \in \Omega_V$,

$$\mu(\sigma|\bar{\sigma}) = Z^{-1}(\bar{\sigma}) \exp(-H(\sigma|\bar{\sigma})) \tag{2.7}$$

for μ almost every $\bar{\sigma}$ (where $Z^{-1}(\bar{\sigma})$ is the obvious normalization factor).

This definition is natural because one expects that if a measure is an equilibrium measure, then the conditional expectation of a configuration in a finite box, given a configuration outside that box, is given by (2.7). Moreover, under condition (2.2) on the interaction, one may develop a fairly general theory of Gibbs states. In fact, it is rather easy to show that all thermodynamic limits of Gibbs measures[5] defined in finite volumes satisfy (2.7). Besides, one can show that the set of Gibbs states is a closed convex set and every Gibbs state can be decomposed uniquely in terms of the extreme points of that set. The latter can be interpreted physically as the pure phases of the system and can always be obtained as limits of finite volume Gibbs measures with appropriate boundary conditions. Finally, expectations values of functions of the spins in those extremal Gibbs states are related in a natural way to derivatives of the free energy with respect to perturbations of the Hamiltonian.

Returning to our example of the Ising model, it is well known that, at low temperatures, for $d \geq 2$, there are (exactly) two extremal translation invariant Gibbs measures corresponding to the Hamiltonian (2.1), μ_+ and μ_- (moreover, in $d \geq 3$, there are also non-translation invariant Gibbs measures describing interfaces between the two pure phases).

3 Renormalization Group transformations

To define our RGT, let $\mathcal{L} = (L\mathbf{Z})^d$, $L \in \mathbf{N}$, $L \geq 2$ and cover $\mathbf{Z}^d$ with disjoint L-*boxes* $B_x = B_0 + x$, $x \in \mathcal{L}$ where B_0 is a box of side L centered around 0. To simplify the notation, we shall write $\mathbf{x}$ for B_x.

The RGT which is simplest to define, even though it is not the most widely used, is the *decimation* transformation: fix all the spins σ_x located at the center of the boxes B_x and sum over all the other spins. Given a measure μ, the renormalized measure μ' is trivial to define : it is just the restriction[6] of μ to the set of spins $\{\sigma_x\}, x \in \mathcal{L}$.

We can generalize this example as follows: associate to each $x \in \mathcal{L}$ a variable $s_x \in \{-1, +1\}$, denote by $\sigma_{\mathbf{x}} = \{\sigma_i\}_{i \in \mathbf{x}}$, and introduce, for $x \in \mathcal{L}$, the probability kernels

$$T_x = T(\sigma_{\mathbf{x}}, s_x),$$

which means that T_x satisfies

$$\begin{aligned} &1)\ T(\sigma_{\mathbf{x}}, s_x) \geq 0 \\ &2)\ \sum_{s_x} T(\sigma_{\mathbf{x}}, s_x) = 1 \end{aligned} \tag{3.1}$$

[5] Under the following notion of convergence : $\mu_n \to \mu$ if $\mu_n(s) \to \mu(s)$ $\forall V$ finite $\forall s \in \Omega_V$.
[6] Also called the projection or the marginal distribution of μ.

In the example of the decimation transformation, $T(\sigma_{\mathbf{x}}, s_x) = \delta(\sigma_x - s_x)$. Other examples include the majority transformation, defined when $|B_x|$ is odd, where $T(\sigma_{\mathbf{x}}, s_x) = 1$ if and only if the majority of the signs of the spins in $\mathbf{x}$ coincide with s_x. Or the Kadanoff transformation, defined, for $p \geq 0$, by

$$T(\sigma_{\mathbf{x}}, s_x) = \frac{\exp(p s_x \sum\limits_{i \in \mathbf{x}} \sigma_i)}{2 \cosh(p \sum\limits_{i \in \mathbf{x}} \sigma_i)}.$$

Note that, when $p \to \infty$, the probability kernel of that transformation converges towards the one of the majority transformation.

For any measure μ on $\{-1,+1\}^{\mathbf{Z}^d}$, we denote by $\mu(\sigma_A)$ the probability of the configuration $\sigma_A \in \{-1,+1\}^A$.

Definition. Given a measure μ on $\{-1,+1\}^{\mathbf{Z}^d}$, the *renormalized measure* μ' on $\Omega = \{-1,+1\}^{\mathcal{L}}$ is defined by:

$$\mu'(s_A) = \sum_{\sigma_{\mathbf{A}}} \mu(\sigma_{\mathbf{A}}) \prod_{x \in A} T(\sigma_{\mathbf{x}}, s_x) \tag{3.2}$$

where $\mathbf{A} = \cup_{x \in A} \mathbf{x}$, $A \subset \mathcal{L}$, $|A| < \infty$, and $s_A \in \Omega_A = \{-1,+1\}^A$.

It is easy to check, using 1) and 2), that μ' is a measure. We shall call the spins σ_i the *internal* spins and the spins s_x the *external* ones (they are also sometimes called the block spins).

Note that we restrict ourselves here, for simplicity, to transformations that map spin $\frac{1}{2}$ models into other spin $\frac{1}{2}$ models, but this restriction is not essential. In particular, the block spin transformation fits into our framework, defining

$$T(\sigma_{\mathbf{x}}, s_x) = \delta(s_x - L^{-\alpha} \sum_{i \in \mathbf{x}} \sigma_i)$$

for some α, the only difference being that s_x does not belong to $\{-1,+1\}$ anymore.

In order to use the RG it is necessary to iterate those transformations and, for that, it is convenient to *rescale*. That is, consider $\mathcal{L}$ as a lattice Z^d of unit lattice spacing, cover it with boxes of side L (i.e. of side L^2 in terms of the original lattice) associate new s spins to each of those boxes etc. Sometimes the RGT turn out to form semigroups (i.e. applying them n times amounts to applying them once with L replaced by L^n) : e.g. the decimation or block spin transformation form semigroups while the majority and the Kadanoff transformations do not.

However, we are not concerned here with the iteration of the transformation but rather with the mathematical status of a single transformation. Can one, given an RGT defined by a kernel T, associate to a Hamiltonian H a renormalized Hamiltonian H'? A natural scheme would go as follows (see the diagram below). Given H, we associate to it its Gibbs measure as in Section 2 and, given T, we have just defined the renormalized measure μ'. If it can be shown that such measures are Gibbs measures for a certain Hamiltonian H', then the latter could be defined as the renormalized Hamiltonian corresponding to H:

$$
\begin{array}{ccc}
H & \xrightarrow{?} & H' \\
\downarrow & & \uparrow ? \\
\mu & \longrightarrow & \mu'
\end{array}
$$

However, as we said in the Introduction, this simple scheme does not work: The main result of [24] is that, for a variety of RGT, including decimation, majority rule, the Kadanoff transformation or the block spin transformation, there is no interaction satisfying a) and b) in Section 2 for which μ'_+ or μ'_- are Gibbs measures, hence no renormalized Hamiltonian H'. We shall now explain intuitively why this is so.

