# ASYMPTOTIC COMPLETENESS FOR A RENORMALIZED NON-RELATIVISTIC HAMILTONIAN IN QUANTUM FIELD THEORY: THE NELSON MODEL.

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Abstract. Scattering theory for the Nelson model is studied. We show Rosen estimates and we prove the existence of a ground state for the Nelson Hamiltonian. Also we prove that it has a locally finite pure point spectrum outside its thresholds. We study the asymptotic fields and the existence of the wave operators. Finally we show asymptotic completeness for the Nelson Hamiltonian.

Résumé. Nous étudions la théorie de diffusion pour le modèle de Nelson. Nous prouvons des estimations de Rosen et nous établissons l'existence d'un état fondamental pour le Hamiltonien de Nelson, nous prouvons également qui possède un spectre purement ponctuel en dehors de ses seuils. Nous étudions les champs asymptotiques et l'existence des opérateurs d'ondes. Finalement nous montrons la complètude asymptotique du Hamiltonien de Nelson.

## Contents



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## 1. Introduction

Recently there has been a renewed interest in quantum field theory models that describe a system of non-relativistic particles interacting with a bosonic field. The main physical example is a non-relativistic atom interacting with photons. For this model the existence of a ground state was established in [4], [2]. The absence of excited states was also shown in [4] using a renormalization group analysis and in [5], [13] using a positive commutator method.

Another result related to the present work is [11] where for a model of a confined atom interacting with massive bosons the asymptotic completeness of wave operators was proved.

In all these works, the model contains an ultraviolet cutoff which switches off the interaction between the nucleons and the bosonic field above a certain momentum scale. This can be justified physically by the fact that nucleons interacting with bosons of very high energy will become relativistic and in such a situation the model will anyhow loose its validity.

With a cutoff these models are free of ultraviolet divergences and hence can be easily constructed rigorously by rather elementary methods.

However the presence of a cutoff implies that the interaction term is now non local and that quantitative results depend on the choice of the cutoff scale. Therefore it would be more satisfactory to remove the ultraviolet cutoff from the model under consideration.

When the interaction term is linear in the field variables, the removal of the ultraviolet cutoff was done long time ago by Nelson [21]. This was probably the first model which was rigorously constructed using a renormalization procedure. It consists in considering cutoff Hamiltonians $H_\kappa$, where $\kappa$ is some ultraviolet cutoff parameter and applying a cutoff-dependent unitary transformation $U_\kappa$. After substracting a divergent self-energy term $E_\kappa$, the sequence of Hamiltonians $U_\kappa(H_\kappa - E_\kappa)U_\kappa^*$ converges in norm resolvent sense to a Hamiltonian $\hat{H}_\infty$ when $\kappa \to \infty$ while $U_\kappa$ converges strongly to a unitary transformation $U_\infty$ (in other words no change of representation is necessary). The Hamiltonian $H := U_\infty^* \hat{H}_\infty U_\infty$ is called the *Nelson Hamiltonian.* After Nelson's paper the Nelson model was studied by Cannon [8] and Fröhlich [14].

In this paper we consider the Nelson model for a confined atom and massive bosons and study its spectral and scattering theory. Our main result is the asymptotic completeness of the wave operators, which implies the unitarity of the $S$ matrix. The strategy and the proofs of our paper follow closely those of [11], which is devoted to a similar model with an ultraviolet cutoff. Nevertheless there are new difficulties coming from the fact that the Nelson Hamiltonian is only defined as the resolvent limit of the cutoff Hamiltonians.

Let us now describe the content of the paper. In Sect. 2 we recall classical notations related to Fock spaces, introduce some definitions and prove some extensions of Glimm-Jaffe's $N_\tau$ estimates.

In Sect. 3 we recall the construction of the Nelson Hamiltonian following [21]. In Sect. 4 we prove the so-called *higher order estimates*, following an idea of Rosen [22]. In Sect. 5 we study the spectral theory for the confined Nelson Hamiltonian. We prove an HVZ theorem and a positive commutator estimate. Sect. 6 is devoted to the scattering theory for the Nelson model. We show the existence of asymptotic fields and that the CCR representation they define is of Fock type, using an argument from [12]. In Sect. 7 we prove various propagation estimates for the Nelson Hamiltonian. Finally the asymptotic completeness of the wave operators is shown in Sect. 8.

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## 2. Presentation of the model

In this section we define the Nelson model. We start with a review of the basic construction and the main notations related to bosonic Fock spaces. For a more detailed exposition we refer the reader to [3],[6],[7]. In Subsect. 2.2 we give some technical estimates obtained by adaptation of $N_\tau$−estimates [16]. Finally in Subsect. 2.3 we introduce the formal Hamiltonian of the interacting system of $P$ confined nonrelativistic particles (nucleons) with a relativistic scalar field (mesons). In order to give sense to the formal Hamiltonian, we put a high-momentum cutoff in the interaction and we show that the cutoff Hamiltonian is a well defined selfadjoint operator.

2.1. **Basic definitions and notations.** Let $\mathfrak{h}$ be a complex Hilbert space. Let $\otimes_s^n \mathfrak{h}$ denote the symmetric n-fold tensor power of $\mathfrak{h}$. We introduce the bosonic Fock space by

$$\Gamma(\mathfrak{h}) := \bigoplus_{n\geq 0} \otimes_s^n \mathfrak{h}.$$

$\otimes_s^0 \mathfrak{h} := \mathbb{C}$, identified as subspace of $\Gamma(\mathfrak{h})$, represents the space of zero-particle states. We denote by $\Omega$ the vector (1,0,...) usually called *vacuum vector* and by $\Gamma_{\mathrm{fin}}(\mathfrak{h})$ the subspace of finite particle states, which is the subspace of finite sum of vectors in $\otimes_s^n \mathfrak{h}$. Among the main operators acting on $\Gamma(\mathfrak{h})$, we will first recall the definitions of the most familiar as *number operator* $N$ given in its spectral decomposition

$$N_{|\otimes_s^n \mathfrak{h}} := n\mathbb{1}.$$

Creation operators, which are unbounded operators densely defined on $\mathcal{D}(N^{\frac{1}{2}})$, are given by

$$a^*(h)_{|\otimes_s^n \mathfrak{h}} := \sqrt{(n+1)} S_{n+1} h \otimes \mathbb{1}_{\otimes_s^n \mathfrak{h}},$$

where $S_n$ denotes the orthogonal projection from $\otimes^n \mathfrak{h}$ into $\otimes_s^n \mathfrak{h}$. The annihilation operator $a(h)$ is the adjoint of $a^*(h)$. We will use the notation $a^\sharp$ for $a$ or $a^*$.

We define the *field operator* by

$$\phi(h) := \frac{1}{\sqrt{2}}(a^*(h) + a(h)),\ h \in \mathfrak{h}.$$

$\phi(h)$ is essentially selfadjoint on $\Gamma_{\mathrm{fin}}(\mathfrak{h})$. We still denote by $\phi(h)$ its closure. The field operators generated a strongly continuous unitary group called *Weyl operators*:

$$W(h) := \mathrm{e}^{\mathrm{i}\phi(\mathrm{h})}.$$

We recall a useful differentiation estimate for $W(h)$:

$$||(W(h_1) - W(h_2))u|| \leq C_\epsilon\, ||h_1 - h_2||^\epsilon((||h_1||^2 + ||h_2||^2)^{\frac{\epsilon}{2}}||u|| + ||(N+1)^{\frac{\epsilon}{2}}u||),\ 0 \leq \epsilon \leq 1, \tag{2.1}$$

$$\lim_{s\to 0} \sup_{||h||\leq c} s^{-1}||(W(sh) - \mathbb{1} - \mathrm{i}\phi(h))(N+1)^{-\frac{1}{2}-\epsilon}|| = 0,\ 0 < \epsilon. \tag{2.2}$$

Let $\mathfrak{h}$ be a Hilbert space. Let $\mathfrak{f} : \mathfrak{h} \to \mathfrak{h}$ be an (unbounded) operator. We denote by $\mathrm{d}\Gamma(\mathfrak{f})$ the amplification of $\mathfrak{f}$ to the whole space $\Gamma(\mathfrak{h})$

$$\mathrm{d}\Gamma(\mathfrak{f})_{|\otimes_s^n \mathfrak{h}} := \sum_{j=1}^n \mathbb{1}^{\otimes(j-1)} \otimes \mathfrak{f} \otimes \mathbb{1}^{\otimes(n-j)}.$$

Let $\mathfrak{h}_i$, $i = 1, 2$ be two Hilbert spaces. Let $\mathfrak{g} : \mathfrak{h}_1 \to \mathfrak{h}_2$ be a bounded operator. We define the operator $\Gamma(\mathfrak{g})$ by

$$\Gamma(\mathfrak{g}) : \mathfrak{h}_1 \to \mathfrak{h}_2,$$

$$\Gamma(\mathfrak{g})_{|\otimes_s^n \mathfrak{h}_1} := \mathfrak{g}^{\otimes(n)}.$$

A less familiar operator is $\mathrm{d}\Gamma(\mathfrak{f}, \mathfrak{g})$, where $\mathfrak{f}, \mathfrak{g}$ are two operators on $\mathfrak{h}_1$ into $\mathfrak{h}_2$. It is defined as in [11]:

$$\mathrm{d}\Gamma(\mathfrak{f}, \mathfrak{g}) : \Gamma(\mathfrak{h}_1) \to \Gamma(\mathfrak{h}_2),$$

$$\mathrm{d}\Gamma(\mathfrak{f}, \mathfrak{g})_{|\otimes_s^n \mathfrak{h}_1} := \sum_{j=1}^n \mathfrak{f}^{\otimes(j-1)} \otimes \mathfrak{g} \otimes \mathfrak{f}^{\otimes(n-j)}.$$

We notice that $\mathrm{d}\Gamma(\mathfrak{f}, \mathfrak{f}) = N\Gamma(\mathfrak{f})$ and if $\mathfrak{h}_1 = \mathfrak{h}_2$, we have $\mathrm{d}\Gamma(\mathbb{1}, \mathfrak{g}) = \mathrm{d}\Gamma(\mathfrak{g})$. If $||\mathfrak{f}|| \leq 1$, the following inequality holds

$$||N^{-\frac{1}{2}}\,\mathrm{d}\Gamma(\mathfrak{f}, \mathfrak{g})u|| \leq ||\mathrm{d}\Gamma(\mathfrak{g}^*\mathfrak{g})^{\frac{1}{2}}u||. \tag{2.3}$$

Let $i_1\,(resp.\, i_2)$ be the injection of $\mathfrak{h}_1\,(resp.\,\mathfrak{h}_2)$ into $\mathfrak{h}_1 \oplus \mathfrak{h}_2$. There exists a unitary transformation $U$ identifying $\Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2)$ with $\Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$, defined as follows

$$Uu \otimes v := \sqrt{\frac{(p+q)!}{p!q!}} S_{p+q} \Gamma(i_1) u \otimes \Gamma(i_2) v, \quad u \in \otimes_s^p \mathfrak{h}_1,\, v \in \otimes_s^q \mathfrak{h}_2.$$

This transformation has the following properties:
*i)*

$$U\Omega \otimes \Omega = \Omega.$$

*ii)* Let $h_1 \in \mathfrak{h}_1, h_2 \in \mathfrak{h}_2$

$$a^\sharp(h_1 \oplus h_2) U = U(a^\sharp(h_1) \otimes 1\!\!1 + 1\!\!1 \otimes a^\sharp(h_2)),$$

$$\phi(h_1 \oplus h_2) U = U(\phi(h_1) \otimes 1\!\!1 + 1\!\!1 \otimes \phi(h_2)).$$

*iii)* Let $\mathfrak{f}_i : \mathfrak{h}_i \to \mathfrak{h}_i$ , $i = 1, 2$ be two operators

$$\mathrm{d}\Gamma(\mathfrak{f}_1 \oplus \mathfrak{f}_2) U = U(\mathrm{d}\Gamma(\mathfrak{f}_1) \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\mathfrak{f}_2)),$$

$$U\,\Gamma(\mathfrak{f}_1) \otimes \Gamma(\mathfrak{f}_2) = \Gamma(\mathfrak{f}_1 \oplus \mathfrak{f}_2).$$

We define the *scattering identification operator I*

$$I : \Gamma_{\mathrm{fin}}(\mathfrak{h}) \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h}) \to \Gamma_{\mathrm{fin}}(\mathfrak{h}),$$

$$Iu \otimes v := \sqrt{\frac{(p+q)!}{p!q!}}\; S_{p+q}\, u \otimes v, \;\; u \in \otimes_s^p \mathfrak{h},\, v \in \otimes_s^q \mathfrak{h}.$$

We can also define $I$ by the following formula:

$$I \prod_{i=1}^{p} a^*(h_i)\Omega \otimes \prod_{i=1}^{q} a^*(g_i)\Omega := \prod_{i=1}^{q} a^*(g_i) \prod_{i=1}^{p} a^*(h_i)\Omega, \quad h_i, g_i \in \mathfrak{h}.$$

Let $\pi$ be the following map

$$\pi : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h},$$
$$(h_0, h_\infty) \to h_0 + h_\infty.$$

Then we can express $I$ as following

$$I = \Gamma(\pi) U.$$

We notice that $I$ is unbounded since $||\pi|| = \sqrt{2}$.
Let $i = (i_0, i_\infty)$ be a pair of maps from $\mathfrak{h}$ to $\mathfrak{h}$. We define

$$I(i) : \Gamma_{\mathrm{fin}}(\mathfrak{h}) \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h}) \to \Gamma_{\mathrm{fin}}(\mathfrak{h}),$$

$$I(i) := I\,\Gamma(i_0) \otimes \Gamma(i_\infty).$$

Let $i = (i_0, i_\infty), j = (j_0, j_\infty)$ be two pairs of maps from $\mathfrak{h}$ to $\mathfrak{h}$. We define

$$\mathrm{d}I(i,j) : \Gamma_{\mathrm{fin}}(\mathfrak{h}) \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h}) \to \Gamma_{\mathrm{fin}}(\mathfrak{h}),$$

$$\mathrm{d}I(i,j) := I\,(\mathrm{d}\Gamma(i_0, j_0) \otimes \Gamma(i_\infty) + \Gamma(i_0) \otimes \mathrm{d}\Gamma(i_0, j_\infty)).$$

If $i_0 i_0^* + i_\infty i_\infty^* \leq 1$ we have the estimates

$$||(N_0 + N_\infty)^{-\frac{1}{2}} \mathrm{d}I^*(i,j) u|| \leq ||\mathrm{d}\Gamma(j_0 j_0^* + j_\infty j_\infty^*)^{\frac{1}{2}} u||, \tag{2.4}$$

$$|(u_2 | \mathrm{d}I^*(i,j) u_1)| \leq ||\mathrm{d}\Gamma(|j_0|)^{\frac{1}{2}} \otimes 1\!\!1\, u_2||\; ||\mathrm{d}\Gamma(|j_0|)^{\frac{1}{2}} u_1|| + ||1\!\!1 \otimes \mathrm{d}\Gamma(|j_\infty|)^{\frac{1}{2}} u_2||\; ||\mathrm{d}\Gamma(|j_\infty|)^{\frac{1}{2}} u_1||. \tag{2.5}$$

For other properties and equivalent definitions of these operators, we refer the reader to [12].

Let $\mathcal{K}$ be an auxiliary Hilbert space. Let $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$. We define an extended creation operator:

$$a^*(v) : \mathcal{K} \otimes \Gamma_{\text{fin}}(\mathfrak{h}) \to \mathcal{K} \otimes \Gamma_{\text{fin}}(\mathfrak{h}),$$

$$a^*(v)_{|\mathcal{K} \otimes \otimes_s^n \mathfrak{h}} := \sqrt{(n+1)}\; \mathbb{1}_{\mathcal{K}} \otimes S_{n+1}\; v \otimes \mathbb{1}_{\otimes_s^n \mathfrak{h}}.$$

$a^*(v)$ is closable densely defined operator since its adjoint $a(v)$ is densely defined. We define the field operator $\phi(v)$ as in the scalar case. When $\mathfrak{h} = L^2(\mathbb{R}^d, dk)$, $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$ can be represented as a function $k \to v(k) \in \mathcal{B}(\mathcal{K})$, such that for $x \in \mathcal{K}$, $v(k)x := vx(k)$, $k$ *a.e.* and

$$\mathcal{K} \times \mathcal{K} \ni (x, y) \to \int (v(k)^*\, v(k)x|\, y)_{\mathcal{K}}\; dk = (vx|vy)_{\mathcal{K} \otimes \mathfrak{h}}$$

is a continuous quadratic form. A stronger condition is $v \in L^2(\mathbb{R}^d, \mathcal{B}(\mathcal{K}))$, i.e.:

$$\int ||v(k)||^2_{\mathcal{B}(\mathcal{K})}\, dk \, < \, \infty.$$

Assume that $[v_1^*(k),\; v_2(k')] = 0$, $\forall k, k'$. Then:

$$\begin{aligned} [a(v_1), a^*(v_2)] &= v_1^*\, v_2 \otimes\; \mathbb{1}_{\Gamma(\mathfrak{h})}, \\ [\phi(v_1), \phi(v_2)] &= \mathrm{i}\mathrm{Im}(v_1^*\, v_2) \,\otimes \mathbb{1}_{\Gamma(\mathfrak{h})}, \\ [\phi(v_1), W(v_2)] &= \mathrm{Im}(v_1^*\, v_2) \otimes\; W(v_2). \end{aligned}$$

2.2. **Technical Estimates.** In this subsection we will collect some technical estimates which are adaptation of Glimm-Jaffe's $N_\tau-$ estimates.
We recall the symbolic annihilation and creation operators in the case of a Fock space constructed over the space of square integrable functions $\mathfrak{h} := L^2(\mathbb{R}^d, dk)$. Let $\Psi \in \Gamma_{\text{fin}}(\mathfrak{h})$ :

$$(a(k)\Psi)^{(n)}(k_1, ..., k_n) := (n+1)^{\frac{1}{2}}\, \Psi^{(n+1)}(k, k_1, ..., k_n),$$

$$(a^*(k)\, \Psi)^{(n)}(k_1, ..., k_n) := n^{-\frac{1}{2}} \sum_{j=1}^{n} \delta(k - k_j)\, \Psi^{(n-1)}(k_1, ..., \hat{k}_j, ..., k_n),$$

where $\hat{k}_j$ means that $k_j$ is omitted. Let $S(\mathbb{R}^d)$ be the Schwartz space. We can define the monomial $a(k_1)...a(k_s)$ as an operator from $\Gamma_{\text{fin}}(S(\mathbb{R}^d))$ into $S(\mathbb{R}^{ds}) \otimes \Gamma_{\text{fin}}(S(\mathbb{R}^d))$.
Let $\mathcal{K}$ be an auxiliary Hilbert space. Let $w$ be unbounded operator from $\mathcal{K} \otimes \mathfrak{h}^{\otimes s}$ into $\mathcal{K} \otimes \mathfrak{h}^{\otimes r}$, with a domain containing $\mathcal{K} \otimes S(\mathbb{R}^{ds})$. A *Wick monomial* with *symbol* $w$ is the following sesquilinear form on $\mathcal{K} \otimes \Gamma_{\text{fin}}(S(\mathbb{R}^d))$

$$W_{r,s} := a^*(k'_1)...a^*(k'_r)\, w\, a(k_1)...a(k_s).$$

Let $\omega$ be a positive regular function satisfying:

$$\begin{cases} \omega \in C^\infty(\mathbb{R}^d), \\ |\partial_k^\alpha \omega(k)| \le c_\alpha (1 + |k|)^{N_\alpha} \;\text{ for } \alpha \in \mathbb{N}^d, \\ \omega(k) \ge m > 0. \end{cases}$$

We set:

$$N_\tau := \mathrm{d}\Gamma(\omega^\tau).$$

**Lemma 2.1.** *Let $r$ be a positive integer and $\tau := (\tau_i)_{1..r}$ be a sequence of real numbers. For $\psi \in \mathcal{D}(\prod_{i=1}^{r} N_{\tau_i}^{\frac{1}{2}})$:*

$$||\prod_{i=1}^{r} N_{\tau_i}^{\frac{1}{2}} \psi||^2 = \sum_{j=1}^{r} \int P^\tau_{r,j}(k_1, ..., k_j)\, ||\prod_{i=1}^{j} a(k_i)\psi||^2 dk_1 ... dk_j, \tag{2.6}$$

*where* $P^{\tau}_{r,j}$ *is a sum of homogeneous functions in the variables* $\omega(k_i)$ *of degree* $\sum_{i=1}^{r}\tau_i$ *and satisfying*

$$P^{\tau}_{r,j}(k_1,...,k_j)=\sum_{(i_s)_{1..r}\in\mathfrak{S}_{r,j}}\omega(k_{i_1})^{\tau_1}...\omega(k_{i_r})^{\tau_r}, \tag{2.7}$$

*where* $\mathfrak{S}_{r,j}$ *is the set, constructed by induction, of surjective maps* $\{1,...,r\}\to\{1,...,j\}$*, such that* $i_s\le s$ *and* $(i_s)_{1..r-1}$ *is in* $\mathfrak{S}_{r-1,j}$ *or in* $\mathfrak{S}_{r-1,j-1}$ *and* $\mathfrak{S}_{j,j+1}=\mathfrak{S}_{1,0}=\emptyset$*.*

**Proof.** $\prod_{i=1}^{j}a(k_i)$ can be defined as operator on $\Gamma_{\rm fin}(S(\mathbb{R}^d))$ into $S(\mathbb{R}^{dj})\otimes\Gamma_{\rm fin}(S(\mathbb{R}^d))$. For $\psi\in\Gamma_{\rm fin}(S(\mathbb{R}^d))$, $(k_1,...,k_j)\to||\prod_{i=1}^{j}a(k_i)\psi||$ is a function in $S(\mathbb{R}^{dj})$. This implies that the r.h.s. of (2.6) is well defined for $\psi\in\Gamma_{\rm fin}(S(\mathbb{R}^d))$ . The hypothesis on $\omega$ imply that $\Gamma_{\rm fin}(S(\mathbb{R}^d))$ is a core for $N_\tau^{\frac12}$ and for $\prod_{i=1}^{r}N_{\tau_i}^{\frac12}$. If the lemma holds for $\psi\in\Gamma_{\rm fin}(S(\mathbb{R}^d))$ then it can be extended to $\psi\in\mathcal{D}(\prod_{i=1}^{r}N_{\tau_i}^{\frac12})$. In fact since $\Gamma_{\rm fin}(S(\mathbb{R}^d))$ is a core for $\prod_{i=1}^{r}N_{\tau_i}^{\frac12}$, then we can extend $\prod_{i=1}^{j}a(k_i)$ to bounded operator from $\mathcal{D}(\prod_{i=1}^{r}N_{\tau_i}^{\frac12})$ into $L^2(\mathbb{R}^{dj},(P^{\tau}_{r,j})^{\frac12}dk)\otimes\Gamma(\mathfrak{h})$. Let us prove the lemma for $\psi\in\Gamma_{\rm fin}(S(\mathbb{R}^3))$ by induction in $r$.
For $r=1$

$$\begin{aligned}||N_{\tau_1}^{\frac12}\psi||^2 &= (\psi|N_{\tau_1}\psi)\\ &= \int\omega(k)^{\tau_1}||a(k)\psi||^2dk.\end{aligned}$$

We see that $P^{\tau_1}_{1,1}(k)=\omega(k)^{\tau_1}$. (2.6), (2.7) are satisfied for $r=1$. Assume that (2.6), (2.7) hold for $r$. Using the fact that $N_\tau$ preserves $\Gamma_{\rm fin}(S(\mathbb{R}^d))$ and the induction hypothesis, we have

$$\begin{aligned}||\prod_{i=1}^{r+1}N_{\tau_i}^{\frac12}\psi||^2 &= \sum_{j=1}^{r}\int P^{\tau}_{r,j}(k_1,...,k_j)\,||\prod_{i=1}^{j}a(k_i)N_{\tau_{r+1}}^{\frac12}\psi||^2dk\\ &= \sum_{j=1}^{r}\int P^{\tau}_{r,j}\,||(N_{\tau_{r+1}}+\sum_{i=1}^{j}\omega(k_i)^{\tau_{r+1}})^{\frac12}\prod_{i=1}^{j}a(k_i)\psi||^2dk\\ &= \sum_{j=1}^{r}\int P^{\tau}_{r,j}\,\{\sum_{i=1}^{j}\omega(k_i)^{\tau_{r+1}}\,||\prod_{i=1}^{j}a(k_i)\psi||^2+||N_{\tau_{r+1}}^{\frac12}\prod_{i=1}^{j}a(k_i)\psi||^2\}dk\\ &= \sum_{j=1}^{r+1}\int[P^{\tau}_{r,j}\sum_{i=1}^{j}\omega(k_i)^{\tau_{r+1}}+P^{\tau}_{r,j-1}\omega(k_j)^{\tau_{r+1}}]\,||\prod_{i=1}^{j}a(k_i)\psi||^2dk,\end{aligned}$$

where $P_{r,r+1}=P_{r,0}=0$. Then we obtain the following iterated relation

$$P^{\tau}_{r+1,j}(k_1,...,k_j)=P^{\tau}_{r,j}(k_1,...,k_j)\sum_{i=1}^{j}\omega(k_i)^{\tau_{r+1}}+P^{\tau}_{r,j-1}(k_1,...,k_{j-1})\omega(k_j)^{\tau_{r+1}}. \tag{2.8}$$

We note that $P^{\tau}_{r,r}(k_1,...,k_r)=\prod_{i=1}^{r}\omega(k_i)^{\tau_i}$. It is easy to see, using induction hypothesis (2.7) for $r$ and (2.8), that $P^{\tau}_{r+1,j}(k_1,...,k_{r+1})=\sum_{(i_s)_{1..r+1}\in\mathfrak{S}_{r+1,j}}\omega(k_{i_1})^{\tau_1}...\omega(k_{i_{r+1}})^{\tau_{r+1}}$. ■

**Corollary 2.2.** *Let $\alpha,\nu$ ,$\tau:=(\tau_i)_{1..r}$ be a sequence of real numbers. For $\psi\in\mathcal{D}(N_\nu^\alpha\prod_{i=1}^{r}N_{\tau_i}^{\frac12})$*

$$||N_\nu^\alpha\prod_{i=1}^{r}N_{\tau_i}^{\frac12}\psi||^2=\sum_{j=1}^{r}\int P^{\tau}_{r,j}(k_1,...,k_j)\;||(N_\nu+\sum_{i=1}^{j}\omega(k_i)^{\nu})^{\alpha}\prod_{i=1}^{j}a(k_i)\psi||^2dk_1...dk_j, \tag{2.9}$$

*where $P^{\tau}_{r,j}$ is the function defined by (2.7).*

**Lemma 2.3.** *Let $p,q\in\mathbb{N}$ and $\tau':=(\tau'_i)_{1..p},\tau:=(\tau_i)_{1..q}$ be two sequences of real numbers. Let $W_{r,s}$ be a Wick monomial such that $r\le p,s\le q$. Then*

$$||(\prod_{i=1}^{p}(N_{\tau'_i}+1)^{-\frac12})\;W_{r,s}\prod_{i=1}^{q}(N_{\tau_i}+1)^{-\frac12}||\le||(P^{\tau'}_{p,r})^{-\frac12}\,w\,(P^{\tau}_{q,s})^{-\frac12}||_{\mathcal{B}(\mathcal{K}\otimes\mathfrak{h}^{\otimes s},\mathcal{K}\otimes\mathfrak{h}^{\otimes r})}.$$

**Proof.** Let $\psi,\phi\in\mathcal{K}\otimes\Gamma_{\rm fin}(S(\mathbb{R}^d))$.

$$\begin{aligned}
|(W_{r,s}\,\psi|\,\phi)| &\le |(w\prod_{i=1}^{s}a(k_i)\psi|\prod_{i=1}^{r}a(k'_i)\phi)_{\mathcal{K}\otimes\mathfrak{h}^{\otimes r}\otimes\Gamma(\mathfrak{h})}|\\
&\le |((P^{\tau'}_{p,r})^{-\frac12}\,w\,(P^{\tau}_{q,s})^{-\frac12}\,(P^{\tau}_{s,q})^{\frac12}\prod_{i=1}^{s}a(k_i)\psi|\,(P^{\tau'}_{p,r})^{\frac12}\prod_{i=1}^{r}a(k'_i)\phi)|\\
&\le ||(P^{\tau'}_{p,r})^{-\frac12}\,w\,(P^{\tau}_{q,s})^{-\frac12}||_{\mathcal{B}(\mathcal{K}\otimes\mathfrak{h}^{\otimes s},\mathcal{K}\otimes\mathfrak{h}^{\otimes r})}\,(\int P^{\tau}_{q,s}||\prod_{i=1}^{s}a(k_i)\psi||^2dk)^{\frac12}\\
&\quad\times(\int P^{\tau'}_{p,r}||\prod_{i=1}^{r}a(k'_i)\phi||^2dk')^{\frac12}\\
&\le ||(P^{\tau'}_{p,r})^{-\frac12}\,w\,(P^{\tau}_{q,s})^{-\frac12}||_{\mathcal{B}(\mathcal{K}\otimes\mathfrak{h}^{\otimes s},\mathcal{K}\otimes\mathfrak{h}^{\otimes r})}\,||\prod_{i=1}^{q}N_{\tau_i}^{\frac12}\psi||\;\times||\prod_{i=1}^{p}N_{\tau'_i}^{\frac12}\phi||.
\end{aligned}$$

This inequality shows that the quadratic form $\prod_{i=1}^{p}(N_{\tau'_i}+1)^{-\frac12}W_{r,s}\prod_{i=1}^{q}(N_{\tau_i}+1)^{-\frac12}$ can be extended to a bounded operator with norm less than $||(P^{\tau'}_{p,r})^{-\frac12}w\,(P^{\tau}_{q,s})^{-\frac12}||_{\mathcal{B}(\mathcal{K}\otimes\mathfrak{h}^{\otimes s},\mathcal{K}\otimes\mathfrak{h}^{\otimes r})}$. ■

**Corollary 2.4.** *Let $v_i\in\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})$; $i=1,...,n$. Then there exists $c>0$ such that*

*i)* $||(N+1)^p\prod_{i=1}^{n}a^\sharp(v_i)(N+1)^{-p-\frac{n}{2}}||\;\le\;c\prod_{i=1}^{n}||v_i||_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}.$

*ii)* $||(N+1)^p\prod_{i=1}^{n}\phi(v_i)(N+1)^{-p-\frac{n}{2}}||\;\le\;c\prod_{i=1}^{n}||v_i||_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}.$

*iii) Let $r,s\in\mathbb{N},(\tau'_i)_{1..r},(\tau_i)_{1..s}$ be sequences of real numbers. Then*

$$||(\prod_{i=1}^{r}(N_{\tau'_i}+1)^{-\frac12})\;W_{r,s}\prod_{i=1}^{s}(N_{\tau_i}+1)^{-\frac12}||\le||\prod_{i=1}^{r}\omega(k'_i)^{-\frac{\tau'_i}{2}}\;w\;\prod_{i=1}^{s}\omega(k_i)^{-\frac{\tau_i}{2}}||_{\mathcal{B}(\mathcal{K}\otimes\mathfrak{h}^{\otimes s},\mathcal{K}\otimes\mathfrak{h}^{\otimes r})}.$$

**Proof.** Clearly $i)$ gives $ii)$. For $p=0$ and $n=1$, $i)$ follows from the last lemma by taking $\omega=1$ and $r=p=1, s=q=0$ or $r=p=0, s=q=1$. For $p,n\in\mathbb{N}$, commutation properties reduces the inequality to the case $p=0, n=1$. $iii)$ is a direct application of Lemma 2.3. ■

We recall now well known estimates, see [22].

