**HOW TO PROVE DYNAMICAL LOCALIZATION**

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**Abstract:** Let $H$ be a self-adjoint operator on $l^2(\mathbf{Z}^d)$ or $L^2(\mathbf{R}^d)$ with pure point spectrum on some interval $I$. We establish general necessary and sufficient conditions for dynamical localization for a given vector and on the interval of energies $I$. The sufficient conditions we obtain improve the existing ones such as SULE or WULE and can be useful in applications.

# 1 Introduction

Let $H$ be a self-adjoint operator on the Hilbert space $\mathcal{H}$ with pure point spectrum on some interval $I \subset \mathbf{R}$. We shall consider the case $\mathcal{H} = l^2(\mathbf{Z}^d)$ as well as $\mathcal{H} = L^2(\mathbf{R}^d)$. Consider the subspace $\mathcal{H}(I)$ of $\mathcal{H}$, $\mathcal{H}(I) = P_I(H)\mathcal{H}$, where $P_I(H)$ is the spectral projector of $H$ onto $I$. It is natural to say that the operator $H$ has dynamical localization on $I$ if for any $p > 0$ and well localised $\psi \in \mathcal{H}$

$$\sup_t r^p(t) \equiv \sup_t \langle \exp(-itH)P_I(H)\psi, |X|^p \exp(-itH)P_I(H)\psi\rangle < +\infty, \tag{1.1}$$

where $X$ is the usual position operator.

The problem of dynamical localization was intensively studied during last years [1]-[8], especially in the case of random discrete and continuous Schrödinger operators (in particular, the Anderson model). The aim of the present paper is not to give a review of the results obtained for concrete models. We are rather interested by general mathematical methods which can be used to prove dynamical localization (1.1) for any self-adjoint operator $H$. We hope, however, that the obtained results (especially sufficient conditions for dynamical localization) will be useful in applications.

Let $\{e_k\}$ be any orthonormal set of eigenfunctions of $H$ complete in $\mathcal{H}(I)$. For any $k$ we have $He_k = \lambda_k e_k$ with $\lambda_k \in I$. Suppose that $\mathcal{H} = l^2(\mathbf{Z}^d)$ (in the last section of the paper we discuss also the continuous case). Let $\psi = \delta_m$ for some $m \in \mathbf{Z}^d$. Then

$$\psi_I(t,n) \equiv (\exp(-itH)P_I(H)\psi)(n) = \sum_k \exp(-it\lambda_k)e_k(n)\overline{e_k(m)}$$

and

$$\sup_t |\psi_I(t,n)| \le W(n,m), \quad \sup_t r^p(t) \le \sum_{n\in\mathbf{Z}^d} |n|^p W^2(n,m), \tag{1.2}$$

where

$$W(n,m) = \sum_k |e_k(n)e_k(m)|.$$

To prove dynamical localization (1.1) for any $\psi = \delta_m$, it is sufficient to show that the function $W(n,m)$ is fast decaying as $|n| \to \infty$ for all $m \in \mathbf{Z}^d$.

What one usually proves for "concrete" Schrödinger operators is the so-called exponential (or mathematical) localization on interval $I$. Namely, for some $\alpha > 0$ and any eigenfunction $e_k \in \mathcal{H}(I)$,

$$|e_k(n)| \le C(k)\exp(-\alpha|n|) \tag{1.3}$$

where $C(k) < +\infty$. If the sum is finite, it is evident that $W(n,m)$ is exponentially decaying in $n$ for any $m$ and (1.1) holds. However, typically the sum has infinitely many terms. In this case the bounds (1.3) give nothing about the behaviour of the sum $W(n,m)$. The well known example of [5] shows that it is quite possible that (1.3) holds, but there are no dynamical localization for $\psi = \delta_0$ and $r^2(t_n)/t_n^{2-\delta} \to +\infty$ for any $\delta > 0$ for some sequence $t_n \to \infty$.

It is clear that to prove dynamical localization, one should control the decay of $|e_k(n)e_k(m)|$ both in $n$ and in $k$. Or, equivalently, one should control the constants $C(k)$ in (1.3). Two approaches have been developed to solve this problem. The first is to estimate rather directly $|e_k(n)e_k(m)|$ and to prove that the sum $W(n,m)$ is fast decaying as $|n| \to \infty$ [7], [8]. For example, one shows that

$$|e_k(n)e_k(m)| \le C(m)a_k \exp(-\alpha|n-m|), \tag{1.4}$$

where $\sum_k a_k = D < +\infty$. Clearly, (1.4) yields

$$W(n,m) \le C(m)D\exp(-\alpha|n-m|)$$

and (1.1) holds for any finite $\psi$. In particular, the condition called WULE was considered [7]. This is (1.4), where $a_k = \|Be_k\|^2$ and $B$ is the operator of multiplication by $(1+|n|^2)^{-\delta/2}$ with $\delta > d/2$. Obviously, as $\sum_k |e_k(n)|^2 \le 1$ for any $n$,

$$\sum_k a_k \le \sum_n (|n|+1)^{-\delta} < +\infty.$$

Another possibility is to have some control of constants $C(k)$ in (1.3). Namely, the following condition called SULE was introduced in [5]. The operator $H$ has SULE on $I$, if $H$ has a complete in $\mathcal{H}(I)$ set $\{e_k\}$ of orthonormal eigenfunctions and there exist $\alpha > 0$ and $n_k \in \mathbf{Z}^d$ such that for any $\delta > 0$,

$$|e_k(n)| \le C(\delta)\exp(\delta|n_k| - \alpha|n-n_k|) \tag{1.5}$$

with some finite constants $C(\delta)$ uniform in $k, n$. One shows that if (1.5) holds, then $e_k(m)$ are fast decaying in $k$ for all $m$. So, it is easy to see that the function $W(n,m)$ is exponentially decaying in $n$ for all $m$ and (1.1) holds (in fact, one controls also the behaviour in $m$, so one proves (1.1) for any exponentially decaying $\psi$). This result was extended to the continuous case in [6] and applied in [6], [4] to prove dynamical localization for some concrete models. Similar ideas are used in [3] to prove strong dynamical localization.

In the present paper we propose a new approach which ameliorates considerably the existing methods. Surprisingly, one can show that if the function $|e_k(n)e_k(m)|$ decays sufficiently fast in $n$ uniformly in $k$, then one should not take special care of decay in $k$ necessary to the convergence of the sum $W(n,m)$. The key point is the following. Due to the fact that the system $\{e_k\}$ is orthonormal in $\mathcal{H}$, the decay in $n$ and decay in $k$ of $|e_k(n)e_k(m)|$ are related. Therefore, one can "sacrify" some decay in $n$ to obtain a decay in $k$ sufficient to the convergence of the sum. One can even allow some growth in $k$ in the bounds for $|e_k(n)e_k(m)|$ provided the decay in $n$ is fast enough (see Theorem 5.3 and Theorem 5.4).

One should say that there is a deep relation between our approach and that of [5]. The main technical ingredient in our consideration is the following result (Theorem 2.2 and Theorem 6.1). Let $\{e_k\}$ be any orthonormal system in $l^2(\mathbf{Z}^d)$. For any $p>0$ define the positive numbers

$$d_k(p)=\sum_n |e_k(n)|^2(|n|+1)^p, \quad \eta_k(p)=\sup_n(|e_k(n)|^2\exp(p|n|))$$

(where it is possible that $d_k(p)=+\infty$, $\eta_k(p)=+\infty$). For any $p>0$, one can reorder $d_k(p),\eta_k(p)$ so that

$$d_k(p)\geq D(p,d)k^{p/d}, \quad \eta_k(p)\geq C(p,d)\exp(\beta k^{1/d}) \tag{1.6}$$

with some positive universal constants $D(p,d),C(p,d),\beta(p,d)$. Considering the SULE condition, one can easily verify that the fact that $|n_k|\geq Ck^{1/d}$ (after reordering) implies $d_k(p)\geq Dk^{p/d}$ and $\eta_k(p)\geq C\exp(\beta k^{1/d})$ for any $p>0$. So, the growth of $|n_k|$ for the systems $\{e_k\}$ with SULE, which plays a key role in the proof of [5], can be considered as a manifestation of a more general result (1.6) valid for any orthonormal system. When proving (1.6), we don't need any assumptions about the form of eigenfunctions $e_k$ or the notion of "centers of localization" $n_k$.

The paper is organised as follows. In Section 2 we prove our main technical result about the growth of the moments $d_k(p)$. In Section 3 we establish necessary conditions for dynamical localization for a given vector $\psi$. In particular, we show (Theorem 3.4) that if

$$\sup_t |X|^p_\psi(t)\equiv\sup_t\langle\exp(-itH)\psi,|X|^p\exp(-itH)\psi\rangle<+\infty, \tag{1.7}$$

for some $p>0$, then the coefficients of the spectral measure of $\psi$ decay sufficiently fast. Namely, if $\mu_\psi=\sum_k a_k\delta_{\lambda_k}$, then

$$\sum_k a_k^{\frac{1}{1+\beta}}<+\infty$$

for any $0<\beta<p/d$. In particular, if (1.7) holds for any $p>0$, then $a_k$ are fast decaying:

$$\sum_k a_k^\nu<+\infty \quad \forall\nu>0.$$

In Section 4 we give some sufficient conditions for dynamical localization for a given vector $\psi$ and $p > 0$ (Theorem 4.2 is the main result of the Section). As a result, we obtain (Corollary 4.3) general necessary and sufficient conditions for dynamical localization for a given vector $\psi$ for all $p > 0$. Namely, let $\psi \in \mathcal{H}_{pp}$. One can always represent it as

$$\psi = \sum_k \psi_k, \quad \psi_k \in \mathcal{H}_{\lambda_k},$$

where $\lambda_k \neq \lambda_s$ for $k \neq s$ and $\mathcal{H}_\lambda$ is an eigenspace of $H$ corresponding to the eigenvalue $\lambda$. Consider the function

$$R_\psi(n) = \sup_k |\psi_k(n)|.$$

Then (1.7) holds for any $p > 0$ iff $R_\psi(n)$ is fast decaying. We show also that the dynamical localization

$$\sup_t |X|^p_\psi(t) < +\infty \quad \forall p > 0$$

is equivalent to the dynamical localization for Cesaro averages:

$$\sup_T 1/T \int_0^T |X|^p_\psi(t) dt < +\infty \quad \forall p > 0.$$

The results of Sections 3 and 4 are well adapted to the case of power law or subexponential decay of eigenfunctions of $H$.

In Section 5 we discuss the problem of dynamical localization on the interval of energies $I$ (in the sense (1.1)). Theorems 5.3 and 5.4 give sufficient conditions that (1.1) hold for any finite $\psi$ and any fast decaying $\psi$ respectively. We give also some bound (Theorem 5.5) which can be used to prove the strong dynamical localization on $I$ considered in [3], [8] (in the case when there is a family of operators $H$ depending on some parameter).