4 Origin of the pathologies

In order to understand the origin of the pathologies, consider the simplest example, namely the decimation transformation (let us emphasize, however, that pathologies occur for many other RG transformations and that, for those transformations, the origin of the pathologies is basically the same as in this rather artificial example). Assume that μ' is a Gibbs measure for a uniformly absolutely summable potential Φ and consider the following consequence of this assumption:

$$
\lim_{N\to\infty} \sup_{\bar{s}^1,\bar{s}^2}{}^{N} \left| \frac{\mu'(s_0|\bar{s}^1)}{\mu'(s_0|\bar{s}^2)} - 1 \right| = 0 \tag{4.1}
$$

where $\sup^N$ means that we take the sup over all $\bar{s}^1, \bar{s}^2$ satisfying

$$
\bar{s}^1_x = \bar{s}^2_x \qquad \forall x \in V_N \equiv [-N, N]^d
$$

So, $\bar{s}^1, \bar{s}^2$ are two "boundary conditions" acting on the spin at the origin (any other fixed site would do of course) that coincide in a box around the origin, V_N, that becomes arbitrarily large (as $N \to \infty$), and are free to differ outside V_N.

To check (4.1), observe that, for any $\bar{s}^1$, $\bar{s}^2$ over which the supremum is taken, we have

$$
\begin{aligned}
& |H(s_0|\bar{s}^1) - H(s_0|\bar{s}^2)| \\
\leq & \sum_X^{0,N} \|\Phi_X\| \equiv \mathcal{E}_N
\end{aligned} \tag{4.2}
$$

where $\sum_X^{0,N}$ runs over all sets X whose contribution to $H(s_0|\bar{s}^1)$ is not cancelled by the corresponding term in $H(s_0|\bar{s}^2)$, i.e. containing 0 but not contained inside V_N: $X \ni 0$, $X \cap V_N^c \neq \emptyset$. The RHS of (4.2) tends to zero, as $N \to \infty$, since it is, by assumption, the tail of the convergent series (2.2)[7].

[7] Note that the bound (4.2) implies that H is a *continuous* function of $\bar{s}$, in the product topology, i.e. for the following notion of convergence: a sequence of configurations $s^n \to s$ if $\forall V$ finite, $\exists n_V$ such that $s^n_x = s_x, \forall x \in V, \forall n \geq n_V$.

Now, it is easy to see, using the definition (2.7) of a Gibbs state, that (4.2) implies

$$e^{-2\mathcal{E}_N} \leq \frac{\mu'(s_0|\bar{s}^1)}{\mu'(s_0|\bar{s}^2)} \leq e^{2\mathcal{E}_N}, \tag{4.3}$$

so that $\mathcal{E}_N \to 0$ implies (4.1).

So, (4.1) means that, for Gibbs measures defined as above, with the interaction satisfying the summability condition (2.2), the conditional probability of the spin at the origin does not depend too much on the value of the boundary conditions $\bar{s}^1, \bar{s}^2$ far away (i.e. outside V_N).

So, to prove that there does not exist a uniformly absolutely summable potential, it is enough to find a sequence of pairs of configurations $(\bar{s}^1_N, \bar{s}^2_N)$, coinciding inside V_N and differing outside V_N, such that

$$\left| \frac{\mu'(s_0|\bar{s}^1_N)}{\mu'(s_0|\bar{s}^2_N)} - 1 \right| \geq \delta \tag{4.4}$$

for some $\delta > 0$ independent of N.

The trick is to construct $\bar{s}^1_N$, $\bar{s}^2_N$ as modifications of s^{alt}, the *alternating configuration*:

$$s^{\text{alt}}_x = (-1)^{|x|} \qquad \forall x \in \mathcal{L} \tag{4.5}$$

where $|x| = \sum_{i=1}^d |x_i|$, i.e. the configuration equal to $+1$ when $|x|$ is even and to -1 when $|x|$ is odd. Now take $\bar{s}^1_N = \bar{s}^2_N = s^{\text{alt}}$ inside V_N and, outside V_N, we take $\bar{s}^1_N$ everywhere equal to $+1$ and $\bar{s}^2_N$ everywhere equal to -1, which we shall call the "all $+$" and the "all $-$" configurations.

To see what this does, let us rewrite the Hamiltonian (2.1) as:

$$-H = J \sum_{\langle ij \rangle, i,j \notin \mathcal{L}} (\sigma_i \sigma_j - 1) + \sum_{x \in \mathcal{L}} \sum_{|i-x|=1} (\sigma_i s_x - 1) \tag{4.6}$$

where the first sum runs over the pairs of nearest neighbours contained in $\mathbf{Z}^d \backslash \mathcal{L}$ and the second sum contains the couplings between the decimated spins (σ) and the "renormalized" ones (s). In this formulation, s can be thought of as being a (random) *external magnetic field* acting on the σ spins. One may also write:

$$\mu'(s_0|\bar{s}^1_N) = \frac{\langle \exp(s_0 \sum_{|i|=1} \sigma_i) \rangle (\bar{s}^1_N)}{\sum_{s_0=\pm 1} \langle \exp(s_0 \sum_{|i|=1} \sigma_i) \rangle (\bar{s}^1_N)} \tag{4.7}$$

where $\langle \cdot \rangle (\bar{s}^1_N)$ denotes the expectation in the Gibbs measure on the σ spins, with a Hamiltonian like (4.6), but with the second sum running only over $x \neq 0$ and with $s = \bar{s}^1_N$ fixed[8].