**Lemma 2.5.** *Let $b$ be a positive operator. Then*

$$\mathrm{d}\Gamma(b^{\alpha_1}) \leq N^{1-\alpha_1}\,\mathrm{d}\Gamma(b)^{\alpha_1},\ \textit{where}\ \alpha_1\leq 1.$$
$$\mathrm{d}\Gamma(b^{\tau_1})^{\alpha_1} \leq \mathrm{d}\Gamma(b^{\tau_2})^{\alpha_2}\,\mathrm{d}\Gamma(b^{\tau_3})^{\alpha_3},$$

*where $\alpha_1=\alpha_2+\alpha_3$, and $\alpha_1\tau_1=\alpha_2\tau_2+\alpha_3\tau_3$.*

Combining Lemma 2.5 and Lemma 2.3 we obtain a slightly more general estimate.

**Lemma 2.6.** *Let $r,s,p,q\in\mathbb{N}$. Let $\tau'_j := ({\tau'}^i_j)_{1..p}$, $\tau_j := (\tau^i_j)_{1..q}$, $j=1..3$ be sequences of real numbers such that ${\tau'}^i_1={\tau'}^i_2+{\tau'}^i_3$, $\tau^i_1=\tau^i_2+\tau^i_3$ . Then*

$$||\prod_{i=1}^{p}(N_{{\tau'}^i_2}+1)^{-\frac12}(N_{{\tau'}^i_3}+1)^{-\frac12}W_{r,s}\prod_{i=1}^{q}(N_{\tau^i_2}+1)^{-\frac12}(N_{\tau^i_3}+1)^{-\frac12}|| \leq ||(P^{\tau'_1}_{p,r})^{-\frac12}\,w\,(P^{\tau_1}_{q,s})^{-\frac12}||.$$

The following estimate is an immediate application of Lemma 2.6.

**Corollary 2.7.** *Let $r,s,\alpha,\beta$ be positive integers such that $\alpha\leq r,\beta\leq s$. Then the following assertion holds*

$$||(N+1)^{-\frac{r-\alpha}{2}}(\mathrm{d}\Gamma(\omega)+1)^{-\frac{\alpha}{2}}\,W_{r,s}\,(N+1)^{-\frac{s-\beta}{2}}(\mathrm{d}\Gamma(\omega)+1)^{-\frac{\beta}{2}}|| \leq$$

$$\inf_{\substack{\{\tau'_i,\tau_i\in[0,1]\}\\ \sum\tau'_i=\alpha,\ \sum\tau_i=\beta\}}} ||\prod_{i=1}^{r}\omega(k'_i)^{-\frac{\tau'_i}{2}}\ w\ \prod_{i=1}^{s}\omega(k_i)^{-\frac{\tau_i}{2}}||_{\mathcal{B}(\mathcal{K}\otimes\mathfrak{h}^{\otimes s},\mathcal{K}\otimes\mathfrak{h}^{\otimes r})}.$$

2.3. **The Nelson model.** The Nelson model [21] describes a system of $P$ nonrelativistic particles coupled to a scalar relativistic field of bosons by a local, translation invariant interaction. It exhibits a relatively mild ultraviolet divergence, and was the first QFT model, on which a renormalization procedure was rigorously carried on.

We consider the atomic Hamiltonian of the system of $P$ nonrelativistic confined particles as follows

$$K := \frac{1}{2M}\sum_{j=1}^{P} D_j^2 + V(\mathrm{x}_1,...,\mathrm{x}_P).$$

It acts on the Hilbert space $L^2(\mathbb{R}^{3P}, d\mathrm{x})$ which we denote in the sequel by $\mathcal{K}$. We assume that $V\in L^2_{\mathrm{loc}}(\mathbb{R}^{3P})$ and $V\geq 0$. Kato's inequality gives that $K$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^{3P})$, see [23, Thm. X.28]. We set $\langle \mathrm{x}\rangle := (|\mathrm{x}|^2+1)^{\frac12}$. We will also assume

$$V \geq c\sum_i \langle \mathrm{x}_i\rangle^{\alpha}\ \ ,\alpha>2.$$

We notice that for $0\leq\beta\leq 1$, $\langle D\rangle^{\beta}(K+1)^{-\frac{\beta}{2}}$, $\langle \mathrm{x}\rangle^{\beta}(K+1)^{-\frac{\beta}{2}}$ are bounded operators.

The boson one particle space is the Hilbert space $\mathfrak{h} := L^2(\mathbb{R}^3, dk)$, where $k$ denotes the boson momentum observable. The boson position observable $-\frac{\nabla_k}{\mathrm{i}}$ will be denotes by the italic letter $x$. This should not be confused with the nucleon position observable denoted by the roman letter x. The free bosonic Hamiltonian is defined by the second quantization of a single boson energy. It acts on the bosonic Fock space $\Gamma(\mathfrak{h})$.

$$H_b := \mathrm{d}\Gamma(\omega),$$
$$\omega(k) := (|k|^2+m^2)^{\frac12},\ \ m>0.$$

It is essentially selfadjoint on $\Gamma_{\mathrm{fin}}(\mathcal{D}(\omega))$.

The Hilbert space of the joint system is $\mathcal{H} := \mathcal{K}\otimes\Gamma(\mathfrak{h})$. The Hamiltonian without interaction is given by

$$H_0 := K\otimes 1\!\!1 + 1\!\!1\otimes H_b.$$

This is a selfadjoint operator since $K \otimes 1\!\!1$ and $1\!\!1 \otimes H_b$ commute on $\mathcal{H}$. The local translation invariant interaction between nucleons and bosons is given by the formal expression

$$\sum_{j=1}^{P} \varphi(\mathrm{x}_j), \quad \text{where}\, \varphi(\mathrm{x}) := [2(2\pi)^3]^{-\frac{1}{2}} \int \mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle}(a^*(k) + a(-k))\frac{dk}{\omega(k)^{\frac{1}{2}}}.$$

The interaction term cannot be defined as an operator on $\mathcal{H}$ with a dense domain. This comes from the fact that $\omega^{-\frac{1}{2}} \notin L^2(\mathbb{R}^3, dk)$, because the integral diverges for large $k$. This phenomenon is known as an *ultraviolet problem*. In order to have a well defined operator, one introduces cutoff interactions:

$$\varphi_\kappa(\mathrm{x}) := [2(2\pi)^3]^{-\frac{1}{2}} \int \mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle}(a^*(k) + a(-k))\frac{\chi_\kappa(k)}{\omega(k)^{\frac{1}{2}}}\, dk,$$

$$I_\kappa := \sum_{j=1}^{P} \varphi_\kappa(\mathrm{x}_j),\; v_\kappa := [(2\pi)^3]^{-\frac{1}{2}} \frac{\chi_\kappa(k)}{\omega(k)^{\frac{1}{2}}}\, \mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle}.$$

Here $\chi$ a is positive function in $C^\infty(\mathbb{R}^3)$ such that $0 \leq \chi(k) \leq 1$, $\chi(k) = 1$ for $|k| \leq 1$ and $\chi(k) = 0$ for $|k| \geq 2$. We set $\chi_\kappa(k) := \chi(\frac{k}{\kappa})$.

**Lemma 2.8.** *One has for* $\alpha \geq 0$

$$i)\, \mathrm{d}\Gamma(\omega)^\alpha \leq H_0^\alpha,$$
$$ii)\, K^\alpha \leq H_0^\alpha.$$

**Proof.** For $\psi \in C_0^\infty(\mathbb{R}^3) \otimes \Gamma_{\mathrm{fin}}(S(\mathbb{R}^3))$, which is a core for $H_0$, one has

$$(\mathrm{d}\Gamma(\omega)\psi, \psi) \leq (H_0\psi, \psi),$$
$$(K\psi, \psi) \leq (H_0\psi, \psi).$$

This means that $\mathrm{d}\Gamma(\omega) \leq H_0$ and $K \leq H_0$. Since $H_0, K$ and $\mathrm{d}\Gamma(\omega)$ commute, the spectral theorem gives the inequalities announced in the lemma. ∎

$I_\kappa$ are well-defined operators on $\mathcal{D}((N+1)^{\frac{1}{2}})$ as long as $\kappa < \infty$ and they are $H_0$-bounded with infinitesimal bound. We set $H_\kappa := H_0 + I_\kappa$.

**Theorem 2.9.** *For* $\kappa < \infty$, $H_\kappa$ *is a selfadjoint operator on* $\mathcal{D}(H_0)$.

**Proof.** Using Corollary 2.4 and Lemma 2.8, we prove for $c$ independent from $\lambda$

$$||I_\kappa(H_0 + \lambda)^{-1}|| \leq c\,\lambda^{-\frac{1}{2}}\, ||v_\kappa||.$$

Then by the Kato-Rellich theorem, one sees that $H_\kappa$ is a selfadjoint operator on $\mathcal{D}(H_0)$. ∎

## 3. Construction of the Nelson Hamiltonian

In this section we recall the construction in [21] of the Nelson Hamiltonian. It consists in applying to the cutoff Hamiltonians $H_\kappa$ a cutoff dependent unitary transformation $U_\kappa$, letting then $\kappa$ to $\infty$.

3.1. **Dressing Transformation.** For a fixed $\kappa_0$ and $\kappa < \infty$, we define

$$g_\kappa(k) \;:=\; -\mathrm{i}\frac{(2\pi)^{-\frac{3}{2}}}{\omega^{\frac{1}{2}}(k)}\,\frac{\chi_\kappa(k) - \chi_{\kappa_0}(k)}{\omega(k) + \frac{k^2}{2M}} \in C_0^\infty(\mathbb{R}^3), \tag{3.1}$$

$$G_\kappa \;:=\; \sum_{j=1}^{P} \mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}_j\rangle}\, g_\kappa(k) \,\in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h}), \tag{3.2}$$

$$C_\kappa \;:=\; \frac{1}{2}\sum_{1\leq j,\ell\leq P} \int |g_\kappa(k)|^2 \,\sin\langle k, (\mathrm{x}_j - \mathrm{x}_\ell)\rangle\, dk \,\in S(\mathbb{R}^{3P}), \tag{3.3}$$

$$E_\kappa := \frac{P}{2(2\pi)^{-3}} \int \frac{1}{\omega(k)} \frac{(\chi_\kappa(k)-\chi_{\kappa_0}(k))^2}{(\omega(k)+\frac{k^2}{2M})}\, dk \;-\frac{P}{(2\pi)^{-3}} \int \frac{\chi_\kappa(k)}{\omega(k)} \frac{\chi_\kappa(k)-\chi_{\kappa_0}(k)}{(\omega(k)+\frac{k^2}{2M})}\, dk, \tag{3.4}$$

$$r_\kappa(\mathrm{x}) := -\mathrm{i}k\mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle} g_\kappa(k), \tag{3.5}$$

$$U_\kappa := \mathrm{e}^{\mathrm{i}\phi(\mathrm{G}_\kappa)+\mathrm{i}\mathrm{C}_\kappa}, \tag{3.6}$$

$$\hat{H}_\kappa := U_\kappa (H_\kappa - E_\kappa) U_\kappa^*. \tag{3.7}$$

In order to simplify the writing of some formulas, we will replace often $r_\kappa(\mathrm{x}_j), v_\kappa(\mathrm{x}_j)$ by $r_\kappa^j, v_\kappa^j$.

**Lemma 3.1.** *For a fixed $\kappa_0$ and for $\kappa_0 < \kappa < \infty$, $\hat{H}_\kappa$ is a selfadjoint operator on the domain $\mathcal{D}(H_0)$ and equal to*

$$\hat{H}_\kappa = H_0 + \sum_{1\le i<j\le P} V_\kappa(\mathrm{x}_i - \mathrm{x}_j) \otimes \mathbb{1} + \hat{I}_\kappa, \tag{3.8}$$

*where*

$$\hat{I}_\kappa := I_{\kappa_0} + \frac{1}{2M}\sum_{j=1}^{P} R_j(r_\kappa(\mathrm{x}_j)),$$
$$R_j(v) := \frac{1}{2}\, a^2(v) + \frac{1}{2}\, {a^*}^2(v) \; + a^*(v)\, a(v) - \sqrt{2} D_j\, a(v) - \sqrt{2} a^*(v) D_j,$$
$$V_\kappa(\mathrm{x}) := \mathrm{Re}\int \omega(k)\, |g_\kappa(k)|^2\, \mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle} dk \; - 2\mathrm{Im}\int \bar{g}_\kappa(k)\, v_\kappa(k) \mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle} dk.$$

**Proof.** We compute $U_\kappa H_\kappa U_\kappa^*$ using commutation relations. We notice that terms coming from $H_b$ and $K$ have been mixed. This creates a translation invariant potential and second order interacting terms:

$$U_\kappa\, H_b U_\kappa^* = H_b - \phi(\mathrm{i}\omega G_\kappa) + V_\kappa^{(1)}(\mathrm{x}_i - \mathrm{x}_j) + \frac{P}{2}\int \omega(k)|g_\kappa(k)|^2\, dk, \tag{3.9}$$

$$\text{where } V_\kappa^{(1)}(\mathrm{x}) := \mathrm{Re}\int \omega(k)|g_\kappa(k)|^2\, \mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle} dk.$$

$$U_\kappa K U_\kappa^* = \frac{1}{2M}\sum_{j=1}^{P} (D_j + \phi(\mathrm{i}k\, g_\kappa(k)\mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}_j\rangle}))^2 + V(\mathrm{x}_1, ..., \mathrm{x}_P). \tag{3.10}$$

$$U_\kappa\, I_\kappa U_\kappa^* = I_\kappa + \sum_{1\le i<j\le P} V_\kappa^{(2)}(\mathrm{x}_i - \mathrm{x}_j) - \frac{P}{(2\pi)^3}\int \frac{\chi_\kappa(k)}{\omega(k)} \frac{\chi_\kappa(k)-\chi_{\kappa_0}(k)}{\omega(k)+\frac{k^2}{2M}}\, dk,$$
$$\text{where } V_\kappa^{(2)}(\mathrm{x}) := -2\mathrm{Im}\int \bar{g}_\kappa(k)\, v_\kappa(k)\, \mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle}\, dk.$$

Using the following computation

$$\begin{aligned}(D_j - \phi(r_\kappa^j))^2 \;=\; & D_j^2 + \tfrac{1}{2}a^2(r_\kappa^j) + \tfrac{1}{2}{a^*}^2(r_\kappa^j) + a^*(r_\kappa^j)\, a(r_\kappa^j) - \sqrt{2} D_j a(r_\kappa^j) - \sqrt{2} a^*(r_\kappa^j) D_j \\ & -\phi(\mathrm{i}k^2\mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}_j\rangle} g_\kappa) + \frac{1}{2}\int k^2 |g_\kappa(k)|^2 dk,\end{aligned}$$

in the second term (3.10) and collecting similar terms together, we obtain (3.8).
It is easy to see, by (3.9)-(3.10), that $U_\kappa$ preserves $\mathcal{D}(H_0)$ for $\kappa < \infty$, hence $\mathcal{D}(\hat{H}_\kappa) = U_\kappa \mathcal{D}(H_0) = \mathcal{D}(H_0)$.■

3.2. **Removal of the ultraviolet cutoff.** We set

$$
\begin{aligned}
g_\infty(k) &:= -\mathrm{i}\frac{(2\pi)^{-\frac{3}{2}}}{\omega^{\frac{1}{2}}(k)}\frac{1-\chi_{\kappa_0}(k)}{(\omega(k)+\frac{k^2}{2M})}\ ,\\
G_\infty &:= \sum_{j=1}^{P} \mathrm{e}^{-\mathrm{i}\langle k, \mathrm{x}_j\rangle} g_\infty .\\
V_\infty &:= \mathrm{Re}\int \omega(k)\,|g_\infty(k)|^2\,\mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle}dk\ -2\mathrm{Im}\int \bar{g}_\infty(k)\,\frac{1}{\omega(k)^{\frac{1}{2}}}\mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle}dk,\\
C_\infty &:= \frac{1}{2}\sum_{1\le j,\ell\le P}\int |g_\infty(k)|^2\,\sin\langle k,(\mathrm{x}_j-\mathrm{x}_\ell)\rangle\,dk,\\
U_\infty &:= \mathrm{e}^{\mathrm{i}\phi(G_\infty)+\mathrm{i}C_\infty}\ .
\end{aligned}
$$

**Lemma 3.2.** *Letting $\kappa$ to $\infty$, we obtain the following limits:*

$$
\begin{aligned}
&i) E_\kappa \to -\infty\\
&ii) g_\kappa \to g_\infty \text{ in } \mathfrak{h},\\
&iii) C_\kappa \to C_\infty \text{ in } L^\infty(\mathbb{R}^{3P}),\\
&iv) V_\kappa \to V_\infty \text{ in } L^\infty(\mathbb{R}^3)+L^s(\mathbb{R}^3) \text{ for } s\in ]2,+\infty[,\\
&v) U_\kappa \to U_\infty \text{ strongly in } \mathcal{B}(\mathcal{H}).
\end{aligned}
$$

**Proof.** $i)$ is obvious. $ii)$ follows from the monotone convergence theorem. $iii)$ follows from $ii)$, since $|g_\kappa|^2$ converges in $L^1(\mathbb{R}^3)$. $V_\kappa^{(1)}$ converges in $L^\infty(\mathbb{R}^3)$, which follows from the fact that its integrand converges in $L^1(\mathbb{R}^3)$. $V_\kappa^{(2)}$ converges in $L^s(\mathbb{R}^3)$ for $s\in ]2,+\infty[$, by using Hausdorff-Young inequality. This proves $iv)$. Using the fact that the map $\mathfrak{h}\ni v\to \mathrm{e}^{\mathrm{i}\phi(v)}$ is strongly continuous and the limits $ii), iii)$, we see that $s-\lim\limits_{k\to\infty} U_\kappa$ exists and is equal to $\mathrm{e}^{\mathrm{i}\phi(G_\infty)+\mathrm{i}C_\infty}$. ■

We give now a lemma which will be useful in this section and in Sect. 4.

**Lemma 3.3.** *For $s\in[0,1]$ , and $v_i\in\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})$, $i=1,2$ we have:*

$$
\begin{aligned}
&i)\ ||(N+1)^{-\frac{s}{2}}\,a(v_1)\,(H_0+1)^{-\frac{1-s}{2}}||\ \le\ ||\omega^{\frac{s-1}{2}}\,v_1||_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}.\\
&ii)\ ||(H_0+1)^{-\frac{s}{2}}\,a^*(v_1)\,(N+1)^{-\frac{1-s}{2}}||\ \le\ ||\omega^{-\frac{s}{2}}\,v_1||_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}.\\
&iii)\ ||(N+1)^{-s}\,a(v_1)\,a(v_2)\,(H_0+1)^{-1+s}||\le\ ||\omega^{-\frac{1-s}{2}}\,v_1||_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}\,||\omega^{-\frac{1-s}{2}}\,v_2||_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}.\\
&iv)\ ||(H_0+1)^{-s}\,a^*(v_1)a^*(v_2)(N+1)^{-1+s}||\le\ ||\omega^{-\frac{s}{2}}\,v_1||_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}\,||\omega^{-\frac{s}{2}}\,v_2||_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}.
\end{aligned}
$$

**Proof.** We see clearly that suitable choice of $r, s, \alpha$, and $\beta$ in Corollary 2.7 gives similar inequalities with $\mathrm{d}\Gamma(\omega)$ in the place of $H_0$. Now using Lemma 2.8 we obtain $i)-iv)$. ■

**Lemma 3.4.** *One has for $v_i \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$, such that $[V, v_i] = 0$, where $V$ is the potential of the atomic Hamiltonian and $s, \beta \in [0,1]$:*

*i)* $||(N+1)^{-\frac{1}{2}} a(v_1)\, (H_0+1)^{-\frac{1}{2}}|| \leq c\, ||(V+1)^{-\frac{s}{2}} \omega^{\frac{s-1}{2}}\, v_1||_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}$.

*ii)* $||(H_0+1)^{-\frac{1}{2}} a^*(v_1)\, (N+1)^{-\frac{1}{2}}|| \leq c\, ||(V+1)^{-\frac{s}{2}} \omega^{\frac{s-1}{2}}\, v_1||_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}$.

*iii)* $||(N+1)^{-1} a(v_1) a(v_2) (H_0+1)^{-\frac{1}{2}}|| \leq c\, ||(V+1)^{-\frac{\beta s}{2}}\, \omega^{\frac{s-1}{4}}\, v_1|| \times ||(V+1)^{-\frac{(1-\beta)s}{2}}\, \omega^{\frac{s-1}{4}}\, v_2||$.

*iv)* $||(H_0+1)^{-\frac{1}{2}} a^*(v_1) a^*(v_2) (N+1)^{-1}|| \leq c\, ||(V+1)^{-\frac{\beta s}{2}}\, \omega^{\frac{s-1}{4}}\, v_1|| \times ||(V+1)^{-\frac{(1-\beta)s}{2}}\, \omega^{\frac{s-1}{4}}\, v_2||$.

*v)* $||(N+1)^{-\frac{1}{2}} (H_0+1)^{-\frac{1}{2}} a^*(v_1) a(v_2) (H_0+1)^{-\frac{1}{2}}|| \leq c$

$||(V+1)^{-\frac{1-s}{2}} \omega^{-\frac{s}{2}} v_1||_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})} \times ||(V+1)^{-\frac{s}{2}} \omega^{-\frac{1-s}{2}} v_2||_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}$.

**Proof.** Using Lemma 3.3, we see that $i) - v)$ are true with $(H_0+1)^{-\frac{1}{2}}$ replaced by $(H_0+1)^{-\frac{1-s}{2}}(V+1)^{-\frac{s}{2}}$. We use then the fact that $(V+1)^{\frac{s}{2}}(H_0+1)^{-\frac{s}{2}}$ is bounded for $s \in [0,1]$. ■

We set for $\kappa < \infty$:

$$B_\kappa(\phi) := \Big( \sum_{1 \leq i < j \leq P} V_\kappa(\mathrm{x}_i - \mathrm{x}_j) + \hat{I}_\kappa \phi | \phi \Big), \quad \phi \in \mathcal{D}(H_0^{\frac{1}{2}}).$$

**Lemma 3.5.** *There exists $\kappa_0, 0 \leq a < 1, 0 \leq b$ independent from $\kappa$, such that for $\kappa \geq 2\kappa_0$*

$$|B_\kappa(\phi)| \leq a||H_0^{\frac{1}{2}} \phi||^2 + b||\phi||^2, \quad \phi \in \mathcal{D}(H_0^{\frac{1}{2}}).$$

**Proof.** One has using Lemma 3.3

$$||(H_0+\lambda)^{-\frac{1}{2}} \hat{I}_\kappa (H_0+\lambda)^{-\frac{1}{2}}|| \leq c(\lambda^{-\frac{1}{2}} ||\omega^{-\frac{1}{2}} v_{\kappa_0}|| + ||\omega^{-\frac{1}{4}} r_\kappa||^2 + ||\omega^{-\frac{1}{2}} r_\kappa||^2 + ||\omega^{-\frac{1}{2}} r_\kappa||). \tag{3.11}$$

We notice that $\omega^{-\frac{1}{4}} r_\kappa$ contains the term $1 - \chi_{\kappa_0}$, for $\kappa \geq 2\kappa_0$, which is arbitrarily small for $\kappa_0$ large enough. Using this fact and (3.11) we see that there exists $\kappa_0$ such that $\hat{I}_\kappa$ has a small $H_0-$form bound for $\kappa > 2\kappa_0$. The integrand of $V_\kappa^{(1)}, V_\kappa^{(2)}$ contain the term $1 - \chi_{\kappa_0}$, then there exist $\kappa_0$ such that $V_\kappa^{(1)}$ (resp. $V_\kappa^{(2)}$) has a small $L^\infty(\mathbb{R}^3)$ (resp. $L^s(\mathbb{R}^3), s > 2$) norm for $\kappa > 2\kappa_0$. After a change of variables to separate the motion of the center of mass, the term $V_\kappa^{(2)}$ becomes $V_\kappa^{(3)} + V_\kappa^{(4)} \in L^\infty + L^q$ where $2 \leq q < \infty$. Using the Sobolev injection $H^1(\mathbb{R}^3) \to L^{q'}(\mathbb{R}^3)$, $2 \leq q' \leq 6$, we obtain, by a convenient choice of $q$ and $q'$, that $V_\kappa^{(2)}$ is $H_0-$form bounded [21]. ■

**Theorem 3.6.** *There is a unique selfadjoint operator $\hat{H}_\infty$ acting on $\mathcal{H}$, satisfying*

$$i) \lim_{\kappa \to \infty} (\hat{H}_\kappa - z)^{-1} = (\hat{H}_\infty - z)^{-1},$$

$$ii) \ s - \lim_{\kappa \to \infty} \mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa} = \mathrm{e}^{-\mathrm{i}t\hat{H}_\infty} \text{ for } t \in \mathbb{R}.$$

*The domain of $\hat{H}_\infty$ satisfies $\mathcal{D}(\hat{H}_\infty) \subset \mathcal{D}(\hat{H}_\infty^{\frac{1}{2}}) = \mathcal{D}(H_0^{\frac{1}{2}})$.*

**Proof.** The proof is based on the Thm. A.1 in the appendix. Let us apply now this theorem with $B_\kappa(\phi) = \Big( \sum_{1 \leq i < j \leq P} V_\kappa(\mathrm{x}_i - \mathrm{x}_j) + \hat{I}_\kappa \phi |\, \phi \Big)$. Using Lemma 3.3, one has for $\phi \in \mathcal{D}(H_0^{\frac{1}{2}})$

$$\begin{aligned} |B_\kappa(\phi) - B_{\kappa'}(\phi)| \ \leq \ & c(||V_\kappa^{(1)} - V_{\kappa'}^{(1)}||_{L^\infty} + ||V_\kappa^{(2)} - V_{\kappa'}^{(2)}||_{L^s} + ||\omega^{-\frac{1}{2}}(r_\kappa - r_{\kappa'})|| \\ & + ||\omega^{-\frac{1}{4}}(r_\kappa - r_{\kappa'})|| + ||\omega^{-\frac{1}{4}}(r_\kappa - r_{\kappa'})||^2) \times ||(H_0+1)^{\frac{1}{2}} \phi||^2. \end{aligned} \tag{3.12}$$

We see clearly, using Lemma 3.2 *ii)* and *iv)* in r.h.s. of (3.12), that $B_\kappa$ satisfies hypothesis of Thm. A.1. The KLMN theorem applied for $B_\infty$, gives that $\mathcal{D}(\hat{H}_\infty^{\frac{1}{2}}) = \mathcal{D}(H_0^{\frac{1}{2}})$. So this proves the theorem. ■

**Definition 3.7.** *The Hamiltonian $H := U_\infty^* \hat{H}_\infty U_\infty$ is called the* Nelson Hamiltonian *and $\hat{H}_\infty$ the* modified Hamiltonian.

**Theorem 3.8.** *One has*

$$i)\ \lim_{\kappa\to\infty}(H_\kappa - E_\kappa - z)^{-1} = (H-z)^{-1},$$
$$ii)\ s-\lim_{\kappa\to\infty} \mathrm{e}^{-\mathrm{i}t(H_\kappa - E_\kappa)} = \mathrm{e}^{-\mathrm{i}tH}\ \textit{for}\ t\in\mathbb{R}.$$
$$iii)\ \mathcal{D}(H)\subset\mathcal{D}(H^{\frac12}) = U_\infty^*\,\mathcal{D}(H_0^{\frac12}).$$

**Proof.** One has

$$\begin{aligned} \|U_\kappa^*(\hat H_\kappa - z)^{-1}U_\kappa - U_\infty^*(\hat H_\infty - z)^{-1}U_\infty\| &\le \|(U_\kappa^* - U_\infty^*)(\hat H_\kappa - z)^{-1}\| \\ &+\|(U_\kappa^* - U_\infty^*)(\hat H_\infty - \bar z)^{-1}\| \\ &+\|(\hat H_\kappa - z)^{-1} - (\hat H_\infty - z)^{-1}\|. \end{aligned} \tag{3.13}$$

Using (3.13), Thm. 3.6 and the fact that the map $h\to W(h)(N+1)^{-\epsilon}$, $\epsilon>0$, is norm continuous, we obtain

$$\begin{aligned} &\|U_\kappa^*(\hat H_\kappa - z)^{-1}U_\kappa - U_\infty^*(\hat H_\infty - z)^{-1}U_\infty\| \le \\ &c(\|(W(G_\kappa) - W(G_\infty))(N+1)^{-\epsilon}\| \\ &+\|(\hat H_\kappa - z)^{-1} - (\hat H_\infty - z)^{-1}\| \\ &+\|\mathrm{e}^{\mathrm{i}C_\kappa} - \mathrm{e}^{\mathrm{i}C_\infty}\|). \end{aligned}$$

The application of Lemma 3.2 completes the proof of *i*). *ii*) follows from the equivalence of the convergence in the strong resolvent sense and the strong convergence of unitary groups (Trotter theorem). *iii*) is obvious.■

Let $\chi'_\kappa$ be another cutoff function and define $H'$ to be the Nelson Hamiltonian constructed using the later cutoff.