In Section 6 we consider exponential dynamical localization:

$$\sup_t |(\exp(-itH)\psi)(t,n)| \leq C\exp(-\gamma|n|), \ \gamma > 0, \tag{1.8}$$

(denoted as $EDL(\psi)$) which is a special case of dynamical localization. In particular, we show that if (1.8) holds for some vector $\psi$, then the coefficients of the spectral measure of $\psi$ decay (after reordering) as follows:

$$a_k \leq C\exp(-\beta k^{1/d})$$

with some $\beta > 0$. We give also (Theorem 6.5) some sufficient conditions for exponential localization on the interval of energies $I$. In particular, if

$$|e_k(n)e_k(m)| \leq C\exp(-\alpha|n-m| + \beta|m|) \tag{1.9}$$

for some $\alpha > 0, \beta > 0$, then $EDL(P_I(H)\psi)$ holds for any exponentially decaying $\psi$. The condition (1.9) is similar to (1.4), but is easier to prove in concrete cases because one don't need the decreasing factors $a_k$. In particular, the SULE condition (1.5) implies immediately (1.9).

In Section 7 we consider the continuous case $\mathcal{H} = L^2(\mathbf{R}^d)$. We show that most of results proved in the discrete case remain true under some additional assumptions about eigenfunctions of $H$. In particular, it is the case if $H = -\Delta + V(x)$ with $V(x)$ bounded from below and $I = (-\infty, K],\ K \in \mathbf{R}$.

One should stress that practically all the results of the paper hold whatever is the multiplicity of the spectrum of $H$.

# 2 Lower bounds for the moments of eigenfunctions

Let $\{a_{kn}\}$ be a double sequence of nonnegative numbers labelled by $k \in \mathbf{N},\ n \in \mathbf{Z}^d$. We shall suppose that there exist two positive constants $A, B$ such that

$$\forall n \in \mathbf{Z}^d,\ \sum_{k=1}^{\infty} a_{kn} \leq A < +\infty, \tag{2.1}$$

$$\forall k \in \mathbf{N},\ \sum_{n \in \mathbf{Z}^d} a_{kn} \geq B > 0. \tag{2.2}$$

For $p > 0$ define the numbers

$$d_k(p) = \sum_{n \in \mathbf{Z}^d} a_{kn}(|n| + 1)^p$$

with the understanding that $d_k(p)$ may be equal to $+\infty$. One can also remark that $d_k(p) \geq B$ for any $k, p$.

**Lemma 2.1** *Let $p > 0$. There exist a positive constants $D(A, B, p, d)$ such that for any $\{a_{nk}\}$ satisfying (2.1), (2.2), one can reorder $d_k(p)$ so that*

$$d_k(p) \geq D k^{\frac{p}{d}}. \tag{2.3}$$

Proof. For any $N > 0$ consider the following set in $\mathbf{N}$:

$$J(N) = \{k \in \mathbf{N} \mid \sum_{n:|n| \geq N} a_{kn} \leq B/2\}$$

and the sum

$$S(N) = \sum_{k \in J(N)} \sum_{|n| < N} a_{kn}.$$

It follows from (2.2) that for any $k \in J(N)$

$$\sum_{|n| < N} a_{kn} = \sum_{n \in \mathbf{Z}^d} a_{kn} - \sum_{|n| \geq N} a_{kn} \geq B - B/2 = B/2.$$

Therefore,

$$S(N) \geq B/2 \mathrm{Card}(J(N)). \tag{2.4}$$

On the other hand, with (2.1) we can estimate:

$$S(N) \leq \sum_{k=1}^{\infty} \sum_{|n|\leq N} a_{kn} \leq A \sum_{|n|\leq N} 1 \leq C(d)A(N+1)^d. \tag{2.5}$$

The bounds (2.4), (2.5) yield

$$\mathrm{Card}(J(N)) \leq C(d)A/B(N+1)^d \tag{2.6}$$

with some constant $C(d)$.

For any $p>0$ consider the following set in $\mathbf{N}$:

$$I(N,p) = \{k \in \mathbf{N} \mid d_k(p) \leq \frac{B}{2}(N+1)^p\}.$$

Since

$$d_k(p) = \sum_{n\in\mathbf{Z}^d} a_{kn}(|n|+1)^p \geq (N+1)^p \sum_{|n|\geq N} a_{kn},$$

for any $k \in I(N,p)$ we have

$$\sum_{|n|\geq N} a_{kn} \leq (N+1)^{-p} d_k(p) \leq \frac{B}{2}.$$

Therefore, $I(N,p) \subset J(N)$ and (2.6) yields

$$\mathrm{Card}(I(N,p)) \leq \mathrm{Card}(J(N)) \leq C(d)A/B(N+1)^d. \tag{2.7}$$

Let $L \geq B$. On can find $N>0$ such that $L = \frac{B}{2}(N+1)^p$. It follows from the definition of the set $I(N,p)$ and (2.7) that

$$\mathrm{Card}(\{k \in \mathbf{N} \mid d_k(p) \leq L\}) \leq C(d)A/B(2L/B)^{\frac{d}{p}} \tag{2.8}$$

The bound (2.8) implies, in particular, that the set $\{k \in \mathbf{N} \mid d_k(p) \leq L\}$ is finite for any $L>0$. Reordering $d_k(p)$ in such a way that they increase with $k$, we obtain the result of the Lemma.

As a direct application of this Lemma, we obtain the following important result.

**Theorem 2.2** *Let $\{e_k\}$, $k \in \mathbf{N}$ be any orthonormal system in $l^2(\mathbf{Z}^d)$ (not necessarily complete). For any $p>0$ define the moments*

$$d_k(p) = \sum_{n\in\mathbf{Z}^d} |e_k(n)|^2(|n|+1)^p.$$

*One can reorder $d_k(p)$ so that*

$$d_k(p) \geq D(p,d)k^{\frac{p}{d}},$$

*where the positive constants $D$ are the same for any system $\{e_k\}$.*

Proof. Let $a_{kn} = |e_k(n)|^2$. Since the system is orthonormal,

$$\forall n \in \mathbf{Z}^d, \ \sum_{k=1}^{\infty} a_{kn} \leq 1, \tag{2.9}$$

$$\forall k \in \mathbf{N}, \ \sum_{n \in \mathbf{Z}^d} a_{kn} = 1.$$

(One has the equality in (2.9) if the system is complete). The result of the Theorem follows directly from Lemma 2.1, where $A = B = 1$.

**Remark 2.3** *The result of the Theorem is optimal since there exist orthonormal systems such that*

$$C_1 k^{p/d} \leq d_k(p) \leq C_2 k^{p/d}.$$

*The simplest example is the canonical basis of $l^2(\mathbf{Z}^d)$ or, more generally, complete systems with SULE [5], where the functions $e_k(n)$ are well localised and fast decaying at infinity.*

One can observe that if the system is not complete, then $d_k(p)$ can grow as fast as you will (taking, for example, $e_k = \delta_{m(k)}$ with fast growing $|m(k)|$). Even if the system is complete but the functions $e_k$ decay not too fast at infinity, it is possible that $d_k(p)$ are fast growing (in particular, $d_k(p) = +\infty$ for any $k$). An interesting problem: is it possible that $d_k(p)$ grow faster than $k^{p/d}$ for some complete systems where all the functions $e_k(n)$ decay fast (for example, exponentially) as $|n| \to +\infty$?

# 3 Localization for a given vector: necessary conditions

Let $H$ be a self-adjoint operator in $\mathcal{H} = l^2(\mathbf{Z}^d)$, $\psi \in \mathcal{H}$, $\|\psi\| = 1$, $\psi(t) = \exp(-itH)\psi$. For any $p > 0, t \in \mathbf{R}$ define the moments of the position operator

$$|X|^p_\psi(t) = \sum_{n \in \mathbf{Z}^d} |\psi(t,n)|^2 (|n|+1)^p.$$

We prefer to take $(|n|+1)^p$ rather than $|n|^p$ to avoid some technical problems in the proofs.

**Definition 3.1** *One has dynamical localization for $\psi$ and the moment of order $p$, if*

$$\sup_t |X|^p_\psi(t) < +\infty.$$

*We shall denote it as $DL(\psi, p)$.*

*One has Cesaro dynamical localization $CDL(\psi, p)$ if*

$$\sup_T \langle |X|^p_\psi \rangle(T) \equiv \sup_T 1/T \int_0^T |X|^p_\psi(t) dt < +\infty.$$

Clearly, $DL(\psi,p) \Rightarrow CDL(\psi,p)$.

**Definition 3.2** *One has dynamical localization (Cesaro dynamical localization) for $\psi$ if $DL(\psi,p)$ (respectively, $CDL(\psi,p)$) holds for any $p>0$. We shall write $DL(\psi)$ and $CDL(\psi)$ respectively.*

Again, $DL(\psi) \Rightarrow CDL(\psi)$.

Let $\mathcal{H}_c$ be the subspace of continuum spectrum of $H$ and $P_c$ the orthogonal projection on it. It is well known that if $P_c\psi \neq 0$, then $|X|^p_\psi(t) \to +\infty$ as $t\to\infty$ for any $p>0$. So, the dynamical localization is possible only if $\psi \in \mathcal{H}_{pp}$ - the subspace of pure point spectrum of $H$. We shall denote by $\lambda$ the eigenvalues of $H$ and by $\mathcal{H}_\lambda$ the corresponding eigenspaces:

$$\mathcal{H}_\lambda = \{\varphi|\ H\varphi = \lambda\varphi\}.$$

Clearly, the subspaces $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$ are mutually orthogonal for $\lambda\neq\mu$ and $\mathcal{H}_{pp} = \oplus_\lambda \mathcal{H}_\lambda$. We shall denote by $P_\lambda$ orthogonal projection on $\mathcal{H}_\lambda$. For any $\psi\in\mathcal{H}_{pp}$ consider the set (at most countable)

$$I(\psi) = \{\lambda|\ P_\lambda\psi \neq 0\}.$$

Then $\psi$ can be written as

$$\psi = \sum_{k=1}^{+\infty} \psi_k,\ \ \psi_k \neq 0,\ \ \psi_k \in \mathcal{H}_{\lambda_k},\ \ \lambda_k \in I(\psi),$$

where $\lambda_k \neq \lambda_s$ for $k\neq s$. (It is possible that the sum is finite, in this case the problem of dynamical localization for the vector $\psi$ is rather trivial). For any $k$ define

$$e_k = \|\psi_k\|^{-1}\psi_k.$$

As $\mathcal{H}_{\lambda_k}$ and $\mathcal{H}_{\lambda_s}$ are mutually orthogonal for $k\neq s$, the system $\mathcal{M}(\psi)\equiv\{e_k\}$ is orthonormal in $\mathcal{H}_{pp}$. Finally, any $\psi\in\mathcal{H}_{pp}$ can be written as

$$\psi = \sum_k \gamma_k e_k,$$

where $\gamma_k = \langle\psi, e_k\rangle$, $He_k = \lambda_k e_k$ and $\mathcal{M}(\psi)=\{e_k\}$ is some orthonormal system of eigenfunctions of $H$ depending on $\psi$. It is obvious that

$$\psi(t) = \sum_k \exp(-it\lambda_k)\gamma_k e_k. \tag{3.1}$$

One can also note that the spectral measure of $\psi$ is equal to

$$\mu_\psi = \sum_k a_k \delta_{\lambda_k}, \tag{3.2}$$

where $a_k = |\gamma_k|^2 = \|\psi_k\|^2 > 0$, $\sum_k a_k = \|\psi\|^2 = 1$.