Now, it is easy to see that the external field s^{alt} has a neutral effect : on average, it does not "push" the σ spins either up or down. On the other hand,

[8] To be precise, the expectation in (4.7) is obtained by taking the infinite volume limit of expectations in finite volumes, with $+$ boundary conditions. See (7.30) below for a general expression of the conditional probabilities for the renormalized measure, from which (4.7) can be deduced via elementary calculations.

the "all +" or "all −" configurations do tend to align the σ spins along their respective directions. Now, think of the effect of $\bar{s}^1_N$: coinciding with the "all +" configuration, outside of V_N, it pushes the σ spins up in that region. But, being neutral inside V_N, it does not exert any particular influence there (one can think of it as being essentially equivalent to a zero field inside V_N). However, the σ spins live on a lattice that, although decimated, is nevertheless connected, so that this spin system, considered on its own, in the absence of any external field, i.e. without the second term in (4.6), has *long range order* (LRO) at low temperatures. Now the mechanism should be obvious : The "field" $\bar{s}^1_N$ pushes the σ spins up outside V_N, the LRO "propagates" this orientation inside V_N (where $\bar{s}^1_N$ is neutral and thus essentially equivalent to a zero field) and, finally, the σ_i, with $|i| = 1$ i.e. the nearest - neighbours of s_0, act as external fields on s_0, see (4.7), and, since they tend to be up, so does s_0. Of course $\bar{s}^2_N$ acts likewise, with up replaced by down; hence the ratio of the conditional probabilities appearing in (4.4) does not tend to 1 as $N \to \infty$ because, by definition of LRO, the effect described here is independent of N.

As stressed in [24], this is the basic mechanism producing "pathologies": for a fixed value of the external spins, the internal ones undergo a phase transition. The complete proof ot course involves a Peierls (or Pirogov-Sinai) type of argument (see [24] for full details as well as for a discussion of other RG transformations) but the intuition, outlined above, should make the result plausible.

5 Connection with the Griffiths singularities

In [49], Griffiths showed that the free energy of dilute ferromagnets[9] is not analytic, as a function of the magnetic field h, at low temperatures and at $h = 0$, even below the percolation treshold for occupied bonds (i.e. with $J \neq 0$). The mechanism is, here too, easy to understand intuitively : for any given, arbitrarily large, but finite region of the lattice, there is a non zero probability that the bonds in that region will all be occupied; since the system is at low temperatures, this produces singularities of the free energy arbitrarily close to $h = 0$. Of course, if the size of the region increases, the probability of this event decreases (very fast). But, if one considers an infinite lattice such events occur with probability one with a non-zero frequency and this is sufficient to spoil analyticity.

A related phenomenon concerns the decay of the pair correlation function which, if we consider a random ferromagnet and denote by $\mathbf{J}$ a realization of the random couplings, satisfies the bound[10]:

$$\langle s_0 \ s_x \rangle(\mathbf{J}) \leq C(\mathbf{J}) \exp(-m|x|) \tag{5.1}$$

where $\sup_{\mathbf{J}} C(\mathbf{J}) = \infty$ (if the distribution of the couplings is not of compact support), although $C(\mathbf{J}) < \infty$ with probability one at high temperatures. So,

[9] Meaning that the coupling constant for a nearest-neighbour bond is equal to J with probability p, with $0 \leq p \leq 1$, and to 0 with probability $1 - p$. The magnetic field is introduced as in (7.1) below.

[10] At high temperatures, $\langle s_0 \rangle(\mathbf{J}) = 0$ with probability one, so we do not need to truncate the expectation which, besides, is positive for ferromagnetic couplings.

the pair correlation function decays, but not uniformly in $\mathbf{J}$. This reflects again the fact that, with some small but non zero probability, the couplings may be arbitrarily large but finite in an arbitrarily large but finite region around the origin and then, in this case, the correlation functions decays only if $|x|$ is sufficiently large so that x is far away from that region.

Since the probability of having large couplings over a large region is small, one can understand why the probability of a large $C(\mathbf{J})$ is small and why $C(\mathbf{J}) < \infty$ with probability one.

To understand the connection with the RG pathologies, start with an untypical $\mathbf{J}$ (e.g. a coupling that is everywhere large), i.e. of probability strictly equal to zero, and construct an event of small but non zero probability by restricting that configuration to a large but finite box, in such a way that this event destroys some property of the non-random system such as analyticity or uniform decay of correlations[11].

Now, think of (4.1) as expressing a form of decay of correlations for the σ spins given some (random) configuration of the s spins[12]. Of course, the expression in (4.1) is not of the form of a decay of a pair correlation function but, if the distribution of the spins σ_i with $|i| = 1$ became independent of the one of the spins outside V_N when $N \to \infty$, then one would expect the distribution of s_0 (on which the σ_i with $|i| = 1$ act as external fields) to become independent of the value of $\bar{s}_x$ for $x \notin V_N$ and, hence, (4.1) to hold. However, if the configuration of the s spin was equal to s^{alt} over the whole lattice, then one would expect the σ spins to have LRO (since, without any external field, they have LRO and the effect of s^{alt} is similar to having no external field). So, what happens with the $\bar{s}^1_N$ and $\bar{s}^2_N$ chosen above, is that putting $\bar{s}^1_N$, $\bar{s}^2_N$ equal to s^{alt} over a large region, one can make the decay of correlation arbitrarily slow, hence show that (4.1) does not hold.

When thinking of $\bar{s}$ as a random field acting on the σ variables, one should keep in mind that the distribution of this random field is nothing but μ'_+ or μ'_-. Now, at low temperatures, typical configurations with respect to μ'_+ (or μ'_-) are just typical configurations of the Ising model, i.e. a "sea" of $+$ spins with some islands of $-$ spins (and islands of $+$ spins within the islands of $-$ spins, etc), with the role of $+$ and $-$ interchanged for μ'_-. Hence the configuration s^{alt} is untypical both with respect to μ'_+ <u>and</u> μ'_- (just like $\mathbf{J}$ large for the random system).

What this suggests is that one might want to prove a weaker property for the renormalized Hamiltonian which, following the analogy with random systems, would be similar to showing that $C(\mathbf{J}) < \infty$ with probability one. The analogous property will be a summability property of the interaction, but not a uniform one, as we had in (2.2). We shall now state this property explicitly.

[11] This is expected to be a general feature of (non trivial) random systems (random magnetic fields, spin glasses, Anderson localization, etc.) although it is often not easy to prove.

[12] See [74] and [75] for a precise formulation of this idea.

6 The renormalized measures as weak Gibbs measures

The basic observation, going back to Dobrushin ([19], see also [20]), which leads to a generalization of the notion of Gibbs measure, is that, in order to define $H(s_V|\bar{s}_{V^c})$, it is not necessary to assume (2.2) ; it is enough to assume the existence of a (suitable) set $\overline{\Omega} \subset \Omega$ on which the following *pointwise* bounds hold:

b') Φ is $\overline{\Omega}$-pointwise absolutely summable:

$$\sum_{X \ni x} |\Phi_X(s_X)| < \infty \quad \forall x \in \mathcal{L}, \forall s \in \overline{\Omega}. \tag{6.1}$$

We shall therefore enlarge the class of "allowed" interactions by dropping the condition (2.2) and assuming (6.1) instead.

