THE KERNEL OF FOCK REPRESENTATIONS OF WICK ALGEBRAS WITH BRAIDED OPERATOR OF COEFFICIENTS

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It is shown that the kernel of the Fock representation of a certain Wick algebra with braided operator of coefficients T, $||T|| \leq 1$, coincides with the largest quadratic Wick ideal. Improved conditions on the operator T for the Fock inner product to be strictly positive are given.

1. Introduction

The problem of positivity of the Fock space inner product is central in the study of the Fock representation of Wick algebras (see [**2**], [**3**], [**5**], [**6**]). The paper [**6**] presents several conditions on the coefficients of the Wick algebra for the Fock inner product to be positive. If the operator of coefficients of the Wick algebra T satisfies the braid condition and the norm restriction $\|T\| \leq 1$, then, as proved in [**2**], the Fock inner product is positive. Moreover if $-1 < T < 1$, it was shown in [**2**] that the Fock inner product is strictly positive. In this article we prove that, for braided T with $\|T\| \leq 1$, the kernel of the Fock inner product coincides with the largest quadratic Wick ideal. In particular this implies that, for $-1 < T \leq 1$, the Fock inner product is strictly positive definite, and the Fock representation is faithful.

This article is organized as follows. In sec. 2 we present definitions of Wick algebras and the Fock representation and show that, in the braided case, the kernel of the Fock representation is generated by the kernel of the Fock inner product. In sec. 3 we prove that if the operator T is braided and $\|T\| \leq 1$, then the kernel of the Fock inner product coincides with the two-sided ideal generated by $\ker(1+T)$. In sec. 4 we combine results obtained in sec. 2 and sec. 3 to examine the C^*-representability of certain Wick algebras or their quotients. All results are illustrated by examples of different kinds of q_{ij}-CCR.

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2. Preliminaries

For more detailed information about Wick algebras and the Fock representation we refer the reader to [**6**]. In this section we present only the basic definitions and properties.

1. The notion of a $*$-algebra allowing Wick ordering (Wick algebra) was presented in the paper [**6**] as a generalization of a wide class of $*$-algebras, including the twisted CCR and CAR algebras (see [**10**]), the q-CCR (see [**4**]) algebra, etc.

Definition 1. *Let $\mathbb{J} = \mathbb{J}_d = \{1, 2, \ldots, d\}$, $T_{ij}^{kl} \in C$, $i, j, k, l \in \mathbb{J}$, be such that $T_{ij}^{kl} = \overline{T_{ji}^{lk}}$. The Wick algebra with the set of coefficients $\{T_{ij}^{kl}\}$ is denoted $W(T)$, and is a $*$-algebra, defined by generators a_i, a_i^*, $i \in \mathbb{J}$, which satisfy the basic relations:*

$$a_i^* a_j = \delta_{ij} 1 + \sum_{k,l=1}^{d} T_{ij}^{kl} a_l a_k^*$$

Definition 2. *Monomials of the form $a_{i_1} a_{i_2} \cdots a_{i_m} a_{j_1}^* a_{j_2}^* \cdots a_{j_k}^*$ are called Wick ordered monomials.*

It was proved in [**6**] that the Wick ordered monomials form a basis for $W(T)$.

Let $\mathcal{H} = \langle e_1, \ldots, e_d \rangle$. Consider the full tensor algebra over $\mathcal{H}$, $\mathcal{H}^*$, denoted by $\mathcal{T}(\mathcal{H}, \mathcal{H}^*)$. Then

$$W(T) \simeq \mathcal{T}(\mathcal{H}, \mathcal{H}^*) \Big/ \Big\langle e_i^* \otimes e_j - \delta_{ij} 1 - \sum T_{ij}^{kl} e_l \otimes e_k^* \Big\rangle .$$

To study the structure of Wick algebras, and the structure of the Fock representation, it is useful to introduce the following operators on $\mathcal{H}^{\otimes n} := \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n}$ (see [**6**]):

$$T\colon \mathcal{H} \otimes \mathcal{H} \mapsto \mathcal{H} \otimes \mathcal{H}, \quad T e_k \otimes e_l = \sum_{i,j} T_{ik}^{lj} e_i \otimes e_j, \ T = T^*,$$

$$T_i\colon \mathcal{H}^{\otimes n} \mapsto \mathcal{H}^{\otimes n}, \quad T_i = \underbrace{1 \otimes \cdots \otimes 1}_{i-1} \otimes T \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-i-1},$$

$$R_n\colon \mathcal{H}^{\otimes n} \mapsto \mathcal{H}^{\otimes n}, \quad R_n = 1 + T_1 + T_1 T_2 + \cdots + T_1 T_2 \cdots T_{n-1},$$

$$P_n\colon \mathcal{H}^{\otimes n} \mapsto \mathcal{H}^{\otimes n}, \quad P_2 = R_2, \ P_{n+1} = (1 \otimes P_n) R_{n+1}.$$

In this article we suppose that the operator T is contractive, i.e., $\|T\| \leq 1$, and satisfies the *braid condition*, i.e., on $\mathcal{H}^{\otimes 3}$ the equality $T_1 T_2 T_1 = T_2 T_1 T_2$ holds. It follows from the definition of T_i that then $T_i T_j = T_j T_i$ if $|i - j| \geq 2$, and for the braided T one has $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$.