Let $d_k(p)$ be the moments of the functions $e_k$:

$$d_k(p) = \sum_{n\in\mathbf{Z}^d} |e_k(n)|^2(|n|+1)^p.$$

The first result we shall prove is a necessary condition for dynamical localization.

**Theorem 3.3** *1. For any $p > 0$,*

$$\liminf_{T\to\infty}\langle |X|_{\psi}^{p}\rangle(T) \geq \sum_{k=1}^{\infty} a_k d_k(p)$$

*(with the convention that if $d_k(p) = +\infty$, for some $k$, then $\sum_k = +\infty$).*

*2. $DL(\psi, p) \Rightarrow CDL(\psi, p) \Rightarrow d_k(p) < +\infty$ for any $k$ and $\sum_k a_k d_k(p) < +\infty$.*

Proof. It follows from (3.1) that

$$|\psi(t,n)|^2 = \sum_{k,m=1}^{\infty} \exp(-it(\lambda_k - \lambda_m))\gamma_k \overline{\gamma_m} e_k(n)\overline{e_m(n)}. \qquad (3.3)$$

The sum in (3.3) is absolutely converging for any $n$ because

$$\sum_k |\gamma_k|^2 = 1, \ \sum_k |e_k(n)|^2 \leq 1.$$

Therefore, for any $N > 0, T \in \mathbf{R}$

$$A(N,T) \equiv 1/T \int_0^T dt \sum_{|n|\leq N} |\psi(t,n)|^2(|n|+1)^p = \sum_{k,m=1}^{\infty} b_{km}(T)\gamma_k\overline{\gamma_m} d_{km}(p,N), \qquad (3.4)$$

where $d_{km}(p,N) = \sum_{|n|\leq N} e_k(n)\overline{e_m(n)}(|n|+1)^p$, $b_{km}(T) = 1$ for $k = m$ and $b_{km}(T) = (\exp(-iT(\lambda_k - \lambda_m)) - 1)/(-iT(\lambda_k - \lambda_m)$ for $k \neq m$. As $A(N,T) \leq \langle |X|_{\psi}^{p}\rangle(T)$, for any $N > 0$ we have the inequality

$$\text{liminf}_{T\to\infty}\langle |X|_{\psi}^{p}\rangle(T) \geq \text{liminf}_{T\to\infty} A(N,T). \qquad (3.5)$$

On the other hand, due to the dominated convergence theorem, one can take the limit in (3.4) for a fixed $N$ as $T \to \infty$:

$$\lim_{T\to\infty} A(N,T) = \sum_k |\gamma_k|^2 \sum_{|n|\leq N} |e_k(n)|^2(|n|+1)^p. \qquad (3.6)$$

As $a_k = |\gamma_k|^2$, it follows from (3.5) and (3.6) that

$$\text{liminf}_{T\to\infty}\langle |X|_{\psi}^{p}\rangle(T) \geq \sum_k a_k \sum_{|n|\leq N} |e_k(n)|^2(|n|+1)^p$$

for any $N > 0$. Taking the limit $N \to +\infty$, we obtain the first statement of the Theorem. The second statement follows directly from the first.

As a corollary of Theorem 3.3, we shall prove a necessary condition for dynamical localization for vector $\psi$ in terms of its spectral measure $\mu_\psi$ given by (3.2).

**Theorem 3.4** *The following statements hold:*

*1. For any $p > 0$,*

$$DL(\psi, p) \Rightarrow CDL(\psi, p) \Rightarrow \sum_k a_k^{\frac{1}{1+\beta}} < +\infty$$

*for all* $0 < \beta < p/d$.

*2.*

$$DL(\psi) \Rightarrow CDL(\psi) \Rightarrow \sum_k a_k^{\nu} < +\infty$$

*for all* $\nu > 0$.

Proof. Assume that $CDL(\psi, p)$ holds. Theorem 3.3 implies

$$\sum_k a_k d_k(p) < +\infty.$$

One can apply the Theorem 2.2 to the orthonormal system $\{e_k\}$ and reorder $d_k(p)$ so that $d_k(p) \geq Dk^{p/d}$. Therefore, after reordering,

$$\sum_k a_k k^{p/d} < +\infty.$$

Let $0 < \beta < p/d,\ r = 1+\beta, r' = 1 + 1/\beta$. Applying the Hölder inequality, one can easily see that

$$\sum_k a_k^{\frac{1}{1+\beta}} \leq \left(\sum_k a_k k^{p/d}\right)^{\frac{1}{1+\beta}} \left(\sum_k k^{-p/(\beta d)}\right)^{\beta/(1+\beta)} < +\infty$$

The fact that $\sum_k a_k^{\frac{1}{1+\beta}} < +\infty$, does not depend of reordering of $\{a_k\}$. The second part of the Theorem is obvious.

**Corollary 3.5** *If $CDL(\psi)$ holds, then the numbers $a_k$ are fast decaying: for any $s > 0$, one can reorder $\{a_k\}$ so that*

$$a_k \leq C(s)k^{-s}.$$

One should stress that the statements of Theorem 3.4 and Corollary 3.5 are weaker than that of Theorem 3.3, because they do not depend on the system $\{e_k\}$, and inevitably, one loses some information about the moments $d_k(p)$. If $d_k(p)$ grow very fast as $k \to \infty$, it is even possible that $\sum_k a_k^{\nu} < +\infty$ for all $\nu > 0$, but $\sum_k a_k d_k(p) = +\infty$ for all $p > 0$.

Finally, in this section we shall give necessary conditions for $DL(\psi, p)$ in terms of projections of $\psi$ on $\mathcal{H}_{\lambda_k}$.

**Lemma 3.6** *Let $\mathcal{M} = \{e_k\}$ be any orthonormal system in $\mathcal{H}$ (in particular, the system $\mathcal{M}(\psi)$), $\psi \in \mathcal{H}$ Define the following function:*

$$R_{\psi,\mathcal{M}}(n) = \sup_k |\gamma_k e_k(n)|, \quad \gamma_k = \langle \psi, e_k \rangle.$$

*Then*

$$\sum_{n \in \mathbf{Z}^d} R^2_{\psi,\mathcal{M}}(n) \leq \|\psi\|^2$$

Proof. As the system $\{e_k\}$ is orthonormal, $\sum_k |\gamma_k|^2 \leq \|\psi\|^2$. Therefore,

$$S = \sum_{k,n} |\gamma_k e_k(n)|^2 = \sum_k |\gamma_k|^2 \leq \|\psi\|^2. \tag{3.7}$$

On the other hand,

$$S = \sum_n \sum_k |\gamma_k e_k(n)|^2 \geq \sum_n \sup_k |\gamma_k e_k(n)|^2 = \sum_n R^2_{\psi,\mathcal{M}}(n) \tag{3.8}$$

The result of the Lemma follows from (3.7)-(3.8).

The function $R_{\psi,\mathcal{M}}(n)$ (especially its decay properties) will play an important role in the next part of the paper. Lemma 3.6 implies that $R_{\psi,\mathcal{M}}$ always lies in $l^2(\mathbf{Z}^d)$. We shall see below that if $DL(\psi,p)$ holds, then $R_{\psi,\mathcal{M}(\psi)}$ decays faster at infinity. On the other hand, in the next section we shall prove that a sufficiently fast decay of $R_{\psi,\mathcal{M}}$ for some $\mathcal{M}$ implies $DL(\psi,p)$.

**Definition 3.7** *We say that a function $f : \mathbf{Z}^d \to \mathbf{C}$ ($f : \mathbf{Z}^d \to \mathbf{R}$) is fast decaying if for any $s > 0$*

$$\sup_n |f(n)|(|n|+1)^s < +\infty.$$

**Theorem 3.8** *The following statements hold:*

1. *$DL(\psi,p) \Rightarrow CDL(\psi,p) \Rightarrow \sum_n R^2_{\psi,\mathcal{M}(\psi)}(n)(|n|+1)^p < +\infty$.*
2. *$DL(\psi) \Rightarrow CDL(\psi) \Rightarrow R_{\psi,\mathcal{M}(\psi)}$ is fast decaying.*

Proof. Assume that $CDL(\psi,p)$ holds. Theorem 3.3 implies that $d_k(p) < +\infty$ for any $k$ and

$$S(p) = \sum_k a_k d_k(p) \leq C(p) < +\infty.$$

On the other hand, it follows from definition of $d_k(p)$ that

$$S(p) = \sum_n (|n|+1)^p \sum_k |\gamma_k e_k(n)|^2 \geq \sum_n (|n|+1)^p \sup_k |\gamma_k e_k(n)|^2 =$$

$$\sum_n (|n|+1)^p R^2_{\psi,\mathcal{M}(\psi)}(n),$$

so we obtain the first statement of the Theorem. The second statement follows directly from the first.

# 4 Localization for a given vector: sufficient conditions

In this section we shall give some sufficient conditions for $DL(\psi,p)$ and $DL(\psi)$.

Let $\mathcal{M} = \{e_k\}$ be any orthonormal system of eigenfunctions of $H$ and $\mathcal{H}_{\mathcal{M}}$ the subspace of $\mathcal{H}_{pp}$ spanned by $\mathcal{M}$. For any $\psi \in \mathcal{H}_{\mathcal{M}}$, we have the identity

$$\psi = \sum_k \gamma_k e_k, \;\; \gamma_k = \langle \psi, e_k \rangle. \tag{4.1}$$

We shall denote as $\lambda_k$ and $d_k(p)$ the eigenvalue and the moments of $e_k(n)$ respectively. In this section we shall consider any systems $\mathcal{M}$ of eigenfunctions of $H$, not necessarily $\mathcal{M}(\psi)$, so it is possible that $\lambda_k = \lambda_m$ for $k \neq m$ if the spectrum of $H$ is not simple. The decomposition

$$\psi = \sum_k \psi_k$$

of the previous section is a special case of (4.1), when $\mathcal{M} = \mathcal{M}(\psi)$.

The simplest sufficient condition for $DL(\psi, p)$ can be given in terms of $d_k(p)$ and $a_k = |\gamma_k|^2$.