However, since we want to define (2.5) for arbitrary volumes V, the set $\overline{\Omega}$ must be defined by conditions that are, in some sense, "at infinity" (this is what we meant by "suitable"). This can be defined precisely by saying that the fact that a configuration s belongs or does not belong to $\overline{\Omega}$ is not affected if we change the values of that configuration on finitely many sites. Sets of configurations having this property are called *tail sets*[13].

Definition. Given a tail set $\overline{\Omega} \subset \Omega$, μ is a *Gibbs measure* for the pair (Φ, $\overline{\Omega}$) if $\mu(\overline{\Omega}) = 1$, and there exists a version of the conditional probabilities that satisfy, $\forall V \subset \mathcal{L}$, $|V|$ finite, $\forall s_V \in \Omega_V$,

$$\mu(s_V|\bar{s}_{V^c}) = Z^{-1}(\bar{s}_{V^c}) \exp(-H(s_V|\bar{s}_{V^c})) \tag{6.2}$$

$\forall \bar{s} \in \overline{\Omega}$.

Since conditional probabilities are defined almost everywhere, this definition looks very similar to the usual one[14], given in Section 2. However, the introduction of the set $\overline{\Omega}$ has some subtle consequences. To see why, consider the (trivial) case, where $L = 1$, and $T = \delta(\sigma_i - s_x)$ with $i = x$, i.e. the "renormalized" system is identical to the original one[15]. Take $\overline{\Omega}$ to be the set of configurations such that all the (usual) Ising contours are finite and each site is surrounded by at most a finite number of contours. Thus configurations in $\overline{\Omega}$ consist of a "sea" of plus or minus spins with small islands of opposite spins, and even smaller islands within islands. Clearly, $\overline{\Omega}$ is a tail set. When $X =$ a contour γ (considered as a set of sites), we let

$$\Phi_X(s_X) = -2\beta|\gamma| \tag{6.3}$$

[13] A (trivial) example of a tail set is the set of configurations such that there exists a finite volume V, outside of which the configuration coincides with a given configuration (e.g. all plus).

[14] However, when condition (2.2) holds, the conditional probabilities can be extended everywhere, and are continuous, in the product topology (see note 6), which is not the case here.

[15] This example was suggested to us by A. Sokal.

for s_X = a configuration making γ a contour, and $\Phi_X(s_X) = 0$ otherwise. Obviously, this Φ satisfies (6.1) but not (2.2). One can write $\overline{\Omega} = \overline{\Omega}_+ \cup \overline{\Omega}_-$, according to the values of the spins in the infinite connected component of the complement of the contours. It is easy to see that μ^+, μ^- are indeed, at low temperatures, Gibbs measures (in the sense considered here) for this new interaction: a Peierls argument shows that $\mu^+(\overline{\Omega}_+) = \mu^-(\overline{\Omega}_-) = 1$, and for $s \in \overline{\Omega}$ the (formal) Hamiltonian (2.1) is $\beta H = 2\beta \sum_\gamma |\gamma|$. Actually, the proof of Theorem 1 below is constructed by using a kind of perturbative analysis around this example. Of course, in this example one could alternatively take $\overline{\Omega} = \Omega$ and Φ = the original nearest-neighbor interaction; this shows the *nonuniqueness* of the pair $(\Phi, \overline{\Omega})$, associated to a single measure, in our generalized Gibbs-measure framework[16]. This will be important when we discuss the significance of the result below for the implementation of the RG.

Before stating our main result we need to detail some conditions on the kernel T.We assume that T is symmetric:

$$T(\sigma_{\mathbf{x}}, s_x) = T(-\sigma_{\mathbf{x}}, -s_x) \tag{6.4}$$

and that

$$0 \leq T(\sigma_{\mathbf{x}}, s_x) \leq e^{-\beta} \tag{6.5}$$

if $\sigma_i \neq s_x$, $\forall i \in \mathbf{x}$.

Note that (3.1, 6.4, 6.5) imply that

$$\overline{T} \equiv T(\{\sigma_i = +1\}_{i \in \mathbf{x}}, +1) = T(\{\sigma_i = -1\}_{i \in \mathbf{x}}, -1) \geq 1 - e^{-\beta} \tag{6.6}$$

So, condition (6.5) means that there is a coupling which tends to align s_x and the spins in the block B_x; this condition is satisfied for the majority, decimation and Kadanoff (with p large) transformations[17].

Theorem 1 *Under assumptions (6.4, 6.5) on T, and for β large enough, there exist disjoint tail sets $\overline{\Omega}_+, \overline{\Omega}_- \subset \Omega$ such that $\mu'_+(\overline{\Omega}_+) = \mu'_-(\overline{\Omega}_-) = 1$ and a translation invariant interaction Φ satisfying b') with $\overline{\Omega} = \overline{\Omega}_+ \cup \overline{\Omega}_-$ such that μ'_+ and μ'_- are Gibbs measures for the pair (Φ, $\overline{\Omega}$).*

Remarks.

1. This result was recently extended in [75] to general projections and to the general framework covered by the Pirogov-Sinai theory [84, 88] (see Section 7 below for a brief discussion of that theory), using percolation techniques. However, our approach also shows that the two renormalized states are Gibbsian with respect to the *same* interaction Φ (while this question is left open in [75]).

2. The analogy with the random systems discussed in the previous section is that instead of having $C(\mathbf{J}) < \infty$ with probability one, we have (6.1) holding with probability one, with respect to the renormalized measure.

[16]While in the usual framework, one can define a notion of "physical equivalence" of interactions so that a measure can be a Gibbs measure for at most one interaction (up to physical equivalence), see [24].

[17]It would be more natural to have, instead of (6.5), $0 \leq T \leq \epsilon$ (with ϵ independent of β but small enough). However, assuming (6.5) simplifies the proofs.

3. Note that in the theory of "unbounded spins" with long range interactions, a set $\overline{\Omega}$ of "allowed" configurations has to be introduced, where a bound like (6.1) holds [47, 63, 65]. Here, of course, contrary to the unbounded spins models, each $\|\Phi_X\|$ is finite. Still, one can think of the size of the regions of alternating signs in the configuration as being analogous to the value of unbounded spins. The analogy with unbounded spins systems was made more precise and used in [78] and [67] to study the thermodynamic properties of the potential above.