Remark 1. These conditions hold for such well-known algebras as q_{ij}-CCR, μ- CCR, μ-CAR (see [**6**]).

The Fock representation of a Wick $*$-algebra is determined by a vector Ω such that $a_i^*\Omega = 0$ for all $i = 1, \ldots, d$ (see [**6**]).

Definition 3 (The Fock representation)**.** *The representation λ_0, acting on the space $\mathcal{T}(\mathcal{H})$ by formulas*

$$\lambda_0(a_i)e_{i_1} \otimes \cdots \otimes e_{i_n} = e_i \otimes e_{i_1} \otimes \cdots \otimes e_{i_n}, \quad n \in \mathbb{N} \cup \{0\},$$
$$\lambda_0(a_i^*)1 = 0,$$

where the action of $\lambda_0(a_i^*)$ on the monomials of degree $n \geq 1$ is determined inductively using the basic relations, is called the Fock representation.

Note that the Fock representation is not a $*$-representation with respect to the standard inner product on $\mathcal{T}(\mathcal{H})$. However, it was proved in [**6**] that there exists a unique Hermitian sesquilinear form $\langle \cdot, \cdot \rangle_0$ on $\mathcal{T}(\mathcal{H})$ such that λ_0 is a $*$-representation on $(\mathcal{T}(\mathcal{H}), \langle \cdot, \cdot \rangle_0)$. This form is called the Fock inner product on $\mathcal{T}(\mathcal{H})$.

The subspaces $\mathcal{H}^{\otimes n}$, $\mathcal{H}^{\otimes m}$, $n \neq m$, are orthogonal with respect to $\langle \cdot, \cdot \rangle_0$, and on $\mathcal{H}^{\otimes n}$ we have the following formula (see [**6**]):

$$\langle X, Y \rangle_0 = \langle X, P_n Y \rangle, \; n \geq 2.$$

So, the positivity of the Fock inner product is equivalent to the positivity of operators P_n, $n \geq 2$, and $\mathcal{I} = \bigoplus_{n \geq 2} \ker P_n$ determines the kernel of the Fock inner product. It was noted in [**6**] that the Fock representation is the GNS representation associated with the linear functional f on a Wick algebra such that $f(1) = 1$ and, for any Wick ordered monomial, $f(a_{i_1} \cdots a_{i_n} a_{j_1}^* \cdots a_{j_m}^*) = 0$. Then for any $X, Y \in \mathcal{T}(\mathcal{H})$ we have (see [**6**]):

$$\langle X, Y \rangle_0 = f(X^* Y).$$

2. In the following proposition we describe the kernel of the Fock representation of a Wick algebra with braided operator T in terms of the Fock inner product.

Proposition 1. *Let $W(T)$ be the Wick algebra with braided operator T, and let the Fock representation λ_0 be positive* (*i.e., the Fock inner product is positive definite*)*. Then* $\ker \lambda_0 = \mathcal{I} \otimes \mathcal{T}(\mathcal{H}^*) + \mathcal{T}(\mathcal{H}) \otimes \mathcal{I}^*$.

Proof. First, we show that $X \in \ker P_m$ implies $X \in \ker \lambda_0$. Indeed, let $Y \in \mathcal{H}^{\otimes n}$; then

$$\lambda_0(X)Y = X \otimes Y.$$

Note that for braided T we have the following decomposition (see [**2**] and sec. 3 for more details):

$$P_{n+m} = P(D_m)(P_m \otimes \mathbf{1}_n),$$

where

$$P(D_m) = \widetilde{R}_{n+m}\widetilde{R}_{n+m-1}\cdots\widetilde{R}_{m+1},$$
$$\widetilde{R}_k = 1 + T_{k-1} + T_{k-2}T_{k-1} + \cdots + T_1T_2\cdots T_{k-1}, \quad k \geq 2.$$

Then

$$P_{n+m}(\lambda_0(X)Y) = P_{n+m}(X \otimes Y) = P(D_m)(P_m X \otimes Y) = 0,$$

and $\lambda_0(X) = 0$ on $\big(\mathcal{T}(\mathcal{H})\,,\,\langle\,\cdot\,,\,\cdot\,\rangle_0\big)$. Therefore $\mathcal{I} \subset \ker\lambda_0$, and since $\ker\lambda_0$ is a $*$-ideal,

$$\mathcal{I} \otimes \mathcal{T}(\mathcal{H}^*) + \mathcal{T}(\mathcal{H}) \otimes \mathcal{I}^* \subset \ker\lambda_0. \tag{1}$$