**Lemma 4.1** *Let $p > 0$. The following statement holds:*

$$\sup_t |X|^p_\psi(t) \leq \sum_n (|n|+1)^p \sup_t |\psi(t,n)|^2 \leq (\sum_k \sqrt{a_k d_k(p)})^2.$$

*So, if the last sum converges, $DL(\psi, p)$ holds.*

Proof. Since

$$\sup_t |\psi(t,n)| = \sup_t |\sum_k \exp(-it\lambda_k)\gamma_k e_k(n)| \leq \sum_k |\gamma_k e_k(n)|, \tag{4.2}$$

the Cauchy-Schwartz inequality yields for any $t$

$$\sum_n (|n|+1)^p \sup_t |\psi(t,n)|^2 \leq \sum_{k,m} |\gamma_k \gamma_m| \sum_n |e_k(n) e_m(n)| (|n|+1)^p \leq \tag{4.3}$$

$$\sum_{k,m} |\gamma_k \gamma_m| \sqrt{d_k(p) d_m(p)} = (\sum_k \sqrt{a_k d_k(p)})^2.$$

If the functions $e_k$ and $e_m$ have only small overlapping for $k \neq m$, one can better estimate the sums

$$d_{km}(p) = \sum_n |e_k(n) e_m(n)| (|n|+1)^p.$$

If $d_{km}(p)$ decay fast when $|k-m| \to \infty$, the sum on the r.h.s of (4.3) can be majorated by $C \sum_k a_k d_k(p)$ (or by something close to it). In this case one obtains a sufficient condition which is close to (or even identical with) the necessary condition given by Theorem 3.3. In particular, this is the case when $\mathcal{M}$ is the canonical basis in $l^2(\mathbf{Z}^d)$.

The sufficient condition of Lemma 4.1 is, however, difficult to apply in the concrete cases, because one should control at the same time the growth of $d_k(p)$ and the decreasing of $a_k$. More practical sufficient condition can be given in terms of the function $R_{\psi,\mathcal{M}}(n)$, defined by

$$R_{\psi,\mathcal{M}}(n) = \sup_k |\gamma_k e_k(n)|, \quad \gamma_k = \langle \psi, e_k \rangle, \quad \sum_k |\gamma_k|^2 = \|\psi\|^2.$$

Later on in this section, $\psi$ and $\mathcal{M}$ are fixed and we omit them in notations.

To prove $DL(\psi, p)$, one shall use the trivial bound (4.2):

$$|\psi(t,n)| \le \sum_k |\gamma_k e_k(n)|.$$

Clearly, any term in the sum is majorated by $R(n)$, and if the sum has a finite number of terms, the sufficiently fast decay of $R(n)$ implies $DL(\psi, p)$. However, typically it is not the case, and one needs some decay in $k$ so that the sum converge. The key moment is the following: one can sacrify some decay in $n$ to obtain a necessary decay in $k$. This is possible due to the growth of the moments $d_k(p)$ given by Theorem 2.2. Surprisingly, one can even allow some growth in $k$ in the bounds for $\gamma_k e_k(n)$. Namely, suppose that for some $\alpha > 0$ the moments $d_k(\alpha)$ are finite for any $k$. Consider the function

$$R(n,\alpha) = \sup_k \frac{|\gamma_k e_k(n)|}{d_k(\alpha)}.$$

For $\alpha = 0$ one has $d_k(0) = 1$ (so, $d_k(\alpha)$ are always finite) and $R(n,0)$ coincides with the function $R(n)$ considered above. For any $s > 0$ define the moments

$$L_\alpha(s) = \sum_n R^2(n,\alpha)(|n|+1)^s,$$

where $L_\alpha(s)$ depend also on $\psi, \mathcal{M}$ and it is possible that $L_\alpha(s) = +\infty$.

**Theorem 4.2** *Let $\psi \in \mathcal{H}_{\mathcal{M}}$, $\alpha > 0$. Suppose that $d_k(\alpha) < +\infty$ for any $k$. The following statements hold:*

*1. Let $\delta \in (0,1)$, $\varepsilon > 0$. Then*

$$\sup_t |\psi(t,n)| \le C(\varepsilon,\delta,d) L_\alpha^{\delta/2}\left(\frac{2\alpha + d(2-\delta)}{\delta} + \varepsilon\right) R^{1-\delta}(n,\alpha) \tag{4.4}$$

*where the constants $C(\varepsilon,\delta,d)$ are universal, i.e. do not depend on $H, \mathcal{M}$ or $\psi$.*

*2. Let $p > 0, \varepsilon > 0$. There exist the universal constants $C(\varepsilon, p, d)$ such that*

$$\sup_t |X|_\psi^p p(t) \le \sum_n (|n|+1)^p \sup_t |\psi(t,n)|^2 \le C(\varepsilon,p,d) L_\alpha(2\alpha + p + 2d + \varepsilon) \tag{4.5}$$

*3. If $R(n,\alpha)$ is fast decaying in $n$, then so is $\sup_t |\psi(t,n)|$ and $DL(\psi)$ holds.*

**Proof.** Let $r = \frac{2\alpha + d(2-\delta)}{\delta} + \varepsilon$. If $L_\alpha(r) = +\infty$, the bound (4.4) is trivially true with any finite constant $C$. Suppose that $L_\alpha(r) < +\infty$.

It follows from definition of $d_k(r)$ and $R_\alpha(n)$ that for any $r > 0$, $k \in \mathbf{N}$,

$$a_k d_k(r) = \sum_n |\gamma_k e_k(n)|^2 (|n|+1)^r \le d_k^2(\alpha) \sum_n R_(^2 n,\alpha)(|n|+1)^r \equiv d_k^2(\alpha) L_\alpha(r).$$

As $L_\alpha(r) < +\infty,\ d_k(r) < +\infty$ for any $k$ such that $a_k = |\gamma_k|^2 \neq 0$. Therefore,

$$|\gamma_k| \leq \left(L_\alpha(r) d_k^2(\alpha)/d_k(r)\right)^{1/2}. \tag{4.6}$$

At the same time,

$$|\gamma_k e_k(n)| \leq d_k(\alpha) R(n, \alpha) \tag{4.7}$$

directly from definition of $R(n, \alpha)$. Using the bounds (4.6)-(4.7), one can estimate:

$$|\psi(t,n)| \leq \sum_k |\gamma_k e_k(n)| \leq \sum_k (d_k(\alpha) R_\alpha(n))^{1-\delta} |\gamma_k e_k(n)|^\delta \leq$$

$$R^{1-\delta}(n,\alpha) L_\alpha^{\delta/2}(r) \sum_k |e_k(n)|^\delta d_k(\alpha) d_k^{-\delta/2}(r), \tag{4.8}$$

where the summation is carried only over $k$ such that $a_k > 0$, so $d_k(r) < +\infty$. Let $s = 2/(2-\delta),\ s' = 2/\delta$. Applying the Hölder inequality, and using the fact that $\sum_k |e_k(n)|^2 \leq 1$, one obtains:

$$S \equiv \left(\sum_k |e_k(n)|^\delta d_k(\alpha) d_k^{-\delta/2}(r)\right)^s \leq \sum_k d_k^{2/(2-\delta)}(\alpha) d_k^{-\delta/(2-\delta)}(r). \tag{4.9}$$

Let $w < q$. Applying the Hölder inequality with $s = q/w, s' = q/(q-w)$, one can estimate:

$$d_k(w) = \sum_n |e_k(n)|^{2/s} (|n|+1)^w |e_k(n)|^{2/s'} \leq$$

$$(\sum_n |e_k(n)|^2 (|n|+1)^{ws})^{1/s} (\sum_n |e_k(n)|^2)^{1/s'} = (d_k(q))^{w/q}. \tag{4.10}$$

The bound (4.10) with $w = \alpha,\ q = r$ and (4.9) imply

$$S \leq \sum_k d_k(r)^{(2\alpha/r - \delta)/(2-\delta)}. \tag{4.11}$$

Now we shall use the result of Theorem 2.2. One can reorder $d_k(r)$ so that

$$d_k(r) \geq D(r,d) k^{r/d}. \tag{4.12}$$

The choice of $r$ and (4.9)-(4.12) yield

$$S \leq C(\varepsilon, \delta, d) < +\infty \tag{4.13}$$

with some universal constants $C$. The first statement of the Theorem follows from (4.8) and (4.13).

To prove the second statement, we shall use the inequality (4.6) with $r = 2\alpha + p + 2d + \varepsilon$ and the bound of Lemma 4.1. Again, if $L_\alpha(r) = +\infty$, there is nothing to prove. Suppose that $L_\alpha(r) < +\infty$. One obtains

$$\sum_n (|n|+1)^p \sup_t |\psi(t,n)|^2 \leq L_\alpha(r) \left(\sum_k \sqrt{\frac{d_k^2(\alpha) d_k(p)}{d_k(r)}}\right)^2. \tag{4.14}$$

Applying twice the bound (4.10), we obtain

$$\sum_n (|n|+1)^p \sup_t |\psi(t,n)|^2 \leq L_\alpha(r) \left( \sum_k d_k(r)^{\frac{2\alpha+p-r}{2r}} \right)^2 . \tag{4.15}$$

Again, by Theorem 2.2 and the choice of $r$, the sum converges and the second statement of the Theorem follows directly from (4.15). The third statement follows directly from the first.

As a direct application of Theorem 4.2 we obtain the necessary and sufficient conditions for $DL(\psi)$.

**Corollary 4.3** *Let $\psi \in \mathcal{H}_{pp}$ and $R(n) = R(n,0)$ be defined with the system $\mathcal{M}(\psi)$ as in the previous section. Then*
*$CDL(\psi) \Leftrightarrow DL(\psi) \Leftrightarrow R(n)$ is fast decaying.*

The result follows from Theorem 3.8 and Theorem 4.2.

# 5 Localization on the interval of energies

In the previous parts of the paper the vector $\psi \in \mathcal{H}_{pp}$ was fixed and we did not suppose anything about decay of $\psi$ or about the set of $\lambda_k$ such that $\|\psi_k\| \neq 0$. In this section we shall consider the set of functions $\psi$ with some decaying properties at infinity and with the energies $\lambda_k$ from some interval $I \subset \mathbf{R}$ (bounded or not). First of all, if $DL(\psi)$ holds, then, in particular, for any $p > 0$

$$|X|^p_\psi(0) = \sum_n |\psi(n)|^2 (|n|+1)^p = C(p) < +\infty,$$

so $\psi$ is fast decaying. Therefore, the largest set of $\psi$ for which one could prove $DL(\psi)$ is the set of fast decaying functions:

$$\mathcal{A} = \{\psi \in \mathcal{H} \mid \psi \text{ fast decaying } \}.$$

We shall also consider the set of finite functions

$$\mathcal{B} = \{\psi \in \mathcal{H} \mid \psi \text{ finite } \},$$

which is the subset of $\mathcal{A}$. The set of $\psi$ exponentially decaying at infinity is intermidiate between $\mathcal{A}$ and $\mathcal{B}$ and will be considered in the next section.

Let $I \subset \mathbf{R}$ be some interval (bounded or not). We shall denote by $\mathcal{H}(I)$ the subspace of $\mathcal{H}_{pp}$ with the energies from $I$

$$\mathcal{H}(I) = \oplus_{\lambda \in I} \mathcal{H}_\lambda$$

and by $P_I(H)$ the orthogonal projection on $\mathcal{H}(I)$.