4. The set $\overline{\Omega} = \overline{\Omega}_+ \cup \overline{\Omega}_-$ is not "nice" topologically: e.g. it has an empty interior (in the usual product topology, defined in footnote 6). Besides, our effective potentials do not belong to a natural Banach space like the one defined by (2.2). However, this underlines the fact that the concept of Gibbs measure is a measure - theoretic notion and the latter often do not match with topological notions.

5. There has been an extensive investigation of this problem of pathologies and Gibbsianness. Martinelli and Olivieri [81, 82] have shown that, in a non-zero external field, the pathologies disappear after sufficiently many decimations. Fernandez and Pfister [34] study the set of configurations that are responsible for those pathologies. They give criteria which hold in particular in a non-zero external field, and which imply that this set is of zero measure with respect to the renormalized measures. Following the work of Kennedy [59], several authors [52, 3, 17, 5] analyze the absence of pathologies near the critical point. Also, if one combines projection with enough decimation, as in [69], then one knows that each of the resulting states is Gibbsian (in the strongest sense, i.e. with interactions satisfying (2.2)), but for different interactions. This in turn implies that non-trivial convex combinations of these states are not quasilocal everywhere, see [26], where other examples of "robust" non-Gibbsianness can be found.

The main remark to be made, however, is that this Theorem, although it clarifies the nature of the pathologies, does not in itself suffices to define the RGT as a nice map between Hamiltonians. Indeed, as we observed above, the pair $(\Phi, \overline{\Omega})$ is not unique, even in the simple case of the nearest-neighbour Ising model. One might try to impose further conditions that might select a unique pair, but that has not been done. Thus, in terms of the diagram at the end of Section 3, the problem has changed: with the approach based on the usual notion of Gibbs state, there was no interaction with respect to which the renormalized measures were Gibbsian. But, with our extended notion, the interaction exists but is not unique and the map from H to H' is still not well defined.

In order to have a nice set of RG transformations, it seems that one has to give up the spin representation of the model and use instead the contour representation. In the next Section, we will sketch how this strategy works.

7 The Renormalization Group method in the contour representation

To motivate the RG approach at low temperatures, consider the following problem. Let the perturbed Ising Hamiltonian be defined by

$$-\beta H = \beta J \sum_{<ij>} (\sigma_i\sigma_j - 1) + \beta J_3 \sum_{<ijk>} \sigma_i\sigma_j\sigma_k + \beta h \sum_i \sigma_i \tag{7.1}$$

where the first sum runs over pairs of nearest neighbours, the second one over all elementary triangles of the lattice (made of two sites adjacent to a third one) and the last term represents the effect of an external magnetic field. We consider J_3 small, i.e. we are interested in a small perturbation of the Ising model.

Now, for $J_3 \neq 0$, this model does no longer have the spin flip symmetry of the Ising model. However it is easy to see that, for J_3 small, one can find $h_0 = h_0(J_3)$ so that the Hamiltonian (7.1) has two ground states, the "all plus" and the "all minus" configurations. A natural question is whether there exists a phase transition in this model, i.e. a line $h(T)$, defined for T small, on which the plus and the minus phases coexist and such that, for $h > h(T)$ (resp. $h < h(T)$) we have only the plus (resp. minus) phase (and such that $h(T) \to h_0$ for $T \to 0$). Of course, for $J_3 = 0, h_0 = 0$ and $h(T) \equiv 0$ by symmetry, but if one looks at standard arguments such as Peierls' argument proving the existence of a phase transition for the Ising model, one sees that they use crucially, at least in their simplest versions, the information, (given by the symmetry of the model), that the phase transition must occur at $h = 0$. For the perturbed model one has, so to speak, to find the value of $h(T)$ for which one will be able to prove that phase coexistence occurs at the same time as one proves that coexistence. It was therefore quite an achievement when Pirogov and Sinai [84, 88] provided a rigorous construction of phase diagrams at low temperatures for a general class of Hamiltonians (including of course (7.1)): basically those with discrete spins, finite range interactions and finitely many ground states. That theory was extended in several directions later (see e.g. [6, 12, 55, 89, 91]) but what interest us here is that it can be rather easily understood using the RG.

Indeed, writing the RGT in the contour language, Gawędzki, Kotecký and Kupiainen [40] were able to rigorously justify the standard picture that, at low temperatures, the Renormalization Group drives the spin system (7.1) towards lower temperatures and towards larger external fields (for $h \neq h(T)$). Using that fact, one can solve the problem of "selecting" $h(T)$ by defining it as the unique initial value of the external field so that the effective field remains bounded under all iterations of the RG (see Sect. 7.3 below). However, instead of discussing in full the RG approach of [40], we shall illustrate it on the simple example of the Ising model in a uniform field, i.e. (7.1) with $J_3 = 0$. This model is trivial[18], but, nevertheless, it can be used to illustrate how one implements the RG and how the effective field flows.

[18] As a consequence of the Lee-Yang theorem, the free energy and the correlation functions are analytic for $Re\ h \neq 0$.

In order to indicate how such ideas extend to disordered systems, we shall also discuss the Ising model with a random external field:

$$-\beta H = \beta J \sum_{\langle ij \rangle} (\sigma_i \sigma_j - 1) + \beta \sum_i h_i \sigma_i$$

where the h_i's are i.i.d. random variables with, say, a Gaussian distribution of mean zero and variance ε^2. This model has a phase transition in $d \geq 3$, which was the subject of some controversy in the early 80's.

We shall explain the Renormalization Group method of [40, 7] in three stages : First, the usual Ising model with no field. Here, one shows only that the temperature flows towards zero. Then, we add a uniform external field (and explain why it grows, except on the transition line) and finally a random field. Finally, we shall make some remarks on the proof of Theorem 1.

7.1 The Renormalization Group transformation for the Ising model

We shall assume that the reader is somewhat familiar with the Peierls' argument and the picture of the low temperature phases that emerges from it. Thus typical configurations of the plus and the minus phases consist of a sea of plus or minus spins with small islands of opposite spins, and even smaller islands within islands. The Renormalization Group transformation consists of two steps:

- integrate out the small scale degrees of freedom of the system (defined below). This operation produces an effective model for the remaining large scale degrees of freedom.
- rescale this effective model so as to produce new small scale degrees of freedom.

And then, iterate the transformation.