To prove the converse inclusion, we need a formula determining the action of $\lambda_0(X^*)$ on $\mathcal{T}(\mathcal{H})$ for any $X \in \mathcal{H}^{\otimes k}$, $k \in \mathbb{N}$. For $k = 1$, $X = e_i$, $i = 1, \ldots, d$, it was proved in [**6**] that:

$$\lambda_0(e_i^*)Y = \mu(e_i^*)R_nY, \quad \forall\, Y \in \mathcal{H}^{\otimes n},$$

where $\mu(e_i^*)\colon \mathcal{T}(\mathcal{H}) \mapsto \mathcal{T}(\mathcal{H})$ is the annihilation operator:

$$\mu(e_i^*)e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} = \delta_{ii_1}e_{i_2} \otimes \cdots \otimes e_{i_n}.$$

Then, using the definition of P_n, it is easy to see that, for $X \in \mathcal{H}^{\otimes n}$ and $Y \in \mathcal{H}^{\otimes n}$,

$$\lambda_0(X^*)Y = \langle X, P_nY\rangle = \langle X, Y\rangle_0.$$

Let now

$$Z = \sum_{i=1}^{n} Y_iX_i^* + \sum_{j=n+1}^{l} Y_jX_j^* \in \ker\lambda_0,$$

where $Y_i \in \mathcal{T}(\mathcal{H})$, $i = 1, \ldots, l$,

$$X_i \in \mathcal{H}^{\otimes m},\ i = 1, \ldots, n\,, \quad X_j \in \mathcal{H}^{\otimes n_j},\ n_j > m\,, \quad j = n+1, \ldots, l.$$

Now (1) implies that we can suppose that the elements X_i are linearly independent modulo $\mathcal{I}$. Denote by $\{\widehat{X}_i,\ i = 1, \ldots, n\} \subset \mathcal{H}^{\otimes m}$ a family dual to the $\{X_i,\ i = 1, \ldots, n\}$ with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_0$, i.e., such that

$$\langle X_i\,,\,P_m\widehat{X}_j\rangle = \langle X_i\,,\,\widehat{X}_j\rangle_0 = \delta_{ij}, \quad i, j = 1, \ldots, n.$$

Since, for any $j = n+1, \ldots, l$ and $i = 1, \ldots, n$,

$$\lambda_0(X_j^*)\widehat{X}_i = 0,$$

we have, in $\big(\mathcal{T}(\mathcal{H})\,,\,\langle\,\cdot\,,\,\cdot\,\rangle_0\big)$,

$$0 = \lambda_0(Z)\widehat{X}_i = Y_i, \quad i = 1, \ldots, n,$$

which implies $Y_i \in \mathcal{I}$, $i = 1, \ldots, n$. The proof can be completed by evident induction. □

Remark 2. In particular, we have shown, for braided T, and for any $X \in \mathcal{H}^{\otimes n}$ and $Y \in \ker P_m$, that

$$X \otimes Y \in \ker P_{n+m}.$$

By similar arguments, $Y \otimes X \in \ker P_{n+m}$, i.e., $\mathcal{I} = \left\langle \bigotimes_{n\geq 2} \ker P_n \right\rangle$ is a two-sided ideal in $\mathcal{T}(\mathcal{H})$.

The two-sided ideal $\mathcal{J} \subset \mathcal{T}(\mathcal{H})$ is called a Wick ideal (see [**6**]) if it satisfies the following condition:

$$\mathcal{T}(\mathcal{H}^*) \otimes \mathcal{J} \subset \mathcal{J} \otimes \mathcal{T}(\mathcal{H}^*)\,. \tag{2}$$

If $\mathcal{J}$ is generated by some subspace of $\mathcal{H}^{\otimes n}$, then $\mathcal{J}$ is called a homogeneous Wick ideal of degree n.

We show that for Wick algebras with braided operator of coefficients, $\mathcal{I}$ is a Wick ideal .