**Proposition 3.9.** *There exists a finite constant $E$, such that*

$$H' = H + E.$$

**Proof.** We define $H''$ (resp.$\hat H''_\kappa$) to be the Nelson Hamiltonian (resp. the cutoff modified Hamiltonian) obtained using the dressing transformation given by $g''_\kappa := -\mathrm{i}\frac{(2\pi)^{-\frac32}}{\omega^{\frac12}(k)}\frac{\chi'_\kappa(k)-\chi_{\kappa_0}(k)}{\omega(k)+\frac{K^2}{2M}}$. It is easy to see that $H' = H'' + E$, where $E$ is a finite constant. Using similar calculus with (3.12), we obtain that $\hat H_\infty = \hat H''_\infty$. Since $U_\infty = U''_\infty$ we have $H = H''$. Then $H' = H + E$. ■

## 4. Higher order estimates

In this section we prove some estimates which allow to bound powers of $N$ and $H_0$ by powers of $H$. They play an important role. Fröhlich has proved a higher estimates in the massless case [14], but they are different from what we intend to prove. Their proofs are based in the following principle of cutoff independence [22].

**Lemma 4.1.** *Let $\{N_j\}$ and $\{H_j\}$ be sequences of operators such that, for $c$ independent of $j$,*

$$\|N_j\psi\| \le c\,\|H_j\psi\|,\ \textit{for}\ \psi\in\mathcal{D}(H_j).$$

*Suppose that $N_j$ is selfadjoint and that $N_j\to N$ in the strong resolvent sense, where $N$ is selfadjoint, and $H_j\to H$ in the strong graph limit. Then*

$$\|N\psi\| \le c\,\|H\psi\|,\ \textit{for}\ \psi\in\mathcal{D}(H).$$

Let $\Gamma$ (*resp.* $\Gamma_j$) denotes the graph of $H$ (*resp.* $H_j$). We recall that $H_j\to H$ in the strong graph sense if for all $(\psi,\varphi)\in\Gamma$, there exist a sequence $(\psi_j,\varphi_j)\in\Gamma_j$ which converges to $(\psi,\varphi)$ in $\mathcal{H}\times\mathcal{H}$.

The following easy first order estimate follows from the proof of Thm. 3.6.

**Lemma 4.2.** *There exists $c>0$ independent for $\kappa$ such that for $\kappa_0\le\kappa\le\infty$*

$$H_0 \le (\hat H_\kappa + c).$$

4.1. **Rosen estimates.** In this subsection we prove higher order estimates using a technique due to Glimm and Jaffe to prove similar estimates for the $Y_2$ and $(\varphi^4)_2$ models. It has been taken up by Rosen in [22] for the general $(\varphi^{2n})_2$ model and Fröhlich [14]. This technique is based in the so called *pull-through formula* which is the identity that we obtain, in a formal way, when we move the resolvent through a product of annihilation operators $a(k_i)$. But some care must be taken when we want to rigorously prove the pull through formula , since we need a dense subspace $\mathcal{H}_0 \subset \mathcal{D}(H_0)$ on which $\prod_i a(k_i)$ acts as an operator and satisfies $\prod_i a(k_i)\mathcal{H}_0 \subset \mathcal{D}(H_0)$ and $\prod_i a(k_i)$ can be defined as operator on $H_0\mathcal{H}_0$. We need also a resolvent control of the commutator of the modified cutoff interaction with $a(k_i)$, which allows to define it as locally integrable function with values in bounded operators on $\mathcal{D}(H_0)$. This requires $H_0\mathcal{H}_0$ to be dense.

Let $\{J_i\}_{1..j+1}$ be a set of disjoint subsets of $\{1, ..., n\}$ so that within each subset $J_i$ the elements are taken in their natural order. We introduce the notation:

$$\begin{aligned} H_\kappa^J &:= [a(k_{i_1}), ..., [a(k_{i_j}), \hat{I}_\kappa]...], \quad \text{where } J = \{i_1, ..., i_j\}, \\ R(z) &:= (z - \hat{H}_\kappa)^{-1}, \\ R_\ell(z) &:= (z - \sum_i \omega(k_i) - \hat{H}_\kappa)^{-1}, \end{aligned}$$

where the sum runs over $i \in J_\ell \cup J_{\ell+1} ... \cup J_{j+1}$.

**Lemma 4.3.** *Let $\alpha$ ,$\tau := (\tau_i)_{1..r}$ be a sequence of real numbers. For $\psi \in \mathcal{D}((H_0+1)^\alpha \prod_{i=1}^r N_{\tau_i}^{\frac{1}{2}})$*

$$(4.1)\quad ||(H_0+1)^\alpha \prod_{i=1}^r N_{\tau_i}^{\frac{1}{2}}\psi||^2 = \sum_{j=1}^r \int P_{r,j}^\tau(k_1, ..., k_j)\ ||(H_0 + \sum_{i=1}^j \omega(k_i) + 1)^\alpha \prod_{i=1}^j a(k_i)\psi||^2 dk_1...dk_j.$$

**Proof.** This lemma is similar to Corollary 2.2. We prove (4.1) for $\psi \in \mathcal{D}(K) \otimes \Gamma_{\rm fin}(S(\mathbb{R}^3))$ and then we extend it to $\psi \in \mathcal{D}((H_0+1)^\alpha \prod_{i=1}^r N_{\tau_i}^{\frac{1}{2}})$. This extends $\prod_{i=1}^j a(k_i)$ to bounded operator from $\mathcal{D}((H_0+1)^\alpha \prod_{i=1}^r N_{\tau_i}^{\frac{1}{2}})$ into $(H_0 + \sum_{i=1}^j \omega(k_i) + 1)^{-\alpha} L^2(\mathbb{R}^{3j}, (P_{r,j}^\tau)^{\frac{1}{2}} dk) \otimes \mathcal{H}$. ■

Using the fact that $(H_0 + \sum_{i=1}^j \omega(k_i) + 1)^{-1}H_0$ is bounded and (4.1), we have for $\psi \in (H_0+1)^{-1}\mathcal{D}(N^{\frac{r}{2}})$

$$\begin{aligned} \int ||N^{\frac{p-r}{2}} H_0 \prod_{i=1}^r a(k_i)\psi||^2 dk &\leq c\int ||(H_0 + \sum_i^r \omega(k_i) + 1)\prod_{i=1}^r a(k_i) N^{\frac{p-r}{2}}\psi||^2 dk \\ &\leq c\int ||\prod_{i=1}^r a(k_j)(H_0+1)N^{\frac{p-r}{2}}\psi||^2 dk \\ &\leq c\,||N^{\frac{p}{2}}(H_0+1)\psi||^2. \end{aligned}$$

Hence $\prod_{i=1}^r a(k_i)$ can be defined as bounded operator from $(H_0+1)^{-1}\mathcal{D}(N^{\frac{p}{2}})$ into $L^2(\mathbb{R}^{3r}, dk) \otimes (H_0 + 1)^{-1}\mathcal{D}(N^{\frac{p-r}{2}})$. It is easy to see, using commutation relations and Lemma 3.1, that $(N+1)^{-\frac{p}{2}} \hat{I}_\kappa N^{\frac{p}{2}} (H_0+1)^{-1}$ is bounded for $\kappa < \infty$, hence $\hat{I}_\kappa$ can be defined as bounded operator from $(H_0+1)^{-1}\mathcal{D}(N^{\frac{p}{2}})$ into $\mathcal{D}(N^{\frac{p}{2}})$. So the commutator $H_\kappa^{\{1\}}$ acts as bounded operator from $(H_0+1)^{-1}\mathcal{D}(N^{\frac{p}{2}})$ into $L^2(\mathbb{R}^3, dk_1) \otimes \mathcal{D}(N^{\frac{p-1}{2}})$. Now for $J := \{1, ..., r\}$ we can prove by induction on $r$ that

$$(4.2)\qquad H_\kappa^{\{J\}} : (H_0+1)^{-1}\mathcal{D}(N^{\frac{p}{2}}) \to L^2(\mathbb{R}^{3r}, \prod_{i=1}^r dk_i) \otimes \mathcal{D}(N^{\frac{p-r}{2}}).$$

For $\#J = 1$ (4.2) is already done. Assume that (4.2) holds for $\#J = r$. Using the fact that $a(k_{r+1})$ maps $L^2(\mathbb{R}^{3r}, \prod_{i=1}^r dk_i) \otimes \mathcal{D}(N^{\frac{p-r}{2}})$ into $L^2(\mathbb{R}^{3(r+1)}, \prod_{i=1}^{r+1} dk_i) \otimes \mathcal{D}(N^{\frac{p-r-1}{2}})$ and maps $(H_0+1)^{-1}\mathcal{D}(N^{\frac{p}{2}})$ into

$L^2(\mathbb{R}^3, dk_{r+1}) \otimes (H_0+1)^{-1}\mathcal{D}(N^{\frac{p-1}{2}})$, we prove (4.2).
A simple computation gives:

$$H_\kappa^{\{1\}} = \frac{1}{\sqrt{2}}\sum_{j=1}^{P} v_{\kappa_0}^j(k_1) + \frac{1}{2M}\sum_{j=1}^{P} r_\kappa^j(k_1)a^*(r_\kappa^j) + r_\kappa^j(k_1)a(r_\kappa^j) - \sqrt{2}D_j r_\kappa^j(k_1), \text{ on } (H_0+1)^{-1}\mathcal{D}(N^{\frac{1}{2}}).$$

$$H_\kappa^{\{1,2\}} = \frac{1}{2M}\sum_{j=1}^{P} r_\kappa^j(k_1) r_\kappa^j(k_2), \text{ on } (H_0+1)^{-1}\mathcal{D}(N).$$

$$H_\kappa^J = 0, \text{ for all } J \text{ such that } \#J \geq 3.$$

Before starting with the pull-through formula we will prove two lemma which will be useful in the sequel.

**Lemma 4.4.** *Let $r$ be an integer and $z \notin \sigma(\hat{H}_\kappa)$, then*

$$(H_0+1)(\hat{H}_\kappa - z)^{-1} : \mathcal{D}(N^{\frac{r}{2}}) \to \mathcal{D}(N^{\frac{r}{2}}),$$

*is a bijective map.*

**Proof.** Since $(N+1)^{-\frac{r}{2}}\hat{I}_\kappa N^{\frac{r}{2}}(H_0+1)^{-1}$ is bounded for $\kappa < \infty$, we see that $(\hat{H}_\kappa - z)(H_0+1)^{-1}$ maps $\mathcal{D}(N^{\frac{r}{2}})$ into $\mathcal{D}(N^{\frac{r}{2}})$. So it is enough to show that $(H_0+1)N^{\frac{r}{2}}(\hat{H}_\kappa - z)^{-1}(1+N)^{\frac{-r}{2}}$ is bounded. We define $[N, \mathrm{i}\hat{H}_\kappa]$ as quadratic form on $\mathcal{D}(H_0)$

$$[N, \mathrm{i}\hat{H}_\kappa] = \sum_{j=1}^{P}\phi(\mathrm{i}\, v_{\kappa_0}^j) + \frac{1}{2M}\sum_{j=1}^{P} \mathrm{i}a^{*^2}(r_\kappa^j) - \mathrm{i}\, a^2(r_\kappa^j) + \mathrm{i}\sqrt{2}D_j\, a(r_\kappa^j) - \mathrm{i}\sqrt{2}a^*(r_\kappa^j)D_j.$$

For $\kappa < \infty$, using Corollary 2.4 and the fact that $\langle D\rangle(K+1)^{-\frac{1}{2}}$ is bounded, we see that $[N, \mathrm{i}\hat{H}_\kappa]$ can be defined as a bounded operator on $\mathcal{D}(H_0)$.
We set $ad_N^\ell . := [N, \mathrm{i}\, ad_N^{\ell-1}.]$ and $ad_N^1 . := [N, \mathrm{i}.]$. So $ad_N^\ell \hat{H}_\kappa$, which is similar to $ad_N^1\hat{H}_\kappa$ is defined by induction in $\ell$ as a bounded operator on $\mathcal{D}(H_0)$ and equal to:

$$ad_N^\ell \hat{H}_\kappa =$$

$$\sum_{j=1}^{P}\phi(\mathrm{i}^\ell\, v_{\kappa_0}^j) + \frac{1}{2M}\sum_{j=1}^{P}\mathrm{i}^\ell 2^{\ell-1}a^{*^2}(r_\kappa^j) - \mathrm{i}^\ell(-2)^{\ell-1}\, a^2(r_\kappa^j) - (-\mathrm{i})^\ell\sqrt{2}D_j\, a(r_\kappa^j) - \mathrm{i}^\ell\sqrt{2}a^*(r_\kappa^j)D_j.$$

Since $\mathcal{D}(N) \supset \mathcal{D}(\hat{H}_\kappa) = \mathcal{D}(H_0)$, the resolvent $(z - \hat{H}_\kappa)^{-1}$ preserves the domain of $N$. This means that the following identity

$$N(z-\hat{H}_\kappa)^{-1} = (z-\hat{H}_\kappa)^{-1}N + (z-\hat{H}_\kappa)^{-1}[N,\hat{H}_\kappa](z-\hat{H}_\kappa)^{-1}, \tag{4.3}$$

holds in the sense of bounded operators on $\mathcal{H}$. Using repeatedly (4.3) we notice that $(z-\hat{H}_\kappa)^{-1}$ preserves $\mathcal{D}(N^p)$ and we obtain on $\mathcal{D}(N^p)$

$$N^p(z-\hat{H}_\kappa)^{-1} = N^{p-1}(z-\hat{H}_\kappa)^{-1}N - \mathrm{i}N^{p-1}(z-\hat{H}_\kappa)^{-1}ad_N^1\hat{H}_\kappa\,(z-\hat{H}_\kappa)^{-1}. \tag{4.4}$$

We move now all factors of $N$ in each term to the right, we obtain the following identity between bounded operators

$$N^p\,(z-\hat{H}_\kappa)^{-1}\,N^{-p} = (z-\hat{H}_\kappa)^{-1} + \sum_{\ell=1}^{k}(z-\hat{H}_\kappa)^{-1}B_\ell(z)N^{-\ell},$$

where $B_\ell(z)$ is a polynomial in $ad_N^j\hat{H}_\kappa\,(z-\hat{H}_\kappa)^{-1}$, $j \leq \ell$. Using Lemma 3.3 with $s=0$, we see that $B_\ell(z)$ is bounded for $\kappa < \infty$. Hence $(z-\hat{H}_\kappa)^{-1}(H_0+1)$ is a bijective map from $\mathcal{D}(N^p)$ into $\mathcal{D}(N^p)$. Using Hadamard's three lines lemma [23] for

$$f(\zeta) := ((z-\hat{H}_\kappa)^{-1}H_0N^{-\zeta}\psi, N^{\bar{\zeta}}\phi), \text{ in } \mathfrak{S} := \{\zeta \in \mathbb{C}, p \leq \mathrm{Im}(\zeta) \leq p+1\},$$

where $\psi, \phi \in \mathcal{K} \otimes \Gamma_{\text{fin}}(\mathfrak{h})$. $f(\zeta)$ is a bounded continuous analytic function on $\mathfrak{S}$, satisfying

$$|f(p)| \leq c\,||\phi||\,||\psi||,$$

$$|f(p+1)| \leq c\,||\phi||\,||\psi||.$$

We obtain

$$|f(\zeta)| \leq c\,||\phi||\,||\psi||, \text{ for } \zeta \in \mathfrak{S}.$$

Let $2p \leq r \leq 2p+2$. Since $\mathcal{K} \otimes \Gamma_{\text{fin}}(\mathfrak{h})$ is a core for $N^{\frac{r}{2}}$ then $(z - \hat{H}_\kappa)^{-1} H_0 (1+N)^{-\frac{r}{2}} \psi \in \mathcal{D}(N^{\frac{r}{2}})$ and we have also

$$||N^{\frac{r}{2}} (z - \hat{H}_\kappa)^{-1} H_0 (1+N)^{-\frac{r}{2}} \psi|| \leq c\,||\psi||,\ \psi \in \mathcal{K} \otimes \Gamma_{\text{fin}}(\mathfrak{h}).$$

Then we have $(H_0+1) N^{\frac{r}{2}} (\hat{H}_\kappa - z)^{-1} (1+N)^{\frac{r}{2}}$ bounded. ■

**Lemma 4.5.** *There exist $\epsilon > 0, b < 0$ and $c$ independent for $\kappa$*

$$i)\ ||R_\ell^{\frac{1}{2}}(b) H_\kappa^{\{1\}}\, R_{\ell+1}^{\frac{1}{2}}(b)|| \leq c(\sum_{j=1}^{P} |v_{\kappa_0}^j(k_1)| + |r_\kappa^j(k_1)\omega(k_1)^{-\epsilon}|).$$

$$ii)\ ||R_\ell^{\frac{1}{2}}(b) H_\kappa^{\{1,2\}}\, R_{\ell+1}^{\frac{1}{2}}(b)|| \leq c(\sum_{j=1}^{P} |r_\kappa^j(k_1)\omega(k_1)^{-\epsilon} r_\kappa^j(k_2)\omega(k_2)^{-\epsilon}|).$$

**Proof.** Let $\mathcal{D}$ be a dense set of analytic vectors for $K$. $H_\kappa^{J_i}$ can be defined on $\mathcal{D} \otimes \Gamma_{\text{fin}}(S(\mathbb{R}^3))$, then $H_\kappa^{J_i} R_{\ell+1}^{\frac{1}{2}}(b)$ is well defined on $\mathcal{D}_\kappa(b) := (\hat{H}_\kappa - b)^{\frac{1}{2}} \mathcal{D} \otimes \Gamma_{\text{fin}}(S(\mathbb{R}^3))$. Furthermore since $\mathcal{D} \otimes \Gamma_{\text{fin}}(S(\mathbb{R}^3))$ is a core for $(\hat{H}_\kappa - b)$, $\mathcal{D}_\kappa(b)$ is dense in $\mathcal{D}((\hat{H}_\kappa - b)^{\frac{1}{2}})$, which is dense in $\mathcal{H}$. Hence it is enough to show $i) - ii)$ in $\mathcal{D}_\kappa(b)$. Lemma 3.3 and Lemma 2.8 give the bounds uniformly in $\kappa$. Then $R_\ell^{\frac{1}{2}}(b)\, H_\kappa^J\, R_{\ell+1}^{\frac{1}{2}}(b)$ extends from $\mathcal{D}_\kappa(b)$ to a bounded operator on $\mathcal{H}$. ■

The following lemma is the *generalized pull through formula*.

**Lemma 4.6.** *The following identity holds for all $\phi \in \mathcal{D}(N^{\frac{r}{2}})$:*

$$\begin{aligned}\prod_{i=1}^{r} a(k_i)(z - \hat{H}_\kappa)^{-1}\phi &= R_1(z)\prod_{i=1}^{r} a(k_i)\phi + \sum_{\text{part.}} R_1(z) H_\kappa^{J_1} R_2(z) ... R_{\ell-1}(z) H_\kappa^{J_{\ell-1}} R_\ell(z) \prod_{j\in J_\ell} a(k_j)\phi \\ &\quad + \sum_{\text{part.}} R_1(z) H_\kappa^{J_1} R_2(z) ... R_{\ell-1}(z) H_\kappa^{J_\ell} R(z)\,\phi.\end{aligned}$$

*The sum in r.h.s. is taken over all the partitions of the set $\{1, ..., r\}$ into ordered subsets.*

**Proof.** We prove this lemma by induction on $r$. By Lemma 4.4, we know that there exist $\psi \in (H_0+1)^{-1}\mathcal{D}(N^{\frac{1}{2}})$ such that $\phi = (z - \hat{H}_\kappa)\psi$. We consider $a(k_1)$ as bounded operator

$$a(k_1) : (H_0+1)^{-1}\mathcal{D}(N^{\frac{1}{2}}) \to (H_0 + \omega(k_1) + 1)^{-1} L^2(\mathbb{R}^3, dk_1) \otimes \mathcal{H}.$$

Then we can write for $\kappa < \infty$

$$a(k_1)\psi = (z - \omega(k_1) - \hat{H}_\kappa)^{-1}(z - \omega(k_1) - \hat{H}_\kappa) a(k_1)\psi.$$

By the justification in the beginning of this subsection, we see that $H_\kappa^{\{1\}}\psi \in L^2(\mathbb{R}^3, dk_1) \otimes \mathcal{H}$. We have the following identity on $L^2(\mathbb{R}^3, dk_1) \otimes \mathcal{H}$

$$(z - \omega(k_1) - \hat{H}_\kappa) a(k_1)\psi = a(k_1)(z - \hat{H}_\kappa)\psi + H_\kappa^{\{1\}}\psi. \tag{4.5}$$

This proves the pull through formula for $r = 1$.
The formula (4.5) can be generalized by induction in $r$.

We claim that for $\psi \in (H_0+1)^{-1}\,\mathcal{D}(N^{\frac{r}{2}})$, $I_r := \{1...r\}$, we have in $L^2(\mathbb{R}^{3r}, \prod_{i=1}^{r} dk_i)\otimes\mathcal{H}$:

$$(4.6)\quad (z-\sum_{i\in I_r}\omega(k_i)-\hat{H}_\kappa)\prod_{i\in I_r}a(k_i)\psi = \prod_{i\in I_r}a(k_i)(z-\hat{H}_\kappa)\psi + H_\kappa^{I_r}\psi + \sum_{I_r=(J_1,J_2)} H_\kappa^{J_1}\prod_{i\in J_2}a(k_i)\psi,$$

where the sum is over all partitions $(J_1, J_2)$ of $I_r$. Let us prove (4.6), by induction on $r$. We have for $\psi \in (H_0+1)^{-1}\mathcal{D}(N^{\frac{r+1}{2}})$:

$$\begin{aligned}(z-\sum_{i\in I_{r+1}}\omega(k_i)-\hat{H}_\kappa)\prod_{i\in I_{r+1}}a(k_i)\psi &= a(k_{r+1})(z-\sum_{i\in I_r}\omega(k_i)-\hat{H}_\kappa)\prod_{i\in I_r}a(k_i)\psi + H_\kappa^{\{r+1\}}\prod_{i\in I_r}a(k_i)\psi \\ &= \prod_{i\in I_{r+1}}a(k_i)(z-\hat{H}_\kappa)\psi + H_\kappa^{\{r+1\}}\prod_{i\in I_r}a(k_i)\psi \\ &\quad + a(k_{r+1})\sum_{I_r=(J_1,J_2)}H_\kappa^{J_1}\prod_{J_2}a(k_i)\psi + a(k_{r+1})H_\kappa^{I_r}\psi.\end{aligned}$$

Moving $a(k_{r+1})$ through $H_\kappa^{J_1}$ and $H_\kappa^{I_r}$ and using the identity

$$a(k_{r+1})H_\kappa^{J_1} = H_\kappa^{J_1}a(k_{r+1}) + H_\kappa^{J_1\cup\{r+1\}},\ \text{on}\ (H_0+1)^{-1}\mathcal{D}(N^{\frac{\#J_1+1}{2}}),$$

we obtain (4.6).
Now assume that the pull through formula holds for $r$, and let us prove it for $r+1$. We have:

$$\prod_{i\in I_{r+1}}a(k_i)(z-\hat{H}_\kappa)^{-1}\phi = (z-\sum_{i\in I_{r+1}}\omega(k_i)-\hat{H}_\kappa)^{-1}(z-\sum_{i\in I_{r+1}}\omega(k_i)-\hat{H}_\kappa)\prod_{i\in I_{r+1}}a(k_i)\psi.$$

Using now the iterated formula (4.6), we obtain:

$$\begin{aligned}\prod_{i\in I_{r+1}}a(k_i)(z-\hat{H}_\kappa)^{-1}\phi &= (z-\sum_{i\in I_{r+1}}\omega(k_i)-\hat{H}_\kappa)^{-1}\prod_{i\in I_{r+1}}a(k_i)\phi \\ &\quad +(z-\sum_{i\in I_{r+1}}\omega(k_i)-\hat{H}_\kappa)^{-1}H_\kappa^{I_{r+1}}(z-\hat{H}_\kappa)^{-1}\phi \\ &\quad +(z-\sum_{i\in I_{r+1}}\omega(k_i)-\hat{H}_\kappa)^{-1}\sum_{I_{r+1}=(J_1,J_2)}H_\kappa^{J_1}\prod_{i\in J_2}a(k_i)(z-\hat{H}_\kappa)^{-1}\phi.\end{aligned}$$

Using the induction hypothesis we obtain:

$$\sum_{I_{r+1}=(J_1,J_2)}H_\kappa^{J_1}\prod_{i\in J_2}a(k_i)(z-\hat{H}_\kappa)^{-1} = \sum_{\text{part.}}R_1(z)H_\kappa^{J_1}R_2(z)...R_{\ell-1}(z)H_\kappa^{J_{\ell-1}}R_\ell(z)\prod_{i\in J_\ell}a(k_i).$$

This completes the proof. ■

**Theorem 4.7.** *Let $\nu\le 1, 0\le\tau$ and $r\in\mathbb{N}$. Then*

$$(4.7)\qquad \|N_\nu^{\frac12}N_{-\tau}^{\frac{r-1}{2}}\psi\| \le c\,\|(\hat{H}_\kappa+b)^{\frac{r}{2}}\psi\|,\quad \textit{for}\ \psi\in\mathcal{D}(\hat{H}_\kappa^{\frac{r}{2}}),$$

*where c,b are constants independent of $\kappa$.*

**Proof.** We prove the theorem by induction on $r$. The case $r=1$ follows from Lemma 4.2. Assume that (4.7) holds for all $j\le r$. Let $\phi\in(H_0+1)^{-\frac12}\mathcal{D}(N^{\frac{r-1}{2}})\subset\mathcal{D}(N^{\frac{r}{2}})$ and $\psi := R(-b)\phi$. Since $\phi\in\mathcal{D}(N^{\frac{r}{2}})$, by Lemma 4.4, we see that $\psi\in\mathcal{D}(N_\nu^{\frac12}N_{-\tau}^{\frac{r}{2}})$. Clearly, we have also $\phi\in\mathcal{D}(N_\nu^{\frac12}N_{-\tau}^{\frac{j-2}{2}})$, $j\le r$.
We have:

$$(4.8)\qquad \|N_\nu^{\frac12}N_{-\tau}^{\frac{r}{2}}\psi\|^2 = \sum_{j=1}^{r}\int P_{r,j}^{-\tau}\|(N_\nu+\sum_{i=1}^{j}\omega(k_i)^\nu)^{\frac12}\prod_{i=1}^{j}a(k_i)R(-b)\phi\|^2dk.$$

(4.8) follows by Corollary 2.2. We recall that $H_\kappa^J$ is an operator valued function in variables $k_i, i \in J$. If we denote by $dJ := dk_{i_1}...dk_{i_p}$ where $J = \{i_1, ..., i_p\}$, then by Lemma 4.5 we have $||R_\ell^{\frac{1}{2}}(-b)H_\kappa^J R_{\ell+1}^{\frac{1}{2}}(-b)|| \in L^2(\mathbb{R}^{3p}, dJ)$. Using the pull through formula in the r.h.s. of (4.8) and the fact that $(N_\nu + \sum_{i=1}^{j} \omega(k_i)^\nu)^{\frac{1}{2}} R_1^{\frac{1}{2}}(-b)$ is bounded, we obtain:

$$\begin{aligned}
||N_\nu^{\frac{1}{2}} N_{-\tau}^{\frac{r}{2}}\psi||^2 \quad &\leq \quad c\sum_{j=1}^{r}\int P_{r,j}^{-\tau}||R_1^{\frac{1}{2}}(-b)\prod_{i=1}^{j}a(k_i)\phi||^2 dk \\
&+c\sum_{j=1}^{r}\sum_{\text{part.}}\int P_{r,j}^{-\tau}\prod_{\ell=1}^{d}||R_\ell^{\frac{1}{2}}(-b)H_\kappa^{J_\ell^j}\, R_{\ell+1}^{\frac{1}{2}}(-b)||^2 \times ||R_{d+1}^{\frac{1}{2}}(-b)\prod_{i\in J_{d+1}^j}a(k_i)\phi||^2\prod_{\ell=1}^{d+1}dJ_\ell^j \\
&+c\sum_{j=1}^{r}\sum_{\text{part.}}\int P_{r,j}^{-\tau}\prod_{\ell=1}^{d'}||R_\ell^{\frac{1}{2}}(-b)H_\kappa^{J_\ell^j}\, R_{\ell+1}^{\frac{1}{2}}(-b)||^2 \times ||R^{\frac{1}{2}}(-b)\phi||^2\prod_{\ell=1}^{d'}dJ_\ell^j \\
&=: \quad I + II + III.
\end{aligned}$$

We recall that $P_{j,j}^{-\tau} = \prod_{i=1}^{j}\omega(k_i)^{-\tau}$ and by (2.7) we notice that

$$P_{r,j}^{-\tau} \leq c\,P_{j,j}^{-\tau}. \tag{4.9}$$

In $(II)$ using (4.9) we can separate the integral in variables $k_i \notin J_{d+1}^j$ and $k_i \in J_{d+1}^j$. Since $||R_\ell^{\frac{1}{2}}(-b)H_\kappa^{J_\ell^j} R_{\ell+1}^{\frac{1}{2}}(-b)|| \in L^2(\mathbb{R}^{3p}, dJ)$, then we obtain:

$$II \leq c\sum_{j=1}^{r}\sum_{\text{part.}}\int\prod_{i\in J_{d+1}^j}\omega(k_i)^{-\tau}\;||R_{d+1}^{\frac{1}{2}}(-b)\prod_{i\in J_{d+1}^j}a(k_i)\phi||^2 dJ_{d+1}^j. \tag{4.10}$$

Now reordering the terms in the r.h.s. of (4.10) and taking into account the fact that $J_{d+1}^j \neq \emptyset$, we have:

$$II \leq c\sum_{j=1}^{r-1}\int P_{j,j}^{-\tau}||(\hat{H}_\kappa + \sum_{i=1}^{j}\omega(k_i)^\nu + b)^{-\frac{1}{2}}\prod_{i=1}^{j}a(k_i)\phi||^2 dk. \tag{4.11}$$

