WAVE EQUATIONS WITH POINT INTERACTIONS IN FINITE ENERGY SPACES

by

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Abstract. *Given the abstract wave equation $\ddot\phi - \Delta_\alpha\phi = 0$, where Δ_α is the Laplace operator with a point interaction of strength α, we define and study $\bar W_\alpha$, the associated wave generator in the phase space of finite energy states. We prove the existence of the phase flow generated by $\bar W_\alpha$, and describe its most relevant properties with particular emphasis on the associated symplectic structure and scattering theory.*

1. INTRODUCTION.

To introduce the problem we begin with a well known example. Given the free scalar, *zero* mass, wave equation

$$\ddot{\phi} - \Delta\phi = 0\,, \tag{1.1}$$

the usual attitude in the literature is to search the solutions in the real Sobolev-Hilbert space $H^2(\mathbb{R}^3)$; in order to fix the notations we recall that $H^s(\mathbb{R}^3)$, $s \in \mathbb{R}$, is defined as the set of tempered distributions with a Fourier transform which is square integrable w.r.t. the measure with density $(1 + |k|^2)^s$. This is a standard mathematical choice but not the more natural one. In fact, equation (1.1) can be written in the first order form

$$\dot{\psi} = W\psi\,, \tag{1.2}$$

where the linear operator

$$W : H^2(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$$

is defined as

$$W\begin{pmatrix}\phi\\ \dot{\phi}\end{pmatrix} = \begin{pmatrix}0 & \mathbb{1}\\ \Delta & 0\end{pmatrix}\begin{pmatrix}\phi\\ \dot{\phi}\end{pmatrix}\,. \tag{1.3}$$

It is well known that equation (1.2) generates a strongly continuous one parameter group of evolution

$$U^t : H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\,.$$

This group group is energy preserving, i.e. there exists an energy form

$$\mathcal{E}(\phi, \dot{\phi}) = \frac{1}{2}\left(\|\dot{\phi}\|_2^2 + \|\sqrt{-\Delta}\phi\|_2^2\right)\,,$$

coinciding with the Hamiltonian of the system, preserved by the flow. Moreover U^t constitutes a group of canonical transformations w.r.t. the symplectic form

$$\omega\left((\phi, \dot{\phi}), (\varphi, \dot{\varphi})\right) := \langle\phi, \dot{\varphi}\rangle - \langle\varphi, \dot{\phi}\rangle$$

($\langle\cdot,\cdot\rangle$ denoting the usual scalar product on $L^2(\mathbb{R}^3)$) and W is nothing but the Hamiltonian vector field corresponding, via ω, to $\mathcal{E}$.

As the form of Hamiltonian $\mathcal{E}$ suggests, a more natural domain for the study of the system described by (1.3) is the space of the finite energy states, which is larger than the original one, because the first component ϕ of such a state is not necessarily square integrable, as instead is implicit in the standard Sobolev environment recalled above. This more suitable description goes as follows.

Let us define (general and more complete definitions will be given in the following section) $\bar{H}^1(\mathbb{R}^3)$ as the completion of the space $C_0^\infty(\mathbb{R}^3)$ in the norm $\|\sqrt{-\Delta}\,\phi\|_2$. Now it is possible to define the new operator $\bar{W}$ on $\bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, the Hilbert space of finite energy states, by

$$\bar{W} : \bar{H}^2(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3) \to \bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) , \qquad \bar{W}(\phi, \dot{\phi}) := (\dot{\phi}, \Delta\phi) \ . \tag{1.4}$$

where

$$\bar{H}^2(\mathbb{R}^3) := \{\phi \in \bar{H}^1(\mathbb{R}^3) : \Delta\phi \in L^2(\mathbb{R}^3)\} .$$

It is an easy matter to verify that $\bar{W}$ is a skew-adjoint operator (see e.g. [P, thm. 2.1.2], [RS, §XI.10]) so that due to Stone theorem it defines a strongly continuous one parameter group of evolution

$$\bar{U}^t : \bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$$

which is trivially energy preserving, just because the energy coincides with the norm of the Hilbert space, and the flow is given by a group of isometric operators. This procedure generalizes to the case in which one considers an abstract wave equation with a positive self-adjoint operator in the place of $-\Delta$ (see [W], [K, §8]).

Here we consider and study in detail the case in which $-\Delta$ is replaced by $-\Delta_\alpha$, the Laplace operator with a point interaction of strength α (see §2 for its precise definition), and construct the corresponding wave generator $\bar{W}_\alpha$; since $-\Delta_\alpha$ is not positive when $\alpha < 0$ one can not directly use the results appearing in [W] and [K].
The abstract wave equation corresponding to Δ_α, i.e.

$$\ddot{\phi} - \Delta_\alpha\phi = 0 , \tag{1.5}$$

was introduced for the first time in [NP1]. There, when ϕ is vector-valued and when $\alpha = -\frac{mc}{e}$ (m the phenomenological mass, c the velocity of light, e the electric charge), it is shown that (1.5) describes the evolution of the electromagnetic field self-interacting with a point particle in dipole approximation (the so called linearized Pauli-Fierz model). Another model connected with the wave equation (1.5), often studied in the fifties' and sixties' literature on exact models in quantum field theory, is the so called "pair theory" (see ([We], [CN], [HT], and references therein). The classical version of this model is the regularized version of the one we study here, and many at the time unanswered questions about its behaviour in the ultraviolet limit find their rigorous collocation in the present work.

In [NP1-3] it is also shown that the Cauchy problem associated to (1.5) is well posed on the phase space $D^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, $D^1(\mathbb{R}^3) \simeq H^1(\mathbb{R}^3) \oplus \mathbb{R}$, (refer to §2 for the definition of $D^1(\mathbb{R}^3)$) and that the corresponding strongly continuous one parameter group of evolution

$$U^t_\alpha : D^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to D^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$$

preserves the energy

$$\mathcal{E}_\alpha(\phi, \dot{\phi}) := \frac{1}{2} \left(\|\dot{\phi}\|_2^2 + F_\alpha(\phi, \phi) \right) ,$$

where F_α denotes the bilinear form corresponding to the self-adjoint operator $-\Delta_\alpha$. Therefore, analogously to the case of the free wave equation, the problem of defining (1.5) on the larger space of finite energy states naturally arises. The theory of delta point interactions was originally developped in the context of nonrelativistic quantum mechanics (see [AGHH] and references therein); this made natural to use $L^2(\mathbb{R}^3)$ as the underlying Hilbert space and so, in order to define the dynamics on the space of finite energy states, one has to modify the original definition of $-\Delta_\alpha$, to allow the elements of its domain being not square integrable. This is done in §3 where we also show (thm. 3.1) that the operators $\bar{W}_\alpha$ here constructed generate an evolution group $\bar{U}^t_\alpha$, a fact that, in the case $\alpha \le 0$, is not immediately evident. So, as an aside result, a conserved energy form exists; this form however is not positive when $\alpha < 0$, and therefore it is not suitable to define the norm of the appropriate phase space.

In the fourth section we treat the Hamiltonian formulation of the wave equations with delta interactions. Here we solve the problem by giving a complex structure $\mathcal{J}_\alpha$ commuting with the operator $\bar{W}_\alpha$. This leads to an equivalent Schrödinger-like first-order formulation which, also in view of a future quantization of the dynamical system under study, plays a key role. The complex structure before mentioned is obtained considering separately the case $\alpha \le 0$ from the other case: in the strictly negative case we obtain an invariant splitting of the phase space and the complex structure in such a way that the Hamiltonian vector field appears separately as a Schrödinger equation both on the stable and unstable part of the phase space; in particular, on the unstable subspace, which is finite dimensional, the Hamiltonian is that of an harmonic repulsor, and the Schrödinger equation is the corresponding ordinary differential equation, as expected.

The last topic treated is the scattering theory for the pair of operators $(\bar{W}_\alpha, \bar{W})$. Being the Hilbert phase spaces for $\bar{W}_\alpha$ and $\bar{W}$ different, resort has to be made to the two Hilbert space scattering theory introduced by Kato in [K]. Using then Birman invariance principle and the trace condition of Birman-Kuroda theorem, we are able to prove the existence of the Möller wave operators and their completeness (thms. 5.1 and 5.2). As a consequence of the machinery needed for the definition of wave operators, one obtains a relation (see (5.3)-(5.5)) between the evolution group (acting on the real Hilbert space of states with finite energy) generated by $\bar{W}_\alpha$ and the unitary group (acting on the complex Hilbert space $L^2_{\mathbb{C}}(\mathbb{R}^3)$, the complexification of $L^2(\mathbb{R}^3)$) generated by $\sqrt{-\Delta_\alpha}$ (in the case $\alpha < 0$ one consider only the positive part of the operator). This can be seen as a variation of the procedure applied in section 3 in the case one uses the standard complex structure on $L^2_{\mathbb{C}}(\mathbb{R}^3)$: indeed the two structures are related in a simple way (see (5.1) and (5.2)). The relations (5.3)-(5.5) could also be used to define the group $\bar{U}^t_\alpha$, the generator of which is easily seen to be $\bar{W}_\alpha$, so providing an alternative proof of the existence of the dynamics.