Proposition 2. *Let T satisfy the braid condition, and $\mathcal{I} = \left\langle \bigoplus_{n\geq 2} \ker P_n \right\rangle$; then*

$$\mathcal{H}^* \otimes \mathcal{I} \subset \mathcal{I} + \mathcal{I} \otimes \mathcal{H}^*. \tag{3}$$

Proof. Note that conditions (2) and (3) are equivalent (see [**6**]). To prove the proposition, it is sufficient to show that, if $X \in \ker P_n$ for some $n \geq 2$, then for any $i = 1, \dots, d$,

$$e_i^* \otimes X \in \ker P_{n-1} + \ker P_n \otimes \mathcal{H}^*.$$

Indeed, for any $X \in \mathcal{H}^{\otimes n}$, we have the following formula (see [**9**]):

$$e_i^* \otimes X = \mu(e_i^*) R_n X + \mu(e_i^*) \sum_{k=1}^{d} T_1 T_2 \cdots T_n (X \otimes e_k) \otimes e_k^*.$$

Then for $X \in \ker P_n$, we have

$$P_{n-1}\mu(e_i^*) R_n X = \mu(e_i^*)(1 \otimes P_{n-1}) R_n X = \mu(e_i^*) P_n X = 0.$$

Note that, for braided T, for any $k = 2, \dots, n$,

$$T_k (T_1 T_2 \cdots T_n) = (T_1 T_2 \cdots T_n) T_{k-1},$$

which implies that

$$(1 \otimes P_n)(T_1 T_2 \cdots T_n) = (T_1 T_2 \cdots T_n)(P_n \otimes 1).$$

Then for any $k = 1, \dots, d$,

$$\begin{aligned} P_n \mu(e_i^*) T_1 T_2 \cdots T_n (X \otimes e_k) &= \mu(e_i^*)(1 \otimes P_n) T_1 T_2 \cdots T_n (X \otimes e_k) \\ &= \mu(e_i^*) T_1 T_2 \cdots T_n (P_n \otimes 1)(X \otimes e_k) = 0. \quad \square \end{aligned}$$

For Wick algebras with braided T, the largest homogeneous ideal of degree n is generated by $\ker R_n$ (see [**6**] and [**9**]), i.e., the condition $\ker R_n \neq \{0\}$ is necessary and sufficient for the existence of homogeneous Wick ideals. In the following proposition we show that the same is true for arbitrary Wick ideals.

Theorem 1. *If $\mathcal{J} \subset \mathcal{T}(\mathcal{H})$ is a non-trivial Wick ideal, then there exists $n \geq 2$ such that* $\ker R_n \neq \{0\}$.

Proof. For any $X \in \mathcal{T}(\mathcal{H})$, by $\deg X$ we denote the highest degree of its homogeneous components. Let $Y \in \mathcal{J}$ be of minimal degree.

$$Y = Y_1 + Y_2 + \cdots + Y_k, \quad Y_i \in \mathcal{H}^{\otimes n_i},\ i = 1, \ldots, k,\ n_i \in \mathbb{N} \cup \{0\}.$$

Suppose that $\deg Y \geq 2$: then for any $i = 1, \ldots, d$, we have

$$e_i^* \otimes Y = \sum_{j=1}^{k} \mu(e_i^*) R_{n_j} Y_j + \mu(e_i^*) \sum_{j=1}^{k} \sum_{l=1}^{d} \widetilde{T}_{n_j} (Y_j \otimes e_l) \otimes e_l^*,$$

where we put $R_0 = 1$, $R_1 = 1$, and

$$\widetilde{T}_k = \begin{cases} T_1 T_2 \cdots T_k, & k \geq 2, \\ T, & k = 1, \\ 1, & k = 0. \end{cases}$$

Then condition (3) implies that for any $i = 1, \ldots, d$,

$$\sum_{j=1}^{k} \mu(e_i^*) R_{n_j} Y_j \in \mathcal{J}.$$

Since the degrees of these elements are less than the degree of Y, we conclude that

$$\sum_{j=1}^{k} \mu(e_i^*) R_{n_j} Y_j = 0, \quad i = 1, \ldots, d,$$

and the independence of the Wick ordered monomials then implies

$$\mu(e_i^*) R_{n_j} Y_j = 0, \quad i = 1, \ldots, d.$$

Let Y_k be the highest homogeneous component of Y; then, by our assumption, $\deg Y_k \geq 2$, and $\sum_{i=1}^{d} e_i \mu(e_i^*) R_{n_k} Y_k = R_{n_k} Y_k = 0$, i.e., $Y_k \in \ker R_{n_k}$.

To complete the proof, note that if $X = \beta + \sum_{i=1}^{d} \alpha_i e_i \in \mathcal{J}$, then for any j, we have

$$e_j^* \otimes X = \alpha_j + \beta e_j^* + \sum_{i=1}^{d} \alpha_i \sum_{k,l=1}^{d} T_{ji}^{kl} e_l \otimes e_k^*,$$

and (3) implies $\alpha_j = 0$, $j = 1, \ldots, d$, $\beta = 0$. $\square$

3. The structure of $\ker P_n$

In this section, we show that for Wick algebras with braided T satisfying the condition $-1 < T \leq 1$, the Fock representation is faithful, and for $-1 \leq T \leq 1$, the kernel of the Fock representation is generated by the largest quadratic Wick ideal (the largest quadratic Wick ideal is the largest homogeneous Wick ideal of degree 2).