**Definition 5.1** *The operator $H$ has $\mathcal{A}$ -dynamical localization on $I$ if for any $\psi \in \mathcal{A}$, we have $DL(P_I(H)\psi)$.*

*$H$ has $\mathcal{B}$-dynamical localization on $I$ if for any $\psi \in \mathcal{B}$ we have $DL(P_I(H)\psi)$.*

Let $\mathcal{M} = \{e_k\}$ be any orthonormal system of eigenfunctions of $H$ complete in $\mathcal{H}(I)$. One can obtain such systems choosing for any eigenvalue $\lambda \in I$ an orthonormal system $\mathcal{M}_\lambda$ complete in $\mathcal{H}_\lambda$ and then taking $\mathcal{M} = \cup_{\lambda\in I}\mathcal{M}_\lambda$. Clearly, $\mathcal{M}$ is unique if the spectrum of $H$ is simple on $I$.

For any $\varphi \in \mathcal{H}(I)$ the identity holds:

$$\varphi = \sum_k \gamma_k e_k, \quad \gamma_k = \langle \varphi, e_k\rangle.$$

Suppose that for some $\alpha \geq 0$

$$\sum_n |g(n)|^2(|n|+1)^\alpha < +\infty$$

for any eigenfunction of $H$ from $\mathcal{H}(I)$ (if $\alpha = 0$, this is always true). Define as in the previous section

$$R_{\varphi,\mathcal{M}}(n,\alpha) = \sup_k \frac{|\gamma_k e_k(n)|}{d_k(\alpha)}.$$

Define also three functions from $\mathbf{Z}^{2d}$ to $\mathbf{R}_+$:

$$F_\alpha(n,m) = \sup_k \frac{|e_k(n)e_k(m)|}{d_k(\alpha)},$$

$$G_\alpha(n,m) = \sup_{g\in\mathcal{L}} \frac{|g(n)g(m)|}{d_g(\alpha)},$$

where

$$\mathcal{L} = \{g \in \mathcal{H} \mid Hg = \lambda g,\ \lambda \in I,\ \|g\| = 1\},\ d_g(\alpha) = \sum_n |g(n)|^2(|n|+1)^\alpha,$$

and

$$U_\alpha(n,m) = \sup_{\tilde g\in\mathcal{K}} |\tilde g(n)\tilde g(m)|,$$

where

$$\mathcal{K} = \{\tilde g \mid H\tilde g = \lambda\tilde g,\ \lambda\in I,\ \text{and}\ \forall n\in\mathbf{Z}^d,\ |\tilde g(n)| \leq (|n|+1)^{-\alpha/2}\}.$$

One can see that

$$F_\alpha(n,m) \leq G_\alpha(n,m) \leq U_\alpha(n,m). \tag{5.1}$$

The first inequality in (5.1) is obvious. To prove the second, for any $g \in \mathcal{L}$ define $\tilde g(n) = (d_g(\alpha))^{-1/2}g(n)$, so that

$$\frac{|g(n)g(m)|}{d_g(\alpha)} = |\tilde g(n)\tilde g(m)|.$$

One verifies that

$$\sum_n |\tilde g(n)|^2(|n|+1)^\alpha = 1,$$

so $\tilde g \in \mathcal{K}$ and the second inequality in (5.1) holds.

**Lemma 5.2** *Let $\alpha \geq 0, \psi \in \mathcal{H}$, $\varphi = P_I(H)\psi$. The inequality holds:*

$$R_{\varphi,\mathcal{M}}(n,\alpha) \leq \sum_{m \in \mathbf{Z}^d} N_\alpha(n,m)|\psi(m)|, \tag{5.2}$$

*where $N_\alpha$ is one of the three functions $F_\alpha, G_\alpha, U_\alpha$.*

Proof. As $\varphi = P_I(H)\psi$ and the system $\mathcal{M}$ is complete in $\mathcal{H}(I)$,

$$\varphi = \sum_k \gamma_k e_k, \quad \gamma_k = \langle \varphi, e_k \rangle = \langle \psi, e_k \rangle.$$

Therefore,

$$\gamma_k = \sum_m \psi(m)\overline{e_k(m)}$$

and

$$\frac{|\gamma_k e_k(n)|}{d_k(\alpha)} \leq \sum_m |\psi(m)| \frac{|e_k(n) e_k(m)|}{d_k(\alpha)}. \tag{5.3}$$

Taking in (5.3) supremum over $k$, we obtain the statement of Lemma for $F_\alpha$. The inequality (5.1) yields the result for $G_\alpha$ and $U_\alpha$.

The bound (5.2) combined with Theorem 4.2 allows us to give sufficient conditions for $\mathcal{B}$- and $\mathcal{A}$-dynamical localization on $I$.

**Theorem 5.3** *The following statements hold:*

*1) Let $\alpha \geq 0$. If one of three functions $F_\alpha(n,m), G_\alpha(n,m), U_\alpha(n,m)$ is fast decaying in $n$ for all fixed $m \in \mathbf{Z}^d$, then $\mathcal{B}$-dynamical localization holds on $I$. In particular, $P_I(H)\psi \in \mathcal{A}$ for any $\psi \in \mathcal{B}$.*

*2) If the spectrum of $H$ is simple on $I$ and $\mathcal{B}$-dynamical localization holds on $I$, then the function $F_0(n,m) = G_0(n,m)$ is fast decaying in $n$ for all fixed $m \in \mathbf{Z}^d$.*

Proof. Let $\psi \in \mathcal{B}$ and $N_\alpha$ be one of three functions $F_\alpha, G_\alpha, U_\alpha$. As the function $\psi$ is finite and $N_\alpha(n,m)$ is fast decaying in $n$ for any $m$, (5.2) implies that $R_{\varphi,\mathcal{M}}(n,\alpha)$ is fast decaying in $n$. The third statement of Theorem 4.2 implies $DL(\varphi)$, so $\mathcal{B}$-dynamical localization holds on $I$.

To prove the second statement of the Theorem, observe that since the spectrum of $H$ is simple on $I$, the system $\mathcal{M}$ is unique and coincides with the set of normalised eigenfunctions of $H$ with eigenvalues from $I$. Therefore, $F_\alpha(n,m) = G_\alpha(n,m)$. Moreover, one sees easily that for any $\varphi \in \mathcal{H}(I)$, $\mathcal{M}(\varphi)$ is a subset of $\mathcal{M}$, where $\mathcal{M}(\varphi)$ was defined in Section 3. Namely,

$$\mathcal{M}(\varphi) = \{e_k\}_{k \in J}, \quad J = \{k \mid \gamma_k \neq 0\}.$$

Since $\gamma_k = 0$ for any $k \notin J$,

$$R_{\varphi,\mathcal{M}(\varphi)}(n,0) = \sup_{k \in J} |\gamma_k e_k(n)| = \sup_k |\gamma_k e_k(n)| = R_{\varphi,\mathcal{M}}(n,0).$$

Let $\psi \in \mathcal{B}$, so that $DL(\varphi)$ holds. By the second statement of Theorem 3.8, the function $R_{\varphi,\mathcal{M}}(n,0)$ is fast decying in $n$. In particular, if $\psi = \delta_m,\ m \in \mathbf{Z}^d$, then $\gamma_k = \overline{e_k(m)}$ and $R_{\varphi,\mathcal{M}}(n,0) = F_0(n,m)$ is fast decainy in $n$, so the second statement of Theorem holds.

As to the $\mathcal{A}$-dynamical localization on $I$, there are many possible sufficient conditions to propose. For example, the following result holds.

**Theorem 5.4** *Let $\alpha \geq 0$ and $N_\alpha$ is one of three functions $F_\alpha, G_\alpha, U_\alpha$. Suppose that one of the two conditions holds:*

*1. For any $s > 0$ there exist two positive finite constants $k(s), C(s)$ such that*

$$N_\alpha(n,m) \leq C(s)(|m|+1)^{k(s)}(|n|+1)^{-s}. \tag{5.4}$$

*2. For any $s > 0$ there exist two positive constants $k(s), C(s)$ such that*

$$N_\alpha(n,m) \leq C(s)(|m|+1)^{k(s)}(|n-m|+1)^{-s}. \tag{5.5}$$

*Then $\mathcal{A}$-dynamical localization holds on $I$.*

Proof. Let $\psi \in \mathcal{A}$, so

$$|\psi(m)| \leq C(r)(|m|+1)^{-r}$$

for any $r > 0$. For any $s > 0$ the bounds (5.2) ans (5.4) yield

$$R_{\varphi,\mathcal{M}}(n,\alpha) \leq C(r)C(s)(|n|+1)^{-s}\sum_m (|m|+1)^{k(s)-r}.$$

Taking $r = k(s) + 2d$, we see that $R_{\varphi,\mathcal{M}}(n,\alpha)$ is fast decaying in $n$. The first statement of the Theorem follows from the third statement of Theorem 4.2. In the case of (5.5) the proof is similar.

Up to now, the operator $H$ was fixed. Suppose that there is a family of self-adjoint operators $H(\theta)$ depending on some parameter $\theta \in \Omega$. We shall assume also that $\mu$ is some probability measure on $\Omega$. Let $I$ be some interval such that the spectrum of $H(\theta)$ is pure point on $I$ for $\mu$-almost all $\theta$. One says that the strong $\mathcal{A}$- dynamical localization (strong $\mathcal{B}$-dynamical localization) holds on $I$ if for any $p > 0$ and $\psi \in \mathcal{A}$ (respectively, $\psi \in \mathcal{B}$),

$$\int_\Omega d\mu(\theta) \sup_t \langle \varphi(t,\theta), |X|^p \varphi(t,\theta)\rangle < +\infty,$$

where

$$\varphi(t,\theta) = \exp(-itH(\theta))P_I(H(\theta))\psi.$$

One can give the following bound which can be used to prove strong dynamical localization.

**Theorem 5.5** *Let $\alpha \geq 0$ be such that for $\mu$-almost all $\theta$, $\sum_n |g(n)|^2(|n|+1)^\alpha < +\infty$ for any eigenfunction $g$ of $H(\theta)$ with eigenvalue from $I$. Let $N_\alpha(n,m,\theta)$ be one of three functions $F_\alpha, G_\alpha, U_\alpha$ (it depends also on the parameter $\theta$). Define*

$$Q(n,\alpha,\theta) = \sum_m |\psi(m)| N_\alpha(n,m,\theta).$$

*For any $p > 0,\ \varepsilon > 0$*

$$\int_\Omega d\mu(\theta) \sup_t \langle \varphi(t,\theta), |X|^p \varphi(t,\theta)\rangle \leq$$

$$C(\varepsilon, p, d) \sum_n (|n|+1)^{p+\alpha+2d+\varepsilon} \int_\Omega d\mu(\theta) Q^2(n,\alpha,\theta),$$

*where, as usually, $\varphi(t,\theta) = \exp(-itH(\theta))P_I(H(\theta))\psi$.*

Proof. The result follows directly from the bound (5.2) of Lemma 5.2 and the second statement of Theorem 4.2, where the constants $C(\varepsilon, p, d)$ do not depend on $\theta$.