Doing the same thing for $(III)$, we obtain:

$$III \leq c\,||R^{\frac{1}{2}}(-b)\phi||^2. \tag{4.12}$$

Collecting (4.11)-(4.12) we obtain

$$\begin{aligned}
||N_\nu^{\frac{1}{2}} N_{-\tau}^{\frac{r}{2}}\psi||^2 \quad &\leq \quad c\,\Big(\sum_{j=1}^{r}\int P_{r,j}^{-\tau}||(\hat{H}_\kappa + \sum_{i=1}^{j}\omega(k_i)^\nu + b)^{-\frac{1}{2}}\prod_{i=1}^{j}a(k_i)\phi||^2 dk \\
&+\sum_{j=1}^{r-1}\int P_{j,j}^{-\tau}||(\hat{H}_\kappa + \sum_{i=1}^{j}\omega(k_i)^\nu + b)^{-\frac{1}{2}}\prod_{i=1}^{j}a(k_i)\phi||^2 dk + ||R^{\frac{1}{2}}(-b)\phi||^2\Big) \\
&\leq \quad c\,\Big(\sum_{j=1}^{r}\int P_{j,j}^{-\tau}||(\hat{H}_\kappa + \sum_{i=1}^{j}\omega(k_i)^\nu + b)^{-\frac{1}{2}}\prod_{i=1}^{j}a(k_i)\phi||^2 dk + ||R^{\frac{1}{2}}(-b)\phi||^2\Big). \qquad (4.13)
\end{aligned}$$

(4.13) follows by (4.9). Now using the fact that $(\hat{H}_\kappa + \sum_{i=1}^{j} \omega(k_i)^\nu + b)^{-\frac{1}{2}}(H_0 + \sum_{i=1}^{j} \omega(k_i)^\nu + b)^{\frac{1}{2}}$ is bounded uniformly in $k_i$ and Corollary 2.2, we obtain

$$
\begin{aligned}
||N_\nu^{\frac{1}{2}} N_{-\tau}^{\frac{r}{2}} R(b)\phi||^2 &\leq c\,(\sum_{j=1}^{r} ||(H_0+b)^{-\frac{1}{2}} N_{-\tau}^{\frac{j}{2}}\phi||^2 + ||R^{\frac{1}{2}}(-b)\phi||^2) && (4.14)\\
&\leq c\,(\sum_{j=1}^{r} ||N_\nu^{\frac{1}{2}} N_{-\tau}^{\frac{j-2}{2}}\phi||^2 + ||R^{\frac{1}{2}}(-b)\phi||^2). && (4.15)
\end{aligned}
$$

In fact, using the fact that $N_{-\tau}(H_0+b)^{-\frac{1}{2}}(1+N_\nu)^{-\frac{1}{2}}$ is bounded , we see that the r.h.s. in (4.14) is less than (4.15), which holds for all $\phi \in (H_0+1)^{-\frac{1}{2}}\mathcal{D}(N^{\frac{r-1}{2}})$. Since $\mathcal{D} \otimes \Gamma_{\text{fin}}(C_0^\infty(\mathbb{R}^3)) \subset (H_0+1)^{-\frac{1}{2}}\mathcal{D}(N^{\frac{r-1}{2}})$ is a core for $N_\nu^{\frac{1}{2}} N_{-\tau}^{\frac{r-2}{2}}$, we see that $(H_0+1)^{-\frac{1}{2}}\mathcal{D}(N^{\frac{r-1}{2}})$ is dense in $\mathcal{D}(N_\nu^{\frac{1}{2}} N_{-\tau}^{\frac{r-2}{2}})$ and hence (4.15) holds for all $\phi \in \mathcal{D}(N_\nu^{\frac{1}{2}} N_{-\tau}^{\frac{r-2}{2}})$. Now let $\phi := (\hat{H}_\kappa + b)\psi, \psi \in \mathcal{D}((\hat{H}_\kappa+b)^{\frac{r+1}{2}})$ and $b>0$. Then $\phi \in \mathcal{D}((\hat{H}_\kappa+b)^{\frac{r-1}{2}}) \subset \mathcal{D}(N_\nu^{\frac{1}{2}} N^{\frac{r-2}{2}})$. The induction hypothesis and (4.15) give

$$||N_\nu^{\frac{1}{2}} N_{-\tau}^{\frac{r}{2}}\psi|| \leq c\,||(\hat{H}_\kappa+b)^{\frac{r+1}{2}}\psi||,\ \psi \in \mathcal{D}((\hat{H}_\kappa+b)^{\frac{r+1}{2}}).$$

■

**Corollary 4.8.** *Let $\gamma \geq 0$ and $\epsilon < \frac{1}{r}$, where $r \in \mathbb{N}$. Then*

$$N_\epsilon^r \leq c\,(\hat{H}_\kappa + b)^r, \tag{4.16}$$

$$H_0^{1-\gamma} N^{r-1+\gamma} \leq c\,(\hat{H}_\kappa + b)^r, \tag{4.17}$$

*where $c$ and $b$ are constants independent of $\kappa$.*

**Proof.** The inequalities follow from Lemma 2.5 and Thm. 4.7. (4.16) follows with $\nu = 1, \tau = \frac{1-\epsilon r}{1-r}$ in Thm. 4.7 and $b = \omega, \tau_1 = \epsilon, \tau_2 = 1, \tau_3 = -\frac{1-\epsilon r}{1-r}, \alpha_1 = r, \alpha_2 = 1, \alpha_3 = r-1$ in Lemma 2.5. (4.17) follows with $\nu = 1, \tau = \frac{\gamma}{r-1}$ in Thm. 4.7 and $b = \omega, \tau_1 = 0, \tau_2 = 1, \tau_3 = -\frac{\gamma}{r-1}, \alpha_1 = r-1+\gamma, \alpha_2 = \gamma, \alpha_3 = r-1$ in Lemma 2.5 and the fact that $\mathrm{d}\Gamma(\omega)^\gamma \leq H_0^{\gamma-1}\,\mathrm{d}\Gamma(\omega)$. ■

Using the principle of the cutoff independence, formulated in Lemma 4.1, we deduce similar estimates for $\hat{H}_\infty$.

**Theorem 4.9.** *Let $\gamma \geq 0$ and $\epsilon < \frac{1}{r}$, where $r \in \mathbb{N}$. $c, b$ are positive constants. Then*

$$
\begin{aligned}
N_\epsilon^r &\leq c\,(\hat{H}_\infty + b)^r,\\
H_0^{1-\gamma} N^{r-1+\gamma} &\leq c\,(\hat{H}_\infty + b)^r.
\end{aligned}
$$

Thm. 4.9 for $\epsilon = 0$ and Hadamard's three lines lemma in [22] give the following corollary.

**Corollary 4.10.** *For $r \in \mathbb{R}^+$ there are c,b positive constants such that*

$$(N+1)^r \leq c\,(\hat{H}_\infty + b)^r.$$

**Corollary 4.11.** *For $r \in \mathbb{R}^+$ there are c,b positive constants such that*

$$(N+1)^r \leq c\,(H + b)^r.$$

**Proof.** If we prove that $N^r U_\infty (N+1)^{-r}$ is bounded for $r$ positive integer, the corollary follows from Corollary 4.10.
We have on $\mathcal{D}(N)$ the identity

$$U_\infty^* N U_\infty = N - \mathrm{i}\phi(\mathrm{i}G_\infty) - \frac{1}{2}||G_\infty||^2.$$

So $U_\infty$ preserves $\mathcal{D}(N)$. By iteration we have on $\mathcal{D}(N^r)$

$$U_\infty^* N^r U_\infty = (N - \mathrm{i}\phi(\mathrm{i}G_\infty) - \frac{1}{2}||G_\infty||^2)^r.$$

Then we obtain the boundness of the operator $N^r U_\infty (N+1)^{-r}$, since

$$(N - \mathrm{i}\phi(\mathrm{i}G_\infty) - \frac{1}{2}||G_\infty||^2)^r (N+1)^{-r},$$

is bounded. ■

4.2. **Number-energy estimates.**

**Lemma 4.12.** *We have uniformly in $\kappa$ for $z$ in a compact set of $\mathbb{C}$ and $m \in \mathbb{N}, \gamma \geq 0$:*

$$i)(H_0+1)^{\frac{1-\gamma}{2}} (N+1)^{\frac{\gamma}{2}+m} (z - \hat{H}_\kappa)^{-k} (N+1)^{-m+k-1} \in O(|\mathrm{Im}(z)|^{c_{m,k}}).$$

$$ii)(N+1)^m (z - \hat{H}_\kappa)^{-k} (N+1)^{-m+k+\frac{\gamma-2}{2}} (H_0+1)^{\frac{1-\gamma}{2}} \in O(|\mathrm{Im}(z)|^{c_{m,k}}).$$

*iii) Let $\chi \in C_0^\infty(\mathbb{R})$ and $n, q \in \mathbb{N}$ :*

$$||N^n \chi(\hat{H}_\kappa) N^q|| < \infty.$$

**Proof.** We recall the identity (4.4) on $\mathcal{D}(N^k)$, which had been proved in the proof of Lemma 4.4

$$N^k (z - \hat{H}_\kappa)^{-1} = N^{k-1}(z - \hat{H}_\kappa)^{-1} N - \mathrm{i} N^{k-1} (z - \hat{H}_\kappa)^{-1} ad_N^1 \hat{H}_\kappa\, (z - \hat{H}_\kappa)^{-1}.$$

We move now all factors of $N$ in each term to the right, we obtain the following identity between bounded operators

$$N^k\, (z - \hat{H}_\kappa)^{-1}\, N^{-k} = (z - \hat{H}_\kappa)^{-1} + \sum_{\ell=1}^{k} (z - \hat{H}_\kappa)^{-\frac{1}{2}} B_\ell(z) (z - \hat{H}_\kappa)^{-\frac{1}{2}} N^{-\ell},$$

where $B_\ell(z)$ is a polynomial in $(z - \hat{H}_\kappa)^{-\frac{1}{2}}\, ad_N^j \hat{H}_\kappa\, (z - \hat{H}_\kappa)^{-\frac{1}{2}}$, $j \leq \ell$. Using Lemma 3.3 and Lemma 4.2, we see that $||B_\ell(z)|| \leq c|\mathrm{Im}(z)|^{-c_\ell}$, uniformly in $\kappa$. Using Corollary 4.8, (4.17), we see that $||H_0^{\frac{1-\gamma}{2}} N^{\frac{\gamma}{2}} (z - \hat{H}_\kappa)^{-\frac{1}{2}}|| \leq |\mathrm{Im}(z)|^{-\frac{1}{2}}$, which proves $i)$ for $k = 1$. For $k \neq 1$, we write

$$H_0^{\frac{1-\gamma}{2}} N^{m+\frac{\gamma}{2}} (z - \hat{H}_\kappa)^{-k} N^{-m+k-1} = H_0^{\frac{1-\gamma}{2}} N^{m+\frac{\gamma}{2}} (z - \hat{H}_\kappa)^{-1} N^{-m} \prod_{\ell=0}^{k} N^{m-\ell} (z - \hat{H}_\kappa)^{-1} N^{-m+\ell-1}.$$

This proves $i)$ for $k \neq 1$. The proof of $ii)$ is similar to $i)$. $iii)$ follows from the higher order estimates in Thm. 4.7 with $\tau = \nu = 0$. ■

We set:

$$\begin{aligned} \mathcal{H}^{\mathrm{ext}} &:= \mathcal{H} \otimes \Gamma(\mathfrak{h}), \ \ \hat{H}_\kappa^{\mathrm{ext}} = \hat{H}_\kappa \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega), \\ N_0 &:= N \otimes \mathbb{1}, \ \ N_\infty := \mathbb{1} \otimes N. \end{aligned}$$

**Lemma 4.13.** *We have uniformly in $\kappa$ and $z$ in a compact set of $\mathbb{C}$ and $m \in \mathbb{N}$*

$$i)\ (N_0 + N_\infty)^m\, (z - \hat{H}_\kappa^{\mathrm{ext}})^{-1}\, (N_0 + N_\infty)^{-m+1} \in O(|\mathrm{Im}(z)|^{-c_m}).$$

$$ii)\ (H_0^{\mathrm{ext}} + 1)^{\frac{1}{2}}\, (N_0 + N_\infty)^m\, (z - \hat{H}_\kappa^{\mathrm{ext}})^{-1}\, (N_0 + N_\infty)^{-m} \in O(|\mathrm{Im}(z)|^{-c_m}).$$

**Proof.** The proof is analogous to the proof of Lemma 4.12. In fact we have

$$\begin{aligned} & ad^\ell_{N_0+N_\infty} \hat{H}_\kappa^{\mathrm{ext}} = ad_N^\ell \hat{H}_\kappa \otimes \mathbb{1}, \\ & (N_0 + N_\infty)^m\, (z - \hat{H}_\kappa^{\mathrm{ext}})^{-1}\, (N_0 + N_\infty)^{-m} = \\ & (z - \hat{H}_\kappa^{\mathrm{ext}})^{-1} + \sum_{\ell=1}^{m} (z - \hat{H}_\kappa^{\mathrm{ext}})^{-\frac{1}{2}} B_\ell(z)\, (z - \hat{H}_\kappa^{\mathrm{ext}})^{-\frac{1}{2}}\, (N_0 + N_\infty)^{-\ell}, \end{aligned}$$

where $B_\ell(z) \in O(|\mathrm{Im} z|^{-c_\ell})$. ■

4.3. **Commutator estimates.** Let $q \in C_0^\infty(\mathbb{R}^3), 0 \leq q \leq 1, q = 1$ near 0. We set $q^R := q(\frac{x}{R})$. We recall that we consider $\mathfrak{h}$ in its momentum representation $L^2(\mathbb{R}^3, dk)$.
We use the following functional calculus formula, see [10], for $\chi \in C_0^\infty(\mathbb{R})$ and $A$ a selfadjoint operator:

$$\chi(A) = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\bar z}\breve{\chi}(z)\ (z-A)^{-1}\ dz \wedge d\bar z, \tag{4.18}$$

where $\breve{\chi}$ is an almost analytic extension of $\chi$, such that

$$\breve{\chi}_{|\mathbb{R}} = \chi,\ |\partial_{\bar z}\breve{\chi}(z)| \leq c_n\ |\mathrm{Im} z|^n,\ n \in \mathbb{N}.$$

**Lemma 4.14.** *Let $\chi \in C_0^\infty(\mathbb{R})$, then one has uniformly for $\kappa \leq \infty$*

$$N^n[\chi(\hat H_\kappa), \Gamma(q^R)]N^m\ \in O(R^{-1}).$$

**Proof.** Commutation relations allow to compute $[\hat H_\kappa, \Gamma(q^R)]$ as quadratic form on $D(H_0)$, which by $N_\tau-$estimates is a bounded operator on $D(H_0)$, when $\kappa < \infty$.
We have

$$\begin{aligned}
[H_0, \Gamma(q^R)] &= \mathrm{d}\Gamma(q^R, [\omega, q^R]),\\
[\phi(v_{\kappa_0}), \Gamma(q^R)] &= a^*((1-q^R)v_{\kappa_0})\Gamma(q^R) - \Gamma(q^R)a((1-q^R)v_{\kappa_0}),\\
[a^2(r_\kappa^j), \Gamma(q^R)] &= -\Gamma(q^R)a((1-q^R)r_\kappa^j)a((1+q^R)r_\kappa^j),\\
[{a^*}^2(r_\kappa^j), \Gamma(q^R)] &= a^*((1-q^R)r_\kappa^j)a^*((1+q^R)r_\kappa^j)\Gamma(q^R),\\
[a^*(r_\kappa^j)a(r_\kappa^j), \Gamma(q^R)] &= -a^*(r_\kappa^j)\Gamma(q^R)a((1-q^R)r_\kappa^j) +\ a^*((1-q^R)r_\kappa^j)\Gamma(q^R)a(r_\kappa^j),\\
[a^*(r_\kappa^j)D_j, \Gamma(q^R)] &= a^*((1-q^R)r_\kappa^j)\Gamma(q^R)D_j,\\
[D_j a(r_\kappa^j), \Gamma(q^R)] &= -D_j\Gamma(q^R)a((1-q^R)r_\kappa^j).
\end{aligned} \tag{4.19}$$

Let $\chi_1 \in C_0^\infty(\mathbb{R})$ such that $\chi_1\chi = \chi$. Using (4.18), we have:

$$\begin{aligned}
&N^n[\chi(\hat H_\kappa), \Gamma(q^R)]N^m\\
&= N^n\chi_1(\hat H_\kappa)[\chi(\hat H_\kappa), \Gamma(q^R)]N^m + N^n[\chi_1(\hat H_\kappa), \Gamma(q^R)]\chi(\hat H_\kappa)N^m\\
&= \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\bar z}\breve{\chi}(z)N^n\chi_1(\hat H_\kappa)(z-\hat H_\kappa)^{-1}\,[\hat H_\kappa, \Gamma(q^R)](z-\hat H_\kappa)^{-1}N^m dz \wedge d\bar z\\
&+\frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\bar z}\breve{\chi}_1(z)N^n\ (z-\hat H_\kappa)^{-1}\,[\hat H_\kappa, \Gamma(q^R)](z-\hat H_\kappa)^{-1}\chi(\hat H_\kappa)N^m dz \wedge d\bar z.
\end{aligned} \tag{4.20}$$

Moving the power of $N$ toward $\chi(\hat H_\kappa), \chi_1(\hat H_\kappa)$ and then using Lemma 4.12 and Corollary 4.8, we see that it is enough to show that for $b > 0$, $(N+1)^{-n}(H_0+b)^{-\frac12}[\hat H_\kappa, \Gamma(q^R)](H_0+b)^{-\frac12} \in O(R^{-1})$, uniformly in $\kappa$, to have the lemma. Using now Lemma 3.4, we obtain

$$\begin{aligned}
&||(N+1)^{-n}(H_0+b)^{-\frac12}[\hat H_\kappa, \Gamma(q^R)](H_0+b)^{-\frac12}|| \leq\\
&c(||(1-q^R)\,(V+1)^{-\frac12}v_{\kappa_0}|| + ||(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{4}}\,(1-q^R)r_\kappa||\,||\omega^{\frac{s-1}{4}}\,(1+q^R)r_\kappa||\\
&+||N^{-1}\mathrm{d}\Gamma(q^R, [\omega, q^R])|| + ||(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}}\,(1-q^R)r_\kappa||\,||(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}}r_\kappa||\\
&+||(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}}(1-q^R)r_\kappa||\,||(K+\mathrm{i})^{-\frac12}D||).
\end{aligned} \tag{4.21}$$

Using the inequality (2.3) recalled in Subsect. 2.1,

$$||N^{-1}\,\mathrm{d}\Gamma(q^R, [\omega, q^R])|| \leq ||[\omega, q^R]||,$$

and the fact that $[\omega, q^R] \in O(R^{-1})$ , we see that $||N^{-1}\mathrm{d}\Gamma(q^R,[\omega,q^R])|| \in O(R^{-1})$. Now for the other kind of terms we will proceed as follows. Since $V \geq \langle \mathrm{x}\rangle^\alpha$, $\alpha > 2$, we can pick $\mu > 0$ and $s < 1$, such that $(V+1)^{-\frac{s}{2}}\langle \mathrm{x}\rangle^{1+\mu}$ is bounded. Then using Lemma A.2, we obtain

$$||\langle \mathrm{x}\rangle^{1+\mu}\omega^{\frac{s-1}{2}}(1-q^R)r_\kappa|| \in O(R^{-1-\mu}),$$

$$||\langle \mathrm{x}\rangle^{1+\mu}\omega^{\frac{s-1}{4}}(1-q^R)r_\kappa|| \in O(R^{-1-\mu}).$$

Hence we have:

$$||(N+1)^{-n}(H_0+b)^{-\frac{1}{2}}[\hat{I}_\kappa,\Gamma(q^R)](H_0+b)^{-\frac{1}{2}}|| \in O(R^{-1-\mu}). \tag{4.22}$$

Then the integrand in (4.20) is $|\mathrm{Im}(z)|^{-2}O(R^{-1})$. This ends the proof. ■

Let $j_0 \in C_0^\infty(\mathbb{R}^3)$, $j_\infty \in C^\infty(\mathbb{R}^3)$, $0 \leq j_0$, $0 \leq j_\infty$ ,$j_0^2 + j_\infty^2 \leq 1$, $j_0 = 1$ near 0. We set for $R \geq 1$, $j^R := (j_0^R, j_\infty^R)$, where $j_0^R := j_0(\frac{x}{R}), j_\infty^R := j_\infty(\frac{x}{R})$. We set $j := j^1$

**Lemma 4.15.** *One has uniformly for* $\kappa \leq \infty$

*i)* $\chi(\hat{H}_\kappa^{\mathrm{ext}})\, I^*(j^R) - I^*(j^R)\,\chi(\hat{H}_\kappa) \in O(R^{-1})$.

*ii) Let* $\chi \in C_0^\infty(\mathbb{R})$ *then*
$(N_0+N_\infty)^n\,(\chi(\hat{H}_\kappa^{\mathrm{ext}})\, I^*(j^R) - I^*(j^R)\,\chi(\hat{H}_\kappa))\,N^m \in O(R^{-1})$.

**Proof.** The proof is similar to the previous one. Instead of (4.19), we use the identities:

$$\begin{aligned}
H_0^{\mathrm{ext}} I^*(j^R) - I^*(j^R)H_0 &= \mathrm{d}I^*(j^R,[\omega,j^R]), \text{ where } [\omega,j^R] = ([\omega,j_0^R],[\omega,j_\infty^R]),\\
\phi(v_{\kappa_0})\otimes \mathbb{1} I^*(j^R) - I^*(j^R)\phi(v_{\kappa_0}) &= \phi((1-j_0^R)v_{\kappa_0})\otimes \mathbb{1} I^*(j^R) - \mathbb{1}\tilde{\otimes}\phi(j_\infty^R v_{\kappa_0})I^*(j^R),
\end{aligned}$$

$$\begin{aligned}
a^*(r_\kappa^j)D_j\otimes \mathbb{1} I^*(j^R) - I^*(j^R)a^*(r_\kappa^j)D_j &= a^*((1-j_0^R)r_\kappa^j)D_j\otimes \mathbb{1} I^*(j^R) - \mathbb{1}\tilde{\otimes}a^*(j_\infty^R r_\kappa^j)D_j I^*(j^R),\\
D_j\,a(r_\kappa^j)\otimes \mathbb{1} I^*(j^R) - I^*(j^R)D_j\,a(r_\kappa^j) &= D_j\,a((1-j_0^R)r_\kappa^j)\otimes \mathbb{1} I^*(j^R) - \mathbb{1}\tilde{\otimes}D_j a(j_\infty^R r_\kappa^j)I^*(j^R),
\end{aligned}$$

$$\begin{aligned}
a^2(r_\kappa^j)\otimes \mathbb{1} I^*(j^R) - I^*(j^R)a^2(r_\kappa^j) &= -I^*(j^R)a((1-j_0^R)r_\kappa^j)a((1+j_0^R)r_\kappa^j),\\
{a^*}^2(r_\kappa^j)\otimes \mathbb{1} I^*(j^R) - I^*(j^R){a^*}^2(r_\kappa^j) &= (-2a^*(j_0^R r_\kappa^j)\otimes a^*(j_\infty^R r_\kappa^j) - \mathbb{1}\tilde{\otimes}{a^*}^2(j_\infty^R r_\kappa^j))\times I^*(j^R),
\end{aligned}$$

$$\begin{aligned}
a^*(r_\kappa^j)a(r_\kappa^j)\otimes \mathbb{1} I^*(j^R) - I^*(j^R)a^*(r_\kappa^j)a(r_\kappa^j) &= a^*(r_\kappa^j)\otimes \mathbb{1} I^*(j^R)a((-1+j_0^R)r_\kappa^j) + \\
&\quad (a^*((1-j_0^R)r_\kappa^j)\otimes \mathbb{1} + \mathbb{1}\tilde{\otimes}a^*(j_\infty^R r_\kappa^j))\\
&\quad \times I^*(j^R)a(r_\kappa^j).
\end{aligned}$$

We notice that for $v \in \mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h}), \mathfrak{h}_1 = \mathfrak{h}_2 = \mathfrak{h}$, we define

$$\mathbb{1}\tilde{\otimes}a^\sharp(v) : \mathcal{K}\otimes\Gamma(\mathfrak{h}_1)\otimes\Gamma(\mathfrak{h}_2) \to \mathcal{K}\otimes\Gamma(\mathfrak{h}_1)\otimes\Gamma(\mathfrak{h}_2)$$

$$\mathbb{1}\tilde{\otimes}a^\sharp(v) := T\,\mathbb{1}_{\Gamma(\mathfrak{h}_1)}\otimes a^\sharp(v)\,T^{-1},$$

where $T$ is the natural identification:

$$T : \mathcal{K}\otimes\Gamma(\mathfrak{h}_1)\otimes\Gamma(\mathfrak{h}_2) \to \Gamma(\mathfrak{h}_1)\otimes\mathcal{K}\otimes\Gamma(\mathfrak{h}_2).$$

Using (2.4) with $j_0^2 + j_\infty^2 \leq 1$, we see that $(H_0^{\mathrm{ext}} I^*(j^R) - I^*(j^R)H_0)(N+1)^{-1}$ is bounded, which shows that $I^*(j^R) : \mathcal{D}(H_0) \to \mathcal{D}(H_0^{\mathrm{ext}})$. We obtain the following identity on $\mathcal{H}$

$$C(z) \;:=\; (z-\hat{H}_\kappa^{\mathrm{ext}})^{-1}I^*(j^R) - I^*(j^R)(z-\hat{H}_\kappa)^{-1}$$

$$= \quad (z - \hat{H}_\kappa^{\mathrm{ext}})^{-1}(\hat{H}_\kappa^{\mathrm{ext}} I^*(j^R) - I^*(j^R)\hat{H}_\kappa)(z - \hat{H}_\kappa)^{-1}.$$

Using (4.18), we obtain:

$$\chi(\hat{H}_\kappa^{\mathrm{ext}})\, I^*(j^R) - I^*(j^R)\, \chi(\hat{H}_\kappa) = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\bar{z}}\breve{\chi}(z)\; C(z)\, dz \wedge d\bar{z}.$$

Using Corollary 4.8 and Lemma 3.4 we obtain for $b < 0$

$$\begin{aligned}
\|C(b)\| \quad \leq \quad & c(\|(N_\infty + N_0)^{-1}\mathrm{d}I^*(j^R, [j^R, \omega])\| + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}}(1 - j_0^R)v_{\kappa_0}\| \\
& + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}} j_\infty^R v_{\kappa_0}\| + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{4}}(1 - j_0^R) r_\kappa\|\, \|\omega^{\frac{s-1}{4}}(1 + j_0^R) r_\kappa\| \\
& + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{4}} j_\infty^R r_\kappa\|\, \|\omega^{\frac{s-1}{4}} j_0^R r_\kappa\| + \|(V+1)^{-\frac{s}{4}}\omega^{\frac{s-1}{4}} j_\infty^R r_\kappa\|^2 \\
& + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}} r_\kappa\| \times \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}}(1 - j_0^R) r_\kappa\| \\
& + (\|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}}(1 - j_0^R) r_\kappa\| \; + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}} j_\infty^R r_\kappa\|)\; \|(K + \mathrm{i})^{-\frac{1}{2}} D\| \\
& + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}} r_\kappa\| \times \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}} j_\infty^R r_\kappa\|).
\end{aligned} \tag{4.23}$$

Applying (2.4) with $[j_0^R, \omega]^2 + [j_\infty^R, \omega]^2 \in O(R^{-2})$, we obtain

$$\|(b - \hat{H}_\kappa^{\mathrm{ext}})^{-1}(H_0^{\mathrm{ext}} I^*(j^R) - I^*(j^R) H_0)(b - \hat{H}_\kappa)^{-1}\| \in O(R^{-1}).$$

For the other terms of $C(b)$, we use (4.23), the fact that we can pick $\mu > 0$ and $s < 1$ such that $(V+1)^{-\frac{s}{2}}\langle \mathrm{x}\rangle^{1+\mu}$ is bounded and Lemma A.2. We obtain:

$$\begin{aligned}
& \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}} f_\epsilon^R r_\kappa\| \in O(R^{-1-\mu}), \\
& \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{4}} f_\epsilon^R r_\kappa\| \in O(R^{-1-\mu}), \\
& \|(V+1)^{-\frac{s}{4}}\omega^{\frac{s-1}{4}} j_\infty^R r_\kappa\|^2 \in O(R^{-1-\mu}),
\end{aligned} \tag{4.24}$$

Where $f_\epsilon^R$ denotes $j_\infty^R$ or $1 - j_0^R$. Then we have

$$\|(b - \hat{H}_\kappa^{\mathrm{ext}})^{-1}(\hat{I}_\kappa \otimes \mathbb{1}\, I^*(j^R) - I^*(j^R)\hat{I}_\kappa)(b - \hat{H}_\kappa)^{-1}\| \in O(R^{-1-\mu}). \tag{4.25}$$

Hence we have $\|C(z)\| \in |\mathrm{Im}(z)|^{-2} O(R^{-1})$. This proves $i)$.

Let $\chi_1 \in C_0^\infty(\mathbb{R})$ such that $\chi_1\chi = \chi$. As in the previous lemma, we have using (4.18):

$$\begin{aligned}
& (N_0 + N_\infty)^n \chi(\hat{H}_\kappa^{\mathrm{ext}}) I^*(j^R) - I^*(j^R)\chi(\hat{H}_\kappa) N^m \\
& = (N_0 + N_\infty)^n \chi_1(\hat{H}_\kappa^{\mathrm{ext}})(\chi(\hat{H}_\kappa^{\mathrm{ext}}) I^*(j^R) - I^*(j^R)\chi(\hat{H}_\kappa)) N^m \\
& + (N_0 + N_\infty)^n (\chi_1(\hat{H}_\kappa^{\mathrm{ext}}) I^*(j^R) - I^*(j^R)\chi_1(\hat{H}_\kappa))\chi(\hat{H}_\kappa) N^m \\
& = \frac{\mathrm{i}}{2}\int_{\mathbb{C}} \partial_{\bar{z}}\breve{\chi}(z)(N_0 + N_\infty)^n \chi_1(\hat{H}_\kappa^{\mathrm{ext}}) C(z) N^m dz \wedge d\bar{z} \\
& + \frac{\mathrm{i}}{2}\int_{\mathbb{C}} \partial_{\bar{z}}\breve{\chi}_1(z)(N_0 + N_\infty)^n C(z)\chi(\hat{H}_\kappa) N^m dz \wedge d\bar{z}.
\end{aligned}$$

Moving $(N_0 + N_\infty)^n$ (resp. $N^m$) toward $\chi(\hat{H}_\kappa)$, (resp. $\chi_1(\hat{H}_\kappa^{\mathrm{ext}})$) in the last expression and then using $i)$ and Lemma 4.13 we prove $ii)$. ■

## 5. Spectral theory for the Nelson Hamiltonian

We study in this section the spectral properties of both Nelson and modified Hamiltonians. In Subsect. 5.1 we prove the existence of ground state for the Nelson Hamiltonian. We use essentially the fact that $H_\kappa - E_\kappa$ are Pauli-Fierz Hamiltonian which converge in the norm resolvent sense to $H$. This is the subject of Thm. 5.1. In Subsect. 5.2 we prove a Mourre estimate for the modified Hamiltonian, which gives that pure point spectrum is locally finite outside its thresholds.