To do this we need some properties of quasimultiplicative maps on the Coxeter group S_n (for more detailed information we refer the reader to [**2**]).

1. Consider S_{n+1} as a Coxeter group, i.e., a group defined as follows: $S_{n+1} = \langle \sigma_i : \sigma_i^2 = e,\ \sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| \geq 2,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ i = 1, \ldots, n\rangle$. In order to study the invertibility of P_n for any family of operators $\{T_i\ ,\ i = 1\ldots, n,\ T \in B(\mathcal{K})\}$, satisfying the conditions

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},\ T_i^* = T_i,\ -1 \leq T_i \leq 1,$$

where $\mathcal{K}$ is a separable Hilbert space, we may define (as in [**2**]) the function

$$\phi\colon S_{n+1} \mapsto B(\mathcal{K})$$

by the formulas

$$\phi(e) = 1, \quad \phi(\sigma_i) = T_i, \tag{4}$$

$$\phi(\pi) = T_{i_1}\cdots T_{i_k}, \tag{5}$$

where $\pi = \sigma_{i_1}\cdots\sigma_{i_k}$ is a reduced decomposition. It was shown in [**2**] that

$$P_{n+1} = P(S_{n+1}) = \sum_{\sigma\in S_{n+1}} \phi(\sigma).$$

Denote by S the set of generators of S_{n+1} as a Coxeter group. Consider, for any $J \subset S$, the set

$$D_J = \{\sigma \in S_{n+1} \mid |\sigma s| = |\sigma| + 1,\ \forall\, s \in J\}.$$

Let W_J be a Coxeter group, generated by J. Then $S_{n+1} = D_JW_J$ (see [**1**]), and $P_{n+1} = P(D_J)P(W_J)$ (see [**2**]). Using the equalities $P_{n+1}^* = P_{n+1}$, $P(W_J)^* = P(W_J)$, we obtain $P_{n+1} = P(W_J)P(D_J)^*$, where for all $M \subset S_{n+1}$,

$$P(M) = \sum_{\sigma\in M} \phi(\sigma).$$

In what follows we use a quasimultiplicative analogue of the Euler-Solomon formula (see [**2**, Lemma 2.6]):

$$\sum_{\substack{J\subset S\\ J\neq S,\, J\neq\emptyset}} (-1)^{|J|}P(D_J) = -(-1)^{|S|}1 + \phi(\sigma_0^{(n+1)}) - P(S_{n+1}), \tag{6}$$

where $\sigma_0^{(n+1)}$ is the unique element of S_{n+1} with maximal possible length of the reduced decomposition.

Remark 3. 1. The element $\sigma_0^{(n+1)}$ of the group S_{n+1} has the form

$$\sigma_0^{(n+1)} = (\sigma_1 \cdots \sigma_n)(\sigma_1 \cdots \sigma_{n-1}) \cdots (\sigma_1\sigma_2)\sigma_1.$$

Set $U_n = \phi(\sigma_0^{(n+1)})$: then

$$U_n = (T_1T_2 \cdots T_n)(T_1T_2 \cdots T_{n-1}) \cdots (T_1T_2)T_1.$$

2. It is easy to see that the operator U_n is selfadjoint, and, taking adjoints, we can rewrite (6) in the following form:

$$\sum_{J\subset S,\, J\neq\emptyset} (-1)^{|J|} P(D_J)^* = (-1)^{n+1}1 + U_n - P_{n+1}. \tag{7}$$

3. Note also that, for all $J \subset S$, the group W_J is isomorphic to S_k for some $k < n$, or to the direct product of some such groups.

2. In what follows we shall use the following properties of the operator U_n.

Proposition 3. $\ker P_{n+1}$ *is invariant with respect to the action of* U_n.

Proof. First we show that for all $J \subset S$,

$$P(D_J^*)\colon \ \ker P_{n+1} \mapsto \ker P_{n+1}.$$

It can be easily obtained from the equality

$$P_{n+1}P(D_J)^* = P(D_J)P(W_J)P(D_J)^* = P(D_J)P_{n+1}.$$

Then by (7), we have

$$U_n - (-1)^n 1\colon \ \ker P_{n+1} \mapsto \ker P_{n+1}.$$ □

Proposition 4. *Let operators* $\{T_i,\ i = 1, \ldots, n\}$ *satisfy the braid condition* $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, $i = 1, \ldots, n-1$, *and* $T_iT_j = T_jT_i$, $|i-j| \geq 2$. *Then*

$$T_kU_n = U_nT_{n+1-k}, \quad \forall\, k = 1, \ldots n. \tag{8}$$

Proof. 1. For $n = 1$ the equality is evident.