2. PRELIMINAIRES.

We start by giving definitions and main properties of the Sobolev type spaces needed in the sequel, and to which we made reference in the introduction. We define the family of pre-Hilbert spaces $\tilde H^s(\mathbb{R}^3)$, $s\in\mathbb{R}$, as the set of tempered distributions with a Fourier transform (denoted by $\hat{}$ or by $\mathcal{F}$) which is square integrable w.r.t. to the measure with density $|k|^{2s}$. The scalar product is defined as

$$\langle\phi_1,\phi_2\rangle_s := \int_{\mathbb{R}^3} dk\,|k|^{2s}\hat\phi_1(k)\,\hat\phi_2(k)\ .$$

Note that, when $s>0$, $H^s(\mathbb{R}^3)\subset\tilde H^s(\mathbb{R}^3)$ and $\tilde H^{-s}(\mathbb{R}^3)\subset H^{-s}(\mathbb{R}^3)$, the embeddings being continuous. Since $|k|^{-2s}$ is locally integrable for any $s<3/2$,

$$\forall s<\frac{3}{2}\,,\qquad L^2(\mathbb{R}^3,|x|^{2s}dx)\subset\mathcal{S}'(\mathbb{R}^3)\,,\qquad \tilde H^s(\mathbb{R}^3)\equiv\mathcal{F}^{-1}(L^2(\mathbb{R}^3,|k|^{2s}dk))$$

and thus $\tilde H^s(\mathbb{R}^3)$ is complete for any $s<3/2$ and coincides with the usual Riesz potential spaces (see e.g. [M, §7.1.2]).

We can then define the isomorphism $(r-s<3/2)$

$$(-\bar\Delta)^{s/2}:\tilde H^r(\mathbb{R}^3)\to\tilde H^{r-s}(\mathbb{R}^3)\,,\qquad \mathcal{F}((-\bar\Delta)^{s/2}\phi)(k):=|k|^s\hat\phi(k)\ .$$

Our notation is justified by observing that, in the case $0<r=s<3/2$, $(-\bar\Delta)^{s/2}$ coincides with the closure of $(-\Delta)^{s/2}:H^s(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$.

Since, contrarily to what happens for the usual Sobolev chain $H^s(\mathbb{R}^3)$, $\tilde H^r(\mathbb{R}^3)$ is not included in $\tilde H^s(\mathbb{R}^3)$ when $r\ge s$, we also define the sequence of spaces

$$\bar H^n(\mathbb{R}^3):=\bigcap_{k=1}^n\tilde H^k(\mathbb{R}^3)\equiv\mathcal{F}^{-1}\left(\bigcap_{k=1}^n L^2(\mathbb{R}^3,|x|^{2k}dx)\right)\ .$$

Obvioulsy $\bar H^n(\mathbb{R}^3)$ is a Hilbert space with norm

$$\|\phi\|_{\bar H^n}:=\left(\sum_{k=1}^n\|(-\bar\Delta)^{k/2}\phi\|_2^2\right)^{1/2}\ .$$

We come now to point interactions; for their general theory of we refer to [AGHH]; here we confine ourselves to the essential definitions and results. The operator $-\Delta_\alpha$ describing a standard point interaction at the origin with strength α is defined as follows. Let us introduce the dense linear subspace of $L^2(\mathbb{R}^3)$

$$D^2_\alpha(\mathbb{R}^3):=\left\{\phi\in L^2(\mathbb{R}^3)\ :\ \phi=\phi_\lambda+Q_\phi G_\lambda\ ,\phi_\lambda\in H^2(\mathbb{R}^3)\ ,\left(\alpha+\frac{\sqrt\lambda}{4\pi}\right)Q_\phi=\phi_\lambda(0)\right\}\ ,$$

where $0<\lambda\neq\mathrm{sign}(\alpha)\,(4\pi\alpha)^2$ and

$$G_\lambda(x)=\frac{e^{-\sqrt\lambda|x|}}{4\pi|x|}\ .$$

The Laplacian with a point interaction with strength α is the operator

$$-\Delta_\alpha : D^2_\alpha(\mathbb{R}^3) \to L^2(\mathbb{R}^3)\,, \qquad -\Delta_\alpha \phi := -\Delta\phi_\lambda - \lambda Q_\phi G_\lambda\ .$$

Its resolvent is given by

$$(-\Delta_\alpha + \lambda)^{-1} = (-\Delta + \lambda)^{-1} + \left(\alpha + \frac{\sqrt{\lambda}}{4\pi}\right)^{-1} G_\lambda \otimes G_\lambda\,,$$

where $G_\lambda \otimes G_\lambda(\phi) := \langle G_\lambda, \phi\rangle\, G_\lambda$.
The bilinear form corresponding to $-\Delta_\alpha$ has domain $D^1(\mathbb{R}^3)\times D^1(\mathbb{R}^3)$,

$$D^1(\mathbb{R}^3) := \{\phi \in L^2(\mathbb{R}^3)\ :\ \phi = \phi_\lambda + Q_\phi G_\lambda\ , \phi_\lambda \in H^1(\mathbb{R}^3),\ Q_\phi \in \mathbb{R}\}\,,$$

and is defined by

$$F_\alpha(\phi,\varphi) := \langle(-\Delta+\lambda)^{1/2}\phi_\lambda, (-\Delta+\lambda)^{1/2}\varphi_\lambda\rangle - \lambda\langle\phi,\varphi\rangle + \left(\alpha + \frac{\sqrt{\lambda}}{4\pi}\right) Q_\phi Q_\varphi$$

(see [T]). Both the expressions for F_α and $-\Delta_\alpha$ contain the arbitrary parameter λ, but contrarily to the appearance, they do not depend on it. Indeed (following [DFT,§2]) the operator and form domain can be defined in the following alternative, and more useful, way. Note that, since $\forall\,\lambda > 0$

$$G_\lambda \in L^2(\mathbb{R}^3)\,, \quad G - G_\lambda \in \bar{H}^2(\mathbb{R}^3)\,, \quad (G - G_\lambda)\,(0) = \frac{\sqrt{\lambda}}{4\pi}\,,$$

where

$$G(x) = \frac{1}{4\pi|x|}\ ,$$

defining

$$\phi_{\text{reg}} := \phi_\lambda + Q_\phi(G_\lambda - G) \in \bar{H}^2(\mathbb{R}^3)$$

we have equivalently

$$D^2_\alpha(\mathbb{R}^3) = \left\{\phi \in L^2(\mathbb{R}^3)\ :\ \phi = \phi_{\text{reg}} + Q_\phi G,\ \phi_{\text{reg}} \in \bar{H}^2(\mathbb{R}^3),\ Q_\phi \in \mathbb{R},\ \alpha\, Q_\phi = \phi_{\text{reg}}(0)\right\}\ .$$

Correspondingly, the form domain is

$$D^1(\mathbb{R}^3) = \left\{\phi \in L^2(\mathbb{R}^3)\ :\ \phi = \phi_{\text{reg}} + Q_\phi G,\ \phi_{\text{reg}} \in \bar{H}^1(\mathbb{R}^3),\ Q_\phi \in \mathbb{R}\right\}$$

so that, with this definition, the singular part of the field is exactly Coulombian. However such a singular field is *not* in the configuration space of the system. The removal of this incongruence will lead, in the following section, to the introduction of the operator $\bar{W}_\alpha$.

With the domains so given we can redefine the operator and the form as

$$-\Delta_\alpha \phi = -\bar{\Delta}\phi_{\rm reg}$$

and

$$F_\alpha(\phi,\varphi) = \langle(-\bar{\Delta})^{1/2}\phi_{\rm reg}, (-\bar{\Delta})^{1/2}\varphi_{\rm reg}\rangle + \alpha\, Q_\phi Q_\varphi \ .$$

Now it is well known (see [AGHH, Chap. I.1]) that $-\Delta_\alpha$ is a selfadjoint operator in $L^2(\mathbb{R}^3)$. An important property is that $-\Delta_\alpha$ is positive only for $\alpha \geq 0$, whereas for $\alpha < 0$ it is only bounded from below; more precisely if $\alpha \geq 0$ (*repulsive* delta interactions) the spectrum of the operator is absolutely continuous and coinciding with $[0, +\infty)$; if $\alpha < 0$ (*attractive* delta interactions) the spectrum is given by $\{-\lambda_0\} \cup [0, +\infty)$, where $-\lambda_0 = -(4\pi\alpha)^2$ is an isolated negative eigenvalue, and the remaining part of the spectrum is absolutely continuous. In the Schrödinger case this eigenvalue corresponds to a bound state, while in the wave case where one has a second order equation in time, it leads to unstable solutions exponentially running away in the past or in the future (see [NP1-3] and reference therein for the meaning of these well known runaway solutions in classical electrodynamics).