5.1. **HVZ theorem.**

**Theorem 5.1.** *One has*

$$\sigma_{\rm ess}(H) = [\inf\sigma(H) + m, +\infty[,$$

*and* $\inf\sigma(H)$ *is a discrete eigenvalue of* $H$.

**Proof.** $H_\kappa$ is an example of a Pauli-Fierz Hamiltonian, see [11, Sect. 3]. The HVZ theorem proved in [11] for the Pauli-Fierz Hamiltonians gives for $\kappa < \infty$

$$\sigma_{\rm ess}(\hat{H}_\kappa) = \sigma_{\rm ess}(H_\kappa - E_\kappa) = [\inf\sigma(\hat{H}_\kappa) + m, +\infty[.$$

Using the fact that $(z - \hat{H}_\kappa)^{-1} \to (z - \hat{H}_\infty)^{-1}$ and [23, Thm. VIII.23], we see that

$$\lim_{\kappa\to\infty} (\inf\sigma(\hat{H}_\kappa)) = \inf\sigma(\hat{H}_\infty). \tag{5.1}$$

Let $\chi \in C_0^\infty(]-\infty, \inf\sigma(\hat{H}_\infty) + m[)$, then $\chi \in C_0^\infty(]-\infty, \inf\sigma(\hat{H}_{\kappa_n}) + m[)$ for a sequence $\kappa_n \to +\infty$. By the HVZ theorem for $\hat{H}_{\kappa_n}$, $\chi(\hat{H}_{\kappa_n})$ is compact and by Thm. 3.6 $\chi(\hat{H}_\infty)$ is compact. So we obtain

$$\sigma_{\rm ess}(\hat{H}_\infty) \subset [\inf\sigma(\hat{H}_\infty) + m, +\infty[.$$

Let us now show that $[\inf\sigma(\hat{H}_\infty) + m, +\infty[\subset \sigma_{\rm ess}(\hat{H}_\infty)$. Let $\lambda > \inf\sigma(\hat{H}_\infty) + m$. By (5.1) and the HVZ theorem for $\hat{H}_\kappa$, there exists a sequence $\kappa_n \to +\infty$ such that $\lambda \in \sigma_{\rm ess}(\hat{H}_{\kappa_n})$, or equivalently $\mu := (\lambda + c)^{-1} \in \sigma_{\rm ess}\big((\hat{H}_{\kappa_n} + c)^{-1}\big)$, $c >> 1$.
Let us show that $\mu \in \sigma_{\rm ess}\big((\hat{H}_\infty + c)^{-1}\big)$, i.e.: $\lambda \in \sigma_{\rm ess}(\hat{H}_\infty)$:
Assume the contrary and let $\chi \in C_0^\infty(\mathbb{R})$, such that $\chi(\mu) = 1, \chi((\hat{H}_\infty + c)^{-1})$ compact. Let $\varphi_{\kappa_n,j}$ be Weyl sequences for $(\hat{H}_{\kappa_n} + c)^{-1}$ at $\mu$ such that

$$||\varphi_{\kappa_n,j}|| = 1, \lim_{j\to\infty} ((\hat{H}_{\kappa_n} + c)^{-1} - \mu)\varphi_{\kappa_n,j} = 0, \text{ and } w - \lim_{j\to\infty} \varphi_{\kappa_n,j} = 0.$$

One has

$$\begin{aligned} ||\chi((\hat{H}_\infty + c)^{-1})\varphi_{\kappa_n,j} - \varphi_{\kappa_n,j}|| &\leq ||\chi((\hat{H}_{\kappa_n} + c)^{-1}) - \chi((\hat{H}_\infty + c)^{-1})|| \\ &\quad + ||\chi((\hat{H}_{\kappa_n} + c)^{-1})\varphi_{\kappa_n,j} - \varphi_{\kappa_n,j}||. \end{aligned}$$

Since $\chi((\hat{H}_\infty + c)^{-1})$ compact, there exists for $\epsilon > 0$, a $\kappa_1$ and $j_1$ such that

$$||\chi((\hat{H}_\infty + c)^{-1})\varphi_{\kappa_1,j_1} - \varphi_{\kappa_1,j_1}|| \leq 2\epsilon,$$

$$||\chi((\hat{H}_\infty + c)^{-1})\varphi_{\kappa_1,j_1}|| \leq \epsilon.$$

We obtain $||\varphi_{\kappa_1,j_1}|| \leq 3\epsilon$, this gives a contradiction, if we choose $\epsilon < \frac{1}{3}$. ■

5.2. **Mourre estimate.** We denote by $b$ the operator acting on $\mathfrak{h}$ defined by

$$b := \frac{1}{2}(\nabla\omega\,.\,D_k + D_k\,.\,\nabla\omega), \text{ on } C_0^\infty(\mathbb{R}^3).$$

[1, Prop. 4.2.3] yields that the closure of $b$ is the infinitesimal generator of the strongly continuous unitary group $U_t$ associated to the vector field $\nabla\omega$ in the following sense

$$U_t F := [\det\nabla\phi_{-t}(x)]^{\frac{1}{2}}\, F(\phi_{-t}(x)),\ F \in S'(\mathbb{R}^3) \tag{5.2}$$

where $\phi_t$ is the flow of the vector field $\nabla\omega$. Moreover $C_0^\infty(\mathbb{R}^3)$ is a core for $b$. We denote in the sequel by $b$ its closure. We set $B := {\rm d}\Gamma(b)$. Clearly $B$ is essentially selfadjoint on $\Gamma_{\rm fin}(C_0^\infty(\mathbb{R}^3))$.
We denote by $\tau$ the set of *thresholds*,

$$\tau := \sigma_{\rm pp}(\hat{H}_\infty) + m\mathbb{N}^*.$$

Let $S$ be a selfadjoint operator on $\mathcal{H}$, we say that $S$ is of class $C^1(B)$, see [1], if the map

$$t \mapsto {\rm e}^{{\rm i}tB}(S - z)^{-1}\,{\rm e}^{-{\rm i}tB},$$

is strongly $C^1$ for some $z \in \mathbb{C} \setminus \sigma(S)$. By [1, Lemma 6.2.9] $S \in C^1(B)$ if and only if the sesquilinear form $[B,(z-S)^{-1}]$ on $\mathcal{D}(B)$ is continuous for the topology of $\mathcal{H}$ i.e:

$$|((S-z)^{-1}\varphi, B\varphi) - (B\varphi,(S-z)^{-1}\varphi)| \le c\,||\varphi||^2 \quad \text{for } \varphi \in \mathcal{D}(B). \tag{5.3}$$

We recall here a well known theorem, see [1, Thm. 6.2.10].

**Theorem 5.2.** *Let $S, B$ two selfadjoint operators acting on Hilbert space. If $S \in C^1(B)$ then*

$$\begin{aligned} &i)\, \mathcal{D}(S) \cap \mathcal{D}(B)\, is\, dense\, in\, \mathcal{D}(S) \\ &ii)\, ([B,S]u,u) \le c\,||Su||^2, u \in \mathcal{D}(S) \cap \mathcal{D}(B) \\ &iii)\, [B,(z-S)^{-1}] = (z-S)^{-1}[B,S](z-S)^{-1} \end{aligned}$$

*where $iii)$ is understood as identity between bounded operators in the following sense:*

$$\mathcal{H} \overset{(z-S)^{-1}}{\longrightarrow} \mathcal{D}(S) \overset{[B,S]}{\longrightarrow} \mathcal{D}(S)^* \overset{(z-S)^{-1}}{\longrightarrow} \mathcal{H}.$$

**Lemma 5.3.** *$\hat{H}_\kappa$ is of class $C^1(B)$ if $\kappa < \infty$.*

**Proof.** we will prove

$$\begin{aligned} &i)\ \mathrm{e}^{\mathrm{i}tB} \text{ preserves } \mathcal{D}(H_0). \\ &ii)\ |(\hat{H}_\kappa u, Bu) - (Bu, \hat{H}_\kappa u)| \le c\,||H_0 u||^2 + ||u||^2,\ u \in \mathcal{D}(H_0) \cap \mathcal{D}(B). \end{aligned}$$

[1, Thm. 6.3.4] and [1, Prop. 6.3.5] give that $i), ii)$ implies $\hat{H}_\kappa \in C^1(B)$.
Let us prove $i)$. It is enough to show that $\mathrm{e}^{\mathrm{i}tB} H_0 \mathrm{e}^{-\mathrm{i}tB}(H_0+1)^{-1}$ is bounded to have $i)$. We have using (5.2)

$$\begin{aligned} \mathrm{e}^{\mathrm{i}tB} \mathrm{d}\Gamma(\omega) \mathrm{e}^{-\mathrm{i}tB} &= \mathrm{d}\Gamma(\mathrm{e}^{\mathrm{i}tb}\omega\mathrm{e}^{-\mathrm{i}tb}) \\ &= \mathrm{d}\Gamma(\omega(\phi_{-\frac{t}{2}}(k))). \end{aligned}$$

Since $\nabla\omega$ is a bounded complete $C^\infty$ vector field we have $|\phi_t(k) - k| \le c\,|t|$ uniformly in $k$. So this implies $|\sum_1^N \omega(\phi_t(k_i)) - \omega(k_i)| \le c\,|t|\,N$, and hence $\mathrm{e}^{\mathrm{i}tB} H_0 \mathrm{e}^{-\mathrm{i}tB}(H_0+1)^{-1}$ is bounded. This prove $i)$.
Let us prove $ii)$. We compute $[\hat{H}_\kappa, \mathrm{i}B]$ :

$$\begin{aligned} &(\mathrm{i}Bu, \hat{H}_\kappa u) - (\mathrm{i}\hat{H}_\kappa u, Bu) = \\ &(\mathrm{d}\Gamma(|\nabla\omega|^2)u,u) - \sum_{j=1}^P (\phi(\mathrm{i}bv^j_{\kappa_0})u,u) + \frac{1}{2M}\sum_{j=1}^P (\mathrm{i}a(r^j_\kappa)a(br^j_\kappa)u,u) \\ &-(\mathrm{i}a^*(r^j_\kappa)a^*(br^j_\kappa)u,u) + (\mathrm{i}a^*(r^j_\kappa)a(br^j_\kappa) - \mathrm{i}a^*(br^j_\kappa)a(r^j_\kappa)u,u) \\ &-\sqrt{2}(\mathrm{i}D_j a(br^j_\kappa)u,u) + \sqrt{2}(\mathrm{i}a^*(br^j_\kappa)D_j u,u). \end{aligned}$$

A simple computation yields

$$\begin{aligned} &i)\langle \mathrm{x}\rangle^{-1} bv_{\kappa_0} \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h}), \\ &ii)\langle \mathrm{x}\rangle^{-1} br_\kappa \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h}) \text{ for } \kappa < \infty, \\ &iii)\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1} br_\kappa \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h}) \text{ for } \epsilon > 0, \text{uniformly in } \kappa. \end{aligned} \tag{5.4}$$

Now using Lemma 3.4 with $\beta = 1$ and (5.4) $i) - ii)$ we obtain for $u \in \mathcal{D}(H_0) \cap \mathcal{D}(B)$

$$\begin{aligned} |(\mathrm{i}Bu, \hat{H}_\kappa u) - (\mathrm{i}\hat{H}_\kappa u, Bu)| \quad \le \quad & c(||r_\kappa||\,||(V+1)^{-\frac{1}{2}} br_\kappa|| + ||(V+1)^{-\frac{1}{2}} bv_{\kappa_0}|| \\ &+||(K+\mathrm{i})^{-\frac{1}{2}} D||\,||(V+1)^{-\frac{1}{2}} br_\kappa||) \times (||H_0 u||^2 + ||u||^2) \end{aligned}$$

Since $V \ge \sum_i \langle \mathrm{x}_i\rangle^\alpha$, $\alpha > 2$, we prove $ii)$. This completes the proof. ∎

**Lemma 5.4.** *$\hat{H}_\infty$ is of class $C^1(B)$.*

**Proof.** Since $\hat{H}_\kappa$ is of class $C^1(B)$ we know by [1, Thm. 6.2.10] that $(z-\hat{H}_\kappa)^{-1} : \mathcal{D}(B) \to \mathcal{D}(B)$ and

$$(z-\hat{H}_\kappa)^{-1}[\hat{H}_\kappa, \mathrm{i}B](z-\hat{H}_\kappa)^{-1} = [(z-\hat{H}_\kappa)^{-1}, \mathrm{i}B], \text{ on } \mathcal{H}.$$

For $\phi \in D(B)$, one has

$$(\mathrm{i}(\hat{H}_\kappa+\mathrm{i})^{-1}\phi, B\phi) - (\mathrm{i}B\phi, (\hat{H}_\kappa-\mathrm{i})^{-1}\phi) = ((\hat{H}_\kappa+\mathrm{i})^{-1}\phi, [\hat{H}_\kappa, \mathrm{i}B](\hat{H}_\kappa-\mathrm{i})^{-1}\phi).$$

Using Lemma 3.3 and (4.17) we obtain

$$\begin{aligned}&|((\hat{H}_\kappa+\mathrm{i})^{-1}\phi, B\phi) - (B\phi, (\hat{H}_\kappa-\mathrm{i})^{-1}\phi)| \le \\ &c\,(||\mathrm{d}\Gamma(|\nabla\omega|^2)(N+1)^{-1}|| + ||\langle k\rangle^{-\epsilon} r_\kappa||\ ||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1} b r_\kappa|| \\ &+||\langle \mathrm{x}\rangle^{-1} b v_{\kappa_0}|| + ||(K+\mathrm{i})^{-\frac{1}{2}} D||\ ||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1} b r_\kappa||)||\phi||^2,\ \epsilon > 0,\end{aligned}$$

where c is independent from $\kappa$. Then letting $\kappa \to \infty$ and using Thm. 3.6 we obtain

$$|((\hat{H}_\infty+\mathrm{i})^{-1}\phi, \mathrm{i}B\phi) - (\mathrm{i}B\phi, (\hat{H}_\infty-\mathrm{i})^{-1}\phi)| \le\ c\,||\phi||^2 \text{ for } \phi \in D(B).$$

This implies $\hat{H}_\infty \in C^1(B)$. ■

**Lemma 5.5.** *We have for* $\chi \in C_0^\infty(\mathbb{R})$

$$i)\ w - \lim_{\kappa\to\infty}[(\hat{H}_\kappa-\mathrm{i})^{-1}, \mathrm{i}B] = [(\hat{H}_\infty-\mathrm{i})^{-1}, \mathrm{i}B],$$

$$ii)\ \lim_{\kappa\to\infty} \chi(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) = \chi(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\chi(\hat{H}_\infty).$$

**Proof.** *i)* follows from the proof of Lemma 5.4. Let us prove *ii)*:

$$\begin{aligned}&\chi(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) - \chi(\hat{H}_{\kappa'})[\hat{H}_{\kappa'}, \mathrm{i}B]\chi(\hat{H}_{\kappa'}) = \\ &(\chi(\hat{H}_\kappa) - \chi(\hat{H}_{\kappa'}))[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) \\ &+\chi(\hat{H}_{\kappa'})([\hat{H}_\kappa, \mathrm{i}B] - [\hat{H}_{\kappa'}, \mathrm{i}B])\chi(\hat{H}_\kappa) \\ &+\chi(\hat{H}_\kappa)[\hat{H}_{\kappa'}, \mathrm{i}B](\chi(\hat{H}_\kappa) - \chi(\hat{H}_{\kappa'})).\end{aligned} \tag{5.5}$$

We first claim that

$$\lim_{\kappa\to\infty} \chi(\hat{H}_\kappa) H_0^{\frac{1}{2}} = \chi(\hat{H}_\infty) H_0^{\frac{1}{2}}. \tag{5.6}$$

To have (5.6), we see using the functional calculus formula (4.18), that it is enough to show that

$$\lim_{\kappa\to\infty} (\hat{H}_\kappa - z)^{-1} H_0^{\frac{1}{2}} = (\hat{H}_\infty - z)^{-1} H_0^{\frac{1}{2}}.$$

For $\kappa, \kappa' < \infty$, we have the following operator identity on $\mathcal{H}$

$$(\hat{H}_\kappa - z)^{-1} - (\hat{H}_{\kappa'} - z)^{-1} = (\hat{H}_\kappa - z)^{-1}(\hat{H}_{\kappa'} - \hat{H}_\kappa)(\hat{H}_{\kappa'} - z)^{-1}.$$

It follows from the proof of Thm. 3.6 that:

$$\lim_{\kappa,\kappa'\to\infty} (H_0+1)^{-\frac{1}{2}}(\hat{H}_{\kappa'} - \hat{H}_\kappa)(H_0+1)^{-\frac{1}{2}} = 0.$$

Then we obtain (5.6). We also claim that:

$$\lim_{\kappa\to\infty} (H_0+1)^{-\frac{1}{2}}[\hat{H}_\kappa, \mathrm{i}B](H_0+1)^{-\frac{1}{2}}(N+1)^{-1} = (H_0+1)^{-\frac{1}{2}}[\hat{H}_\infty, \mathrm{i}B](H_0+1)^{-\frac{1}{2}}(N+1)^{-1}. \tag{5.7}$$

In fact, using Lemma 3.4 and the fact that $V \ge c\sum_i \langle x_i\rangle^2$, we get:

$$||(H_0+\mathrm{i})^{-\frac{1}{2}} a^*(br_{\kappa'} - br_\kappa) a^*(r_{\kappa'})(H_0+\mathrm{i})^{-\frac{1}{2}}\, N^{-1}|| \le ||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}(br_{\kappa'} - br_\kappa)||\ ||\langle k\rangle^{-\epsilon} r_{\kappa'}||,$$

$$||(H_0+\mathrm{i})^{-\frac{1}{2}} a(r_{\kappa'} - r_\kappa) a(br_{\kappa'})(H_0+\mathrm{i})^{-\frac{1}{2}}\, N^{-1}|| \le ||\langle k\rangle^{-\epsilon}(r_{\kappa'} - r_\kappa)||\ ||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1} br_{\kappa'}||,$$

$$||(H_0+\mathrm{i})^{-\frac{1}{2}} a^*(br_{\kappa'} - br_\kappa) a(r_{\kappa'})(H_0+\mathrm{i})^{-\frac{1}{2}}\, N^{-1}|| \le ||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}(br_{\kappa'} - br_\kappa)||\ ||\langle k\rangle^{-\epsilon} r_{\kappa'}||,$$

$$||(H_0+\mathrm{i})^{-\frac{1}{2}}a^*(br_{\kappa'}-br_\kappa)D_j(H_0+\mathrm{i})^{-\frac{1}{2}}N^{-1}|| \le ||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}(br_{\kappa'}-br_\kappa)||\;||(K+1)^{-\frac{1}{2}}D||,$$

$$||(H_0+\mathrm{i})^{-\frac{1}{2}}D_ja(br_{\kappa'}-br_\kappa)(H_0+\mathrm{i})^{-\frac{1}{2}}N^{-1}|| \le ||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}(br_{\kappa'}-br_\kappa)||\;||(K+1)^{-\frac{1}{2}}D||.$$

Using these estimates and (5.4), we obtain (5.7). Now using (5.5),(5.6) and (5.7) we obtain *ii*). ■

We have to prove a localization estimate for $[\hat{H}_\kappa, \mathrm{i}B]$ similar to the one in the Lemma 4.15.

**Lemma 5.6.** *We have uniformly in $\kappa$:*

$$\chi(\hat{H}^{\mathrm{ext}}_\kappa)\big([\hat{H}^{\mathrm{ext}}_\kappa, \mathrm{i}B^{\mathrm{ext}}]I^*(j^R) - I^*(j^R)[\hat{H}_\kappa, \mathrm{i}B]\big)\chi(\hat{H}_\kappa) \in o(R^0).$$

**Proof.** We set

$$C(z) := (z-\hat{H}^{\mathrm{ext}}_\kappa)^{-1}([\hat{H}^{\mathrm{ext}}_\kappa, \mathrm{i}B^{\mathrm{ext}}]I^*(j^R) - I^*(j^R)[\hat{H}_\kappa, \mathrm{i}B])(z-\hat{H}_\kappa)^{-1},$$

where $B^{\mathrm{ext}} := B\otimes 1\!\!1 + 1\!\!1\otimes B$. A simple computation gives

$$\begin{aligned}
[H_0^{\mathrm{ext}}, \mathrm{i}B^{\mathrm{ext}}]I^*(j^R) - I^*(j^R)[H_0, \mathrm{i}B] &= \mathrm{d}I^*(j^R, [|\nabla\omega|^2, j^R]),\\
a(r^j_\kappa)a(br^j_\kappa)\otimes 1\!\!1 I^*(j^R) - I^*(j^R)a(r^j_\kappa)a(br^j_\kappa) &= I^*(j^R)(a((j_0^R-1)r^j_\kappa)a(j_0^R br^j_\kappa)\\
&+ a(r^j_\kappa)a((j_0-1)br^j_\kappa)),\\
a^*(r^j_\kappa)a^*(br^j_\kappa)\otimes 1\!\!1 I^*(j^R) - I^*(j^R)a^*(r^j_\kappa)a^*(br^j_\kappa) &= -(a^*(j_0^R r^j_\kappa)\otimes a^*(j_\infty^R br^j_\kappa) + a^*(j_0^R br^j_\kappa)\\
&\otimes a^*(j_\infty^R r^j_\kappa) + 1\!\!1\otimes a^*(j_\infty^R r^j_\kappa)a^*(j_\infty br^j_\kappa)).
\end{aligned}$$

We have also similar identities for $a^*a, aD_j, D_ja^*$, replacing $r^j_\kappa$ by $br^j_\kappa$, in the proof of Lemma 4.15. As in the proof of Lemma 4.15 we use Corollary 4.8 and Lemma 3.3.
We have for $b<0$:

$$\begin{aligned}
&||C(b)||\\
&\le c(||(N_0+N_\infty)^{-1}\mathrm{d}I^*(j^R, [|\nabla\omega|^2, j^R])||\\
&+||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}(1-j_0^R)r_\kappa||\;||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}j_0^R br_\kappa||\\
&+||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}(1-j_0^R)br_\kappa||\;||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}j_0^R r_\kappa||\\
&+||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}j_0^R r_\kappa||\;||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}j_\infty^R br_\kappa||\\
&+||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}j_0^R br_\kappa||\;||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}j_\infty^R r_\kappa||\\
&+||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}j_\infty^R br_\kappa||\;||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}j_\infty^R r_\kappa||\\
&+||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}br_\kappa||\;||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}(1-j_0^R)r_\kappa||\\
&+||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}r_\kappa||\;||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}(1-j_0^R)br_\kappa||\\
&+||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}r_\kappa||\;||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}j_\infty^R br_\kappa||\\
&+||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}br_\kappa||\;||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}j_\infty^R r_\kappa||\\
&+||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}(1-j_0^R)br_\kappa|| + ||\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1}j_\infty^R br_\kappa||).
\end{aligned}$$

Using Lemma A.2, Lemma 3.3 and 5.4 *iii*) we obtain $C(z)\in |\mathrm{Im}(z)|^{-2}o(R^0)$ uniformly in $\kappa$. ■

**Theorem 5.7.** *The following three assertions hold:*
*i) Let* $\lambda \in \mathbb{R} \setminus \tau$. *Then there exist* $\epsilon > 0, C_0 > 0$ *and compact operator* $K_0$ *such that*

$$\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(\hat{H}_\infty) \geq \ C_0 \, \mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(\hat{H}_\infty) + K_0.$$

*ii) For all* $[\lambda_1, \lambda_2]$ *such that* $[\lambda_1, \lambda_2] \cap \tau = \emptyset$, *one has*

$$dim \, \mathbb{1}^{\mathrm{pp}}_{[\lambda_1,\lambda_2]}(\hat{H}_\infty) < \infty.$$

*Consequently* $\sigma_{\mathrm{pp}}(\hat{H}_\infty)$ *can accumulate only at* $\tau$, *which is a closed countable set.*
*iii) Let* $\lambda \in \mathbb{R} \setminus (\tau \cup \sigma_{\mathrm{pp}}(\hat{H}_\infty))$. *Then there exists* $\epsilon > 0, C_0 > 0$ *such that*

$$\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(\hat{H}_\infty) \geq \ C_0 \, \mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(\hat{H}_\infty).$$

**Proof.** We set

$$d(\lambda) := \inf\{\sum_{i=1}^{n} |\nabla\omega(k_i)|^2 \mid \ \tau + \sum_{i=1}^{n} \omega(k_i) = \lambda, \ n = 1, 2..., \ \tau \in \sigma_{\mathrm{pp}}(\hat{H}_\infty)\},$$

$$\tilde{d}(\lambda) := \inf\{\sum_{i=1}^{n} |\nabla\omega(k_i)|^2 \mid \ \tau + \sum_{i=1}^{n} \omega(k_i) = \lambda, \ n = 0, 1, ..., \ \tau \in \sigma_{\mathrm{pp}}(\hat{H}_\infty)\}.$$

$$\Delta^\mu_\lambda := [\lambda - \mu, \lambda + \mu], \ \mu > 0,$$

$$d^\mu(\lambda) := \inf_{\nu \in \Delta^\mu_\lambda} d(\nu),$$

$$\tilde{d}^\mu(\lambda) := \inf_{\nu \in \Delta^\mu_\lambda} \tilde{d}(\nu),$$

$$E_0 := \inf \sigma(\hat{H}_\infty).$$

We will follow the logic of the proof of Mourre estimate in the case of a Pauli-Fierz Hamiltonian [11]. Let us recall the statements that we will prove by induction in $n$:
$H_1(n)$: Let $\epsilon > 0$ and $\lambda \in [E_0, E_0 + nm[$. There exists a compact operator $K_0$, an interval $\Delta \ni \lambda$ such that

$$\mathbb{1}_\Delta(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\mathbb{1}_\Delta(\hat{H}_\infty) \geq (d(\lambda) - \epsilon)\mathbb{1}_\Delta(\hat{H}_\infty) + K_0.$$

$H_2(n)$: Let $\epsilon > 0$ and $\lambda \in [E_0, E_0 + nm[$. There exists an interval $\Delta \ni \lambda$ such that

$$\mathbb{1}_\Delta(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\mathbb{1}_\Delta(\hat{H}_\infty) \geq (\tilde{d}(\lambda) - \epsilon)\mathbb{1}_\Delta(\hat{H}_\infty).$$

$H_3(n)$: Let $\mu > 0, \epsilon_0 > 0$. There exists $\delta > 0$ such that for all $\lambda \in [E_0, E_0 + nm - \epsilon_0]$, one has

$$\mathbb{1}_{\Delta^\delta_\lambda}(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\mathbb{1}_{\Delta^\delta_\lambda}(\hat{H}_\infty) \geq (\tilde{d}^\mu(\lambda) - \epsilon)\mathbb{1}_{\Delta^\delta_\lambda}(\hat{H}_\infty).$$

$S_1(n)$: $\tau$ is closed countable set in $[E_0, E_0 + nm]$.
$S_2(n)$: For all $\lambda_1 \leq \lambda_2 \leq E_0 + nm$ with $[\lambda_1, \lambda_2] \cap \tau = \emptyset$, one has

$$dim \, \mathbb{1}^{\mathrm{pp}}_{[\lambda_1,\lambda_2]}(\hat{H}_\infty) < \infty.$$

The sketch of the proof is given by

$$S_2(n-1) \Rightarrow S_1(n),$$
$$S_1(n), H_3(n-1) \Rightarrow H_1(n),$$
$$H_1(n) \Rightarrow H_2(n),$$
$$H_2(n) \Rightarrow H_3(n),$$
$$H_1(n) \Rightarrow S_2(n).$$

$H(1)$ and $S(1)$ are immediate because the spectrum of $\hat{H}_\infty$ is discrete in $[E_0, E_0 + m[$. $S_2(n-1) \Rightarrow S_1(n)$ is obvious. $H_1(n) \Rightarrow H_2(n), H_2(n) \Rightarrow H_3(n)$ follow using arguments in [9],[20]. The implication $H_1(n) \Rightarrow S_2(n)$

is based in the Virial theorem which holds since $\hat{H}_\infty \in C^1(B)$, see [1, Prop. 7.2.10]. So we have only to prove the implication $(S_1(n), H_3(n-1) \Rightarrow H_1(n))$.
Let $\chi \in C_0^\infty(\mathbb{R})$

$$\chi(\hat{H}_\kappa) = I(j^R)1\!\!1_{\{0\}}(N_\infty)I^*(j^R)\chi(\hat{H}_\kappa) + I(j^R)1\!\!1_{[1,\infty[}(N_\infty)I^*(j^R)\chi(\hat{H}_\kappa) \tag{5.8}$$

$$= \Gamma(q^R)\chi(\hat{H}_\kappa) + I(j^R)1\!\!1_{[1,\infty[}(N_\infty)\chi(\hat{H}_\kappa^{\mathrm{ext}})I^*(j^R) + o(R^0). \tag{5.9}$$

(5.8) follows from the fact that $I(j^R)I^*(j^R) = 1\!\!1$ and $I(j^R)$ is bounded. (5.9) follows from the fact that:

$$\Gamma(q^R) = I(j^R)1\!\!1_{\{0\}}(N_\infty)I^*(j^R), \quad q^R = (j^R)^2; \quad I(j^R) = \Gamma(j^R)U,$$

and Lemma 4.15. We notice that the term $\Gamma(q^R)\chi(\hat{H}_\kappa)$ is compact since $\Gamma(q^R)(H_0+1)^{-\frac{1}{2}}$ is compact which is proved in [11, Lemma 4.2].
Let $\lambda \in [E_0, E_0 + nm[$. Since $S_2(n-1) \Rightarrow S_1(n)$, the set $\tau$ is closed in $[E_0, E_0 + nm]$, which gives $d(\lambda) = \sup_{\mu>0} d^\mu(\lambda)$. So we can choose $\mu$ such that $d^\mu(\lambda) \geq d(\lambda) - \frac{\epsilon}{3}$. $H_3(n-1)$ gives

$$1\!\!1_{\Delta^\delta_{\lambda_1}}(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]1\!\!1_{\Delta^\delta_{\lambda_1}}(\hat{H}_\infty) \geq (\tilde{d}^\mu(\lambda_1) - \frac{\epsilon}{3})1\!\!1_{\Delta^\delta_\lambda}(\hat{H}_\infty), \lambda_1 < E_0 + (n-1)m.$$