2. Suppose that (8) holds for any $n \leq m$. Note that

$$U_{m+1} = T_1T_2 \cdots T_{m+1}U_m.$$

Then, for $1 < k \leq m+1$, we have

$$\begin{aligned}
T_kU_{m+1} &= T_k(T_1T_2 \cdots T_{m+1})U_m \\
&= T_1T_2 \cdots T_{k-2}T_kT_{k-1}T_kT_{k+1} \cdots T_{m+1}U_m \\
&= T_1T_2 \cdots T_{k-2}T_{k-1}T_kT_{k-1}T_{k+1} \cdots T_{m+1}U_m \\
&= (T_1T_2 \cdots T_{m+1})T_{k-1}U_m \\
&= T_1T_2 \cdots T_{m+1}U_mT_{m+1-(k-1)} \\
&= U_{m+1}T_{m+2-k}.
\end{aligned}$$

In particular, for $k = m + 1$ we have $T_{m+1}U_{m+1} = U_{m+1}T_1$. Then taking adjoints, we obtain the required equality for $k = 1$. □

3. Now we can formulate the main result of this paper.

Theorem 2. *Let $W(T)$ be a Wick algebra with braided operator T satisfying the norm bound $\|T\| \leq 1$. Then for any $n \geq 2$, we have*

$$\ker P_{n+1} = \sum_{k+l=n-1} \mathcal{H}^{\otimes k} \otimes \ker(1+T) \otimes \mathcal{H}^{\otimes l} = \sum_{k=1}^{n} \ker(1+T_k).$$

Proof. In fact, we shall prove the following: Let $T_1, T_2, \dots, T_n \in B(\mathcal{K})$, where $\mathcal{K}$ is a finite-dimensional Hilbert space, be selfadjoint contractions satisfying the relations

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},\ 1 \leq i \leq n-1, \quad T_iT_j = T_jT_i,\ |i-j| \geq 2.$$

Then

$$\ker P_{n+1} = \sum_{k=1}^{n} \ker(1+T_k). \tag{9}$$

(It follows trivially from the decomposition $P_{n+1} = P(D_{\{k\}})(1+T_k)$ that $\sum_{k=1}^{n} \ker(1+T_k) \subset \ker P_{n+1}$).

We proceed using induction.

The case $n = 2$**.**

In this case $P_2 = 1 + T$.

The case $n \mapsto n+1$**.**

It follows from $P_{n+1} = P(W_J)P(D_J)^*$ that

$$P(D_J)^* \colon \ker P_{n+1} \mapsto \ker P(W_J),$$

i.e., $\operatorname{ran}(P(D_J)^*|_{\ker P_{n+1}}) \subset \ker P(W_J)$. Moreover, it is obvious that for any $J \subset S$, $\ker P(W_J) \subset \ker P_{n+1}$. Therefore, by (7), we have the following inclusion:

$$\operatorname{ran}(U_n - (-1)^n 1)|_{\ker P_{n+1}} \subset \sum_{J \subset S,\, J \neq \emptyset} \ker P(W_J).$$

Since, for $J \subset S$, the group $W_J = W_{J_1} \times \cdots \times W_{J_k}$, where $W_{J_l} \simeq S_{n_l}$ with $n_l < n+1$, we have a decomposition into the product of pairwise commuting selfadjoint operators

$$P(W_J) = P(W_{J_1}) \cdots P(W_{J_k}).$$

Therefore

$$\ker P(W_J) = \sum_{l=1}^{k} \ker P(W_{J_l}) \subset \sum_{i=1}^{n} \ker(1+T_i),$$

where the last inclusion is obtained from the assumption of induction. So,

$$\operatorname{ran}(U_n - (-1)^n 1)|_{\ker P_{n+1}} \subset \sum_{i=1}^{n} \ker(1 + T_i). \tag{10}$$

Consider the operator $1 - U_n^2$. Since $U_n = U_n^*\colon \ker P_{n+1} \mapsto \ker P_{n+1}$, then

$$\ker P_{n+1} = \operatorname{ran}(1 - U_n^2)|_{\ker P_{n+1}} + \ker(1 - U_n^2)|_{\ker P_{n+1}}.$$

Moreover, since $\operatorname{ran}(1 - U_n^2) \subset \operatorname{ran}(U_n - (-1)^n 1)$, using (10), we have the inclusion

$$\ker P_{n+1} \subset \sum_{k=1}^{n} \ker(1 + T_i) + \ker(1 - U_n^2)|_{\ker P_{n+1}}.$$

To finish the proof it remains only to show that

$$\ker(1 - U_n^2) \cap \ker P_{n+1} \subset \sum_{i=1}^{n} \ker(1 + T_i).$$

To this end, we may present $1 - U_n^2$ in the form

$$\begin{aligned} 1 - U_n^2 &= 1 - T_1 T_2 \cdots T_n U_{n-1}^2 T_n \cdots T_2 T_1 \\ &= (1 - T_1^2) + T_1(1 - T_2^2)T_1 \\ &\quad + \cdots \\ &\quad + T_1 T_2 \cdots T_{n-1}(1 - T_n^2) T_{n-1} \cdots T_2 T_1 \\ &\quad + T_1 T_2 \cdots T_n (1 - U_{n-1}^2) T_n \cdots T_2 T_1. \end{aligned}$$