We now come to the wave generator associated to the standard delta operator. Its domain and action are given by

$$W_\alpha : D^2_\alpha(\mathbb{R}^3) \oplus D^1(\mathbb{R}^3) \to D^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \ ,$$

$$W_\alpha \begin{pmatrix} \phi \\ \dot\phi \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} \\ \Delta_\alpha & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \dot\phi \end{pmatrix} \ . \tag{2.6}$$

By considering the Hilbert space structure given by $D^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \simeq H^1(\mathbb{R}^3) \oplus \mathbb{R} \oplus L^2(\mathbb{R}^3)$ this operator is the generator of a strongly continuous group of operators

$$U^t_\alpha : D^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to D^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \ .$$

In the case $\alpha \geq 0$ this is an immediate conseguence of the skew-adjointness of W_α with respect to the positive energy scalar product on the phase space given by

$$\langle\langle(\phi,\dot\phi),(\varphi,\dot\varphi)\rangle\rangle_\alpha := \langle\dot\phi,\dot\varphi\rangle + F_\alpha(\phi,\varphi) \ . \tag{2.7}$$

More precisely one has the following result (the proof being a straighforward calculation):

Theorem 2.1. *For any $\alpha \in \mathbb{R}$, with respect to the scalar product $\langle\langle\cdot,\cdot\rangle\rangle_\beta$, $\beta \geq 0$, one has*

$$D(W^*_\alpha) = D(W_\beta)$$

and

$$W^*_\alpha(\phi,\dot\phi) = -\left(\dot\phi_{\rm reg} + \frac{\alpha}{\beta}\, Q_{\dot\phi} G, \bar{\Delta}\phi_{\rm reg}\right) \ , \qquad \beta > 0 \ ,$$

$$W_\alpha^*(\phi,\dot\phi)=-\left(\dot\phi_{\rm reg}+Q_{\dot\phi}G,\bar\Delta\phi_{\rm reg}\right)\equiv W_\beta(\phi,\dot\phi)\,,\qquad \beta=0=\alpha\ .$$

In the case $\alpha<0$ the operator W_α is readily proven to be a generator by considering the operator

$$W_{\alpha,\lambda}(\phi,\dot\phi):=(\dot\phi,(\Delta_\alpha-\lambda)\phi_{\rm reg})\,,$$

where $\lambda>\lambda_0$. This, being now $-\Delta_\alpha+\lambda$ positive, is skew-adjoint with respect to the scalar product

$$\langle\langle(\phi,\dot\phi),(\varphi,\dot\varphi)\rangle\rangle_{\alpha,\lambda}:=\langle\dot\phi,\dot\varphi\rangle+F_\alpha(\phi,\varphi)+\lambda\langle\phi,\varphi\rangle\ , \tag{2.8}$$

and so it generates a group of isometries (w.r.t. the Hilbert structure given by (2.8)). The original operator W_α, being a perturbation of the previous one by a bounded operator, also generates a strongly continuous group of operators on the phase space (which however are no more isometries).

We now describe an alternative way to prove that W_α, $\alpha<0$, is a generator. Such a different method will play a key role in the next sections. As we already said before in the case $\alpha<0$ the self-adjoint operator $-\Delta_\alpha$ has a negative eigenvalue $-\lambda_0$ (with corresponding normalized eigenvector $4\pi\sqrt{-2\alpha}\,G_{\lambda_0}$) which gives rise to the runaway solutions of the wave equation associated to W_α. Proceeding as in [NP3, §4] (note that there we worked with the different decomposition $\phi=\phi_{\lambda_0}+Q_\phi G_{\lambda_0}$) we consider the linear operator

$$W_\alpha^{\rm nr}:[D^2_\alpha(\mathbb{R}^3)]_{\rm nr}\oplus[D^1(\mathbb{R}^3)]_{\rm nr}\to[D^1(\mathbb{R}^3)]_{\rm nr}\oplus[L^2(\mathbb{R}^3)]_{\rm nr}\,,$$

$$W_\alpha\begin{pmatrix}\phi\\ \dot\phi\end{pmatrix}=\begin{pmatrix}0 & \mathbb{1}\\ \Delta_\alpha^{\rm nr} & 0\end{pmatrix}\begin{pmatrix}\phi\\ \dot\phi\end{pmatrix}\,,$$

where, given any vector subspace $\mathcal{V}\subseteq L^2(\mathbb{R}^3)$, we have defined the corresponding "non runaway" subspace $[\mathcal{V}]_{\rm nr}$ by

$$[\mathcal{V}]_{\rm nr}:=\{\ \phi\in\mathcal{V}\ :\ \langle\phi,G_{\lambda_0}\rangle=0\ \}\,,$$

and

$$\Delta_\alpha^{\rm nr}:=(\Delta_\alpha)_{|[D^2_\alpha(\mathbb{R}^3)]_{\rm nr}}\equiv P_{\rm nr}\cdot(\Delta_\alpha)_{|[D^2_\alpha(\mathbb{R}^3)]_{\rm nr}}\ ,$$

$P_{\rm nr}$ being the orthogonal projector onto $[L^2(\mathbb{R}^3)]_{\rm nr}$. By simple calculations one has (see [NP3,§4])

$$[D^1(\mathbb{R}^3)]_{\rm nr}=\left\{\phi\in D^1(\mathbb{R}^3)\ :\ Q_\phi=-8\pi\sqrt{\lambda_0}\,\langle\phi_{\rm reg},G_{\lambda_0}\rangle\right\}$$
$$[D^2_\alpha(\mathbb{R}^3)]_{\rm nr}=\left\{\phi\in D^2_\alpha(\mathbb{R}^3)\ :\ \phi_{\rm reg}(0)=-2\lambda_0\,\langle\phi_{\rm reg},G_{\lambda_0}\rangle\right\}$$

and

$$\begin{aligned}\Delta_\alpha^{\rm nr}\phi=&\bar\Delta\phi_{\rm reg}-8\pi\sqrt{\lambda_0}\,\langle\bar\Delta\phi_{\rm reg},G_{\lambda_0}\rangle G_{\lambda_0}\\ =&\bar\Delta\phi_{\rm reg}+8\pi\sqrt{\lambda_0}\,\langle(-\bar\Delta+\lambda_0)\phi_{\rm reg},G_{\lambda_0}\rangle G_{\lambda_0}-8\pi\sqrt{\lambda_0}\,\lambda_0\,\langle\phi_{\rm reg},G_{\lambda_0}\rangle G_{\lambda_0}\\ =&\bar\Delta\phi_{\rm reg}+12\pi\sqrt{\lambda_0}\,\phi_{\rm reg}(0)\,G_{\lambda_0}\,.\end{aligned}$$

The *non-negative* quadratic form associated to $-\Delta_\alpha^{\rm nr}$ is then

$$F_\alpha^{\rm nr}(\phi,\varphi)=\langle(-\bar\Delta)^{1/2}\phi_{\rm reg},(-\bar\Delta)^{1/2}\varphi_{\rm reg}\rangle+8\pi\,\lambda_0^{3/2}\langle\phi_{\rm reg},G_{\lambda_0}\rangle\langle\varphi_{\rm reg},G_{\lambda_0}\rangle$$

and $W_\alpha^{\rm nr}$ is skew-adjoint w.r.t. the scalar product

$$\langle\langle(\phi,\dot\phi),(\varphi,\dot\varphi)\rangle\rangle_\alpha^{\rm nr}:=\langle\dot\phi,\dot\varphi\rangle+F_\alpha^{\rm nr}(\phi,\varphi)\ .$$

The strongly continuous one parameter group of evolution generated by $W_\alpha^{\rm nr}$ preserves the *non-negative* energy

$$\mathcal{E}_\alpha^{\rm nr}(\phi,\dot\phi):=\frac{1}{2}\,\Big(\langle\dot\phi,\dot\phi\rangle+F_\alpha^{\rm nr}(\phi,\phi)\Big)$$

with coincides with the Hamiltonian of the sistem w.r.t. the symplectic form ω (see [NP3, thm. 4.2] for an alternative Hamiltonian picture).

Since $\Delta_\alpha G_{\lambda_0}=\lambda_0 G_{\lambda_0}$, and

$$D^2_\alpha(\mathbb{R}^3)\simeq[D^2_\alpha(\mathbb{R}^3)]_{\rm nr}\oplus\mathbb{R}\,,\quad D^1(\mathbb{R}^3)\simeq[D^1(\mathbb{R}^3)]_{\rm nr}\oplus\mathbb{R}\,,\quad L^2(\mathbb{R}^3)\simeq[L^2(\mathbb{R}^3)]_{\rm nr}\oplus\mathbb{R}\,,$$

we can write

$$W_\alpha=W_\alpha^{\rm nr}\times\Lambda_0\,,$$

where

$$\Lambda_0:\mathbb{R}^2\to\mathbb{R}^2\,,\qquad\Lambda_0(x,\dot x):=(\dot x,\lambda_0\,x)\,.$$

Therefore W_α, $\alpha<0$, is a generator and

$$U_\alpha^t=e^{tW_\alpha^{\rm nr}}\times e^{t\Lambda_0}\,.$$

Here and below, given two linear operators $A_1:D(A_1)\to\mathfrak{H}_1$ and $A_2:D(A_2)\to\mathfrak{H}_2$, $A_1\times A_2:D(A_1)\times D(A_2)\to\mathfrak{H}_1\oplus\mathfrak{H}_2$ denotes the the linear operator defined by

$$A_1\times A_2\,(\phi_1,\phi_2):=(A_1\phi_1,A_2\phi_2)\,.$$

In conclusion, for any $\alpha\in\mathbb{R}$, U_α^t is a group of canonical transformation w.r.t. the symplectic form ω, and W_α is the Hamiltonian vector field corresponding to the energy

$$\mathcal{E}_\alpha(\phi,\dot\phi)=\frac{1}{2}\,\Big(\|\dot\phi\|_2^2+F_\alpha(\phi,\phi)\Big)\equiv\mathcal{E}(\phi_{\rm reg},\dot\phi)+\frac{\alpha}{2}\,Q_\phi^2\ .$$

Let us remark that the flow U_α^t can be explicitly calculated (see [NP1, thm. 3.1]).