Replacing $\lambda_1$ with $\lambda - \mathrm{d}\Gamma(\omega(k))$, we obtain

$$\begin{aligned}
&1\!\!1_{\Delta^\delta_\lambda}(\hat{H}_\infty + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega(k)))([\hat{H}_\infty, \mathrm{i}B] + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega))1\!\!1_{\Delta^\delta_\lambda}(\hat{H}_\infty + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega(k)))1\!\!1_{[1,\infty[}(N_\infty) \\
&\geq 1\!\!1_{\Delta^\delta_\lambda}(\hat{H}_\infty + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega(k)))(\tilde{d}^\mu(\lambda - 1\!\!1 \otimes \mathrm{d}\Gamma(\omega(k))) + 1\!\!1 \otimes \mathrm{d}\Gamma(|\nabla\omega|^2) - \frac{\epsilon}{3})1\!\!1_{[1,\infty[}(N_\infty) \\
&\geq (d^\mu(\lambda) - \frac{\epsilon}{3})1\!\!1_{\Delta^\delta_\lambda}(\hat{H}_\infty + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega(k)))(\hat{H}_\infty + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega))1\!\!1_{[1,\infty[}(N_\infty) \\
&\geq (d^\mu(\lambda) - \frac{2\epsilon}{3})1\!\!1_{\Delta^\delta_\lambda}(\hat{H}_\infty + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega(k)))(\hat{H}_\infty + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega))1\!\!1_{[1,\infty[}(N_\infty)
\end{aligned}$$

Let $\chi \in C_0^\infty(\mathbb{R}), \chi_1 \in C_0^\infty(\mathbb{R})$ such that $\chi_1\chi = \chi$. One has uniformly in $\kappa$:

$$\chi(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa)$$

$$\begin{aligned}
&= \Gamma(q^R)\chi(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) \\
&+ I^*(j^R)1\!\!1_{[1,+\infty[}(N_\infty)\chi(\hat{H}_\kappa^{\mathrm{ext}})I^*(j^R)\chi_1(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) + o(R^0)
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
&= \Gamma(q^R)\chi(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) \\
&+ I^*(j^R)1\!\!1_{[1,+\infty[}(N_\infty)\chi(\hat{H}_\kappa^{\mathrm{ext}})I^*(j^R)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) + o(R^0)
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
&= \Gamma(q^R)\chi(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) \\
&+ I^*(j^R)1\!\!1_{[1,+\infty[}(N_\infty)\chi(\hat{H}_\kappa^{\mathrm{ext}})[\hat{H}_\kappa^{\mathrm{ext}}, \mathrm{i}B^{\mathrm{ext}}]\chi(\hat{H}_\kappa)I^*(j^R) + o(R^0).
\end{aligned} \tag{5.12}$$

(5.10) follows by (5.9). Lemma 4.15 *i)* gives (5.11) and (5.12) follows by Lemma 5.6.
Letting $\kappa \to \infty$ in the expression (5.12) *ii)*, using Lemma 5.5, we obtain:

$$\chi(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\chi(\hat{H}_\infty) = K_1(R) + I^*(j^R)1\!\!1_{[1,+\infty[}(N_\infty)\chi(\hat{H}_\infty^{\mathrm{ext}})[\hat{H}_\infty^{\mathrm{ext}}, \mathrm{i}B]\chi(\hat{H}_\infty^{\mathrm{ext}})I^*(j^R) + o(R^0),$$

where $K_1(R)$ is a compact operator. This gives for $\chi$ such that $\mathrm{supp}\chi \subset [\lambda - \delta, \lambda + \delta[$

$$\chi(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\chi(\hat{H}_\infty) \geq (d(\lambda) - \frac{2\epsilon}{3})\chi^2(\hat{H}_\infty) + K_1(R) + o(R^0).$$

Choosing $R$ large enough, we obtain $H_1(n)$. Properties *ii)*, *iii)* are standard consequences of *i)*. ■

## 6. Construction of the wave operators

**6.1. Asymptotic fields.** In this subsection we prove the existence of asymptotic fields using the Cook method, see e.g.[19]. We set $h_t := \mathrm{e}^{-\mathrm{i}t\omega(k)}h$, for $h \in \mathfrak{h}$ and we denote by $\mathfrak{h}_0$ the space $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. We introduce Heisenberg derivatives:

$$\mathbb{D}_\kappa := \partial_t + \mathrm{i}[H_\kappa, .],$$
$$\mathbb{D} := \partial_t + \mathrm{i}[H, .].$$

**Theorem 6.1.** *i) For $h \in \mathfrak{h}$ the strong limits*

$$W^+(h) := s - \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} W(h_t) \mathrm{e}^{-\mathrm{i}tH} \tag{6.1}$$

*exist and are called* asymptotic Weyl operators.
*ii) Furthermore*

$$W^+(h)(H+\mathrm{i})^{-1} = \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} W(h_t)(H+\mathrm{i})^{-1}\mathrm{e}^{-\mathrm{i}tH}. \tag{6.2}$$

*iii) The map*

$$\mathfrak{h} \ni h \mapsto W^+(h) \text{ is strongly continuous,}$$
$$\mathfrak{h} \ni h \mapsto W^+(h)(H+1)^{-1} \text{ is norm continuous.}$$

*iv) The Weyl commutation relations hold:*

$$W^+(h)\, W^+(g) = \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(h|g)} W^+(h+g),$$
$$W^+(h)^* = W^+(-h).$$

*v) The Hamiltonian preserves the asymptotic Weyl operators:*

$$\mathrm{e}^{\mathrm{i}tH} W^+(h) \mathrm{e}^{-\mathrm{i}tH} = W^+(h_{-t}).$$

**Proof.** We have the relation on $\mathcal{H}$

$$W(h_t) = \mathrm{e}^{-\mathrm{i}tH_0} W(h) \mathrm{e}^{\mathrm{i}tH_0}.$$

Hence we can define $\partial_t W(h_t)$ as quadratic form on $\mathcal{D}(H_0)$

$$\partial_t W(h_t) = -\mathrm{i}[H_0, W(h_t)]. \tag{6.3}$$

Using (6.3) and Thm. 2.9 we have, since $\mathrm{Im}(h_t|v_\kappa) \in \mathcal{B}(\mathcal{K})$, the following identity on $\mathcal{H}$

$$\partial_t(\mathrm{e}^{\mathrm{i}tH_\kappa} W(h_t) \mathrm{e}^{-\mathrm{i}tH_\kappa}) = \mathrm{i}\mathrm{e}^{\mathrm{i}tH_\kappa} \mathrm{Im}(h_t|v_\kappa) W(h_t) \mathrm{e}^{-\mathrm{i}tH_\kappa}. \tag{6.4}$$

We will first prove (6.1) and (6.2) for $h \in \mathfrak{h}_0$ then we extend to $h \in \mathfrak{h}$. Let $h \in \mathfrak{h}_0$, we notice in this case that $(h_s|v_\kappa) = (h_s|v_{\kappa_1})$, for $\kappa \geq \kappa_1$. Since $\mathrm{e}^{\mathrm{i}tH_\kappa}$ is a strongly continuous unitary group and using the inequality (2.1) we see that $t \mapsto \mathrm{e}^{\mathrm{i}tH_\kappa}\mathrm{Im}(h_t|v_{\kappa_1})W(h_t)\mathrm{e}^{-\mathrm{i}tH_\kappa}$ is strongly measurable. Hence by integrating (6.4) we obtain on $\mathcal{H}$ the identity:

$$\mathrm{e}^{\mathrm{i}tH_\kappa} W(h_t) \mathrm{e}^{-\mathrm{i}tH_\kappa} = W(h) + \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}sH_\kappa}\mathrm{Im}(h_s|v_{\kappa_1}) W(h_s) \mathrm{e}^{-\mathrm{i}sH_\kappa} ds,\ h \in \mathfrak{h}_0. \tag{6.5}$$

Using Thm. 3.8 and the convergence dominated theorem, letting $\kappa \to \infty$ in (6.5), we obtain

$$\mathrm{e}^{\mathrm{i}tH} W(h_t) \mathrm{e}^{-\mathrm{i}tH} = W(h) + \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}sH}\mathrm{Im}(h_s|v_{\kappa_1}) W(h_s) \mathrm{e}^{-\mathrm{i}sH} ds,\ h \in \mathfrak{h}_0. \tag{6.6}$$

Moreover

$$\mathrm{e}^{\mathrm{i}tH} W(h_t)(H+\mathrm{i})^{-1} \mathrm{e}^{-\mathrm{i}tH} = W(h)(H+\mathrm{i})^{-1} + \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}sH}\mathrm{Im}(h_s|v_{\kappa_1}) W(h_s)(H+\mathrm{i})^{-1} \mathrm{e}^{-\mathrm{i}sH} ds. \tag{6.7}$$

Clearly there exists $\epsilon > 0$ such that $\sum_i \langle \mathrm{x}_i\rangle^{1+\epsilon}(H+\mathrm{i})^{-1}$ is bounded since $\sum_i \langle \mathrm{x}_i\rangle^{1+\epsilon}(\hat{H}_\infty+\mathrm{i})^{-1}$ is bounded. Lemma A.2 gives that $\sum_i \langle \mathrm{x}_i\rangle^{-1-\epsilon}\mathrm{Im}(h_s|v_{\kappa_1}) \in O(s^{-1-\epsilon})$. Then the existence of the limits (6.2) and consequently (6.1) for $h \in \mathfrak{h}_0$ follows.
Let $h \in \mathfrak{h}$ and $h_n \in \mathfrak{h}_0$ a sequence such that $\lim\limits_{n\to\infty} h_n = h$ in $\mathfrak{h}$. Corollary 4.11 and inequality (2.1) gives

$$||(\mathrm{e}^{\mathrm{i}sH}W(h_s)\mathrm{e}^{-\mathrm{i}sH} - \mathrm{e}^{\mathrm{i}tH}W(h_t)\mathrm{e}^{-\mathrm{i}tH})\,(H+\mathrm{i})^{-1}|| \leq$$

$$c\,(||(\mathrm{e}^{\mathrm{i}sH}W(h_{n,s})\mathrm{e}^{-\mathrm{i}sH} - \mathrm{e}^{\mathrm{i}tH}W(h_{n,t})\mathrm{e}^{-\mathrm{i}tH})(H+\mathrm{i})^{-1}|| + ||h_n - h||^\epsilon).$$

This inequality gives the existence of (6.2) and (6.1) for $h \in \mathfrak{h}$. This shows $i)$, $ii)$. Using (2.1) and Corollary 4.11 we have

$$||\mathrm{e}^{\mathrm{i}sH}(W(h_s) - W(g_s))\mathrm{e}^{-\mathrm{i}sH}(H+\mathrm{i})^{-1}|| \leq c\,||h-g||^{-\epsilon}. \tag{6.8}$$

Taking the limit $s \to \infty$ in (6.8) we obtain

$$||(W^+(h) - W^+(g))(H+1)^{-1}|| \leq c\,||h-g||^{-\epsilon}.$$

This proves $iii)$. The rest follows from simple computations. ■

**Theorem 6.2.** *The five following assertions hold:*
*i) There exist selfadjoint operators $\phi^+(h)$, called* asymptotic fields, *such that*

$$W^+(h) = \mathrm{e}^{\mathrm{i}\phi^+(h)}\ \text{for}\ h \in \mathfrak{h}.$$

*ii) For $h_i \in \mathfrak{h}, i = 1...n$. We have $\mathcal{D}((H+\mathrm{i})^{\frac{n}{2}}) \subset \mathcal{D}(\prod_1^n \phi^+(h_i))$ and*

$$\prod_{i=1}^n \phi^+(h_i)(H+\mathrm{i})^{-\frac{n}{2}} = \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\prod_{i=1}^n \phi^+(h_i)\mathrm{e}^{-\mathrm{i}tH}(H+\mathrm{i})^{-\frac{n}{2}}.$$

*iii) The map*

$$(h_1,...,h_n) \mapsto \prod_{i=1}^n \phi^+(h_i)(H+\mathrm{i})^{-\frac{n}{2}}\ \textit{is norm continuous.}$$

*iv) The commutation relations hold as quadratic forms on $\mathcal{D}(\phi^+(h)) \cap \mathcal{D}(\phi^+(g))$*

$$[\phi^+(h), \phi^+(g)] = \mathrm{i}\mathrm{Im}(h|g).$$

*v) We have*

$$\mathrm{e}^{\mathrm{i}tH}\phi^+(h)\mathrm{e}^{-\mathrm{i}tH} = \phi^+(h_{-t}).$$

**Proof.** Since $s \to W^+(sh)$ is strongly continuous using Thm. 6.1 $iii)$ , $i)$ follows from Stone's theorem. We intend to show the existence of the following limit for $h_i \in \mathfrak{h}$, $i = 1...n$

$$\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\prod_{i=1}^n \phi(h_{i,t})\mathrm{e}^{-\mathrm{i}tH}(H+1)^{-\frac{n}{2}}. \tag{6.9}$$

Let $h_i \in \mathfrak{h}_0$, $i = 1...n$. As in the previous proof we have the following identity as quadratic form on $\mathcal{D}(H_0)$ which extends as an operator identity on $\mathcal{H}$:

$$\partial_t(\prod_{i=1}^n \phi(h_{i,t})(N+1)^{-\frac{n}{2}}) = -\mathrm{i}[H_0, \prod_{i=1}^n \phi(h_{i,t})](N+1)^{-\frac{n}{2}}. \tag{6.10}$$

Now we compute the derivative

$$\partial_t(\mathrm{e}^{\mathrm{i}tH_\kappa}\prod_{i=1}^n \phi(h_{i,t})(H_\kappa - E_\kappa + \mathrm{i})^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH_\kappa}u, v) =$$

$$(\partial_t \prod_{i=1}^n \phi(h_{i,t})(H_\kappa - E_\kappa + \mathrm{i})^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH_\kappa}u, \mathrm{e}^{-\mathrm{i}tH_\kappa}v)$$

$$+(\prod_{i=1}^n \phi(h_{i,t})(H_\kappa - E_\kappa + \mathrm{i})^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH_\kappa}u, H_\kappa\mathrm{e}^{-\mathrm{i}tH_\kappa}v)$$

$$-(H_\kappa(H_\kappa - E_\kappa + \mathrm{i})^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH_\kappa}u, \prod_{i=1}^n \phi(h_{i,t})\mathrm{e}^{-\mathrm{i}tH_\kappa}v).$$

We have $(h_{i,t}|v_\kappa) = (h_{i,t}|v_{\kappa_1})$ for $\kappa \geq \kappa_1$. Hence we have on $\mathcal{H}$

$$\partial_t(\mathrm{e}^{\mathrm{i}tH_\kappa}\prod_{i=1}^n \phi(h_{i,t})(H_\kappa - E_\kappa + \mathrm{i})^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH_\kappa}) =$$

$$\mathrm{i}\mathrm{e}^{\mathrm{i}tH_\kappa}\sum_{j=1}^n\prod_{i\neq j}^n \mathrm{Im}(h_{j,t}|v_{\kappa_1})\phi(h_{i,t})(H_\kappa - E_\kappa + \mathrm{i})^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH_\kappa} \tag{6.11}$$

Letting $\kappa \to \infty$ in (6.11) we obtain

$$\partial_t(\mathrm{e}^{\mathrm{i}tH}\prod_{i=1}^n \phi(h_{i,t})(H + \mathrm{i})^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH}) = \mathrm{i}\mathrm{e}^{\mathrm{i}tH}\sum_{j=1}^n\prod_{i\neq j}^n \mathrm{Im}(h_{j,t}|v_{\kappa_1})\phi(h_{i,t})(H + \mathrm{i})^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH} \tag{6.12}$$

Since $\sum_i \langle \mathrm{x}_i\rangle^{-1-\epsilon}\mathrm{Im}(h_{i,t}|v_{\kappa_1}) \in O(t^{-1-\epsilon})$, the dominated convergence theorem gives the existence of (6.9) for $h \in \mathfrak{h}_0$.
Let $h_i \in \mathfrak{h}, i = 1...n$ and $h_{i,\ell} \in \mathfrak{h}_0$ sequences such that $\lim_\ell h_{i,\ell} = h_i$, in $\mathfrak{h}$. Using Corollary 2.4 and Corollary 4.8, we obtain the inequality

$$||(\mathrm{e}^{\mathrm{i}sH}\prod_{i=1}^n \phi(h_{i,s})\mathrm{e}^{-\mathrm{i}sH} - \mathrm{e}^{\mathrm{i}tH}\prod_{i=1}^n \phi(h_{i,t})\mathrm{e}^{-\mathrm{i}tH})(H+1)^{-\frac{n}{2}}|| \leq$$

$$c\,(\sum_{i=1}^n ||h_i - h_{i,\ell}|| + ||(\mathrm{e}^{\mathrm{i}sH}\prod_{i=1}^n \phi(h_{i,\ell,s})\mathrm{e}^{-\mathrm{i}sH} - \mathrm{e}^{\mathrm{i}tH}\prod_{i=1}^n \phi(h_{i,\ell,t})\mathrm{e}^{-\mathrm{i}tH})(H+1)^{-\frac{n}{2}}||).$$

Cauchy criterion for the convergence, proves the existence of the limit (6.9).
To complete *ii*) it suffices to show by induction in $n$ the existence of following limit

$$\lim_{s\to 0}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}(\frac{1}{s}(W(sh_{1,t}) - 1)\prod_{i=2}^n \phi(h_{i,t}) - \mathrm{i}\prod_{i=1}^n \phi(h_{i,t}))(H+1)^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH}u = 0,\ u \in \mathcal{H}. \tag{6.13}$$

We first prove (6.13) for $u \in \mathcal{D}((H+1)^{-\epsilon}), \epsilon > 0$, then by an argument of density we obtain (6.13) for $u \in \mathcal{H}$.
We recall (2.2):

$$\lim_{s\to 0}\sup_{||h||\leq c} s^{-1}||(W(sh) - 1\!\!1 - \mathrm{i}\phi(h))(N+1)^{-\frac{1}{2}-\epsilon}|| = 0.$$

We see that

$$\lim_{s\to 0}\sup_{t\in\mathbb{R}} ||(\frac{1}{s}(W(sh_{1,t}) - 1)\prod_{i=2}^n \phi(h_{i,t}) - \mathrm{i}\prod_{i=1}^n \phi(h_{i,t}))(H+1)^{-\frac{n}{2}-\epsilon}|| = 0.$$

This completes the proof of *ii*). *iii*) follows from *ii*) and Corollary 2.4 *ii*). *iv*) follows from the properties of CCR representations. ■

**Definition 6.3.** *We define the* asymptotic creation *and* annihilation operators *on* $\mathcal{D}(\phi^+(h)) \cap \mathcal{D}(\phi^+(\mathrm{i}h))$

$$a^+(h) := \frac{1}{\sqrt{2}}\left(\phi^+(h) - \mathrm{i}\phi^+(\mathrm{i}h)\right),$$

$$a^{+*}(h) := \frac{1}{\sqrt{2}}\left(\phi^+(h) + \mathrm{i}\phi^+(\mathrm{i}h)\right),$$

*We denote by* $a^{+\sharp}(h)$ *the operator* $a^{+*}(h)$ *or* $a^+(h)$.

We formulate now a theorem which follows from the previous one.

**Theorem 6.4.** *i)* $a^{+*}(h)$ *and* $a^+(h)$ *are closed operators.*
*ii) For* $h_i \in \mathfrak{h}, i = 1...n$. *We have* $\mathcal{D}((H+\mathrm{i})^{\frac{n}{2}}) \subset \mathcal{D}(\prod_{i=1}^{n} a^{+\sharp}(h_i))$ *and*

$$\prod_{i=1}^{n} a^{+\sharp}(h)(H+\mathrm{i})^{-\frac{n}{2}} = \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} \prod_{i=1}^{n} a^{\sharp}(h_t)\mathrm{e}^{-\mathrm{i}tH}(H+\mathrm{i})^{-\frac{n}{2}}.$$

*iii) The map*

$$(h_1, ..., h_n) \mapsto \prod_{i=1}^{n} a^{+\sharp}(h)(H+\mathrm{i})^{-\frac{n}{2}} \text{ is norm continuous.}$$

*iv) The commutation relations hold as quadratic form on* $\mathcal{D}(a^+(h)) \cap \mathcal{D}(a^+(g))$

$$[a^+(h), a^{+*}(g)] = (h|g)\mathbb{1},$$

$$[a^+(h), a^+(g)] = [a^{+*}(h), a^{+*}(g)] = 0.$$

*v) We have*

$$\mathrm{e}^{\mathrm{i}tH} a^{+\sharp}(h)\mathrm{e}^{-\mathrm{i}tH} = a^{+\sharp}(h_{-t}).$$

Similar results hold for the modified Hamiltonian $\hat{H}_\infty$. We formulate this in the following theorem.

**Theorem 6.5.** *i) For* $h \in \mathfrak{h}$ *the following limit exists*

$$\begin{aligned} \hat{W}^+(h) : &= U_\infty^* W^+(h) U_\infty \\ &= s - \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty} W(h_t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty}. \end{aligned}$$

*ii)* $h \to \hat{W}^+(h)$ *is a CCR representation. We denote by* $\hat{a}^{*+}(h), \hat{a}^+(h)$ *the* creation *and* annihilation operators *associated to this representation.*
*iii) For* $h_i \in \mathfrak{h}, i = 1...n$. *We have* $\mathcal{D}((\hat{H}_\infty+\mathrm{i})^{\frac{n}{2}}) \subset \mathcal{D}(\prod_{i=1}^{n} \hat{a}^{+\sharp}(h_i))$ *and*

$$\prod_{i=1}^{n} \hat{a}^{+\sharp}(h)(\hat{H}_\infty+\mathrm{i})^{-\frac{n}{2}} = \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty} \prod_{i=1}^{n} a^{\sharp}(h_t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty}(\hat{H}_\infty+\mathrm{i})^{-\frac{n}{2}},$$

*where* $\hat{a}^{+\sharp}(h)$ *denote either* $\hat{a}^{+*}(h)$ *or* $\hat{a}^+(h)$.
*iv) The map*

$$(h_1, ..., h_n) \mapsto \prod_{i=1}^{n} \hat{a}^{+\sharp}(h)(\hat{H}_\infty+\mathrm{i})^{-\frac{n}{2}} \text{ is norm continuous.}$$

*v) We have*

$$\mathrm{e}^{\mathrm{i}t\hat{H}_\infty} \hat{a}^{+\sharp}(h)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty} = \hat{a}^{+\sharp}(h_{-t}).$$

**Proof.** The existence of the strong limit follows from (6.1) and the fact that

$$U_\infty^* W(h_t) U_\infty = \mathrm{e}^{-\mathrm{i} Im(G_\infty|h_t)} W(h_t),$$
$$w - \lim_{t\to+\infty} h_t = 0.$$

This prove *i*). Thm. 6.1 *iv*) and *i*) give

$$\hat{W}^+(h)\,\hat{W}^+(g) = \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(h|g)}\hat{W}^+(h+g),$$
$$\hat{W}^+(h)^* = \hat{W}^+(-h).$$

This proves CCR representation. *iii*) is a consequence of Thm. 6.2 *ii*) and the fact that

$$U_\infty^* \phi(h_t) U_\infty = \phi(h_t) + \mathrm{Im}(G_\infty|h_t).$$

The rest follows from Thm. 6.4. ■

6.2. **Wave operators.** We recall the construction of the Fock subrepresentation of a CCR representation. Details can be found in [6], [12]. Let $\mathfrak{g}$ be pre-Hilbert space, and denote by $\bar{\mathfrak{g}}$ its completion. We define the space of *vacua*

$$\mathcal{K}_\pi := \{u \in \mathcal{H}\,|a_\pi(h)u = 0,\, h \in \mathfrak{g}\}.$$

**Proposition 6.6.** *i)* $\mathcal{K}_\pi$ *is a closed space.*
*ii)* $\mathcal{K}_\pi$ *is contained in the set of analytic vectors of* $\phi_\pi(h), h \in \mathfrak{g}$.

Let $\mathcal{H}_\pi := \mathcal{K}_\pi \otimes \Gamma(\bar{\mathfrak{g}})$. We define

$$\Omega_\pi : \mathcal{K}_\pi \otimes \Gamma_{\mathrm{fin}}(\bar{\mathfrak{g}}) \to \mathcal{H}$$
$$\Omega_\pi \psi \otimes \phi(h)^p \Omega := \phi_\pi(h)^p \psi, \quad h \in \mathfrak{g}, \quad \psi \in \mathcal{K}_\pi.$$

**Proposition 6.7.** *The map* $\Omega_\pi$ *extends to an isometric map*

$$\Omega_\pi : \mathcal{H}_\pi \to \mathcal{H},$$

*satisfies* $\Omega_\pi 1\!\!1 \otimes a^\sharp(h) = a_\pi^\sharp(h), \quad h \in \mathfrak{g}$.

Thm. 6.1 shows that asymptotic Weyl operators define a CCR representation. Then we define the space of *vacua* in our case

$$\mathcal{K}^+ := \{u \in \mathcal{H}|\ \ a^+(h)u = 0,\, h \in \mathfrak{h}\}.$$

We denote by $\mathcal{H}^+$ the space $\mathcal{K}^+ \otimes \Gamma(\mathfrak{h})$.