# 6 Exponential dynamical localization

In this section we shall consider an important special case of dynamical localization. Let $\psi \in l^2(\mathbf{Z}^d),\ \psi(t) = \exp(-itH)\psi$. We shall say that the exponential dynamical localization holds for $\psi$, if there exist $\gamma > 0, C > 0$ such that

$$\sup_t |\psi(t,n)| \leq C \exp(-\gamma |n|).$$

We shall note it as $EDL(\psi)$. Clearly, $EDL(\psi)$ implies $DL(\psi)$. To establish necessary and sufficient conditions for $EDL(\psi)$ we shall need the following version of Theorem 2.2.

**Theorem 6.1** *Let $\{e_k\}$ be any orthonormal system in $l^2(\mathbf{Z}^d),\ \gamma > 0$. For any $k$ define the numbers*

$$\eta_k(\gamma) = \sup_n (|e_k(n)|^2 \exp(\gamma|n|)).$$

*One can reorder $\eta_k(\gamma)$ so that*

$$\eta_k(\gamma) \geq D \exp(\beta k^{1/d})$$

*with some universal positive constants $D(\gamma, d),\ \beta(\gamma, d)$.*

Proof. We shall follow the proof of Lemma 2.1 with $a_{kn} = |e_k(n)|^2$ and $A = B = 1$. Let $N > 0$, then for the set

$$J(N) = \{k \mid \sum_{|n|>N} |e_k(n)|^2 \leq 1/2\}$$

we have

$$\mathrm{Card}(J(N)) \leq C(d)(N+1)^d.$$

Let $L > 0$. Consider the following set in $\mathbf{N}$:

$$I(L) = \{k \mid \eta_k(\gamma) \leq L\}.$$

It follows from definition of $\eta_k(\gamma)$ that for any $k \in I(N)$, $|e_k(n)|^2 \leq L\exp(-\gamma|n|)$. Therefore, for any $\nu > 0$,

$$\sum_{|n|>N} |e_k(n)|^2 \leq C(\nu, d)L\exp(-(\gamma-\nu)N).$$

Let $L$ be such that $C(\nu, d)L\exp(-(\gamma-\nu)N) = 1/2$. Then $I(L) \subset J(N)$ and

$$\mathrm{Card}(I(L)) \leq \mathrm{Card}(J(N)) \leq C(d)(N+1)^d \leq C(\gamma, \nu, d)\log^d L \tag{6.1}$$

for any $L \geq L_0(\gamma, \nu, d)$. The result of the theorem follows directly from (6.1).

With this result we can obtain a necessary condition for $EDL(\psi)$ in terms of projections $\psi_k$ and in terms of coefficients of the spectral measure of $\psi$. Let $\mathcal{M}(\psi)$ be the orthonormal system of eigenfunctions of $H$ defined in section 2 and $R_{\psi,\mathcal{M}(\psi)}(n) = \sup_k |\gamma_k e_k(n)|$, where $\gamma_k = \langle \psi, e_k \rangle$, $He_k = \lambda_k e_k$. The spectral measure of $\psi$ can be written as

$$\mu_\psi = \sum_k a_k \delta_{\lambda_k}.$$

**Theorem 6.2** *Suppose that*

$$\sup_t |\psi(t,n)| \leq C\exp(-\alpha|n|) \tag{6.2}$$

*for some $\alpha > 0$. Then*

$$\sup_k |\gamma_k|^2 \eta_k(2\alpha) < +\infty, \tag{6.3}$$

*or, equivalently,*

$$R_{\psi,\mathcal{M}(\psi)}(n) \leq C\exp(-\alpha|n|). \tag{6.4}$$

*One can reorder $a_k$ so that*

$$a_k \leq C\exp(-\beta k^{1/d})$$

*with some positive $C, \beta$.*

**Proof.** The proof of the first statement is made in [5]. Since

$$\psi(t,n) = \sum_s \exp(-it\lambda_s)\gamma_s e_s(n),$$

then for any $k, n$ we have

$$1/T \int_0^T \psi(t,n)\exp(it\lambda_k)dt \to \gamma_k e_k(n) \tag{6.5}$$

as $T \to \infty$. The bound (6.2) and (6.5) yield

$$|\gamma_k e_k(n)| \leq C\exp(-\alpha|n|)$$

for any $k, n$, which implies (6.4) and (6.3).

Next, it follows from (6.3) and Theorem 6.1 that after reordering

$$a_k \equiv |\gamma_k|^2 \le C(\eta_k(2\alpha))^{-1} \le C\exp(-\beta k^{1/d})$$

with some positive $C, \beta$.

In the following statement we shall use the same notations as in Theorem 4.2. As usually, $\mathcal{M} = \{e_k\}$ is any orthonormal system of eigenfunctions of $H$. Moreover, for $\delta > 0$ we define

$$R_{\psi,\mathcal{M}}(n,\delta) = \sup_k \frac{|\gamma_k e_k(n)|}{\eta_k(\delta)}, \tag{6.6}$$

supposing that $\eta_k(\delta) < +\infty$ for any $k$.

**Theorem 6.3** *Let $\psi \in \mathcal{H}_{\mathcal{M}}$ The following statements hold with universal constants $C$:*

*1. If*

$$R_{\psi,\mathcal{M}}(n,0) \le C\exp(-\alpha|n|)$$

*for some $\alpha > 0$, then*

$$\sup_t |\psi(t,n)| \le C(\alpha,d)(|n|^d + 1)\exp(-\alpha|n|).$$

*2. Suppose that $\eta_k(\delta) < +\infty$ for some $\delta > 0$ for any $k$ and*

$$R_{\psi,\mathcal{M}}(n,\delta) \le C\exp(-\alpha|n|),$$

*where $\alpha > \delta$. Then for any $\nu :\ 0 < \nu < \alpha - \delta$,*

$$\sup_t |\psi(t,n)| \le C(\nu,\alpha,d)\exp(-\nu|n|).$$

*In particular, in both cases $EDL(\psi)$ holds.*

Proof. Since

$$\sup_k |\gamma_k e_k(n)| \le C\exp(-\alpha|n|),$$

we obtain

$$\sup_k (|\gamma_k|^2 \eta_k(2\alpha)) < +\infty.$$

The result of Theorem 6.1 yields after reordering

$$|\gamma_k| \le C\exp(-\beta k^{1/d})$$

with some $C, \beta > 0$. Now $\psi(t,n)$ can be estimated in a usual way. For any $n \in \mathbf{Z}^d$ and $B > 0$,

$$|\psi(t,n)| \le \sum_k |\gamma_k e_k(n)| \le \sum_{k \le B|n|^d} C\exp(-\alpha|n|) + \sum_{k > B|n|^d} |\gamma_k| \le$$

$$CB|n|^d \exp(-\alpha|n|) + K\exp(-\beta/2(B|n|^d)^{1/d}). \tag{6.7}$$

Taking in (6.7) $B$ so that $\beta/2B^{1/d} = \alpha$, we obtain the first statement of the Theorem.

To prove the second statement of the Theorem we shall need a bound relating $\eta_k(\alpha)$ and $\eta_k(\nu)$ for $\nu \leq \alpha$. It follows from definition of $\eta_k(\alpha)$ that

$$|e_k(n)|^2 \leq \eta_k(\alpha)\exp(-\alpha|n|)$$

for any $k, n$. At the same time $|e_k(n)|^2 \leq 1$. Therefore,

$$|e_k(n)|^2 \leq \min\{1, \eta_k(\alpha)\exp(-\alpha|n|)\}. \tag{6.8}$$

We shall use the elementary inequality

$$\min\{1, z\} \leq z^s, \quad z \geq 0, \ 0 < s < 1. \tag{6.9}$$

The bounds (6.8) and (6.9) where $s = \nu/\alpha$ yield

$$|e_k(n)|^2 \leq \eta_k^s(\alpha)\exp(-\nu|n|),$$

and finally

$$\eta_k(\nu) \leq (\eta_k(\alpha))^{\nu/\alpha}. \tag{6.10}$$

This bound is very similar to the bound (4.10) for the moments $d_k(r)$. Now we can end the proof. For any $k, n$,

$$|\gamma_k e_k(n)| \leq C\eta_k(\delta)\exp(-\alpha|n|).$$

Therefore,

$$|\gamma_k|\sqrt{\eta_k(2\alpha)} \leq C\eta_k(\delta). \tag{6.11}$$

Next, as

$$|\psi(t,n)| \leq \sum_k |\gamma_k e_k(n)|,$$

one has

$$A \equiv \sup_n \left(\exp(\nu|n|)\sup_t |\psi(t,n)|\right) \leq \sum_k |\gamma_k|\sqrt{\eta_k(2\nu)}. \tag{6.12}$$

The bounds (6.11) and (6.12) imply

$$A \leq C\sum_k \eta_k(\delta)\eta_k^{1/2}(2\nu)\eta_k^{-1/2}(2\alpha). \tag{6.13}$$

Using twice the bound (6.10), we obtain from (6.13):

$$A \leq C\sum_k (\eta_k(2\alpha))^{(\delta+\nu-\alpha)/(2\alpha)}.$$

As $\delta + \nu < \alpha$, by Theorem 6.1 the sum converges and is bounded by some universal constant, so the second statement of Theorem holds.

The result of the Theorem can be used to give sufficient conditions for exponential dynamical localization on the interval of energies $I$. Consider the set of exponentially decaying function in $\mathcal{H}$:

$$\mathcal{C} = \{\psi \mid \exists r > 0: \ |\psi(n)| \leq C\exp(-r|n|)\}.$$

Clearly, $\mathcal{B} \subset \mathcal{C} \subset \mathcal{A}$, where $\mathcal{A}$ and $\mathcal{B}$ were defined in the previous section.

**Definition 6.4** *The operator $H$ has exponential dynamical localization on $I$, if for any $\psi \in \mathcal{C}$, we have $EDL(P_I(H)\psi)$.*

Using the result of Theorem 6.3, one can give sufficient conditions for $EDL$ on the interval $I$. For the sake of simplicity, we restrict ourselves to the first statement of this theorem. One could, however, if necessary, give also a more general sufficient condition based on the second statement of Theorem 6.3.