3. THE OPERATOR $\bar{W}_\alpha$.

Now we would like to mimic the construction of the energy space for the usual wave generator and the extension of the operator itself, to the case of delta point interactions. To this end, let us define the linear operator

$$\bar{W}_\alpha : \bar{D}^2_\alpha(\mathbb{R}^3)\oplus D^1(\mathbb{R}^3) \to \bar{D}^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)\,, \qquad \bar{W}_\alpha(\phi,\dot{\phi}) := (\dot{\phi}, \bar{\Delta}\phi_{\rm reg})\,,$$

where

$$\bar{D}^2_\alpha(\mathbb{R}^3) := \left\{\phi=\phi_{\rm reg}+Q_\phi G,\ \phi_{\rm reg}\in \bar{H}^2(\mathbb{R}^3),\ Q_\phi\in\mathbb{R},\ \alpha\, Q_\phi=\phi_{\rm reg}(0)\right\}\,,$$

$$\bar{D}^1(\mathbb{R}^3) := \left\{\phi=\phi_{\rm reg}+Q_\phi G,\ \phi_{\rm reg}\in \bar{H}^1(\mathbb{R}^3),\ Q_\phi\in\mathbb{R}\right\}\,.$$

Analogously to the free case $\bar{D}^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$ describes now the space of finite energy states. Moreover the Coulombian singularity G *is* now an element of the configuration space $\bar{D}^1(\mathbb{R}^3)$.

Introducing the Hilbert space structure given by $\bar{D}^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)\simeq \bar{H}^1(\mathbb{R}^3)\oplus\mathbb{R}\oplus L^2(\mathbb{R}^3)$ we want now to show that also in this case $\bar{W}_\alpha$ generates a strongly continuous one parameter group of evolution. When $\alpha>0$, considering, similarly to the case of W_α, the scalar product

$$\langle\langle(\phi,\dot{\phi}),(\varphi,\dot{\varphi})\rangle\rangle_\alpha := \langle\dot{\phi},\dot{\varphi}\rangle + \langle(-\bar{\Delta})^{1/2}\phi_{\rm reg},(-\bar{\Delta})^{1/2}\varphi_{\rm reg}\rangle + \alpha\, Q_\phi Q_\varphi\ ,$$

one can prove that $\bar{W}_\alpha$ is skew-adjoint and so it is a generator. Note that when $\alpha=0$, contrarily to situation discussed in the previous paragraph, $\langle\langle\cdot,\cdot\rangle\rangle_\alpha$ is no more a scalar product, being annihilated by the zero energy eigenvector $(G,0)$ (this fact has to be compared with the presence of a zero energy resonance for $-\Delta_0$). In order to show that also in the case $\alpha\le 0$ $\bar{W}_\alpha$ is a generator one can not use the same strategy as before consisting in a translation, since the scalar product (2.8) is now ill-defined, $\bar{D}^1(\mathbb{R}^3)$ being not a subset of $L^2(\mathbb{R}^3)$. So the perturbation argument fails and we are forced to proceed in an alternative way. The decomposition of W_α, $\alpha<0$, introduced at the end of the previous section is our starting point: we simply extend it to the case of $\bar{W}_\alpha$. Therefore we define, when $\alpha<0$,

$$[\bar{D}^2_\alpha(\mathbb{R}^3)]_{\rm nr} = \left\{\phi\in\bar{D}^2_\alpha(\mathbb{R}^3)\ :\ \phi_{\rm reg}(0)=-2\lambda_0\,\langle\phi_{\rm reg},G_{\lambda_0}\rangle\right\}$$
$$[\bar{D}^1(\mathbb{R}^3)]_{\rm nr} = \left\{\phi\in\bar{D}^1(\mathbb{R}^3)\ :\ Q_\phi=-8\pi\sqrt{\lambda_0}\,\langle\phi_{\rm reg},G_{\lambda_0}\rangle\right\}\,,$$

and

$$\bar{W}^{\rm nr}_\alpha : [\bar{D}^2_\alpha(\mathbb{R}^3)]_{\rm nr}\oplus[D^1(\mathbb{R}^3)]_{\rm nr}\to[\bar{D}^1(\mathbb{R}^3)]_{\rm nr}\oplus[L^2(\mathbb{R}^3)]_{\rm nr}\,,$$
$$\bar{W}^{\rm nr}_\alpha(\phi,\dot{\phi}) := (\dot{\phi},\bar{\Delta}^{\rm nr}_\alpha\phi)\,,$$

$$\bar\Delta^{\rm nr}_\alpha : [\bar D^2_\alpha(\mathbb{R}^3)]_{\rm nr} \to [L^2(\mathbb{R}^3)]_{\rm nr}\,,$$
$$\bar\Delta^{\rm nr}_\alpha\phi := \bar\Delta\phi_{\rm reg} + 12\pi\sqrt{\lambda_0}\,\phi_{\rm reg}(0)\,G_{\lambda_0}\,.$$

With such definitions $\bar W^{\rm nr}_\alpha$ results skew-adjoint with respect to the scalar product on $[\bar D^1(\mathbb{R}^3)]_{\rm nr}\oplus[L^2(\mathbb{R}^3)]_{\rm nr}$ given by

$$\begin{aligned}\langle\langle(\phi,\dot\phi),(\varphi,\dot\varphi)\rangle\rangle^{\rm nr}_\alpha :=& \langle\dot\phi,\dot\varphi\rangle + \langle(-\bar\Delta)^{1/2}\phi_{\rm reg},(-\bar\Delta)^{1/2}\varphi_{\rm reg}\rangle\\ &+8\pi\,\lambda_0^{3/2}\langle\phi_{\rm reg},G_{\lambda_0}\rangle\langle\varphi_{\rm reg},G_{\lambda_0}\rangle\,.\end{aligned}$$

Moreover, since

$$\bar D^2_\alpha(\mathbb{R}^3)\simeq[\bar D^2_\alpha(\mathbb{R}^3)]_{\rm nr}\oplus\mathbb{R}\,,\quad \bar D^1(\mathbb{R}^3)\simeq[D^1(\mathbb{R}^3)]_{\rm nr}\oplus\mathbb{R}\,,$$

similarly to the case of W_α we have

$$\bar W_\alpha = \bar W^{\rm nr}_\alpha\times\Lambda_0\,.$$

For the case $\alpha=0$ a similar decomposition is possible by using the projection onto the subspace orthogonal to the eigenvector $(G,0)$. Indeed, defining

$$\bar W_{(0)} : \bar H^2_0(\mathbb{R}^3)\oplus D^1(\mathbb{R}^3)\to\bar H^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)\,,\qquad \bar W_{(0)}(\phi,\dot\phi) := (\dot\phi_{\rm reg},\bar\Delta\phi)\,,$$

where $\bar H^2_0(\mathbb{R}^3) := \big\{\phi\in\bar H^2(\mathbb{R}^3)\ :\ \phi(0)=0\big\}$, the operator $\bar W_{(0)}$ is skew-adjoint with respect to the scalar product

$$\langle\langle(\phi,\dot\phi),(\varphi,\dot\varphi)\rangle\rangle_{(0)} := \langle\dot\varphi,\dot\varphi\rangle + \langle(-\bar\Delta)^{1/2}\phi,(-\bar\Delta)^{1/2}\varphi\rangle$$

and, since $\bar D^1(\mathbb{R}^3)\simeq\bar H^1(\mathbb{R}^3)\oplus\mathbb{R}$, the following decomposition holds:

$$\bar W_0 = \bar W_{(0)}\times 0\,.$$

We can now state our result regarding the existence of dynamics:

Theorem 3.1. *$\bar W_\alpha$ is a closed operator coinciding with the closure of W_α. It generates a strongly continuous group of evolution*

$$\bar U^t_\alpha : \bar D^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)\to\bar D^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)\,.$$

which can be defined as

$$\bar U^t_\alpha(\phi,\dot\phi) = \lim_{n\uparrow\infty}U^t_\alpha(\phi_n,\dot\phi)\,,$$

where $\{\phi_n\}_1^\infty\subset D^1(\mathbb{R}^3)$ is any sequence such that $\phi_n\to\phi$ in $\bar D^1(\mathbb{R}^3)$.