**Proposition 6.8.** *The following three assertions hold:*
*i)* $\mathcal{K}^+$ *is closed* $H$*-invariant space.*
*ii) For* $h_i \in \mathfrak{h}, i = 1...n$*. One has*

$$\mathcal{K}^+ \subset \mathcal{D}(\prod_{i=1}^n a^{+*}(h_i)).$$

*iii)*

$$1\!\!1_{\mathrm{pp}}(H) \subset \mathcal{K}^+.$$

**Proof.** The fact that $\mathcal{K}^+$ is $H-$invariant follows from Thm. 6.4 *v*). *i*) and *ii*) follow by Prop. 6.7. Let us prove now *iii*). Let $u \in \mathcal{H}$ such that $Hu = Eu$, one has

$$\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} a(h_t) \mathrm{e}^{-\mathrm{i}tH} u = 0,$$

since $s - \lim_{t\to\infty} a(h_t) = 0$ and $\mathrm{e}^{\mathrm{i}tH} a(h_t) \mathrm{e}^{-\mathrm{i}tH} u = (E+\mathrm{i})\,\mathrm{e}^{\mathrm{i}t(H-E)} a(h_t)(H+\mathrm{i})^{-1}u$. This means $a^+(h)u = 0$. ■

We define

$$H^+ := H_{|\mathcal{K}^+} \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega),$$

and the *wave operator*

$$\Omega^+ : \mathcal{H}^+ \to \mathcal{H},$$
$$\Omega^+ \psi \otimes \prod_{i=1}^{n} a^*(h_i)\Omega := \prod_{i=1}^{n} a^{+*}(h_i)\psi, \text{ for } \psi \in \mathcal{K}^+, \quad h_i \in \mathfrak{h}, \, i = 1...n.$$

**Theorem 6.9.** $\Omega^+$ *is a unitary map satisfying*

$$a^{+\sharp}(h)\Omega^+ = \Omega^+ \, 1\!\!1 \otimes a^\sharp(h), \text{ for } h \in \mathfrak{h},$$

$$H\Omega^+ = \Omega^+ H^+.$$

**Proof.** Prop. 6.8 gives that $\Omega^+$ is isometric and satisfies properties announced in the theorem. Let prove that $\Omega^+$ is unitary. Using [12, Thm. 3.3], it suffices to show that the CCR representation $h \to W^+(h)$ admits a densely defined number operator. For each finite dimensional space $\mathfrak{f} \subset \mathfrak{h}$, we define as quadratic form the following expression

$$n^+_{\mathfrak{f}}(u) := \sum_{i=1}^{\dim \mathfrak{f}} ||a^+(h_i)u||^2, \quad \{h_i\} \text{ is an orthonormal basis of } \mathfrak{f}, \, u \in \mathcal{H}.$$

Now we show that $n_+(u) := \sup\limits_{\mathfrak{f}} \, n^+_{\mathfrak{f}}(u)$ is densely defined:

$$\begin{aligned} ||n^+_{\mathfrak{f}}(u)||^2 &\leq \lim_{t\to+\infty} \sum_{i=1}^{\dim\mathfrak{f}} ||a(h_{i,t})\mathrm{e}^{-\mathrm{i}tH}u||^2 \\ &\leq \lim_{t\to+\infty} (\mathrm{e}^{-\mathrm{i}tH}u, N\mathrm{e}^{-\mathrm{i}tH}u). \end{aligned}$$

We conclude using Corollary 4.11 that $n^+(u) \leq c\,||(H+b)^{\frac{1}{2}}u||$. Thus $\mathcal{D}(H^{\frac{1}{2}}) \subset \mathcal{D}(n^+)$ and $\mathrm{Ran}\Omega^+ = \mathcal{H}$. ■

We define an *extended wave operator*

$$\hat{\Omega}^{\mathrm{ext},+} : \bigoplus_{n=0}^{\infty} \mathcal{D}((\hat{H}_\infty + 1)^{\frac{n}{2}}) \otimes \otimes_s^n \mathfrak{h} \to \mathcal{H},$$
$$\hat{\Omega}^{\mathrm{ext},+} \psi \otimes \prod_{i=1}^{n} a^*(h_i)\Omega := \prod_{i=1}^{n} \hat{a}^{+*}(h_i)\psi, \quad \psi \in \mathcal{D}((\hat{H}_\infty + 1)^{\frac{n}{2}}).$$

We set

$$\hat{\mathcal{K}}^+ := U^*_\infty \mathcal{K}^+, \quad \hat{\mathcal{H}}^+ := \hat{\mathcal{K}}^+ \otimes \Gamma(\mathfrak{h}).$$

Then we have a wave operator of the modified Hamiltonian:

$$\hat{\Omega}^+ : \hat{\mathcal{H}}^+ \to \mathcal{H},$$

$$\hat{\Omega}^+ \psi \otimes \prod_{i=1}^{n} a^*(h_i)\Omega := \prod_{i=1}^{n} \hat{a}^{+*}(h_i)\psi, \text{ for } \psi \in \hat{\mathcal{K}}^+, \quad h_i \in \mathfrak{h}, \, i = 1...n.$$

We notice that $\hat{\Omega}^{\mathrm{ext},+}_{|\hat{\mathcal{H}}^+} = \hat{\Omega}^+$. This suggests to treat sometimes $\hat{\Omega}^{\mathrm{ext},+}$ as a partial isometry. Another construction of the extended wave operator is given by the following theorem, see [11, Thm. 5.7]:

**Theorem 6.10.** *i) Let $u \in \mathcal{D}(\hat{\Omega}^{\mathrm{ext},+})$. Then one has*

$$\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty} I \mathrm{e}^{-\mathrm{i}t\hat{H}_\infty^{\mathrm{ext}}} u = \hat{\Omega}^{\mathrm{ext},+} u,$$

*where $I$ is the scattering identification operator defined in the Subsect. 2.1.*
*ii) Let $\chi \in C_0^\infty(\mathbb{R})$. $\mathrm{Ran}\,\chi(\hat{H}_\infty^{\mathrm{ext}}) \subset \mathcal{D}(\hat{\Omega}^{\mathrm{ext},+})$ and $I\chi(\hat{H}_\infty^{\mathrm{ext}}), \hat{\Omega}^{\mathrm{ext},+}\chi(\hat{H}_\infty^{\mathrm{ext}})$ are bounded operators. Moreover*

$$\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty} I \mathrm{e}^{-\mathrm{i}t\hat{H}_\infty^{\mathrm{ext}}} \chi(\hat{H}_\infty^{\mathrm{ext}}) = \hat{\Omega}^{\mathrm{ext},+}\chi(\hat{H}_\infty^{\mathrm{ext}}).$$

## 7. Propagation estimates

We make the following notations for the Heisenberg derivatives

$$\mathrm{d}_0 := \partial_t + \mathrm{i}[\omega(k), .],$$
$$\mathbb{D}_0 := \partial_t + \mathrm{i}[\mathrm{d}\Gamma(\omega), .],$$
$$\hat{\mathbb{D}}_\kappa := \partial_t + \mathrm{i}[\hat{H}_\kappa, .].$$

**Proposition 7.1.** *Let $\chi \in C_0^\infty(\mathbb{R})$. For $R' > R > 1$, there exists $c$ independent from $\kappa$ such that we have for $\kappa \le \infty$*

$$\int_1^\infty ||\mathrm{d}\Gamma(1\!\!1_{[R,R']}(\frac{|x|}{t}))^{\frac{1}{2}}\chi(\hat{H}_\kappa)\,\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u||^2\frac{dt}{t} \le c\,||u||^2.$$

**Proof.** We use a standard method in scattering theory of the N-body problem [15],[24]. It is based on a technical lemma, see e.g. [10, lemma B.4.1].
Let $F \in C^\infty(\mathbb{R})$ be a cutoff function equal to 1 near $\infty$, to 0 near the origin, with $F'(s) \ge 1\!\!1_{[R,R']}(s)$. We consider the observable

$$\Phi(t) := \chi(\hat{H}_\kappa)\mathrm{d}\Gamma(F(\frac{|x|}{t}))\chi(\hat{H}_\kappa).$$

By Lemma A.3, it is enough to show that

$$\hat{\mathbb{D}}_\kappa\Phi(t) \ge t^{-1}C_0\ \ \chi(\hat{H}_\kappa)\mathrm{d}\Gamma(F'(\frac{|x|}{t}))\,\chi(\hat{H}_\kappa) + O(t^{-1-\mu}) \tag{7.1}$$

uniformly in $\kappa$ to have the inequality.
One has

$$\hat{\mathbb{D}}_\kappa\Phi(t) = \chi(\hat{H}_\kappa)\ \mathrm{d}\Gamma(\mathrm{d}_0 F(\frac{|x|}{t}))\ \chi(\hat{H}_\kappa) + \chi(\hat{H}_\kappa)[\hat{I}_\kappa, \mathrm{i}\,\mathrm{d}\Gamma(F(\frac{|x|}{t}))]\chi(\hat{H}_\kappa).$$

Using the fact that $\mathrm{d}_0 F(\frac{|x|}{t}) \ge \frac{c_0}{t}\ F'(\frac{|x|}{t}) + O(t^{-2})$, it is sufficient to show that the second term in the previous identity is $O(t^{-1-\mu}), \mu > 0$ uniformly in $\kappa$, to have (7.1).
By simple commutation relations we obtain:

$$\begin{aligned}
[\phi(v_{\kappa_0}), \mathrm{d}\Gamma(F(\frac{|x|}{t}))] &= \mathrm{i}\phi(\mathrm{i}F(\frac{|x|}{t})v_{\kappa_0}),\\
[a^{*^2}(r_\kappa^j), \mathrm{d}\Gamma(F(\frac{|x|}{t}))] &= 2\,a^*(r_\kappa^j)a^*(F(\frac{|x|}{t})r_\kappa^j),\\
[a^2(r_\kappa^j), \mathrm{d}\Gamma(F(\frac{|x|}{t}))] &= -2\,a(r_\kappa^j)a(F(\frac{|x|}{t})r_\kappa^j),\\
[a^*(r_\kappa^j)a(r_\kappa^j), \mathrm{d}\Gamma(F(\frac{|x|}{t}))] &= a^*(r_\kappa^j)a(F(\frac{|x|}{t})r_\kappa^j) + a^*(F(\frac{|x|}{t})r_\kappa^j)a(r_\kappa^j), \qquad (7.2)\\
[D_j\,a(r_\kappa^j), \mathrm{d}\Gamma(F(\frac{|x|}{t}))] &= D_j\,a(F(\frac{|x|}{t})r_\kappa^j),\\
[a^*(r_\kappa^j)\,D_j, \mathrm{d}\Gamma(F(\frac{|x|}{t}))] &= a^*(F(\frac{|x|}{t})r_\kappa^j)\,D_j.
\end{aligned}$$

Using the functional calculus formula (4.18) and Corollary 4.8, it is enough to estimate $(N+1)^{-n}(H_0+c)^{-\frac{1}{2}}[\hat{I}_\kappa, \mathrm{i}\,\mathrm{d}\Gamma(F(\frac{|x|}{t}))](H_0+c)^{-\frac{1}{2}}, c>0$. Using (7.2) and Lemma 3.4, we obtain:

$$
\begin{aligned}
&||(N+1)^{-n}(H_0+c)^{-\frac{1}{2}}[\hat{I}_\kappa, \mathrm{i}\,\mathrm{d}\Gamma(F(\frac{|x|}{t}))](H_0+c)^{-\frac{1}{2}}|| \leq \\
&c(||(V+1)^{-\frac{1}{2}}F(\frac{|x|}{t})v_{\kappa 0}|| + ||\omega^{\frac{s-1}{4}}r^j_\kappa||\;||(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{4}}F(\frac{|x|}{t})r^j_\kappa|| \\
&+||(V+1)^{\frac{s-1}{2}}\omega^{-\frac{s}{2}}r^j_\kappa||\;||(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}}F(\frac{|x|}{t})r^j_\kappa|| \qquad (7.3)\\
&+||D\,(K+c)^{-\frac{1}{2}}||\;||(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}}F(\frac{|x|}{t})r^j_\kappa||).
\end{aligned}
$$

It remains to see that the terms $||(V+1)^{-\frac{1}{2}}F(\frac{|x|}{t})v_{\kappa 0}||$, $||(V+1)^{-\frac{s}{2}}\ \omega^{-\frac{s-1}{2}}\ F(\frac{|x|}{t})r_\kappa||$ and $||(V+1)^{-\frac{s}{2}}\ \omega^{-\frac{s-1}{4}}$ $F(\frac{|x|}{t})r_\kappa||$ are integrable for $(1-s)$ small enough. They are $O(t^{-1-\mu})$ by (4.24). Then using Lemma A.3 we finish the proof of the estimate announced in the proposition. ∎

**Proposition 7.2.** *Let $\chi \in C_0^\infty(\mathbb{R})$, $0<c_0<c_1$, and*

$$\Theta_{[c_0,c_1]}(t) := \mathrm{d}\Gamma(\langle \frac{x}{t} - \nabla\omega(k), 1\!\!1_{[c_0,c_1]}(\frac{x}{t})(\frac{x}{t}-\nabla\omega(k))\rangle).$$

*One has uniformly in $\kappa \leq \infty$*

$$\int_1^\infty ||\Theta_{[c_0,c_1]}(t)^{\frac{1}{2}}\chi(\hat{H}_\kappa)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u||^2\frac{dt}{t} \leq c||u||^2.$$

**Proof.** Let $R_0(x) \in C^\infty$ be a function such that:

$$
\begin{aligned}
&R_0(x) = 0, \text{ for } |x| \leq \frac{c_0}{2},\\
&R_0(x) = \frac{1}{2}x^2 + c, \text{ for } |x| \geq 2c_1,\\
&\nabla^2_x R_0 \geq 1\!\!1_{[c_0,c_1]}(|x|).
\end{aligned}
$$

We choose $c_1>2$, $c_2>c_1+1$ and we define the function $R(x) := F(|x|)R_0(x)$ where $F(s)=1,\ if\ s\leq c_1, F(s)=0,\ if\ s\geq c_2$.
We set

$$b(t) := R(\frac{x}{t}) - \frac{1}{2}(\langle\nabla R(\frac{x}{t}), \frac{x}{t} - \nabla\omega(k)\rangle + hc).$$

We consider the observable

$$\Phi(t) := \chi(\hat{H}_\kappa)\mathrm{d}\Gamma(b(t))\chi(\hat{H}_\kappa).$$

Pseudodifferential calculus gives

$$\chi(\hat{H}_\kappa)\mathbb{D}_0\mathrm{d}\Gamma(b(t))\chi(\hat{H}_\kappa) \geq \chi(\hat{H}_\kappa)(\frac{1}{t}\Theta_{[c_0,c_2]}(t) - \frac{1}{t}\mathrm{d}\Gamma(1\!\!1_{[v_{max}+1,c_2]}(\frac{|x|}{t})))\chi(\hat{H}_\kappa) + O(t^{-2}).$$

The first term will serves in the application of Lemma A.4 and the second is integrable along the evolution using Prop. 7.1. To complete the proof of the proposition, it suffices to show uniformly in $\kappa$ that:

$$\chi(\hat{H}_\kappa)[\hat{I}_\kappa, \mathrm{i}\,\mathrm{d}\Gamma(b(t))]\chi(\hat{H}_\kappa) \in O(t^{-1-\mu}), \mu>0. \qquad (7.4)$$

As in the Prop. 7.1, using (7.2) and (7.3), we see that (7.4) is bounded by a sum of terms $||(V+1)^{-\frac{1}{2}}b(t)v_{\kappa 0}||, ||(V+1)^{-\frac{s}{2}}\ \omega^{-\frac{1-s}{2}}\ b(t)r_\kappa||$ and $||(V+1)^{-\frac{s}{2}}\ \omega^{-\frac{1-s}{4}}\ b(t)r_\kappa||$. By (4.24) these terms are $O(t^{-1-\mu}), \mu>0$, for $(1-s)$ small enough. We end the proof by using lemma A.3. ∎

**Proposition 7.3.** *Let* $0 < c_0 < c_1$, $J \in C_0^\infty(\{c_0 < |x| < c_1\})$, $\chi \in C_0^\infty(\mathbb{R})$. *For* $1 \le i \le 3$, *one has uniformly in* $\kappa \le \infty$

$$\int_1^\infty ||\mathrm{d}\Gamma(|\ J(\frac{x}{t})(\frac{x_i}{t} - \partial_i\omega(k)) + hc\ |)^{\frac{1}{2}}\,\chi(\hat{H}_\kappa)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u||^2\frac{dt}{t} \le c||u||^2.$$

**Proof.** We set

$$A := (\frac{x}{t} - \nabla\omega(k))^2 + t^{-\delta},$$
$$b(t) := J(\frac{x}{t})A^{\frac{1}{2}}J(\frac{x}{t}).$$

Let $J_1 \in C_0^\infty(\{c_0 < |x| < c_1\})$, $0 \le J \le 1$, $J = 1$ near the support of $J_1$. We consider the observable

$$\Phi(t) := -\chi(\hat{H}_\kappa)\mathrm{d}\Gamma(b(t))\chi(\hat{H}_\kappa).$$

One has

$$\chi(\hat{H}_\kappa)\mathbb{D}_0\mathrm{d}\Gamma(b(t))\chi(\hat{H}_\kappa) = -\chi(\hat{H}_\kappa)\mathrm{d}\Gamma(\mathrm{d}_0 b(t))\chi(\hat{H}_\kappa),$$

and we have using [11, Lemma 6.4]

$$\chi(\hat{H}_\kappa)\mathbb{D}_0\mathrm{d}\Gamma(b(t))\chi(\hat{H}_\kappa) \ge \frac{c_0}{t}\chi(\hat{H}_\kappa)\mathrm{d}\Gamma(|\ J_1(\frac{x}{t})(\frac{x_i}{t} - \partial_i\omega(k)) + hc\ |)\chi(\hat{H}_\kappa)$$
$$-\tfrac{c}{t}\chi(\hat{H}_\kappa)\mathrm{d}\Gamma(\langle\tfrac{x}{t} - \nabla\omega, J_2(\tfrac{x}{t})(\tfrac{x}{t} - \nabla\omega)\rangle)\chi(\hat{H}_\kappa) + O(t^{-1-\mu}).$$

The second term is integrable along the evolution by Prop. 7.2. It's enough to show that

$$\chi(\hat{H}_\kappa)[\hat{I}_\kappa, \mathrm{i}\,\mathrm{d}\Gamma(b(t))]\chi(\hat{H}_\kappa) \in O(t^{-1-\mu}), \mu > 0, \text{ uniformly in } \kappa$$

This follows by using (7.3), the fact that $J(\frac{x}{t})A^{\frac{1}{2}} \in O(1)$ and Lemma A.2. Using Lemma A.3 we end the proof. ■

**Proposition 7.4.** *Let* $\chi \in C_0^\infty(\mathbb{R})$, *supported in* $\mathbb{R} \setminus (\tau \cup \sigma_{\mathrm{pp}}(\hat{H}_\infty))$. *There exist* $\epsilon > 0$, $C$ *independent in* $\kappa$ *and a sequence of* $\hat{H}_\kappa$ *such that for* $\kappa \le \infty$, *we have*

$$\int_1^\infty ||\Gamma(\mathbb{1}_{[0,\epsilon]}(\frac{|x|}{t}))\chi(\hat{H}_\kappa)\,\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u||^2\frac{dt}{t} \le C||u||^2.$$

**Proof.** We notice that the Prop. 7.4 is a minimal velocity estimate for a sequence of $\hat{H}_\kappa$ which is uniform in $\kappa$. Let $\chi$ supported near $\lambda$ such that $\lambda \in \mathbb{R} \setminus (\tau \cup \sigma_{\mathrm{pp}}(\hat{H}_\infty))$. Then there exists a sequence $\hat{H}_\kappa$ such that $\lambda \in \mathbb{R} \setminus (\tau \cup \sigma_{\mathrm{pp}}(\hat{H}_\kappa))$. Lemma 5.5 in Mourre estimate section gives

$$\chi(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) \ge c_\kappa\chi^2(\hat{H}_\kappa).$$

Let $\epsilon > 0$. Let $q \in C_0^\infty(|x| \le 2\epsilon)$ such that $0 \le q \le 1, q(x) = 1$, if $|x| \le \epsilon$.
We set

$$\Phi_\kappa(t) := \chi(\hat{H}_\kappa)\Gamma(q^t)\frac{B}{t}\Gamma(q^t)\chi(\hat{H}_\kappa).$$

The Heisenberg derivative of $\Phi_\kappa(t)$ is

$$\begin{aligned}
\hat{\mathbb{D}}_\kappa\Phi_\kappa(t) &= \chi(\hat{H}_\kappa)\mathrm{d}\Gamma(q^t, \mathrm{d}_0 q^t)\,\tfrac{B}{t}\Gamma(q^t)\chi(\hat{H}_\kappa) + hc \\
&\quad +\chi(\hat{H}_\kappa)[\hat{I}_\kappa, \mathrm{i}\Gamma(q^t)]\,\frac{B}{t}\Gamma(q^t)\chi(\hat{H}_\kappa) + hc \\
&\quad +t^{-1}\chi(\hat{H}_\kappa)\Gamma(q^t)[\hat{H}_\kappa, \mathrm{i}B]\Gamma(q^t)\chi(\hat{H}_\kappa) \\
&\quad -t^{-1}\chi(\hat{H}_\kappa)\Gamma(q^t)\frac{B}{t}\Gamma(q^t)\Gamma(q^t)\chi(\hat{H}_\kappa) \\
&=: R_\kappa^1 + R_\kappa^2 + R_\kappa^3 + R_\kappa^4.
\end{aligned}$$

We claim that

$$R_\kappa^2 \in O(t^{-1-\mu}),\ \mu > 0. \tag{7.5}$$

To prove this, we use the estimates (4.22) in the proof of Lemma 4.14 to obtain

$$\chi(\hat{H}_\kappa)[\hat{I}_\kappa, \mathrm{i}\Gamma(q^t)](\mathrm{d}\Gamma(\omega)+1)^{-\frac{1}{2}} \in O(t^{-1-\mu}), \mu > 0. \tag{7.6}$$

To prove (7.5) it suffices by Corollary 4.8 to show that

$$(\mathrm{d}\Gamma(\omega)+1)^{\frac{1}{2}}\frac{B}{t}\Gamma(q^t)(\mathrm{d}\Gamma(\omega)+1)^{-\frac{1}{2}}(N+1)^{-2} \in O(1). \tag{7.7}$$

A simple explicit calculus gives

$$\begin{aligned}(\mathrm{d}\Gamma(\omega)+1)^{\frac{1}{2}}\frac{B}{t}\Gamma(q^t) &= \frac{B}{t}\Gamma(q^t)(\mathrm{d}\Gamma(\omega)+1)^{\frac{1}{2}} + [(\mathrm{d}\Gamma(\omega)+1)^{\frac{1}{2}}, \frac{B}{t}]\Gamma(q^t) \\ &\quad + \frac{B}{t}[(\mathrm{d}\Gamma(\omega)+1)^{\frac{1}{2}}, \Gamma(q^t)].\end{aligned} \tag{7.8}$$

Clearly $\frac{B}{t}\Gamma(q^t)(N+1)^{-2} \in O(1)$ and

$$[(\mathrm{d}\Gamma(\omega)+1)^{\frac{1}{2}}, \frac{B}{t}]\Gamma(q^t) = (\mathrm{d}\Gamma(\omega)+1)^{-\frac{1}{2}}\mathrm{d}\Gamma(|\nabla\omega|^2)\Gamma(q^t) \in O(1).$$

To estimate the last term in (7.8), we write on the n-particle sector:

$$\begin{aligned}[(\mathrm{d}\Gamma(\omega)+1)^{\frac{1}{2}}, \Gamma(q^t)]_{|\mathcal{K}\otimes^s_n\mathfrak{h}} &= \sum_{j=1}^{n}\prod_{i=1}^{j-1} q(\frac{x_i}{t})[(\sum_{i=1}^{n}\omega(k_i)+1)^{\frac{1}{2}}, q(\frac{x_j}{t})]\prod_{i=j+1}^{n} q(\frac{x_i}{t}) \\ &=: \sum_{j=1}^{n} R_j(t).\end{aligned}$$

Pseudodifferential calculus gives that

$$x_k R_j(t) \in O(1), \text{uniformly in } k, j.$$

This proves (7.7). Now (7.6) and (7.7) imply that $R^2_\kappa \in O(t^{-1-\nu})$, for $\kappa < \infty$.
We consider now $R^1_\kappa$. We have:

$$\mathbf{d}_0 q^t = -\frac{1}{2t}\langle\frac{x}{t} - \nabla\omega(k), \nabla q(\frac{x}{t})\rangle + hc + r^t =: \frac{1}{t}g^t + r^t,$$

where $r^t \in O(t^{-2})$. We have using (2.3) and Corollary 4.8:

$$||\chi(\hat{H}_\kappa)\mathrm{d}\Gamma(q^t, r^t)\frac{B}{t}\Gamma(q^t)\chi(\hat{H}_\kappa)|| \in O(t^{-2}).$$

We set

$$B_1 := \chi(\hat{H}_\kappa)\mathrm{d}\Gamma(q^t, g^t)(N+1)^{-\frac{1}{2}},\ B_2 := (N+1)^{\frac{1}{2}}\frac{B}{t}\Gamma(q^t)\chi(\hat{H}_\kappa).$$

So we obtain the inequality

$$R^1_\kappa \geq -\epsilon_0^{-1}t^{-1}B_1B_1^* - \epsilon_0 t^{-1}B_2B_2^*.$$

Using arguments in [11, Prop. 6.5], we obtain

$$-B_2B_2^* \geq -C_1\chi(\hat{H}_\kappa)\Gamma(q^t)^2\chi(\hat{H}_\kappa) - Ct^{-1},$$

$$\int_1^\infty ||B_1\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u|| \leq C||u||^2.$$

Using Lemma 4.14 and Thm. 5.7, we have

$$R^3_\kappa \geq C_0t^{-1}\chi(\hat{H}_\kappa)\Gamma(q^t)^2\chi(\hat{H}_\kappa) - Ct^{-2}.$$

We have

$$-R^4_\kappa \leq C_2\frac{\epsilon}{t}\chi(\hat{H}_\kappa)\Gamma(q^t)^2\chi(\hat{H}_\kappa) + Ct^{-2}.$$

Collecting the four terms, we obtain

$$\begin{aligned}\hat{\mathrm{ID}}_\kappa \phi_\kappa(t) &\geq -\epsilon_0 t^{-1} B_2 B_2^* + R_\kappa^2 + R_\kappa^3 + R_\kappa^4 \\ &\geq (-\epsilon_0 C_1 - \epsilon C_2) t^{-1} \chi(\hat{H}_\kappa)\Gamma(q^t)^2 \chi(\hat{H}_\kappa) + C t^{-2} \\ &\geq \tilde{C}_0^\kappa \chi(\hat{H}_\kappa)\Gamma(q^t)^2\chi(\hat{H}_\kappa) - R(t),\end{aligned}$$

where $R(t)$ is integrable. By Lemma A.3 we obtain the inequality announced in the proposition for $\chi$ supported near a one energy level $\lambda$. Then we complete the proof for an arbitrary $\chi$ using a standard argument, see e.g.[10, Prop. 4.4.7]. ■

## 8. Asymptotic completeness

In this section we prove the main result of this paper , which is the asymptotic completeness of the Nelson Hamiltonian. This is the subject of Thm. 8.5, where we prove

$$1\!\!1_{\rm pp}(H) = \mathcal{K}^{\pm}.$$

**Theorem 8.1.** *Let $q, \tilde{q} \in C_0^\infty(\mathbb{R}^3)$ such that $0 \leq q, \tilde{q} \leq 1$, $q, \tilde{q} = 1$ on a neighborhood of zero and $q^t := q(\frac{x}{t})$*
*i) The following limits exist*

$$\begin{aligned}\Gamma^+(q) &:= s - \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty}\,\Gamma(q^t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty} \\ &= \lim_{\kappa\to\infty} s - \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\kappa}\,\Gamma(q^t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}\end{aligned}$$

*ii) We have*

$$\begin{aligned}&\Gamma^+(q\tilde{q}) = \Gamma^+(q)\Gamma^+(\tilde{q}). \\ &0 \leq \Gamma^+(q) \leq \Gamma^+(\tilde{q}) \leq 1\!\!1, \text{ if } 0 \leq q \leq \tilde{q}. \\ &[\hat{H}_\infty, \Gamma^+(q)] = 0.\end{aligned}$$

*iii) We have*

$$\mathrm{Ran}\Gamma^+(q) \subset \hat{\mathcal{K}}^+.$$

**Proof.** It is sufficient using a density argument and Lemma 4.14 to show for $\chi \in C_0^\infty(\mathbb{R})$ the existence of the limit

$$s - \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty}\chi(\hat{H}_\infty)\,\Gamma(q^t)\chi(\hat{H}_\infty)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty}. \tag{8.1}$$

Using Lemma A.4, we see that as for all asymptotic limits these amounts to bound Heisenberg derivatives uniformly in $\kappa$. We have on $\mathcal{H}$:

$$\begin{aligned}\partial_t(\mathrm{e}^{\mathrm{i}t\hat{H}_\kappa}\chi(\hat{H}_\kappa)\,\Gamma(q^t)\chi(\hat{H}_\kappa)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}) &= \mathrm{e}^{\mathrm{i}t\hat{H}_\kappa}\chi(\hat{H}_\kappa)\mathrm{d}\Gamma(q^t, \mathrm{d}\!\!1_0 q^t)\chi(\hat{H}_\kappa)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa} \\ &\quad + \mathrm{e}^{\mathrm{i}t\hat{H}_\kappa}\chi(\hat{H}_\kappa)[\hat{I}_\kappa, \mathrm{i}\Gamma(q^t)]\chi(\hat{H}_\kappa)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}.\end{aligned}$$

By (4.22) we have uniformly in $\kappa$:

$$\chi(\hat{H}_\kappa)[\hat{I}_\kappa, \mathrm{i}\Gamma(q^t)]\chi(\hat{H}_\kappa) \in O(t^{-1-\epsilon}). \tag{8.2}$$

We use now an argument introduced in [11]:

$$\begin{aligned}&\mathrm{d}\!\!1_0 q^t = \tfrac{1}{t} g^t + r^t, \\ &\text{where } g^t = -\frac{1}{2}((\frac{x}{t} - \partial\omega(k))\,\partial q(\frac{x}{t}) + hc), \text{ and } r^t \in O(t^{-2}).\end{aligned}$$

The estimate (2.3) and Corollary 4.8 give uniformly in $\kappa$

$$||\chi(\hat{H}_\kappa)\mathrm{d}\Gamma(q^t, \mathrm{d}\!\!1_0 r^t)\chi(\hat{H}_\kappa)|| \in O(t^{-2}). \tag{8.3}$$

The other term will be estimated:

$$|(\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u, \chi(\hat{H}_\kappa)\mathrm{d}\Gamma(q^t, g^t)\chi(\hat{H}_\kappa)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u)| \le ||\mathrm{d}\Gamma(|g^t|)^{\frac{1}{2}}\chi(\hat{H}_\kappa)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u||^2, u \in \mathcal{H}. \tag{8.4}$$

Using (8.2)-(8.4), Prop. 7.3 and Lemma A.4, we obtain the existence of the limit (8.1).

The first statement in *ii*) follows from the fact that

$$\Gamma(q^t\tilde{q}^t) = \Gamma(q^t)\Gamma(\tilde{q}^t).$$

The second statement follows by

$$0 \le \Gamma(q^t) \le \Gamma(\tilde{q}^t) \le 1 \; if \; 0 \le q \le \tilde{q}.$$

The last statement is a consequence of *i*) and Lemma 4.14.

Let us prove *iii*). Since $\hat{H}_\infty, \Gamma^+(q)$ commute, $\Gamma^+(q)$ preserves $\mathcal{D}(\hat{H}_\infty)$. Consequently $\mathcal{D}(\hat{H}_\infty) \cap \mathrm{Ran}\Gamma^+(q)$ is dense in $\mathrm{Ran}\Gamma^+(q)$. Since $\hat{\mathcal{K}}^+$ is closed, it is enough to show that $\mathcal{D}(\hat{H}_\infty) \cap \mathrm{Ran}\Gamma^+(q) \subset \hat{\mathcal{K}}^+$ to prove *iii*). Let $v \in \mathcal{D}(\hat{H}_\infty) \cap \mathrm{Ran}\Gamma^+(q)$, $v = \Gamma^+(q)u$.

$$\begin{aligned}
(\hat{H}_\infty + b)^{-1}\hat{a}^+(h)\Gamma^+(q)u &= \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty}(\hat{H}_\infty + b)^{-1}a(h_t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty}\Gamma^+(q)u \\
&= \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty}(\hat{H}_\infty + b)^{-1}a(h_t)\Gamma(q^t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty}u \\
&= \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty}(\hat{H}_\infty + b)^{-1}\Gamma(q^t)a(q^t h_t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty}u.
\end{aligned}$$

If $h \in \mathfrak{h}_0$, by a stationary phase argument we have $q^t h_t \in o(1), t \to +\infty$. Using the fact that $h \to (\hat{H}_\infty + \mathrm{i})^{-1}a(h_t)$ is continuous, we obtain $\hat{a}^+(h)v = 0$ for all $h \in \mathfrak{h}$. This ends the proof. ■

**Corollary 8.2.** *Let* $\{q_n\} \in C_0^\infty(\mathbb{R}^3)$ *be a decreasing sequence of functions such that* $0 \le q_n \le 1$, $q = 1$ *on a neighborhood of* 0 *and* $\cap_{n=1}^\infty \mathrm{supp}\, q_n = \{0\}$. *Then the following limit exist and it does not depend in the choose of the sequence*

$$i)\; P_0^+ := \lim_{n\to\infty} \Gamma^+(q_n),$$

$$ii)\; \mathrm{Ran}P_0^+ \subset \hat{\mathcal{K}}^+.$$

*Moreover* $P_0^+$ *is an orthogonal projection.*

**Proof.** The existence of the limit *i*) follows from Thm. 8.1 *ii*) and Lemma A.5. The independence from the choose of the sequence follows from the fact that there exists an index $m_n$ such that $q_n \ge \tilde{q}_{m_n}, \tilde{q}_n \ge q_{m_n}$; $\lim_{n\to\infty} m_n = +\infty$ and Thm. 8.1 *ii*). *ii*) is a consequence of Thm. 8.1 *iii*) and *i*). ■

Let $j_0 \in C_0^\infty(\mathbb{R}^3), 0 \le j_0, 0 \le j_\infty, j_0^2 + j_\infty^2 \le 1, j_0 = 1$ near 0. Set $j := (j_0, j_\infty)$ and $j^t := (j_0^t, j_\infty^t)$, where $j_0^t := j_0(\frac{x}{t}), j_\infty^t := j_\infty(\frac{x}{t})$. We recall that $I(j^t)$ is the operator introduced in Subsect. 2.1.