Since $\|T\| \le 1$ implies that $\|T_i\| \le 1$, $i = 1, \dots n$, and $\|U_k\| \le 1$, $k \ge 2$, then we have a sum of non-negative operators, and $v \in \ker(1 - U_n^2)$ implies that

$$\begin{aligned} &T_1^2 v = v, \\ &T_2^2 T_1 v = T_1 v, \\ &\vdots \\ &T_n^2 T_{n-1} \cdots T_2 T_1 v = T_{n-1} \cdots T_2 T_1 v. \end{aligned}$$

However, $T_k U_n = U_n T_{n+1-k}$ implies that $T_k U_n^2 = U_n^2 T_k$, and, consequently,

$$T_k\colon \ker(1 - U_n^2) \mapsto \ker(1 - U_n^2), \quad k = 1, \dots, n.$$

Moreover, since the restriction of T_1 to $\ker(1 - U_n^2)$ is an involution,

$$\operatorname{ran}(T_1)|_{\ker(1 - U_n^2)} = \ker(1 - U_n^2),$$

and, for any $v \in \ker(1 - U_n^2)$, we have $T_2^2 v = v$. By the same arguments, we obtain that

$$\forall\, v \in \ker(1 - U_n^2), \quad T_i^2 v = v,\ i = 1, \dots, d.$$

Let now $v \in \ker(1-U_n^2) \cap \ker P_{n+1}$; then, for any $k = 1, \dots, n$,

$$\begin{aligned} P_{n+1}T_k v &= P(D_{\{k\}})(1+T_k)T_k v \\ &= P(D_{\{k\}})(T_k + T_k^2)v \\ &= P(D_{\{k\}})(1+T_k)v = P_{n+1}v = 0. \end{aligned}$$

Therefore T_k maps $\ker(1-U_n^2) \cap \ker P_{n+1}$ onto itself for any $k = 1, \dots, n$. This fact implies that, for any $\sigma \in S_{n+1}$, we have $\phi(\sigma)v \in \ker(1-U_n^2) \cap \ker P_{n+1}$, and

$$\forall\, k = 1, \dots, n,\ \forall\, \sigma \in S_{n+1}, \qquad (1-T_k^2)\phi(\sigma)v = 0.$$

For convenience, we fix the set S_{n+1}, and set $v_i := \phi(\pi_i)v$ for $\pi_i \in S_{n+1}$ ($\pi_1 := \mathrm{id}$ and $v_1 = v$). Then the condition $P_{n+1}v = 0$ takes the form

$$\sum_{k=1}^{n!} v_k = 0. \tag{11}$$

Finally, for any pair $i \neq j$ there exist generators $\sigma_{i_1}, \dots, \sigma_{i_m} \in S$ such that

$$\pi_j = \sigma_{i_1} \cdots \sigma_{i_m} \pi_i$$

and

$$v_j = T_{i_1} \cdots T_{i_m} v_i.$$

Note that, if $v_k = T_r v_l$ for some $r = 1, \dots, n$, then $T_r^2 v_k = v_k$ implies that $v_k - v_l \in \ker(1+T_r)$. Therefore, for any $i \neq j$,

$$v_i - v_j \in \sum_{k=1}^{n} \ker(1+T_k).$$

In particular, for any $j = 2, \dots, n!$,

$$v_1 - v_j \in \sum_{k=1}^{n} \ker(1+T_k).$$

Then from (11), we have

$$n!\, v = n!\, v_1 \in \sum_{k=1}^{n} \ker(1+T_k),$$

and therefore

$$v \in \sum_{k=1}^{n} \ker(1+T_k). \qquad \square$$

Remark 4. Evidently the proof does not depend on the dimension of $\mathcal{K}$. Indeed, in the case when $\mathcal{K}$ is infinite-dimensional, the linear subspace in (9) is replaced by its closure. I.e., if $\mathcal{K}$ is a separable Hilbert space and

$\{T_i,\ i = 1, \ldots, n\}$ are selfadjoint contractions satisfying the braid conditions, then

$$\ker P_{n+1} = \overline{\sum_{k=1}^{n} \ker(1 + T_k)}.$$

As a corollary we have an improved version of the result of Bożejko and Speicher (see [**2**]).