Proof. $\bar W_\alpha$ is a generator since it is skew-adjoint when $\alpha>0$ and $\bar W_0=\bar W_{(0)}\times 0$, $\bar W_\alpha=\bar W^{\rm nr}_\alpha\times\Lambda_0$, $\alpha<0$, where both $\bar W_{(0)}$ and $\bar W^{\rm nr}_\alpha$ are skew-adjoint. Therefore $\bar W_\alpha$ is closed. By its definition $\bar{W}_\alpha$ is equal to W_α on $D^2_\alpha(\mathbb{R}^3)\oplus D^1(\mathbb{R}^3)$ and so it coincides with the closure of W_α if $D^2_\alpha(\mathbb{R}^3)\oplus D^1(\mathbb{R}^3)$ is a core. This is proven as follows: analogously to the case of W_α, any $\phi\in\bar{D}^2_\alpha(\mathbb{R}^3)$ admits the representation

$$\phi=\phi_\lambda+Q_\phi G_\lambda$$

where

$$\phi_\lambda=\phi_{\mathrm{reg}}-Q_\phi(G_\lambda-G)\in\bar{H}^2(\mathbb{R}^3)$$

and

$$\left(\alpha+\frac{\sqrt{\lambda}}{4\pi}\right)Q_\phi=\phi_\lambda(0)\ .$$

Consider then a sequence ϕ^n_λ in $H^2(\mathbb{R}^3)$ and define

$$\phi_n:=\phi^n_\lambda+Q_nG_\lambda\ \in D^2_\alpha(\mathbb{R}^3)\ ,$$

where

$$Q_n:=\left(\alpha+\frac{\sqrt{\lambda}}{4\pi}\right)^{-1}\phi^n_\lambda(0)\ .$$

Now if ϕ^n_λ converges in $\bar{H}^2(\mathbb{R}^3)$ to ϕ_λ, we have that Q_n converges to Q_ϕ, thanks to the continuous embedding of $\bar{H}^2(\mathbb{R}^3)$ in $C^0_b(\mathbb{R}^3)$ (see e.g. [M, §5.6.2]).

Being $\bar{W}_\alpha$ equal to W_α on $D^2_\alpha(\mathbb{R}^3)\oplus D^1(\mathbb{R}^3)$ the same is true for the corresponding groups of evolution. Since $D^2_\alpha(\mathbb{R}^3)\oplus D^1(\mathbb{R}^3)$ is dense in $D^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$ which is continuously embedded in $\bar{D}^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$, one has the equality $\bar{U}^t_\alpha(\phi,\dot\phi)=U^t_\alpha(\phi,\dot\phi)$ for any $\phi\in D^1(\mathbb{R}^3)$. The proof is then concluded by the denseness of $D^1(\mathbb{R}^3)$ in $\bar{D}^1(\mathbb{R}^3)$. $\square$

Let us remark, since $\bar{W}_\alpha$ is the closure of W_α, our construction coincides, in the case $\alpha\ge 0$, with the abstract one given in [W] (see also [K, §8] for a similar construction). Moreover, since $W_\alpha=W^{\mathrm{nr}}_\alpha\times\Lambda_0$ when $\alpha<0$, one has that $\bar{W}^{\mathrm{nr}}_\alpha$ is the closure of W^{nr}_α.

4. THE SYMPLECTIC STRUCTURE.

The standard symplectic structure recalled in the introduction,

$$\omega\left((\phi,\dot\phi),(\varphi,\dot\varphi)\right):=\langle\phi,\dot\varphi\rangle-\langle\varphi,\dot\phi\rangle$$

it is not well defined on the phase space finite energy states, i.e $\bar{D}^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$. This requires a different approach to the Hamiltonian description of the dynamical system described in the previous paragraph. The problem shows up already in the case of the free wave equation, with the phase space $\bar{H}^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$; usually in the standard literature on infinite dimensional Hamiltonian systems (see e.g. [CM]) only the easier case of the free field with strictly positive mass is explicitely discussed.

We recall that (see [S], [CM]) when the Hilbert space carries a complex structure $\mathcal{J}$, it is possible to complexify the space in such a way that the imaginary part of the complex scalar product turns out to be a symplectic form, while the real part is the old (real) scalar product, coinciding with the energy. Any skew-adjoint operator A *commuting* with $\mathcal{J}$ remains skew-adjoint within the complex Hilbert space, so that $iA := \mathcal{J} \cdot A$ is self-adjoint. Therefore, since $e^{tA} \equiv e^{-it(iA)}$, A generates a strongly continuous group of unitary (hence symplectic) transformations. More precisely, collecting the known results on the subject (see e.g. [CM, §2.6, §2.7], [S, chap. II]), we state the following:

Theorem 4.1. *Let A be an injective skew-adjoint operator on the real Hilbert space $\mathfrak{H}$ with inner product $\langle\cdot,\cdot\rangle$. Then the closure of the densely defined linear operator*

$$A \cdot (-A^2)^{-1/2} : \mathrm{Range}\,(A) \to \mathfrak{H}$$

defines a complex structure $\mathcal{J}$ commuting with A. Defining, for any $\psi \in \mathfrak{H}$, the multiplication by the complex number i as

$$i\,\psi := \mathcal{J}\psi\,,$$

$\mathfrak{H}$ becomes a complex Hilbert space with Hermitean inner product

$$[\psi_1, \psi_2] := \langle\psi_1, \psi_2\rangle + i\,\langle\psi_1, \mathcal{J}\psi_2\rangle\ .$$

The strongly continuous one parameter group $U^t := e^{tA}$ is a group of symplectic transformations relatively to the symplectic form

$$\Omega(\psi_1, \psi_2) := \mathrm{Im}\,[\psi_1, \psi_2]$$

and the linear vector field

$$A : D(A) \to \mathfrak{H}$$

is Hamiltonian with associated densely defined Hamiltonian function

$$\mathcal{H} : D(\mathcal{Q}) \to \mathbb{R}\,, \qquad \mathcal{H}(\psi) := \frac{1}{2}\,\mathcal{Q}(\psi, \psi)\,,$$

where $\mathcal{Q}$ denotes the quadratic form associated to the self-adjoint operator $\mathcal{J} \cdot A$.

A wide class of examples is obtained by the following construction, which is a simple consequence of the above theorem. Let us consider an injective nonnegative self-adjoint operator

$$B : D\,(B) \to \mathfrak{K}$$

on the Hilbert space $\mathfrak{K}$ and let us consider the closure of

$$\begin{pmatrix} 0 & \mathbb{1} \\ -B^2 & 0 \end{pmatrix}$$

on the Hilbert space $\mathfrak{H} = \bar{D}(B) \oplus \mathfrak{K}$, where $\bar{D}(B)$ is the completion of $D(B)$ with respect to the norm $\|u\|_B := \|Bu\|_{\mathfrak{K}}$. In the case in which this closure is injective, the complex structure $\mathcal{J}$ given by the previous theorem is

$$\mathcal{J}_B : \bar{D}(B) \oplus \mathfrak{K} \to \bar{D}(B) \oplus \mathfrak{K}\,, \qquad \mathcal{J}_B(u,v) = \left(\bar{B}^{-1}v, \bar{B}u\right)\,,$$

where $\bar{B}$ and $\bar{B}^{-1}$ are the closures respectively of B and its inverse $B^{-1} : \operatorname{Range} B \to \mathfrak{K}$. This allows to endow $\mathfrak{H}$ with the structure of a complex Hilbert space, which we continue to call $\mathfrak{H}$; precisely, defined a generic element as $w := (u,v) \in \bar{D}(B) \oplus \mathfrak{K}$, the Hermitean scalar product in $\mathfrak{H}$ is

$$[w_1, w_2]_B := \langle\langle w_1, w_2\rangle\rangle_B + i\,\langle\langle w_1, \mathcal{J}_B w_2\rangle\rangle_B\,,$$

where

$$\langle\langle w_1, w_2\rangle\rangle_B := \langle \bar{B}u_1, \bar{B}u_2\rangle + \langle v_1, v_2\rangle\,.$$

On the product $\mathfrak{H} \times \mathfrak{H}$ we have the symplectic form

$$\Omega_B : \bar{D}(B) \oplus \mathfrak{K} \times \bar{D}(B) \oplus \mathfrak{K} \to \mathbb{R} \qquad \Omega_B\,(w_1, w_2) = \langle\langle w_1, \mathcal{J} w_2\rangle\rangle\,.$$

With respect to the complex variable w the wave equation

$$\ddot{u} = -\bar{B}^2 u \tag{4.1}$$

assumes the Schrödinger-like form

$$-i\,\dot{w} = \bar{B}w\,.$$

Moreover such an equation is Hamiltonian w.r.t. the symplectic form Ω_B and the densely defined Hamiltonian function

$$\mathcal{H}_B : D(\bar{B}^{3/2}) \times D(\bar{B}^{1/2}) \to \mathbb{R}\,, \quad \mathcal{H}_B(w) = \frac{1}{2}\,\left(\|\bar{B}^{1/2}v\|^2 + \|\bar{B}^{3/2}u\|^2\right)\,,$$

where the operator $\bar{B}^s$ is defined as the closure of B^s.