As in the previous section, $\mathcal{M} = \{e_k\}$ is some orthonormal system of eigenfunctions of $H$ complete in $\mathcal{H}(I)$ and

$$F_0(n,m) = \sup_k |e_k(n)e_k(m)|,$$

$$G_0(n,m) = \sup_{g\in\mathcal{L}} |g(n)g(m)|,$$

$$F_0(n,m) \le G_0(n,m).$$

**Theorem 6.5** *Let $N$ be one of the two functions $F_0, G_0$. Let $\psi \in \mathcal{C}$ so that*

$$|\psi(m)| \le C\exp(-r|m|)$$

*for some $r > 0$. As usually, $\varphi = P_I(H)\psi,\ \varphi(t) = \exp(-itH)\varphi$. The following statements hold:*

*1. Suppose that there exist $\alpha > 0,\ \beta > 0$ such that*

$$N(n,m) \le C\exp(-\alpha|n| + \beta|m|).$$

*Then*

$$\sup_t |\varphi(t,n)| \le C(\gamma)\exp(-\gamma|n|),$$

*where $0 < \gamma < \min\{\alpha, \alpha r/\beta\}$.*

*2. Suppose that there exist $\alpha > 0,\ \beta > 0$ such that*

$$N(n,m) \le C\exp(-\alpha|n-m| + \beta|m|).$$

*Then*

$$\sup_t |\varphi(t,n)| \le C(\gamma)\exp(-\gamma|n|), \tag{6.14}$$

*where $0 < \gamma < \min\{\alpha, \alpha r/(\alpha+\beta)\}$.*

*In particular, in both cases $EDL$ holds on $I$.*

Proof. We shall give it for the second statement of the Theorem, for the first the things being similar. As in the proof of Lemma 5.2, we have the bound

$$R_{\varphi,\mathcal{M}}(n,0) \le \sum_m N(n,m)|\psi(m)|.$$

Since $N(n,m) \le 1$ for any $n, m$,

$$N(n,m) \le N^s(n,m) \le C^s\exp(-s\alpha|n-m| + s\beta|m|)$$

for all $s \in [0,1]$ (the argument is similar to (6.8)-(6.9)). Therefore,

$$R_{\varphi,\mathcal{M}}(n,0) \le C(s)\sum_m \exp(-s\alpha|n-m| - (r-s\beta)|m|).$$

If $r \ge \alpha+\beta$, we take $s=1$, and for $r<\alpha+\beta$, we take $s=r/(\alpha+\beta)$. In both cases we obtain the bound

$$R_{\varphi,\mathcal{M}}(n,0) \le C(\gamma)\exp(-\gamma|n|) \tag{6.15}$$

for all $0<\gamma<\min\{\alpha, \alpha r/(\alpha+\beta)\}$. The bound (6.14) follows directly from (6.15) and the first statement of Theorem 6.3.

As an example where this theorem can be directly applied consider operators with SULE on $I$. Namely, suppose that there exists an orthonormal system $\mathcal{M}=\{e_k\}$ of eigenfunctions of $H$ complete in $\mathcal{H}(I)$ such that for some $n_k \in \mathbf{Z}^d$ for any $\delta>0$

$$|e_k(n)| \le C(\delta)\exp(\delta|n_k| - \alpha|n-n_k|), \tag{6.16}$$

where $\alpha>0$ and the constants $C(\delta)$ are uniform in $k,n$. It follows from (6.16) that

$$|e_k(n)e_k(m)| \le C^2(\delta)\exp(2\delta|n_k| - \alpha(|n-n_k|+|m-n_k|)).$$

Using the elementary inequalities

$$|n_k| \le |m| + |m-n_k|, \quad |n-n_k|+|m-n_k| \ge |m-n|,$$

one can easily show that

$$F_0(n,m) = \sup_k |e_k(n)e_k(m)| \le C^2(\delta)\exp(2\delta|m| - (\alpha-2\delta)|n-m|)$$

for any $\delta>0$. The second statement of Theorem 6.5 implies $EDL$ on $I$. Moreover, if $|\psi(m)| \le C\exp(-r|m|)$, then

$$\sup_t |\varphi(t,n)| \le C(\gamma)\exp(-\gamma|n|)$$

where $0<\gamma<\min\{\alpha, r\}$.

# 7 Adaptation to the continuous case

Most of results of the previous sections remain valid in the case of $L^2(\mathbf{R}^d)$ provided the result of Theorem 2.2 holds. However, one cannot expect that Theorem 2.2 is true in the continuous case in such a generality. For example, in the case of $L^2(R)$, define the moments

$$d_k(p) = \int_{\mathbf{R}} |e_k(x)|^2(|x|+1)^p dx.$$

It is sufficient to take any orthonormal system $\{e_k(x)\}$ in $L^2([-1,1])$ and to put $e_k(x)=0$ for $|x|>1$. For such a system $d_k(p) \le 2^p$ for any $k$. However, if the functions $e_k(x)$ do not oscillate fast, the same phenomenon of "repulsion" of eigenfunctions occurs and one can show the result similar to that of Theorem 2.2. The main result of this section is the following.

**Theorem 7.1** *Let $\{e_k\}, k \in \mathbf{N}$ be an orthonormal system in $L^2(\mathbf{R}^d)$ such that*

$$\lim_{R \to +\infty} \sup_k \int_{|u|>R} |\widehat{e}_k(u)|^2 du = 0, \tag{7.1}$$

*where $\widehat{e}_k$ is the Fourier transform of $e_k$. Then for any $p > 0$ one can reorder the moments*

$$d_k(p) = \int_{\mathbf{R}^d} (|x|+1)^p |e_k(x)|^2 dx$$

*so that*

$$d_k(p) \geq D(p,d) k^{p/d}$$

*with some positive constants $D(p,d)$ depending on the system $\{e_k\}$.*

Proof. To prove the Theorem, we shall discretize the problem and use the same technical Lemma 2.1 as in the discrete case. For any $n = (n_1, ..., n_d) \in \mathbf{Z}^d$, $\varepsilon > 0$ define the cube of size $\varepsilon$ in $\mathbf{R}^d$:

$$K_n(\varepsilon) = \{x = (x_1, ..., x_d) \in \mathbf{R}^d \mid x_j \in [\varepsilon n_j, \varepsilon(n_j+1)),\ j = 1, ..., d\}.$$

It is clear that the cubes are disjoint and $\mathbf{R}^d = \cup_n K_n(\varepsilon)$. Let $x \in K_n(\varepsilon)$. Then $C_1(|n|+1) \leq |x|+1 \leq C_2(|n|+1)$ with some constants $C_1(\varepsilon), C_2(\varepsilon)$. As

$$d_k(p) = \int_{\mathbf{R}^d} (|x|+1)^p |e_k(x)|^2 dx = \sum_{n \in \mathbf{Z}^d} \int_{K_n(\varepsilon)} (|x|+1)^p |e_k(x)|^2 dx,$$

we obtain that

$$C_1^p(\varepsilon) w_k(p) \leq d_k(p) \leq C_2^p(\varepsilon) w_k(p), \tag{7.2}$$

where

$$w_k(p) = \sum_n (|n|+1)^p \int_{K_n(\varepsilon)} |e_k(x)|^2 dx \equiv \sum_n (|n|+1)^p |g_k(n)|^2.$$

One could try to apply Lemma 2.1 taking $a_{kn} = |g_k(n)|^2$. It is obvious that $\sum_n |g_k(n)|^2 = \|e_k\|^2 = 1$, so the condition (2.2) is satisfied. However, it is not clear whether $\sum_k |g_k(n)|^2 \leq A < +\infty$. To avoid this problem, we shall consider rather the quantities

$$h_k(n) = \int_{K_n(\varepsilon)} e_k(x) dx.$$

By the Cauchy-Schwartz inequality,

$$|h_k(n)|^2 \leq \varepsilon^d |g_k(n)|^2. \tag{7.3}$$

Therefore, (7.2) implies

$$d_k(p) \geq \varepsilon^{-d} C_1^p(\varepsilon) \sum_n (|n|+1)^p |h_k(n)|^2, \tag{7.4}$$

and to prove the Theorem it is sufficient to show that the numbers $a_{kn} = |h_k(n)|^2$ verify the conditions of Lemma 2.1 for some $\varepsilon > 0$. One can represent $h_k(n)$ as
$h_k(n) = \langle e_k, \eta_n \rangle_{L^2(\mathbf{R}^d)}$, where $\eta_n$ is the caracteristic function of $K_n(\varepsilon)$. Since the system $\{e_k\}$ is orthonormal,

$$\sum_k |h_k(n)|^2 \leq \|\eta_n\|^2 = \varepsilon^d,$$

so (2.1) holds with $A = \varepsilon^d$.

To prove (2.2) is more difficult. We shall show that the numbers $\varepsilon^{-d}|h_k(n)|^2$ are close to $|g_k(n)|^2$ for $\varepsilon$ small enough if the condition (7.1) is satisfied. Using $\sum_n |g_k(n)|^2 = 1$, we shall prove (2.2) for some $B(\varepsilon) > 0$ if $\varepsilon$ is small enough. We need the following technical result.

**Lemma 7.2** *Let $\psi \in L^2(\mathbf{R}^d)$, $\|\psi\| = 1$. For any $n \in \mathbf{Z}^d, \varepsilon > 0$ define*

$$\Delta_n(\varepsilon) = \int_{K_n(\varepsilon)} |\psi(x)|^2 dx - \varepsilon^{-d} \left| \int_{K_n(\varepsilon)} \psi(x) dx \right|^2$$

*($\Delta_n(\varepsilon) \geq 0$ by Cauchy-Schwartz inequality).*

*There exists some universal constant $C(d)$ such that for any $\varepsilon > 0, R > 0$*

$$0 \leq \sum_{n \in \mathbf{Z}^d} \Delta_n(\varepsilon) \leq C(d) \left( R^2 \varepsilon^2 + \int_{|u|>R} |\widehat{\psi}(u)|^2 du \right)^{1/2}.$$

Proof. One can represent $\Delta_n(\varepsilon)$ as

$$\Delta_n(\varepsilon) = \varepsilon^{-d} \int_{K_n(\varepsilon)} dx \ \psi(x) \int_{K_n(\varepsilon)} dy \ (\overline{\psi(x) - \psi(y)}). \tag{7.5}$$

Applying twice the Cauchy-Schwartz inequality (to the integral over $y$ and to the integral over $x$), we obtain from (7.5):

$$\Delta_n^2(\varepsilon) \leq \varepsilon^{-d} \int_{K_n(\varepsilon)} dx |\psi(x)|^2 \int_{K_n(\varepsilon)} \int_{K_n(\varepsilon)} dx dy |\psi(x) - \psi(y)|^2.$$

Therefore,

$$\sum_n \Delta_n(\varepsilon) \leq \varepsilon^{-d/2} \left( \sum_n \int_{K_n(\varepsilon)} dx |\psi(x)|^2 \right)^{1/2} \left( \sum_n \int_{K_n(\varepsilon)} \int_{K_n(\varepsilon)} dx dy |\psi(x) - \psi(y)|^2 \right)^{1/2} =$$

$$\varepsilon^{-d/2} \left( \sum_n \int_{K_n(\varepsilon)} \int_{K_n(\varepsilon)} dx dy |\psi(x) - \psi(y)|^2 \right)^{1/2}. \tag{7.6}$$