To be concrete, let

$$Z^+(\Lambda) = \sum_\sigma \exp(-\beta H(\sigma|+)) \tag{7.2}$$

with $H(\sigma|+)$ given by (2.1), be the partition function of the Ising model, in Λ, with plus boundary conditions and no external field ($h = 0$). Now introduce *contours* , i.e., identify a spin configuration with the set of nearest-neighbour pairs $< ij >$ where $\sigma_i \neq \sigma_j$ and decompose this set into connected components. Then, the standard low temperature expansion takes the form:

$$Z^+(\Lambda) = \sum_\Gamma \prod_{\gamma \in \Gamma} \rho(\gamma) \tag{7.3}$$

where

$$\rho(\gamma) = \exp(-2\beta J|\gamma|), \tag{7.4}$$

$|\gamma|$ = number of bounds in γ, and the sum runs over all families Γ of pairwise disjoint contours γ. Define a contour to be *small* if diam (γ) is less than L and *large* otherwise.

The small contours are part of the small scale degrees of freedom of the systems. L is a fixed number, whose choice is rather arbitrary. Later it will be chosen large compared to various geometric constants. The larger L is, the larger β has to be chosen for the proof to work (at least in less trivial situations than in this example). L is the renormalization scale; that is, we shall integrate all degrees of freedom on scale less than L and then rescale everything by L^{-1} in order to be able to iterate the transformation.

Observe that, if we restrict the sum in (7.3) to small contours, then the logarithm of Z^+ certainly has a convergent low temperature expansion. One does not even need a Peierls' estimate on the number of contours of a given length to show that. Indeed, there are only finitely many (small) contours per unit volume and each of them has an activity less than $\exp(-2\beta J)$. So, this is the most standard type of Mayer expansion, with a finite number of "species" of particles (the small contours). Note, however, that, if one uses this argument, one needs β large if L is large, because, the larger L, the larger the number of small scale contours.

Now, write

$$Z^+(\Lambda) = \sum_{\Gamma}{}^{l} \prod_{\gamma\in\Gamma} \rho(\gamma) \sum_{\widetilde{\Gamma}}{}^{\Gamma,s} \prod_{\widetilde{\gamma}\in\widetilde{\Gamma}} \rho(\widetilde{\gamma}) \tag{7.5}$$

where the first sum, $\sum^l$, runs over families of large contours and the second one, $\sum^{\Gamma,s}$, is over families of small contours compatible with the large ones (Γ), i.e. not overlapping with them. This equals:

$$Z^+(\Lambda) = \sum_{\Gamma}{}^{l} \prod_{\gamma\in\Gamma} \rho(\gamma) Z^s(\Lambda\backslash\Gamma) \tag{7.6}$$

where $Z^s(\Lambda\backslash\Gamma)$ is the partition function in $\Lambda\backslash\Gamma$ with the sum in (7.3) restricted to families of small contours.

This partition function actually factorizes over connected components of $\Lambda\backslash\Gamma$, but we shall not use that fact. We only use the fact that, since we have a convergent expansion, we may write $\log Z^s(\Lambda\backslash\Gamma)$ as the sum of a bulk term, and a boundary one, which themselves are sums of "polymers" (which can be taken simply as connected sets, see e.g. [60]):

$$\begin{aligned}\log Z^s(V) &= \sum_{C\subseteq V} \varphi^T(C) \\ &= \beta S|V| + f_{\partial V} \end{aligned}\tag{7.7}$$

where

$$S = \lim_{V\to\infty} (\beta|V|)^{-1} \log Z^s(V) = \sum_{C\ni x} \frac{\varphi^T(C)}{\beta|C|}$$

One has the bounds:

$$|\varphi^T(C)| \le e^{-\beta J|C|} \tag{7.8}$$

$$|f_{\partial V}| \le e^{-\beta J} |\partial V| \tag{7.9}$$

$$|S| \leq e^{-\beta J}. \tag{7.10}$$

Inserting (7.7) into (7.6), with $V = \Lambda \backslash \Gamma$, we get:

$$Z^+(\Lambda) = e^{\beta S|\Lambda|} \sum_{\Gamma}{}^{l} \prod_{\gamma \in \Gamma} \rho(\gamma) e^{U_\Lambda(\Gamma)} \tag{7.11}$$

where

$$U_\Lambda(\Gamma) = \sum_{C \subset \Lambda \backslash \Gamma} \varphi^T(C) - \beta S|\Lambda| \tag{7.12}$$

is an "interaction" between the large contours. The constant term, $\beta S|\Lambda|$, which is the bulk free energy of the small contours, may be taken outside of the sum. So, in the partition function (7.11), the sum runs only over large contours; the latter are interacting, but weakly, as one can see from (7.8).

Before iterating, we need to rescale so as to produce new small contours. This is done by *blocking* the large contours. This last operation will also integrate out the remaining small scale degrees of freedom, namely the small wiggles in the large contours. To do that, cover the lattice with disjoint boxes of side L. Define the coarse grained contour γ' corresponding to γ to be the set of boxes intersected by γ. Write $[L^{-1}\gamma] = \gamma'$. And let the weight $\rho'(\gamma')$, of the coarse grained contour, be:

$$\rho'(\gamma') = \sum_{\substack{\gamma \\ [L^{-1}\gamma]=\gamma'}} \rho(\gamma) \tag{7.13}$$

So, we simply resum the weights of all the contours γ mapped onto γ' by the coarse graining. Then, if we neglect for a moment the interaction $U_\Lambda(\Gamma)$ in (7.11), we may write:

$$Z^+(\Lambda) e^{-S\beta|\Lambda|} \cong \sum_{\Gamma'} \prod_{\gamma' \in \Gamma'} \rho'(\gamma') \tag{7.14}$$

That is, the original partition function can be written as a new partition function, of essentially the same form as the previous one, but with new contour weights $\rho'(\gamma')$. The main property of these weights is that they sastisfy the bound:

$$0 < \rho'(\gamma') < e^{-\beta'|\gamma'|} \tag{7.15}$$

with $|\gamma'|$ = number of boxes in γ' and

$$\beta' = L^{d-1} c \beta. \tag{7.16}$$

This is easy to see, since, on the first scale, contours are closed $d-1$ dimensional surfaces and thus each box in γ' contains approximately L^{d-1} bonds of the original contour. This is not exactly true for all boxes, hence the constant c in (7.16), which is independent of L and $|\gamma'|$ (for a proof, see [35]). A convenient way to write (7.16) is

$$\beta' = L^{d-1-\alpha} \beta \tag{7.17}$$

i.e. we assume that L is large enough so that $L^{-\alpha}$ is smaller than c. This is one of the places in this argument where one assumes that L is large enough.