The strongly continuous symplectic group of operators obtained by solving the equation (4.1) preserves the energy $\mathcal{E}_B(u,v) := \frac{1}{2}\,[w,w]_B$.

An immediate example is given by the choice $B = \sqrt{(-\Delta)} : H^1(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, corresponding to the standard wave equation and leading to the complex structure

$$\mathcal{J} : \bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\,,$$

$$\mathcal{J}(\phi, \dot{\phi}) := \left((-\bar{\Delta})^{-1/2}\dot{\phi}, -(-\bar{\Delta})^{1/2}\phi\right)\,.$$

Other concrete examples are obtained when the operator B^2 is a point interaction, more precisely $B = \sqrt{(-\Delta_\alpha)} : D^1(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ with $\alpha > 0$. In this case the corresponding complex structure is given by

$$\mathcal{J}_\alpha : \bar{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \bar{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\,,$$

$$\mathcal{J}_\alpha(\phi, \dot{\phi}) := \left((-\bar{\Delta}_\alpha)^{-1/2}\dot{\phi}, -(-\bar{\Delta}_\alpha)^{1/2}\phi\right)\,.$$

The same procedure is not directly applicable to the cases $\alpha \le 0$, due to the lack of skewadjointness and injectivity for the operator $\bar{W}_\alpha$. A natural way out is to project the operator on the subspace of absolute continuity and to apply the abstract scheme to this projection. This works well for the case $\alpha < 0$, whereas the case $\alpha = 0$ deserves a different treatment.
Here are the details of the two constructions.

In the case $\alpha < 0$ we have seen in section 3 that $\bar{W}_\alpha^{\rm nr}$ is skew-adjoint, w.r.t. the scalar product $\langle\langle\cdot,\cdot\rangle\rangle_\alpha^{\rm nr}$, and one-to-one. Therefore we can apply to it thm. 4.1 (or better the successive example with $B = \sqrt{-\Delta_\alpha^{\rm nr}}$) obtaining the complex structure $\mathcal{J}_\alpha$ commuting with W_α, $\alpha < 0$, defined as

$$\mathcal{J}_\alpha := \mathcal{J}_\alpha^{\rm nr} \times j\,,$$

where

$$\mathcal{J}_\alpha^{\rm nr} : [\bar{D}^1(\mathbb{R}^3)]_{\rm nr} \oplus [L^2(\mathbb{R}^3)]_{\rm nr} \to [\bar{D}^1(\mathbb{R}^3)]_{\rm nr} \oplus [L^2(\mathbb{R}^3)]_{\rm nr}\,,$$

$$\mathcal{J}_\alpha^{\rm nr}(\phi, \dot{\phi}) = \left((-\bar{\Delta}_\alpha^{\rm nr})^{-1/2}\dot{\phi}, -(-\bar{\Delta}_\alpha^{\rm nr})^{1/2}\phi\right)$$

and

$$j : \mathbb{R}^2 \to \mathbb{R}^2\,, \qquad j(x, \dot{x}) := (\dot{x}, -x)\,.$$

Here, analogously to the case $\alpha > 0$, the linear operators

$$(-\bar{\Delta}_\alpha^{\rm nr})^{1/2} : [\bar{D}^1(\mathbb{R}^3)]_{\rm nr} \to [L^2(\mathbb{R}^3)]_{\rm nr}\,,$$

and

$$(-\bar{\Delta}_\alpha^{\rm nr})^{-1/2} : [L^2(\mathbb{R}^3)]_{\rm nr} \to [\bar{D}^1(\mathbb{R}^3)]_{\rm nr}\,,$$

are defined as the closures of

$$(-\Delta_\alpha^{\rm nr})^{1/2} : [D^1(\mathbb{R}^3)]_{\rm nr} \subset [\bar{D}^1(\mathbb{R}^3)]_{\rm nr} \to [L^2(\mathbb{R}^3)]_{\rm nr}$$

and

$$(-\Delta_\alpha^{\rm nr})^{-1/2} : \text{Range } \left((-\Delta_\alpha^{\rm nr})^{1/2}\right) \subset [L^2(\mathbb{R}^3)]_{\rm nr} \to [D^1(\mathbb{R}^3)]_{\rm nr}$$

respectively.

We have then the complex Hilbert space of the couples

$$(\psi, z) := \left((\phi, \dot{\phi}), (x, \dot{x})\right) \in [\bar{D}^1(\mathbb{R}^3)]_{\rm nr} \oplus [L^2(\mathbb{R}^3)]_{\rm nr} \oplus \mathbb{R}^2\,,$$

with the Hermitean scalar product

$$[(\psi_1, z_1), (\psi_2, z_2)]_\alpha := \langle\langle \psi_1, \psi_2 \rangle\rangle_\alpha^{\mathrm{nr}} + (z_1, z_2) + i\, \left(\langle\langle \psi_1, \mathcal{J}_\alpha^{\mathrm{nr}} \psi_2 \rangle\rangle_\alpha^{\mathrm{nr}} + (z_1, j z_2) \right) ,$$

where

$$(z_1, z_2) := \dot{x}_1 \dot{x}_2 + x_1 x_2 \,.$$

The associated symplectic form is

$$\Omega_\alpha : [\bar{D}^1(\mathbb{R}^3)]_{\mathrm{nr}} \oplus [L^2(\mathbb{R}^3)]_{\mathrm{nr}} \oplus \mathbb{R}^2 \to \mathbb{R},$$

$$\Omega_\alpha\left((\psi_1, z_1), (\psi_2, z_2)\right) = \langle\langle \psi_1, \mathcal{J}_\alpha^{\mathrm{nr}} \psi_2 \rangle\rangle_\alpha^{\mathrm{nr}} + (z_1, j z_2) \ .$$

With respect to the complex variables (ψ, z) the wave equation corresponding to W_α takes the Schrödinger-like form

$$\begin{cases} -i\,\dot{\psi} =(-\bar{\Delta}_\alpha^{\mathrm{nr}})^{1/2}\psi \\ -i\,\dot{z} =L_0 z \end{cases} \qquad L_0 := \begin{pmatrix} -\lambda_0 & 0 \\ 0 & 1 \end{pmatrix} ,$$

and such an equation is Hamiltonian w.r.t. the symplectic form Ω_α and the densely defined Hamiltonian function

$$\mathcal{H}_\alpha : D(\mathcal{Q}_\alpha^{\mathrm{nr}}) \oplus \mathbb{R}^2 \to \mathbb{R}, \qquad \mathcal{H}_\alpha(\psi, z) = \frac{1}{2}\, \mathcal{Q}_\alpha^{\mathrm{nr}}(\psi, \psi) + \frac{1}{2}\, \left(\dot{x}^2 - \lambda_0 x^2\right) ,$$

where

$$\mathcal{Q}_\alpha^{\mathrm{nr}}(\phi, \dot{\phi}) = \frac{1}{2}\, \left(\|(-\bar{\Delta}_\alpha^{\mathrm{nr}})^{1/4}\dot{\phi}\|_2^2 + \|(-\bar{\Delta}_\alpha^{\mathrm{nr}})^{3/4}\phi\|_2^2 \right)$$

is the quadratic form associated to the self-adjoint operator $\mathcal{J}_\alpha^{\mathrm{nr}} \cdot W_\alpha^{\mathrm{nr}}$.

If $(\phi, \dot{\phi}) \in \bar{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ has the orthogonal decomposition $(\phi, \dot{\phi}) \equiv \psi + z\tilde{G}$, where $\tilde{G}$ denotes the normalized eigenvector corresponding to λ_0, then

$$\mathcal{E}_\alpha(\phi, \dot{\phi}) = \frac{1}{2}\, [(\psi, 0), (\psi, 0)]_\alpha + \frac{1}{2}\, \mathcal{H}_\alpha(0, z) \,.$$

Therefore, being

$$\bar{U}_\alpha^t = e^{t\bar{W}_\alpha^{\mathrm{nr}}} \times e^{t\Lambda_0}$$

a strongly continuous group of unitary and symplectic transformations, the energy is conserved by the flow.

We come now to the case $\alpha = 0$. In this case, in order to apply thm. 4.1, which requires injectivity, it is necessary to project onto the subspace orthogonal to the eigenvector $(G, 0)$. Being $\bar{W}_{(0)}$ one-to-one and skew-adjoint w.r.t. the scalar product $\langle\langle \cdot, \cdot \rangle\rangle_{(0)}$, one can then apply thm. 4.1 thus obtaining a one parameter group $\bar{U}_{(0)}^t$ of symplectic transformations such that

$$\bar{U}_0^t = \bar{U}_{(0)}^t \times \mathbb{I}$$

and so $\bar{U}_0^t$ preserves the energy $\mathcal{E}_0(\phi,\dot{\phi})=\mathcal{E}(\phi_{\rm reg},\dot{\phi})$.