**Theorem 8.3.** *i*) *The following limits exist*

$$s - \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty^{\mathrm{ext}}}I^*(j^t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty} =: W^+(j),$$

$$s - \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty}I(j^t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty^{\mathrm{ext}}} = W^+(j)^*.$$

*ii*) *For a bounded Borel function F, we have*

$$W^+(j)\; F(\hat{H}_\infty) = F(\hat{H}_\infty^{\mathrm{ext}})W^+(j).$$

*iii*) *Let* $q_0, q_\infty \in C^\infty(\mathbb{R}^3)$, $\nabla q_0$, $\nabla q_\infty \in C_0^\infty(\mathbb{R}^3)$, $0 \le q_0, q_\infty \le 1$, $q_0 = 1$ *near* 0. *Set* $\tilde{j} := (q_0 j_0, q_\infty j_\infty)$. *Then*

$$\Gamma^+(q_0) \otimes \Gamma(q_\infty(\nabla\omega(k)))W^+(j) = W^+(\tilde{j}).$$

*iv*) *Let* $q \in C^\infty(\mathbb{R}^3)$, $\nabla q \in C_0^\infty(\mathbb{R}^3)$, $0 \le q \le 1$, $q = 1$ *near* 0. *Then*

$$W^+(j)\Gamma^+(q) = W^+(qj), \;\; \text{where } qj = (qj_0, qj_\infty).$$

*v) Let $\tilde{j} = (\tilde{j}_0, \tilde{j}_\infty)$ be another pair satisfying the conditions stated before the theorem. Then*

$$W^+(\tilde{j})^* W^+(j) = \Gamma(\tilde{j}_0 j_0 + \tilde{j}_\infty j_\infty),$$

*in particular if $j_0^2 + j_\infty^2 = 1$, then $W^+(j)$ is isometric.*
*vi) Let $j_0 + j_\infty = 1$. If $\chi \in C_0^\infty(\mathbb{R})$, then*

$$\hat{\Omega}^{\text{ext},+}\chi(\hat{H}_\infty^{\text{ext}})W^+(j) = \chi(\hat{H}_\infty).$$

**Proof.** To prove $i)$ we use the same arguments as in Thm. 8.1. Using Lemma A.4, it is enough to prove the existence of the limit

$$s - \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty^{\text{ext}}}\chi(\hat{H}_\infty^{\text{ext}})I^*(j^t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty}\chi(\hat{H}_\infty), \tag{8.5}$$

for some $\chi \in C_0^\infty(\mathbb{R})$. We compute

$$\begin{aligned}\partial_t(\mathrm{e}^{\mathrm{i}t\hat{H}_\kappa^{\text{ext}}}\chi(\hat{H}_\kappa^{\text{ext}})I^*(j^t)\chi(\hat{H}_\kappa)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}) &= \mathrm{e}^{\mathrm{i}t\hat{H}_\kappa^{\text{ext}}}(\chi(\hat{H}_\kappa^{\text{ext}})\mathfrak{D}_0 I^*(j^t)\chi(\hat{H}_\kappa)\\ &\quad +\mathrm{i}\chi(\hat{H}_\kappa^{\text{ext}})(\hat{I}_\kappa \otimes 1\!\!1 I^*(j^t) - I^*(j^t)\hat{I}_\kappa)\chi(\hat{H}_\kappa))\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa},\end{aligned}$$

where $\mathfrak{D}_0$ is the asymmetric Heisenberg derivative $\partial_t + \mathrm{i}H_0^{\text{ext}}. - .\mathrm{i}H_0$. We have $\mathfrak{D}_0 I^*(j^t) = \mathrm{d}I^*(j^t, \mathfrak{d}_0 j^t)$. Pseudodifferential calculus gives

$$\begin{aligned}\mathfrak{d}_0 j^t &= \frac{1}{t}g^t + r^t,\\ g^t = (g_0^t, g_\infty^t),\ \ g_\epsilon^t &= -\frac{1}{2}((\frac{x}{t} - \partial\omega(k))\partial j_\epsilon(\frac{x}{t}) + hc), \epsilon = 0, \infty\end{aligned}$$

with $r^t \in O(t^{-2})$. Using Corollary 4.8 and (2.4) we obtain

$$||\chi(\hat{H}_\kappa^{\text{ext}})\mathrm{d}I^*(j^t, r^t)\chi(\hat{H}_\kappa)|| \in O(t^{-2}). \tag{8.6}$$

Using now (2.5) with $u_i^t := \mathrm{e}^{\mathrm{i}t\hat{H}_\kappa}u_i$, one obtain

$$\begin{aligned}&|(u_1^t|\chi(\hat{H}_\kappa^{\text{ext}})\mathrm{d}I^*(j^t, g^t)\chi(\hat{H}_\kappa)u_2^t)|\\ &\leq ||\mathrm{d}\Gamma(|g_0^t|)^{\frac{1}{2}} \otimes 1\!\!1\chi(\hat{H}_\kappa^{\text{ext}})u_2^t||\ ||\mathrm{d}\Gamma(|g_0^t|)^{\frac{1}{2}}\chi(\hat{H}_\kappa)u_1^t||\\ &+||(1\!\!1 \otimes \mathrm{d}\Gamma(|g_\infty^t|)^{\frac{1}{2}})\chi(\hat{H}_\kappa^{\text{ext}})u_2^t||\ ||\mathrm{d}\Gamma(|g_\infty^t|)^{\frac{1}{2}}\chi(\hat{H}_\kappa)u_1^t||.\end{aligned}$$

Then the $\kappa-$uniform integrability of the term $\chi(\hat{H}_\kappa^{\text{ext}})\mathfrak{D}_0 I^*(j^t)\chi(\hat{H}_\kappa)$ follows using Prop. 7.1. Using (4.25) we obtain uniformly in $\kappa$

$$\chi(\hat{H}_\kappa^{\text{ext}})(\hat{I}_\kappa \otimes 1\!\!1 I^*(j^t) - I^*(j^t)\hat{I}_\kappa)\chi(\hat{H}_\kappa) \in O(t^{-1-\mu}).$$

Then the existence of the limit in $i)$ follows.
$ii)$ follows by Lemma 4.15. $iii)$ follows using the fact that

$$\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\omega)}\Gamma(q^t)\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(\omega)} = \Gamma(q(\nabla\omega)),$$

$$\Gamma(q_0^t) \otimes \Gamma(q_\infty^t)I^*(j^t) = I^*(\tilde{j}^t).$$

$iv)$ is true since

$$I^*(j^t)\Gamma(q^t) = I^*((jq)^t).$$

$v)$ is a consequence of the fact

$$I(\tilde{j}^t)I^*(j^t)\Gamma(\tilde{j}_0^t j_0^t + \tilde{j}_\infty^t j_\infty^t).$$

$vi)$ One has

$$\hat{H}_\infty^{\text{ext}} 1\!\!1_{[k,\infty[}(N_\infty) \geq mk + E_0.$$

Let $\chi \in C_0^\infty(\mathbb{R})$. There exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\chi(\hat{H}_\infty^{\text{ext}})1\!\!1_{]n,\infty[}(N_\infty) = 0. \tag{8.7}$$

We have

$$\hat{\Omega}^{\text{ext},+}\chi(\hat{H}_\infty^{\text{ext}})W^+(j) = \hat{\Omega}^{\text{ext},+}1\!\!1_{[0,n]}(N_\infty)\chi(\hat{H}_\infty^{\text{ext}})W^+(j) \tag{8.8}$$

$$= s - \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty} I 1\!\!1_{[0,n]}(N_\infty)\chi(\hat{H}_\infty^{\text{ext}})I^*(j^t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty} \tag{8.9}$$

$$= s - \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty} I 1\!\!1_{[0,n]}(N_\infty)I^*(j^t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty}\chi(\hat{H}_\infty) \tag{8.10}$$

(8.8) follows from (8.7). (8.9) follows from the limit $i)$ and Thm. 6.10. Lemma 4.15 and the boundeness of the operator $I1\!\!1_{[0,n]}(N_\infty)(N_0+1)^{-\frac{n}{2}}$ gives (8.10). We use now an estimate proved in [12]:

$$||I1\!\!1_{]n,\infty[}(N_\infty)I^*(j^t)(N+1)^{-1}|| \leq (n+1)^{-1}. \tag{8.11}$$

Since $II^*(j^t) = 1\!\!1$, letting $n \to \infty$ we obtain $\hat{\Omega}^{\text{ext},+}\chi(\hat{H}_\infty^{\text{ext}})W^+(j) = \chi(\hat{H}_\infty)$. This completes the proof. ∎

**Theorem 8.4.** *Let $j_n = (j_{0,n}, j_{\infty,n})$ be a sequence satisfying the hypothesis stated in the beginning of Thm. 8.3 such that $j_0 + j_\infty = 1$ and for any $\epsilon > 0$ there exists $m, \forall n > m$, $\mathrm{supp} j_{0,n} \subset [-\epsilon, \epsilon]$. Then*

$$\hat{\Omega}^{+*} = w - \lim_{\kappa\to\infty} W^+(j_n),$$
$$\hat{\mathcal{K}}^+ = \mathrm{Ran} P_0^+,$$

**Proof.** Let $q \in C_0^\infty(\mathbb{R}), 0 \leq q \leq 1$ and $q = 1$ in a neighborhood of zero such that $qj_{0,n} = j_{0,n}$ for $n$ large enough. Using Thm. 8.3 $iii)$ and Corollary 8.2 we obtain

$$\Gamma^+(q) \otimes 1\!\!1\, W^+(j_n) = W^+(j_n),$$

$$w - \lim_{n\to\infty} P_0^+ \otimes 1\!\!1\, W^+(j_n) - W^+(j_n) = 0. \tag{8.12}$$

Let $\chi \in C_0^\infty(\mathbb{R})$. We have

$$\hat{\Omega}^{+*}\chi(\hat{H}_\infty) = \hat{\Omega}^{+*}\hat{\Omega}^{\text{ext},+}\chi(\hat{H}_\infty^{\text{ext}})W^+(j_n) \tag{8.13}$$

$$= w - \lim_{n\to\infty} \hat{\Omega}^{+*}\hat{\Omega}^{\text{ext},+}\chi(\hat{H}_\infty^{\text{ext}})P_0^+ \otimes 1\!\!1\, W^+(j_n) \tag{8.14}$$

$$= w - \lim_{n\to\infty} P_0^+ \otimes 1\!\!1\chi(\hat{H}_\infty^{\text{ext}})W^+(j_n) \tag{8.15}$$

$$= w - \lim_{n\to\infty} \chi(\hat{H}_\infty^{\text{ext}})W^+(j_n) \tag{8.16}$$

$$= w - \lim_{n\to\infty} W^+(j_n)\chi(\hat{H}_\infty). \tag{8.17}$$

(8.13) follows from Thm. 8.3 $iv)$. (8.14) follows by (8.12). (8.15) is true since $P_0^+$ commutes with $\hat{H}_\infty^{\text{ext}}$ and that $\mathrm{Ran}P_0^+ \subset \hat{\mathcal{K}}^+, \hat{\Omega}^{\text{ext},+}1\!\!1_{\hat{\mathcal{K}}^+} \otimes 1\!\!1 = \hat{\Omega}^+$ and $\hat{\Omega}^{+*}\hat{\Omega}^+ = 1\!\!1_{\hat{\mathcal{K}}^+} \otimes 1\!\!1$. (8.16) follows from the fact that $P_0^+$ commutes with $\hat{H}_\infty^{\text{ext}}$ and (8.12). (8.17) is Thm. 8.3 $ii)$. So we conclude by a density argument that

$$\hat{\Omega}^{+*} = w - \lim_{n\to\infty} W^+(j_n),$$

$$P_0^+ \otimes 1\!\!1\, \hat{\Omega}^{+*} = \hat{\Omega}^{+*}.$$

So we obtain

$$\mathrm{Ran}\hat{\Omega}^{+*} = \hat{\mathcal{K}}^+ \otimes \Gamma(\mathfrak{h}) \subset \mathrm{Ran}P_0^+ \otimes \Gamma(\mathfrak{h}) \subset \hat{\mathcal{K}}^+ \otimes \Gamma(\mathfrak{h}).$$

Hence we prove that $\hat{\mathcal{K}}^+ = \mathrm{Ran}P_0^+$. ∎

**Theorem 8.5.** *We have*

$$1\!\!1_{\text{pp}}(H) = \mathcal{K}^\pm.$$

**Proof.** By Prop. 6.8 *iii)* we have

$$\mathbb{1}_{\rm pp}(\hat{H}_\infty) \subset \hat{\mathcal{K}}^+.$$

Then it suffices to show that $\hat{\mathcal{K}}^+ \subset \mathbb{1}_{\rm pp}(\hat{H}_\infty)$. Prop. 7.4 gives the existence of $\epsilon > 0$ and a sequence $\hat{H}_\kappa$ such that

$$\int_1^{+\infty} ||\Gamma(q^t)\chi(\hat{H}_\kappa)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u||^2 \frac{\mathrm{d}t}{t} \leq C||u||^2,$$

where $\chi \in C_0^\infty(\mathbb{R} \setminus (\tau \cap \sigma_{\rm pp}(\hat{H}_\infty)))$ and $q \in C_0^\infty([-\epsilon, \epsilon])$, $q = 1$ for $|x| < \frac{\epsilon}{2}$. Thm. 8.1 gives that

$$||\Gamma(q^t)\chi(\hat{H}_\kappa)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u|| \to ||\Gamma^+(q)\chi(\hat{H}_\infty)u|| = 0,$$

then $\Gamma^+(q)\chi(\hat{H}_\infty) = 0$. So we have $\mathrm{Ran}P_0^+ \subset \mathbb{1}_{\tau\cap\sigma_{\rm pp}(\hat{H}_\infty)}(\hat{H}_\infty)$. Thm. 5.7 gives that $\tau$ is a closed countable set and $\sigma_{\rm pp}(\hat{H}_\infty)$ can accumulate only at $\tau$, so $\mathbb{1}_{\rm pp}(\hat{H}_\infty) = \mathbb{1}_{\tau\cap\sigma_{\rm pp}(\hat{H}_\infty)}(\hat{H}_\infty)$. This proves $\mathbb{1}_{\rm pp}(\hat{H}_\infty) = \hat{\mathcal{K}}^\pm$. Then we prove the theorem. ■

## Appendix A.

The following theorem follows from the KLMN theorem and [23, Thm. VIII.25].

**Theorem A.1.** *Let $H_0$ be a positive selfadjoint operator on $\mathcal{H}$. Let for $\kappa \leq \infty$, $B_\kappa$ be quadratic forms on $D(H_0^{\frac{1}{2}})$ such that*

$$|B_\kappa(\psi,\psi)| \leq a\,||H_0^{\frac{1}{2}}\psi||^2 + b\,||\psi||^2,$$

*where $a < 1$ uniformly in $\kappa$ and $B_\kappa \to B_\infty$ on $D(H_0^{\frac{1}{2}})$. Then*
*i) There exist for $\kappa \leq \infty$ selfadjoint operators $H_\kappa$ with $D(H_\kappa) \subset D(H_0^{\frac{1}{2}})$ and*

$$(H_\kappa\psi,\psi) = B_\kappa(\psi,\psi) + (H_0^{\frac{1}{2}}\psi, H_0^{\frac{1}{2}}\psi),\ \psi \in D(H_\kappa),$$

*ii)*

$$\lim_{\kappa\to\infty}(z - H_\kappa)^{-1} = (z - H_\infty)^{-1},$$

*iii)*

$$s - \lim_{\kappa\to\infty} \mathrm{e}^{-\mathrm{i}tH_\kappa} = \mathrm{e}^{-\mathrm{i}tH_\infty}.$$

**Lemma A.2.** *Let $F \in C^\infty(\mathbb{R})$, equal to 0 near the origin and bounded near $\infty$. we denote by $F^R$ the derivative operator $F(\frac{|x|}{R})$. We recall that* x *denote the nucleon position observable. One has uniformly in $\kappa$*

$$i)\ ||\langle \mathrm{x}\rangle^{-s}F^R v_{\kappa_0}|| \in O(R^{-s}),$$
$$ii) ||\langle \mathrm{x}\rangle^{-s}\omega(k)^{-\epsilon}F^R r_\kappa|| \in O(R^{-s}), \epsilon > 0$$

**Proof.** We have $\langle \mathrm{x}\rangle^{-s}v_{\kappa_0} \in H^s(\mathbb{R}^3, \mathcal{B}(\mathcal{K}))$. Since we have

$$||R^s F(\frac{|D_k|}{R})\langle \mathrm{x}\rangle^{-s}v_{\kappa_0}|| \leq c|||D_k|^s\langle \mathrm{x}\rangle^{-s}v_{\kappa_0}||$$

we obtain *i)*. Let us prove *ii)*. Pseudodifferential calculus gives

$$||\omega(k)^{-\epsilon}F(\frac{|D_k|}{R} \geq 1)r_\kappa|| \leq c||F(\frac{|D_k|}{R} \geq \frac{1}{2})\omega(k)^{-\epsilon}r_\kappa|| + \frac{c}{R^2}||\omega(k)^{-\epsilon}r_\kappa||.$$

Then we obtain *ii)* using *i)* for $\omega^{-\epsilon}r_\kappa$, which is $L^2(\mathbb{R}^3, dk)$ uniformly in $\kappa$. ■

Let $\mathcal{H}$ a Hilbert space. Let $\{H_\kappa\}$ be a sequence of selfadjoint operators on a common domain $\mathcal{D} \subset \mathcal{H}$. We suppose that $s - \lim_{\kappa\to\infty} \mathrm{e}^{-\mathrm{i}tH_\kappa} = \mathrm{e}^{-\mathrm{i}tH_\infty}$, where $H_\infty$ is a selfadjoint operator. We have for $\chi \in C_0^\infty(\mathbb{R})$, using [23, Thm. VIII.20] and [23, Thm. VIII.21]

$$s - \lim_{\kappa\to\infty} \chi(H_\kappa) = \chi(H_\infty).$$

**Lemma A.3.** *Let $t \to B_t \in \mathcal{B}(\mathcal{H})$ and $\chi \in C_0^\infty(\mathbb{R})$ such that for $\kappa \le \infty$*

$$||B_t^* B_t\, \chi(H_\kappa)|| \le c_t,$$

*where $c_t$ is $\kappa-$independent locally integrable function in $t$. If there exist a constant $c$ independent of $\kappa$ such that for $\kappa < \infty$*

$$\int_1^\infty ||B_t\, \chi(H_\kappa)\mathrm{e}^{-\mathrm{i}tH_\kappa} u||^2 dt \le c\, ||u||^2,\ u \in \mathcal{H},$$

*then:*

$$\int_1^\infty ||B_t\, \chi(H_\infty)\mathrm{e}^{-\mathrm{i}tH_\infty} u||^2 dt \le c\, ||u||^2.$$

**Proof.** We have to prove, uniformly in $T$, that:

$$\int_1^T ||B_t\, \chi(H_\infty)\mathrm{e}^{-\mathrm{i}tH_\infty} u||^2 dt \le c\, ||u||^2. \tag{A.1}$$

We apply the dominated convergence theorem. It is enough to show that

$$\lim_{\kappa\to\infty} ||B_t\, \chi(H_\kappa)\mathrm{e}^{-\mathrm{i}tH_\kappa} u||^2 = ||B_t\, \chi(H_\infty)\mathrm{e}^{-\mathrm{i}tH_\infty} u||^2,$$

since we have $||B_t\, \chi(H_\kappa)\mathrm{e}^{-\mathrm{i}tH_\kappa} u||^2 \le c'\, ||B_t^* B_t\, \chi(H_\kappa)||\ ||u||^2$. It is easy to see that to prove (A.1) it suffices to show

$$w - \lim_{\kappa\to\infty} \chi(H_\kappa) B_t^* B_t \chi(H_\kappa) = \chi(H_\infty) B_t^* B_t \chi(H_\infty).$$

This follows from the hypothesis $||B_t^* B_t \chi(H_\kappa)|| \le c_t$ for $\kappa \le \infty$. ∎

Let $\mathcal{H}_i, i = 1, 2$ be two Hilbert spaces. Let $H_{i,\kappa}, i = 1, 2$ be two sequences of selfadjoint operators on $\mathcal{H}_i$, such that $H_{i,\kappa}$ converge in the strong resolvent sense to $H_{i,\infty}$.
The Lemma A.4 follows from the proof of Lemma A.3 and [10, Lemma B.4.2]

**Lemma A.4.** *Let $t \to C(t) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\chi \in C_0^\infty(\mathbb{R})$. We suppose that the asymmetric Heisenberg derivatives:*

$$D_\kappa C(t) := \partial_t C(t) + \mathrm{i}(H_{1,\kappa} C(t) - C(t) H_{2,\kappa}),$$

*satisfies for $\kappa < \infty$, the following hypothesis:*

$$i)\ D_\kappa C(t) = B(t) + R_\kappa(t).$$

$$ii)\ \chi(H_{2,\kappa})\, R_\kappa(t)\, \chi(H_{1,_\kappa k})|| \le c\, t^{-1-\epsilon},\ \epsilon > 0,\ \ uniformly\ \ for\ \kappa < \infty$$

$$iii)\ ||\chi(H_{2,\kappa}) B(t)|| \le c_t,\ ||B(t)\, \chi(H_{1,\kappa})|| \le c_t,\ \ uniformly\ \ for\ \kappa \le \infty,$$

*where $c_t$ is $\kappa-$independent locally integrable function in $t$.*

$$iv)\ |(u_2|B(t)u_1)| \le c \sum_{j=1}^n ||B_{2,j}(t)u_2||\ ||B_{1,j}(t)u_1||, where$$

$$\int_1^\infty ||B_{i,j}(t)\, \chi(H_{i,\infty})\mathrm{e}^{-\mathrm{i}tH_{i,\infty}} u||^2 dt \le c\, ||u||^2, i = 1, 2\ \ and\ \ j = 1...n.$$

*Then the following limit exists*

$$s-\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH_{2,\infty}}\chi(H_{2,\infty})\,C(t)\,\chi(H_{1,\infty})\mathrm{e}^{-\mathrm{i}tH_{1,\infty}}.$$

**Proof.** Let $u\in\mathcal{H}_1, v\in\mathcal{H}_2$. We have:

$$\begin{aligned}
&(u|\mathrm{e}^{\mathrm{i}t_2H_{2,\infty}}\chi(H_{2,\infty})C(t_2)\chi(H_{1,\infty})\mathrm{e}^{\mathrm{i}t_2H_{1,\infty}})-(u|\mathrm{e}^{\mathrm{i}t_1H_{2,\infty}}\chi(H_{2,\infty})C(t_1)\chi(H_{1,\infty})\mathrm{e}^{-\mathrm{i}t_1H_{1,\infty}}v)\\
&=\lim_{\kappa\to\infty}(u|\mathrm{e}^{\mathrm{i}t_2H_{2,\kappa}}\chi(H_{2,\kappa})C(t_2)\chi(H_{1,\kappa})\mathrm{e}^{-\mathrm{i}t_2H_{1,\kappa}}v)-(u|\mathrm{e}^{\mathrm{i}t_1H_{2,\kappa}}\chi(H_{2,\kappa})C(t_1)\chi(H_{1,\kappa})\mathrm{e}^{-\mathrm{i}t_1H_{1,\kappa}}v)\\
&=\lim_{\kappa\to\infty}\int_{t_1}^{t_2}(\mathrm{e}^{-\mathrm{i}t_1H_{2,\kappa}}u|\chi(H_{2,\kappa})\mathbb{D}_\kappa C(t)\chi(H_{1,\kappa})\mathrm{e}^{-\mathrm{i}t_1H_{1,\kappa}}v)dt\\
&=\lim_{\kappa\to\infty}\int_{t_1}^{t_2}(\mathrm{e}^{-\mathrm{i}t_1H_{2,\kappa}}u|(B(t)+R(t))\chi(H_{1,\kappa})\mathrm{e}^{-\mathrm{i}t_1H_{1,\kappa}}v)dt.
\end{aligned}$$

Using Lebesgue dominated convergence theorem and, as in the proof of Lemma A.3, the fact that

$$iii)\Rightarrow w-\lim_{\kappa\to\infty}\chi(H_{2,\kappa})B(t)\chi(H_{1,\kappa})=\chi(H_{2,\infty})B(t)\chi(H_{1,\infty}),$$

we obtain

$$\lim_{\kappa\to\infty}\int_{t_1}^{t_2}(\mathrm{e}^{-\mathrm{i}t_1H_{2,\kappa}}u|B(t)\chi(H_{1,\kappa})\mathrm{e}^{-\mathrm{i}t_1H_{1,\kappa}}v)dt=\int_{t_1}^{t_2}(\mathrm{e}^{-\mathrm{i}t_1H_{2,\infty}}u|B(t)\chi(H_{1,\kappa})\mathrm{e}^{-\mathrm{i}t_1H_{1,\infty}}v)dt. \tag{A.2}$$

We have by $i)$:

$$\lim_{\kappa\to\infty}\int_{t_1}^{t_2}(\mathrm{e}^{-\mathrm{i}t_1H_{2,\kappa}}u|B(t)\chi(H_{1,\kappa})\mathrm{e}^{-\mathrm{i}t_1H_{1,\kappa}}v)dt\le c\,t_2^{-\epsilon}\,||v||\,||u||. \tag{A.3}$$

Using $iv)$ we obtain:

$$\begin{aligned}
&\lim_{\kappa\to\infty}\int_{t_1}^{t_2}(\mathrm{e}^{-\mathrm{i}t_1H_{2,\infty}}u|B(t)\chi(H_{1,\infty})\mathrm{e}^{-\mathrm{i}t_1H_{1,\infty}}v)dt\\
&\le c\sum_{j=1}^{n}\int_{t_1}^{t_2}||B_{2,j}(t)\mathrm{e}^{-\mathrm{i}tH_{2,\infty}}u||^2\,dt\times\int_{t_1}^{t_2}||B_{1,j}(t)\mathrm{e}^{-\mathrm{i}tH_{1,\infty}}u||^2\,dt \qquad (A.4)\\
&\le c\,||u||\,||v||.
\end{aligned}$$

(A.2), (A.3) and (A.4) give the existence of the claimed limit. ∎

We recall here a convergence lemma of positive operators, see e.g. [11, Lemma A.3]

**Lemma A.5.** *Let $Q_n$ be a sequence of commuting selfadjoint operators. If*

$$0\le Q_n\le 1,\quad Q_{n+1}\le Q_n,\quad Q_{n+1}Q_n=Q_{n+1}.$$

*Then there exist $Q$ a projection*

$$Q=s-\lim_n Q_n.$$

## References

[1] W.O. Amrein, A. Boutet de Monvel, V. Georgescu, Commutator Methods and Spectral Theory of N-Body Hamiltonian, Birkhäuser, 1996.
[2] A. Arai, M. Hirokawa, *On the existence and uniqueness of ground states of a generalized spin-boson model*, J. Funct. Anal. **151** (1997) 455-503.
[3] F.A. Berezin, The Method of Second Quantization, Academic Press, 1966.
[4] V. Bach, J. Fröhlich, I. M. Sigal, *Quantum electrodynamics of confined non-relativistic particles*, Adv. Math. **137** (1998) 299-395.
[5] V. Bach, J. Fröhlich, I. M. Sigal, A. Soffer, *Positive commutators and spectrum of nonrelativistic QED*, preprint 1997.
[6] O. Brattelli, D. Robinson, Operator Algebras and Quantum Statistical Mechanics, Springer, Berlin, 1981.

[7] J. C. Baez, I. E. Segal, Z. Zhou, Introduction to Algebraic and Constructive Quantum Field Theory, Princeton series in Physics, 1992.
[8] J. T. Cannon, *Quantum field theoretic properties of a model of Nelson: domain and eigenvector stability for perturbed linear operators*, J. Funct. Anal. **8** (1971) 101-152.
[9] H. L. Cycon, R. Froese, W. Kirsch, B. Simon, Schrödinger Operators with Applications to Quantum Mechanics and Global Geometry, Springer, 1987.
[10] J. Dereziński, C. Gérard, Scattering Theory of Classical and Quantum N-particle Systems, Texts and Monographs in Phys. Springer, 1997.
[11] J. Dereziński, C. Gérard, *Asymptotic Completeness in Quantum Field Theory. Massive Pauli-Fierz Hamiltonians*, Reviews in Math. Phys. **11** (1999) 383-450.
[12] J. Dereziński, C. Gérard, *Spectral and Scattering Theory of Spatially cut-off $P(\phi)_2$ Hamiltonians*, preprint Ecole Polytechnique, 1998.
[13] J, Dereziński, V. Jaksic, *Specrtal theory of Pauli-Fierz Hamiltonians I*, preprint.
[14] J. Fröhlich, *Existence of Dressed Electron States in a Class of Persistent Models*, Fortschritte der Physik, **22** (1974) 159-198.
[15] G. M. Graf, *Asymptotic Completeness for N-Body Short-Range Quantum Systems: A New Proof*, Comm. Math. phys. **123** (1990) 73-101.
[16] J. Glimm, A. Jaffe, Collected Papers, Quantum Field Theory and Statistical Mechanics Expostions, vol. I, Birkhäuser, 1985.
[17] E. P. Gross, Ann. Phys. **19** (1962) 219.
[18] O. W. Greenberg, S. S. Schweber, Nuovo Cimento s, **378** ( 1958).
[19] R. Høegh-Krohn, *Asymptotic limits in some models of quantum field theory* , J. Math. Phys. **9** (1968) 2075-2079.
[20] E. Mourre, *Absence of singular continuous spectrum for certain selfadjoint operators*, Comm. Math. phys. **78** (1981) 519-567.
[21] E. Nelson, *Interaction of Nonrelativistic particles with a Quantized Scalar Field,* J. Math. Phys. **5** (1964) 1190-1197.
[22] L. Rosen, *The $(\Phi^{2n})_2$ Quantum Field Theory: Higher Order Estimates,* Comm. Pure Appl. Math. **24** (1971) 417-457.
[23] M. Reed, B. Simon, Methods of Modern Mathematical Physics, vol. I and II, 1976, vol. III, 1979, vol IV, 1978, Academic Press, London.
[24] I. M. Sigal, A. Soffer, *The N-particle scattering problem: asymptotic completeness for the short-range quantum systems*, Ann. of Math. **125** (1987) 35-108.

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