Now, one should say what to do with the neglected $U_\Lambda(\Gamma)$[19]. Simply, one may expand it in a Mayer series, before the blocking. Using (7.8) one shows that the weights of the Mayer graphs are small. Then, in the blocking, one collects together contours and Mayer graphs joining them. The new contour weights are not necessarily positive, because the weight of the Mayer graphs may be of any sign, and also, instead of (7.15), (7.16), we shall get

$$|\rho'(\gamma')| \leq \exp(-\widetilde{\beta}|\gamma|) \tag{7.18}$$

with

$$\widetilde{\beta} = L^{1-\alpha}\beta \tag{7.19}$$

because the Mayer graphs do not form closed surfaces, just connected sets (so they scale linearly with L). This last remark would have to be made even if we did not have to Mayer expand the interaction, because, after one iteration, one gets contours, in (7.14), which are "thin", i.e. are made of a linear sequence of boxes and are no longer closed surfaces. They come from blocking the contours on the first scale that fitted just into that set of boxes.

If we are only interested in the Ising model these two remarks do not really matter: a bound like (7.18) is sufficient. We may iterate the procedure, all expansions converge even for complex weights and the temperature $T = \beta^{-1}$ runs towards zero.

However, when one deals with the random field model, it becomes necessary to distinguish between β and $\widetilde{\beta}$ and to keep all contour weights positive. We refer the reader to [7] for these modifications of the expansion.

7.2 The Renormalization Group transformation for a uniform field

Let us now turn our attention to the Ising model in a non-zero constant external field. The partition function can be written in terms of contours as:

$$Z^+(\Lambda) = \sum_{\Gamma} \prod_{\gamma\in\Gamma} \rho(\gamma) \exp[B(V^+(\Gamma) - V^-(\Gamma))] \tag{7.20}$$

where $B = \beta h$ and $V^\pm(\Gamma)$ is the number of sites where the spins are equal to ± 1, given that the set of contours is Γ and given the (plus) boundary conditions. Of course the field prevents us from doing the same expansion as before. Indeed, the "interaction" between the contours and the field is quite "non-local": erasing a contour changes the field everywhere inside the contour and not only on its boundary.

However, suppose first that the field is "large", namely that

$$|B| > K \tag{7.21}$$

where K is a constant depending only on the lattice dimension. Then, there exist a standard convergent large field expansion. The polymers of that expansion are simply connected sets where $h\sigma_x < 0$. The conclusion is that for $|B|$

[19]See also [68] for an alternative treatment of this part.

large enough, the model can be completely analyzed and one can show that the system is in the single-phase region, either plus or minus (i.e. the Gibbs measure is unique).

Now, for $|B|$ smaller than K, we may proceed as follows: since, by the isoperimetric inequality,

$$\begin{aligned} V(\gamma) &= \text{volume enclosed by } \gamma \\ &\leq c|\gamma|^{d/d-1} \end{aligned} \tag{7.22}$$

and $|\gamma| \leq c'$ $(\mathrm{diam}(\gamma))^{d-1}$ we have, for small contours ($\mathrm{diam}(\gamma) \leq L$), the bound

$$V(\gamma) \leq \tilde{c}L|\gamma|. \tag{7.23}$$

This means that, for

$$|B|L << \beta J \tag{7.24}$$

which holds, for $|B| \leq K$ and β large, we have

$$\rho(\gamma)\exp(|B|V(\gamma)) \leq \exp(-\beta J|\gamma|) \tag{7.25}$$

(7.25) means that the volume inside γ is controlled by the boundary term $|\gamma|$ and this can be used to show that the small contour expansion, used in section 7.1 above, converges.

Therefore, we may integrate out the small contours, block the large ones as before and see what the effective model on the next scale looks like. The main results are:

- The contour weights for the blocked large contours are (almost) as before.
- There is a new external field given by:

$$B' = L^d(B + S(B)) \tag{7.26}$$

L^d comes from the volume of one box, $S(B)$ is a free energy term coming from the small contours and is bounded by:

$$O(e^{-\beta J}). \tag{7.27}$$

This is shown using again (7.25), (7.8).
Combining (7.26), (7.27), we see that the map from B to B' is expanding, i.e. $|\frac{dB'}{dB}| > 1$ (the derivative is actually of order L^d). Using this, and the fact that, for large B, there is a convergent expansion, we can understand how the coexistence line $h(T)$ is "selected" (see [40] for details)[20]. Let $I_0 = \{B| \;\; |B| \leq K\}$ be such that, for $|B| > K$, i.e. $B \notin I_0$, we are in the large field regime, and therefore we have a unique phase (either plus or minus). Define $I_1 = \{B|B' \in I_0\}$ and, inductively, $I_{n+1} = \{B|B' \in I_n\}$. Then for $B \notin I_n$, we are in the large field region after n RG steps, and therefore,

[20] Of course, for the Ising model, one knows that $h(T) = 0$; but the argument outlined here applies to models like (7.1).

not on the transition line (to show this, one has to extend the large field analysis to the effective models produced by the RG transformations; but this is straightforward, see [40]). Since $|\frac{dB'}{dB}| > 1$, we see that the intervals I_n form a family of imbedded intervals whose length is strictly decreasing and their unique intersecton point $\beta h(T) = B(T) = \cap_n I_n$ is therefore the point of phase coexistence.

Finally, note that, in the spin language, the arguments given here justify the following picture: if we put a small external field h in a box V of volume $|V| = N^d$, then, the phase transition (occuring at $h = 0$) persists as long as $|h| < O(N^{-1})$ (see (7.24) with L replaced by N). That is, depending on whether we have plus or minus boundary conditions on V, most spins in V will be plus or minus. But, for larger V's the field dominates the effect of the boundary conditions and most spins are aligned along the direction of h.

7.3 The Renormalization Group transformation and the random field

To understand the phase diagram of the RFIM, consider the following intuitive argument due to Imry and Ma [56]. In order to test the stability of the plus (or minus) phase, compute the energetic cost due to the introduction of the opposite phase (say minus inside plus) inside a large volume $|V| = N^d$. First, there is a boundary term, basically coming from the contour energy which scales like N^{d-1}, and a random term coming from the field, which scales as $N^{d/2}$ (since it is a sum of N^d independent identically distributed random variables of mean zero). Since the latter contribution is of a random sign, one expects that it will overwhelm the boundary term unless $d-1 > \frac{d}{2}$ i.e. unless $d > 2$.