One can observe that $|x - y| \leq \varepsilon\sqrt{d}$ for any $x, y \in K_n(\varepsilon)$. Therefore,

$$\sum_n \int_{K_n(\varepsilon)} \int_{K_n(\varepsilon)} dx dy |\psi(x) - \psi(y)|^2 \leq \sum_n \int_{K_n(\varepsilon)} dx \int_{\mathbf{R}^d} dy |\psi(x) - \psi(y)|^2 F(|x - y| \leq \varepsilon\sqrt{d})$$

$$= \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} dxdy |\psi(x) - \psi(y)|^2 F(|x-y| \leq \varepsilon \sqrt{d}), \tag{7.7}$$

where $F$ is the caracteristic function of the set $\{(x,y) \mid |x-y| \leq \varepsilon\sqrt{d}\}$. The bounds (7.6)-(7.7) imply

$$\sum_n \Delta_n(\varepsilon) \leq \varepsilon^{-d/2} L^{1/2}(\delta), \tag{7.8}$$

where $L(\delta) = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} |\psi(x) - \psi(y)|^2 F(|x-y| \leq \delta)$ and $\delta = \varepsilon\sqrt{d}$. Changing the variable $z = y - x$ in the integral, we obtain in Fourier representation

$$L(\delta) = \int_{|z|\leq\delta} dz \int_{\mathbf{R}^d} du |\widehat{\psi}(u)|^2 |1 - e^{i\langle z,u\rangle}|^2. \tag{7.9}$$

Let $R > 0$. The integral over $u$ can be written as

$$\int_{\mathbf{R}^d} du |\widehat{\psi}(u)|^2 |1 - e^{i\langle z,u\rangle}|^2 = I_1(z) + I_2(z),$$

where in $I_1(z)$ and $I_2(z)$ one integrates over $u: \ |u| \leq R$ and over $u: \ |u| > R$ respectively. Using the elementary bound $|e^{iw} - 1| \leq C|w|, \ w \in \mathbf{R}$, we estimate

$$I_1(z) \leq C|z|^2 \int_{|u|\leq R} du |u|^2 |\widehat{\psi}(u)|^2 \leq C|z|^2 R^2 \|\widehat{\psi}\|^2 = C|z|^2 R^2. \tag{7.10}$$

As to $I_2(z)$, trivially

$$I_2(z) \leq 4 \int_{|u|>R} du |\widehat{\psi}(u)|^2. \tag{7.11}$$

The bounds (7.9)-(7.11) imply

$$L(\delta) \leq CR^2\delta^{d+2} + C\delta^d \int_{|u|>R} du |\widehat{\psi}(u)|^2. \tag{7.12}$$

Finally, (7.8) and (7.12) yield the statement of the Lemma.

Now we can finish the proof of the Theorem. Let $\{e_k\}$ be an orthonormal system verifying (7.1). The bound of the Lemma 7.2 applied to $e_k$ yields

$$\sum_n (|g_k(n)|^2 - \varepsilon^{-d}|h_k(n)|^2) \leq C(d) \left( R^2\varepsilon^2 + \int_{|u|>R} du |\widehat{e}_k(u)|^2 \right)^{1/2}. \tag{7.13}$$

Using the condition (7.1), it is easy to see that one can choose $R > 0$ big enough and $\varepsilon > 0$ small enough so that the r.h.s. of (7.13) be smaller than $1/2$ for any $k \in \mathbf{N}$. As $\sum_n |g_k(n)|^2 = 1$ for any $k$, (7.13) yields for such $\varepsilon$:

$$\sum_n |h_k(n)|^2 \geq \varepsilon^d/2$$

for any $k$, and (2.2) holds for $a_{kn} = |h_k(n)|^2$ with $B = \varepsilon^d/2$. The proof of the Theorem is completed.

Remark. One can note that the choice of $\varepsilon$ depends on the system $\{e_k\}$, so, unlike the discrete case, the constants $D(p,d)$ are not necessarily universal in the continuous case.

An important example where the condition (7.1) is satisfied, is given by the following

**Theorem 7.3** *Let $H = -\Delta + V(x)$ be an operator in $L^2(\mathbf{R}^d)$ self-adjoint on $H^2(\mathbf{R}^d)$, where $V(x)$ is a real function bounded from below: $V(x) \geq -M$ for a.e. $x \in R^d$. Let $K \in \mathbf{R}$ and $\{e_k\}$ be any orthonormal family of eigenfunctions of $H$ with eigenvalues $\lambda_k \leq K$ for all $k$. Then for any $p > 0$ one can reorder the moments $d_k(p)$ so that*

$$d_k(p) \geq D(p, d, K+M)k^{p/d}$$

*with universal positive constant $D$ depending only on $p, d$ and $K + M$.*

Proof. For any $k \in \mathbf{N}$ we have

$$He_k(x) = -\Delta e_k(x) + V(x)e_k(x) = \lambda_k e_k(x).$$

Therefore,

$$\langle -\Delta e_k, e_k \rangle = \int_{\mathbf{R}^d} dx(\lambda_k - V(x))|e_k(x)|^2 \leq (K+M)\|e_k\|^2 = K + M. \tag{7.14}$$

On the other hand,

$$\langle -\Delta e_k, e_k \rangle = \int_{\mathbf{R}^d} du|u|^2|\hat{e}_k(u)|^2 \geq R^2 \int_{|u|>R} du|\hat{e}_k(u)|^2 \tag{7.15}$$

for any $R > 0$. The bounds (7.14)-(7.15) imply

$$\sup_k \int_{|u|>R} du|\hat{e}_k(u)|^2 \leq (K+M)/R^2,$$

so (7.1) is satisfied. Moreover, it is clear from the proof of Theorem 7.1 that one can choose $R > 0$ and $\varepsilon > 0$ depending only on $d$ and $K + M$ so that the r.h.s of (7.13) be smaller then $1/2$ for any $k$. That means that the constants $A = \varepsilon^d$ and $B = \varepsilon^d/2$ in Lemma 2.1 depend only on $d$ and $K + M$ but not on the choice of the system $\{e_k\}$. This gives us the result of the Theorem.

All the results of Section 3 hold if the orthonormal system $\mathcal{M}(\psi)$ satisfies the condition (7.1). The proof of Theorem 3.3 is essentially the same (one considers $\int_{|x|\leq N} dx$ instead of $\sum_{|n|\leq N}$ in the proof). The proofs of Theorem 3.4 and Corollary 3.5 do not change. The results of Lemma 3.6 and Theorem 3.8 hold with the function $R_{\psi,\mathcal{M}}(n)$ defined as follows:

$$R_{\psi,\mathcal{M}}(n) = \sup_k |\gamma_k g_k(n)|, \quad \gamma_k = \langle \psi, e_k \rangle, \quad n \in \mathbf{N},$$

where $|g_k(n)|^2 \equiv \int_{K_n(1)} dx|e_k(x)|^2$.

The sufficient conditions for $DL(\psi, p)$ and $DL(\psi)$ in the continuous case are based on the following version of Lemma 4.1

$$|X|_\psi^p(t) \leq C \sum_{n\in\mathbf{Z}^d} (|n|+1)^p \int_{K_n(1)} |\psi(t,x)|^2 \leq \left(\sum_k \sqrt{a_k w_k(p)}\right)^2, \tag{7.16}$$

where $w_k(p) = \sum_n (|n|+1)^p |g_k(n)|^2$. The numbers $w_k(p)$ are equivalent to the moments $d_k(p)$ due to (7.2), so the lower bounds $w_k(p) \geq Dk^{p/d}$ hold. The result similar to

Theorem 4.2 can be easily obtained. One should only replace $R(n,\alpha)$ by $\sup_k \frac{|\gamma_k g_k(n)|}{w_k(\alpha)}$, $\sup_t |\psi(t,n)|$ by $\sum_k |\gamma_k g_k(n)|$, $d_k(p)$ by $w_k(p)$, and $e_k(n)$ by $g_k(n)$. The only difference is the following: one does not have the bound $\sum_k |g_k(n)|^2 \le 1$ which was valid for $e_k(n)$ in the discrete case. Therefore the bounds in the statements 1 and 2 of the Theorem one can prove are slightly weaker than in the discrete case. The statement 3 of the Theorem and the result of Corollary 4.3 remain true. For the sake of completeness, let us give a direct proof of the third statement of Theorem 4.2 in the continuous case (this proof is valid also in the discrete case). For simplicity we shall take $\alpha = 0$.

**Theorem 7.4** *Let $\mathcal{M} = \{e_k\}$ be some orthonormal system in $L^2(\mathbf{R}^d)$ verifying the conditions of Theorem 7.1. For a given vector $\psi \in \mathcal{H}_{\mathcal{M}}$ consider the function $R(n) = \sup_k |\gamma_k g_k(n)|$, where $|g_k(n)|^2 = \int_{K_n(1)} |e_k(x)|^2 dx \le 1$, $\quad n \in \mathbf{Z}^d$ and $\gamma_k = \langle \psi, e_k \rangle$. If the function $R(n)$ is fast decaying then $DL(\psi)$ hold.*

Proof. As the function $R(n)$ is fast decaying, for any $r > 0$

$$|\gamma_k|^2 w_k(r) = \sum_n (|n|+1)^r |\gamma_k g_k(n)|^2 \le \sum_n (|n|+1)^r R^2(n) \le C(r) < +\infty.$$

On the other hand, $w_k(r) \ge D k^{r/d}$ after reordering. Therefore,

$$|\gamma_k| \le C(m) k^{-m} \quad \text{for any } m > 0. \tag{7.17}$$

Next, as

$$|\psi(t,x)| \le \sum_{k=1}^{\infty} |\gamma_k e_k(x)|,$$

for any $n \in \mathbf{Z}^d$ the bound holds:

$$\int_{K_n(1)} |\psi(t,x)|^2 dx \le \left( \sum_{k=1}^{\infty} |\gamma_k g_k(n)| \right)^2 \equiv S^2(n). \tag{7.18}$$

Reorder the terms in the sum so that (7.17) hold. Then

$$S(n) = \left( \sum_{k=1}^{|n|^2} + \sum_{k=|n|^2+1}^{\infty} \right) |\gamma_k g_k(n)| \le$$

$$|n|^2 R(n) + \sum_{k=|n|^2+1}^{\infty} |\gamma_k| \le |n|^2 R(n) + C(m)(|n|^2+1)^{1-m} \tag{7.19}$$

for any $m > 0$. The bounds (7.16), (7.18) and (7.19) yield $DL(\psi, p)$ for all $p > 0$. The proof is completed.

Most of results of Section 5 can be adapted to the continuous case. It is sufficient to take $g_k(n)$ instead of $e_k(n)$ and $(\int_{K_n(1)} |\psi(x)|^2)^{1/2}$ instead of $\psi(n)$. The results of Theorem 5.3 and Theorem 5.4 are true if the system $\mathcal{M}$ complete in $\mathcal{H}(I)$ satisfies the conditions of Theorem 7.1. In particular, this is the case if $H = -\Delta + V(x)$ with $V(x)$

bounded from below and $I = (-\infty, K]$. The result similar to that of Theorem 5.5 can be proved in the case $H(\theta) = -\Delta + V(x,\theta)$, where $V(x,\theta) \geq -M$ for $\mu$-a.e. $\theta$ and a.e.$x$. The constants in the bounds will depend on $\varepsilon, p, d$ and $K + M$.

The main results of Section 6 can be also generalized to the continuous case in a similar way.

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