Since $(-\Delta_0)^{1/2}\phi = (-\bar{\Delta})^{1/2}\phi_{\rm reg}$ (note that this equality holds true only in the case $\alpha=0$) one has

$$\phi_{\rm reg}=(-\bar{\Delta})^{-1/2}\cdot(-\Delta_0)^{1/2}\phi$$

and so, when $(\phi,\dot{\phi})\in\bar{H}_0^2(\mathbb{R}^3)\oplus D^1(\mathbb{R}^3)$,

$$\mathcal{J}\cdot\bar{W}_{(0)}(\phi,\dot{\phi})=\left((-\bar{\Delta})^{1/2}\phi,(-\Delta_0)^{1/2}\dot{\phi}\right)\equiv\left((-\bar{\Delta}_0)^{1/2}\phi,(-\Delta_0)^{1/2}\dot{\phi}\right).$$

Moreover $\mathcal{J}$ commutes with $\bar{W}_{(0)}$ (see (5.6) in the next section) and so $\mathcal{J}$ coincides with the complex structure associated to $\bar{W}_{(0)}$ by thm. 4.1. With respect to the complex variable $\psi=(\phi,\dot{\phi})$ the wave equation corresponding to $\bar{W}_{(0)}$ assumes the Schrödinger-like form

$$-i\dot{\psi}=(-\bar{\Delta}_0)^{1/2}\psi\,.$$

We summarize the results obtained in the following

Theorem 4.2. *For every $\alpha\in\mathbb{R}\backslash\{0\}$ there exists a symplectic form*

$$\Omega_\alpha:\bar{D}^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)\times\bar{D}^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)\to\mathbb{R}$$

with respect to which the vector field

$$\bar{W}_\alpha:\bar{D}_\alpha^2(\mathbb{R}^3)\times D^1(\mathbb{R}^3)\to\bar{D}^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$$

is Hamiltoniam. Moreover for $\alpha\le 0$ the analogous result occurs for the reduced vector fields

$$\bar{W}_\alpha^{\rm nr}:[\bar{D}_\alpha^2(\mathbb{R}^3)]_{\rm nr}\times[\bar{D}^1(\mathbb{R}^3)]_{\rm nr}\to[\bar{D}^1(\mathbb{R}^3)]_{\rm nr}\oplus[L^2(\mathbb{R}^3)]_{\rm nr}$$

and

$$\bar{W}_{(0)}:\bar{H}_0^2(\mathbb{R}^3)\oplus D^1(\mathbb{R}^3)\to\bar{H}^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)\,.$$

For every $\alpha\in\mathbb{R}$ the evolution group $\bar{U}_\alpha^t$ preserves the energy

$$\mathcal{E}_\alpha(\phi,\dot{\phi})=\frac{1}{2}\left(\|\dot{\phi}\|_2^2+F_\alpha(\phi,\phi)\right)\,.$$

5. SCATTERING THEORY.

The Hilbert spaces where the operators $\bar{W}_\alpha$ and $\bar{W}$ act on, respectively $\bar{D}^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$ and $\bar{H}^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$, are different (also as sets), and so one is forced to use a two Hilbert space formulation to treat scattering theory for the pair $(\bar{W}_\alpha,\bar{W})$. We refer to the seminal paper by Kato [K] for the relevant constructions and results in scattering

theory with two Hilbert spaces. Our approach will follow the lines of the construction given in [K, §8-9] (also see [B, §3.5]).
From now on, given the real vector space $L^2(\mathbb{R}^3)$, we will denote by $L^2_{\mathbb{C}}(\mathbb{R}^3)$ the complex vector space

$$L^2_{\mathbb{C}}(\mathbb{R}^3) := \left\{\phi_1 + i\phi_2,\ \phi_1,\phi_2 \in L^2(\mathbb{R}^3)\right\} .$$

We begin introducing the isometries

$$C : \bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to L^2_{\mathbb{C}}(\mathbb{R}^3)\,, \qquad C(\phi,\dot{\phi}) \equiv C_0(\phi,\dot{\phi}) := (-\bar{\Delta})^{1/2}\phi - i\dot{\phi}\,,$$

$$C_\alpha : \bar{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to L^2_{\mathbb{C}}(\mathbb{R}^3)\,, \qquad C_\alpha(\phi,\dot{\phi}) := (-\bar{\Delta}_\alpha)^{1/2}\phi - i\dot{\phi}\,, \quad \alpha > 0\,,$$

$$C_\alpha : [\bar{D}^1(\mathbb{R}^3)]_{\rm nr} \oplus [L^2(\mathbb{R}^3)]_{\rm nr} \to [L^2_{\mathbb{C}}(\mathbb{R}^3)]_{\rm nr}\,, \qquad C_\alpha((\phi,\dot{\phi})) := (-\bar{\Delta}^{\rm nr}_\alpha)^{1/2}\phi - i\dot{\phi}\,, \quad \alpha < 0\,.$$

These isometries lead to the following relations:

$$\mathcal{J} = C^{-1}\cdot i\,C\,, \tag{5.1}$$

$$\mathcal{J}_\alpha = C_\alpha^{-1}\cdot i\,C_\alpha\,, \quad \alpha > 0\,, \qquad \mathcal{J}^{\rm nr}_\alpha = C_\alpha^{-1}\cdot i\,C_\alpha\,, \quad \alpha < 0\,, \tag{5.2}$$

$$\bar{U}^t = C^{-1}\cdot e^{it\sqrt{-\Delta}}\cdot C\,,$$

$$\bar{U}^t_0 = C^{-1}\cdot e^{it\sqrt{-\Delta_0}}\cdot C \times \mathbb{1}\,, \tag{5.3}$$

$$\bar{U}^t_\alpha = C_\alpha^{-1}\cdot e^{it\sqrt{-\Delta_\alpha}}\cdot C_\alpha\,, \quad \alpha > 0\,, \tag{5.4}$$

$$\bar{U}^t_\alpha = C_\alpha^{-1}\cdot e^{it\sqrt{-\Delta^{\rm nr}_\alpha}}\cdot C_\alpha \times e^{t\Lambda_0}\,, \quad \alpha < 0\,. \tag{5.5}$$

Note that the two equalities

$$\mathcal{J} = C^{-1}\cdot i\,C\,, \qquad \bar{U}^t_{(0)} = C^{-1}\cdot e^{it\sqrt{-\Delta_0}}\cdot C \tag{5.6}$$

imply, as we stated in the previous section, that $\mathcal{J}$ commutes with $\bar{W}_{(0)}$.
Moreover the relations (5.3)-(5.5) provide an alternative construction of the dynamics generated by $\bar{W}_\alpha$. In fact one could use such relations as definitions of $\bar{U}^t_\alpha$ and then check by differentiating with respect to the time parameter that this evolution group is generated by the operator $\bar{W}_\alpha$.
We introduce now the identification operators

$$J_\alpha : \bar{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\,,$$

$$J_\alpha(\phi,\dot{\phi}) := \begin{cases} (\,(-\bar{\Delta})^{-1/2}\cdot(-\bar{\Delta}_\alpha)^{1/2}\phi,\dot{\phi}\,)\,, & \text{for } \alpha > 0 \\ (\phi_{\rm reg},\dot{\phi}) \equiv (\,(-\bar{\Delta})^{-1/2}\cdot(-\bar{\Delta}_0)^{1/2}\phi,\dot{\phi}\,)\,, & \text{for } \alpha = 0 \\ (\,(-\bar{\Delta})^{-1/2}\cdot(-\bar{\Delta}^{\rm nr}_\alpha)^{1/2}\cdot\Pi_{\rm nr}\phi,\dot{\phi}\,)\,, & \text{for } \alpha < 0\,, \end{cases}$$

where $\Pi_{\rm nr}$ denotes the projection

$$\Pi_{\rm nr} : \bar{D}^1(\mathbb{R}^3) \to [\bar{D}^1(\mathbb{R}^3)]_{\rm nr}\,,$$

and

$$J'_\alpha : \bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \bar{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\,,$$

$$J'_\alpha(\phi,\dot\phi) := \begin{cases} (\,(-\bar\Delta_\alpha)^{-1/2}\cdot(-\bar\Delta)^{1/2}\phi,\dot\phi\,)\,, & \text{for } \alpha>0\\ (\phi,\dot\phi)\,, & \text{for } \alpha=0\\ (\,(-\bar\Delta_\alpha^{\mathrm{nr}})^{-1/2}\cdot P_{\mathrm{nr}}\cdot(-\bar\Delta)^{1/2}\phi,\dot\phi\,)\,, & \text{for } \alpha<0\,. \end{cases}$$

We can then define the Möller wave operators

$$\Omega_\pm(\bar W,\bar W_\alpha;J_\alpha) := \text{s-}\lim_{t\to\pm\infty}\bar U^{-t}\cdot J_\alpha\cdot\bar U^t_\alpha\cdot P_{\mathrm{ac}}(\bar W_\alpha)\,,$$

$$\Omega_\pm(\bar W_\alpha,\bar W;J'_\alpha) := \text{s-}\lim_{t\to\pm\infty}\bar U^{-t}_\alpha\cdot J'_\alpha\cdot\bar U^t\,,$$

where

$$P_{\mathrm{ac}}(\bar W_\alpha) : \bar{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \bar{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\,,$$

$$P_{\mathrm{ac}}(\bar W_\alpha)(\phi,\dot\phi) := \begin{cases} (\phi,\dot\phi)\,, & \text{for } \alpha>0\\ (\phi_{\mathrm{reg}},\dot\phi)\,, & \text{for } \alpha=0\\ (\Pi_{\mathrm{nr}}\phi,P_{\mathrm{nr}}\dot\phi)\,, & \text{for } \alpha<0\,. \end{cases}$$