Coming to the random field, we see that, when the variance h^2 is small, the fields will be small enough so that the small contour expansion converges, with large probability. This last provision means that there is a (random) set where the field is too large for the expansion to converge. Before discussing what to do about this set, consider the result of the small contour expansion in the small field region: we have a new random field given by

$$\beta' h'_x = \beta \left(\sum_{y \in \mathbf{x}} h_y + L^d S(h_{\mathbf{x}}) \right) \tag{7.28}$$

where the sum runs over the L^d sites in the box centered at x. This sum replaces the first term, $L^d B = L^d \beta h$ in (7.26). And the blocked contours are roughly as before.

Now, compute the variance of h': since we have a sum of L^d independent random variables in the first term, multiplied by $\beta/\beta' = L^{-(d-1-\alpha)}$ (see (7.17))[21],

$$\overline{(h')^2} = L^{-(d-2-2\alpha)} h^2. \tag{7.29}$$

This is, in disguise, the scaling coming from the Imry-Ma argument, up to the $L^{-2\alpha}$ factor.

[21] It is here that we need, for contours with an interior, that β flows according to (7.17) and not (7.19).

The second term in (7.28), $L^d S(h_{\mathbf{x}})$, is small, by (7.8), and does not affect much this calculation (one can show that it a also sum of independent random variables of mean zero). Thus, the variance of the field runs down. This means that we should be able to iterate the transformation, and eventually converge to the trivial fixed point $T = h = 0$.

We still have to discuss the large field regions (for details, see [7]). There, we simply do not expand. Physically, the large fields tend to make some contours probable, and therefore it is impossible to expand around a contour-free configuration. But the main observation is that a large field can only produce a contour of a size which depends on the value of that large field. This size grows like a power of the value of the field. But large fields are unprobable, and their probability goes to zero like a Gaussian when they get large (this Gaussian, of course, controls the power growth). So, for typical configurations of the random field, one can essentially treat the large field regions, and the contours in them, as a free very dilute gas (whose "particles" are these regions). Moreover, large fields may appear on all scales since h'_x may be large, even if all the h_y's in the block $\mathbf{x}$ are small (e.g. if they all have the same sign) but the Gaussian has a smaller and smaller variance, by (7.29), thus making large fields more and more unprobable. So, using probabilistic estimates, one shows that the large fields do not destroy the results of the perturbative analysis, expressed by (7.28), (7.29).

7.4 Sketch of the proof of Theorem 1.

Consider μ'_+; using (3.2), one shows that the conditional probabilities $\mu'_+(s_V|\bar{s}_{V^c})$ can be obtained through the following limit, if it exists:

$$\mu'_+(s_V|\bar{s}_{V^c}) = \lim_{\Lambda_2 \uparrow \mathcal{L}} \lim_{\Lambda_1 \uparrow \mathbf{Z}^d} \frac{Z^+_{\Lambda_1}(s_V \vee \bar{s}_{V^c \cap \Lambda_2})}{\sum_{\widetilde{s}_V} Z^+_{\Lambda_1}(\widetilde{s}_V \vee \bar{s}_{V^c \cap \Lambda_2})}. \tag{7.30}$$

Where

$$Z^+_{\Lambda_1}(s_{\Lambda_2}) = \sum_{\sigma_{\Lambda_1}} \prod_{x \in \Lambda_2} T_x e^{-\beta H(\sigma_{\Lambda_1}|+)} \tag{7.31}$$

where $H(\sigma_{\Lambda_1}|+)$ is defined as in (2.1), but with the sum restricted to $i \in \Lambda_1$, with $\sigma_j = +1$, $\forall j \in \Lambda_1^c$. The conditional probabilities $\mu'_-(s_V|\bar{s}_{V^c})$ can be obtained by similar formulas, with $+$ replaced by $-$.

So, in order to define the renormalized Hamiltonian (see (2.7)), it is enough to control (via suitable expansions) $\log \frac{Z^+_{\Lambda_1}(s_{\Lambda_2})}{Z^+_{\Lambda_1}(s'_{\Lambda_2})}$, for configurations s_{Λ_2}, s'_{Λ_2} that are restrictions of a given configuration in $\overline{\Omega}_+$.

To see how this is done, consider first the s configuration where all $s_x = +1$ (or all $s_x = -1$). For β large, this acts as a large external field on the σ spins, and we can perform a large field expansion[22] similar to the one mentioned in Section 7.2. Now, consider typical configurations with respect to μ'_+ (or μ'_-),

[22]Note that, here, we use the fact that the decimation scale b is fixed and that we choose β large depending on b.

which, by definition, will be the configurations in $\overline{\Omega}$. Those typical configurations consist of a sea of $+$ (or $-$) spins with few islands surrounded by "contours", i.e. adjacent blocks B_x, B_y with $s_x \neq s_y$. These contours form a dilute gas and the logarithm of the partition function can approximately be written as a sum of terms, each of which is local, i.e. is the logarithm of a partition function associated to the set of spins situated in the neighbourhood of such a contour. Calling $\phi(\gamma)$ such a term, we obtain the effective interactions by a formula similar to (6.3), with $2\beta|\gamma|$ replaced by $\phi(\gamma)$ (to which one must add terms representing interactions between contours; the latter, however, are exponentially decreasing with the distance betwen contours). This allows us (via a rather complicated inductive argument) to construct interactions that satisfy the bound (6.1) for all $s \in \overline{\Omega}$ (see [11] for details).

8 Concluding Remarks

1. Although at low temperatures the pathologies can be understood as explained above, their existence leaves open some questions (like the possibility of a global RG analysis for all the values of the parameters of the model) and indicates some new interesting problems. For example, one expects to find many natural occurences of weak Gibbs states, in particular in some probabilistic cellular automata, where the stationary measures can be seen as projections of Gibbs measures [64], see also [79, 29, 62, 76] for further concrete examples. Therefore, from a theoretical point of view, it would be interesting to develop the theory of weak Gibbs states and to see which properties following from the usual definition extend to that larger framework. For a discussion of possible extensions of the standard theory, see [77, 78, 79, 27, 66, 67].

2. In many rigorous applications of the RG method (some of which were mentioned in the Introduction) one encounters a so-called "large field problem". These are regions of the lattice where the fields are large and where the renormalized Hamiltonian is not easy to control, because H tends to be large also; however, these large field regions can be controlled because they are very unprobable (since $\exp(-H)$ is small). Thus, the people who actually used the RG to prove theorems encountered a problems quite similar to the pathologies (and to the large random fields in the random field Ising model), and treated them in a way similar to the way the pathologies are treated here.

3. Maybe the last word of the (long) discussion about the pathologies is that the RG is a powerful tool, and a great source of inspiration, both for heuristic and rigorous ideas. But that does not mean that it should be taken too literally.

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