Proposition 5. *If the operator T satisfies the braid condition, and $-1 < T \leq 1$, then $P_n > 0$, $n \geq 2$, i.e., the Fock inner product is strictly positive, and the Fock representation acts in the whole space $\mathcal{T}(\mathcal{H})$.*

Proof. Recall that if T is braided and $\|T\| \leq 1$ then $P_n \geq 0$ (see [**2**]). It remains only to show that $\ker P_n = \{0\}$ for $-1 < T \leq 1$. This fact trivially follows from our theorem since in this case $\ker(1 + T) = \{0\}$. □

4. Corollaries and examples

We summarize the results obtained above in the following proposition.

Proposition 6. *If $W(T)$ is a Wick algebra with braided operator of coefficients T satisfying the norm bound $\|T\| \leq 1$, then the following three statements hold.*

1. *The kernel of the Fock representation is generated by the largest quadratic Wick ideal. In particular, if $-1 < T \leq 1$, then the Fock representation is faithful.*
2. *For any $n \geq 2$ we have the inclusion $\mathcal{I}_n \subset \mathcal{I}_2$.*
3. *If $-1 < T \leq 1$, then $W(T)$ has no non-trivial Wick ideals.*

Example 1. Consider the q-CCR algebra based on a Hilbert space $\mathcal{H}$ and the relations

$$a_i^* a_i = 1 + q a_i a_i^*, \quad i = 1, \ldots, d,$$
$$a_i^* a_j = q a_j a_i^*, \quad i \neq j, \quad 0 < q < 1.$$

We pick an orthogonal basis (e_i) in $\mathcal{H}$, and then T is determined on this basis by the formulas

$$T e_i \otimes e_j = q e_j \otimes e_i, \quad \|T\| < 1.$$

It is evident that T is braided. Then by the proposition, we cannot have any Wick ideals in $W(T)$.

It was proved in [**2**] that for braided T satisfying the norm bound $\|T\| < 1$, the Fock representation is bounded. Therefore we may consider the C^*-algebra generated by operators of the Fock representation.

Recall that a $*$-algebra is called C^*-representable if it can be realized as a $*$-subalgebra of a certain C^*-algebra (see for example [**7**]). Combining the results of Theorem 2 and Proposition 5, we obtain the following statement.

Proposition 7. *If $W(T)$ is a Wick algebra with braided operator of coefficients T satisfying the norm bound $\|T\| < 1$, then $W(T)$ is C^*-representable.*

Suppose that, in the case of braided T with $\|T\| = 1$ and $\ker(1+T) \neq \{0\}$, the Fock representation is bounded. Then Theorem 2 implies that the quotient $W(T)/\mathcal{I}_2$ is C^*-representable.

Example 2. Consider the following type of q_{ij}-CCR (see [**2**]):

$$\begin{aligned} a_i^* a_i &= 1 + q_i a_i a_i^*, \quad i = 1, \dots, d, \quad 0 < q_i < 1, \\ a_i^* a_j &= \lambda_{ij} a_j a_i^*, \quad i \neq j, \ |\lambda_{ij}| = 1, \ \lambda_{ij} = \overline{\lambda}_{ij}. \end{aligned}$$

The corresponding T is braided, $\|T\| = 1$, and

$$\ker(1+T) = \langle a_j a_i - \lambda_{ij} a_i a_j, \ i < j \rangle.$$

Moreover, the Fock representation of this algebra is bounded. Then, as noted above, the $*$-algebra generated by the relations

$$\begin{aligned} a_i^* a_i &= 1 + q_i a_i a_i^*, \quad i = 1, \dots, d, \quad 0 < q_i < 1, \\ a_i^* a_j &= \lambda_{ij} a_j a_i^*, \quad i \neq j, \ |\lambda_{ij}| = 1, \ \lambda_{ij} = \overline{\lambda}_{ij}, \\ a_j a_i &= \lambda_{ij} a_i a_j, \quad i < j \end{aligned}$$

is C^*-representable.

A description of the irreducible representations of these relations can be found for example in [**8**, sec. 2.4].

Note that, if $\|T\| = 1$, then the operators of the Fock representation can be unbounded.

Example 3. Consider the following Wick algebra:

$$\begin{aligned} a_i^* a_i &= 1 + a_i a_i^*, \quad i = 1, \dots, d, \\ a_i^* a_j &= q a_j a_i^*, \quad i \neq j, \quad -1 < q < 1. \end{aligned}$$

The corresponding T is determined by the formulas

$$T e_i \otimes e_i = e_i \otimes e_i, \quad T e_j \otimes e_i = q e_i \otimes e_j, \ i \neq j, \ i = 1, \dots, d.$$

It is easy to see that T is braided and $-1 < T \leq 1$. So, the Fock representation of this algebra is faithful. Note that, if we consider the complement of $\mathcal{T}(\mathcal{H})$ with respect to the Fock inner product, then the operators of the Fock representation are unbounded.

For the definition and properties of representations of $*$-algebras by unbounded operators, see for example [**11**].

Unbounded representations of Wick algebras will be considered in more detail later.

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