Concerning the existence of such wave operators we have the following

Theorem 5.1. *The Möller wave operators*

$$\Omega_\pm(\bar W,\bar W_\alpha;J_\alpha) := \text{s-}\lim_{t\to\pm\infty}\bar U^{-t}\cdot J_\alpha\cdot\bar U^t_\alpha\cdot P_{\mathrm{ac}}(\bar W_\alpha)\,,$$

$$\Omega_\pm(\bar W_\alpha,\bar W;J'_\alpha) := \text{s-}\lim_{t\to\pm\infty}\bar U^{-t}_\alpha\cdot J'_\alpha\cdot\bar U^t$$

exist, are complete and are mutually adjoint isometries, i.e.

$$\mathrm{Range}\,\Omega_+(\bar W,\bar W_\alpha;J_\alpha) = \mathrm{Range}\,\Omega_-(\bar W,\bar W_\alpha;J_\alpha) = \mathbb{1}_{\bar H^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)}\,,$$

$$\mathrm{Range}\,\Omega_+(\bar W_\alpha,\bar W;J'_\alpha) = \mathrm{Range}\,\Omega_-(\bar W_\alpha,\bar W;J'_\alpha) = \mathrm{Range}\,P_{\mathrm{ac}}(\bar W_\alpha)\,,$$

$$\Omega_\pm(\bar W,\bar W_\alpha;J_\alpha)^*\cdot\Omega_\pm(\bar W,\bar W_\alpha;J_\alpha) = P_{\mathrm{ac}}(\bar W_\alpha)\,,$$

$$\Omega_\pm(\bar W_\alpha,\bar W;J'_\alpha)^*\cdot\Omega_\pm(\bar W_\alpha,\bar W;J'_\alpha) = \mathbb{1}_{\bar H^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)}\,,$$

$$\Omega_\pm(\bar W,\bar W_\alpha;J_\alpha)^* = \Omega_\pm(\bar W_\alpha,\bar W;J'_\alpha)\,.$$

Proof. With the above definitions one has

$$\Omega_\pm(\bar W,\bar W_\alpha;J_\alpha) = C^{-1}\cdot\Omega_\pm(\sqrt{-\Delta},\sqrt{H_\alpha};I_\alpha)\cdot C_\alpha\cdot P_{\mathrm{ac}}(\bar W_\alpha)\,,$$

$$\Omega_\pm(\bar W_\alpha,\bar W;J'_\alpha) = C_\alpha^{-1}\cdot\Omega_\pm(\sqrt{H_\alpha},\sqrt{-\Delta};I'_\alpha)\cdot C$$

where

$$\Omega_\pm(\sqrt{-\Delta},\sqrt{H_\alpha};I_\alpha):=\text{s-}\lim_{t\to\pm\infty} e^{-it\sqrt{-\Delta}}\cdot I_\alpha\cdot e^{it\sqrt{H_\alpha}}\,,$$

$$\Omega_\pm(\sqrt{H_\alpha},\sqrt{-\Delta};I'_\alpha):=\text{s-}\lim_{t\to\pm\infty} e^{-it\sqrt{H_\alpha}}\cdot I'_\alpha\cdot e^{it\sqrt{-\Delta}}\,.$$

Here $I'_\alpha=P_{\rm ac}(-\Delta_\alpha)$, I_α is its left inverse, and

$$H_\alpha=\begin{cases}-\Delta_\alpha\,, & \text{for } \alpha\ge 0\\ -\Delta_\alpha^{\rm nr}\,, & \text{for } \alpha<0\end{cases}$$

By Birman invariance principle one has

$$\Omega_\pm(\sqrt{-\Delta},\sqrt{H_\alpha};I_\alpha)=\Omega_\pm(-\Delta,H_\alpha;I_\alpha)$$

and

$$\Omega_\pm(\sqrt{H_\alpha},\sqrt{-\Delta};I'_\alpha)=\Omega_\pm(H_\alpha,-\Delta;I'_\alpha)\,.$$

Therefore one has the identities

$$\begin{aligned}&\Omega_\pm(\sqrt{-\Delta},\sqrt{H_\alpha};I_\alpha)=\text{s-}\lim_{t\to\pm\infty} e^{it\Delta}\cdot I_\alpha\cdot e^{itH_\alpha}\\ &=\text{s-}\lim_{t\to\pm\infty} e^{it\Delta}\cdot e^{-it\Delta_\alpha}\cdot P_{\rm ac}(-\Delta_\alpha)=\Omega_\pm(-\Delta,-\Delta_\alpha)\end{aligned}$$

and

$$\begin{aligned}&\Omega_\pm(\sqrt{H_\alpha},\sqrt{-\Delta};I'_\alpha)=\text{s-}\lim_{t\to\pm\infty} e^{-itH_\alpha}\cdot I'_\alpha\cdot e^{-it\Delta}\\ &=\text{s-}\lim_{t\to\pm\infty} P_{\rm ac}(-\Delta_\alpha)\cdot e^{it\Delta_\alpha}\cdot e^{-it\Delta}=P_{\rm ac}(-\Delta_\alpha)\cdot\Omega_\pm(-\Delta_\alpha,-\Delta)\\ &=\Omega_\pm(-\Delta_\alpha,-\Delta)\,.\end{aligned}$$

In conclusion one obtains the equalities

$$\Omega_\pm(\bar W,\bar W_\alpha;J_\alpha)=C^{-1}\cdot\Omega_\pm(-\Delta,-\Delta_\alpha)\cdot C_\alpha\cdot P_{\rm ac}(\bar W_\alpha)\,,$$

$$\Omega_\pm(\bar W_\alpha,\bar W;J'_\alpha)=C_\alpha^{-1}\cdot\Omega_\pm(-\Delta_\alpha,-\Delta;\,)\cdot C$$

and the proof is concluded since the wave operators $\Omega_\pm(-\Delta,-\Delta_\alpha)$, and $\Omega_\pm(-\Delta_\alpha,-\Delta)$ exist, are complete and are mutually adjoint isometries. This is proven (see [AGHH, appendix E]) by the Birman-Kuroda theorem being the resolvent difference

$$(-\Delta_\alpha+z)^{-1}-(-\Delta+z)^{-1}$$

a rank one (hence trace class) operator. □

The previous theorem holds true also with the different (α-independent and much simpler and natural) couple of identification operators defined by

$$J:\bar D^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)\to\bar H^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)\,,\qquad J(\phi,\dot\phi):=(\phi_{\rm reg},\dot\phi)\,,$$

$$J' : \bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \bar{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\,, \qquad J'(\phi, \dot{\phi}) := (\phi, \dot{\phi})\,.$$

This is true by [K, thms. 10.3 and 10.5] since the condition 10.1 in [K] is verified with $m = M = 1$. In our situation such a condition simply reads

$$\forall \phi \in H^1(\mathbb{R}^3) \cap \operatorname{Range} P_{\text{ac}}(-\Delta_\alpha)\,, \qquad \|\sqrt{H_\alpha}\,\phi\|_2 = \|\sqrt{-\Delta}\,\phi\|_2\,.$$

In more detail one has the following

Theorem 5.2. *J is $(\bar{U}^t_\alpha, \pm)$-equivalent to J_α, i.e.*

$$\text{s-}\lim_{t\to\pm\infty} (J_\alpha - J)\cdot \bar{U}^t_\alpha \cdot P_{\text{ac}}(\bar{W}_\alpha) = 0\,,$$

Therefore

$$\Omega_\pm(\bar{W}, \bar{W}_\alpha; J) := \text{s-}\lim_{t\to\pm\infty} \bar{U}^{-t}\cdot J \cdot \bar{U}^t_\alpha \cdot P_{\text{ac}}(\bar{W}_\alpha)$$

exist and are equal to $\Omega_\pm(\bar{W}, \bar{W}_\alpha; J_\alpha)$.

J' is a $(\bar{U}^t_\alpha, \pm)$-asymptotic left-inverse to J, i.e.

$$\text{s-}\lim_{t\to\pm\infty} (J'\cdot J - \mathbb{1})\cdot \bar{U}^t_\alpha \cdot P_{\text{ac}}(\bar{W}_\alpha) = 0\,,$$

thus

$$\Omega_\pm(\bar{W}_\alpha, \bar{W}; J') := \text{s-}\lim_{t\to\pm\infty} \bar{U}^{-t}_\alpha \cdot J' \cdot \bar{U}^t$$

exist and are equal to $\Omega_\pm(\bar{W}, \bar{W}_\alpha; J)^* \equiv \Omega_\pm(\bar{W}_\alpha, \bar{W}; J'_\alpha)$.

Acknowledgments

We are grateful to Gianfausto Dell'Antonio for several discussions and remarks.

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