Unitary Representations and Osterwalder-Schrader Duality

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Dedicated to the memory of Irving E. Segal

Abstract. The notions of reflection, symmetry, and positivity from quantum field theory are shown to induce a duality operation for a general class of unitary representations of Lie groups. The semisimple Lie groups which have this c-duality are identified and they are placed in the context of Harish-Chandra's legacy for the unitary representations program. Our paper begins with a discussion of path space measures, which is the setting where reflection positivity (Osterwalder-Schrader) was first identified as a useful tool of analysis.

Le plus court chemin entre deux vérités dans
le domaine réel passe par le domaine complexe.

—Jacques Hadamard

Introduction

In this paper, we present an idea which serves to unify the following six different developments:

(i) reflection positivity of quantum field theory,
(ii) the role of reflection positivity in functional integration,
(iii) the spectral theory of unitary one-parameter groups in Hilbert space,
(iv) an extension principle for operators in Hilbert space,
(v) the Bargmann transform, and
(vi) reflection positivity and unitary highest weight modules for semisimple Lie groups.

The emphasis is on (vi), but the common thread in our paper is the unity of the six areas, which, on the face of it, may perhaps appear to be unrelated. We also stress the interconnection between the six projects, and especially the impact on (vi) from (i)–(v).

Readers who may not be familiar with all six of the subjects (i)–(vi) may wish to consult the bibliography; for example [**17**] is an excellent background reference on (i)–(ii), [**15**] is especially useful on (ii), and [**1**] covers the theory underlying (iii).

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Area (iv) is covered in [**40**] and [**45**], while [**20**] and [**78**] cover (v). Background references on (vi) include [**46**], [**41**], and [**42**].

Our main point is to show how the concepts of reflection, symmetry and positivity, which are central notions in quantum field theory, are related to a duality operation for certain unitary representations of semisimple Lie groups. On the group level this duality corresponds to the c-duality of causal symmetric spaces, a duality relating the "compactly causal" spaces to the "non-compactly causal" spaces. On the level of representations one starts with a unitary representation of a group G with an involution τ corresponding to a non-compactly causal space, then uses an involution on the representation space satisfying a certain positivity condition on a subspace to produce a contraction representation of a semigroup $H\exp(C)$, where $H = G^\tau$ and C is an H-invariant convex cone lying in the space of τ-fixed points in the Lie algebra. Now a general result of Lüscher and Mack and the co-authors can be used to produce a unitary representation of the c-dual group G^c on the same space. (See [**33**], [**31**], [**60**], and [**74**] for these terms.)

We aim to address several target audiences: workers in representation theory, mathematical physicists, and specialists in transform theory. This diversity has necessitated the inclusion of a bit more background material than would perhaps otherwise be called for: certain ideas are typically explained slightly differently in the context of mathematical physics from what is customary among specialists in one or more of the other areas.

The symmetry group for classical mechanics is the *Euclidean motion group* $E_n = SO(n) \times_{\mathrm{sp}} \mathbb{R}^n$, where the subscript $_{\mathrm{sp}}$ stands for semidirect product. Here the action of $(A, \mathbf{x}) \in E_n$ on $\mathbb{R}^n$ is given by $(A, \mathbf{x}) \cdot \mathbf{v} = A(\mathbf{v}) + \mathbf{x}$, that is $SO(n)$ acts by rotations and $\mathbb{R}^n$ acts by translations. The connected symmetry group for the space-time of relativity is the *Poincaré group* $P_n = SO_o(n-1,1) \otimes_{sp} \mathbb{R}^n$. Let x_n stand for the time coordinate, that is $t = x_n$. Those two symmetry groups of physics are related by transition to imaginary time, that is multiplying x_n by i. This in particular changes the usual Euclidean form $(\mathbf{x} \mid \mathbf{y}) = x_1y_n + \cdots + x_ny_n$ into the Lorentz form $q_{n-1,1}(\mathbf{x}, \mathbf{y}) = x_1y_1 + \cdots + x_{n-1}y_{n-1} - x_ny_n$, which is invariant under the group $SO_o(n-1,1)$. Those groups and the corresponding geometry can be related by the c-duality. Define an involution $\tau\colon E_n \to E_n$ by

$$\tau(A, \mathbf{x}) = (I_{n-1,1}AI_{n-1,1}, I_{n-1,1}\mathbf{x}), \qquad I_{n-1,1} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix}.$$

The differential $\tau\colon \mathfrak{e}_n \to \mathfrak{e}_n$ is given by the same formula, and τ is an involution on $\mathfrak{e}_\mathfrak{n}$. Let

$$\begin{aligned} \mathfrak{h} &:= \{X \in \mathfrak{e}_n \mid \tau(X) = X\} \simeq \mathfrak{so}(n-1) \times_{sp} \mathbb{R}^{n-1} \\ \mathfrak{q} &:= \{X \in \mathfrak{e}_n \mid \tau(X) = -X\} \end{aligned}$$

The c-dual Lie algebra $\mathfrak{e}^c$ is defined by:

$$\mathfrak{e}_n^c := \mathfrak{h} \oplus i\mathfrak{q}\,.$$

A simple calculation shows that $\mathfrak{e}_n \simeq \mathfrak{p}_n$. Let $G = E_n$ and let G^c denote the simply connected Lie group with Lie algebra $\mathfrak{e}^\mathfrak{c}$. Then $G^c = \tilde{P}_n$, the universal covering group of P_n.

Given that a physical system is determined by a *unitary representation* $(\pi, \mathbf{H}(\pi))$ of the symmetry group, in our case E_n and P_n, the problem is reduced to use "analytic continuation" to move unitary representations of E_n to unitary representations of P_n by passing over to imaginary time. This idea was used in the paper by J. Fröhlich, K. Osterwalder, and E. Seiler in [**15**], see also [**50**], to *construct quantum field theoretical systems using Euclidean field theory.* In this paper we will give some general constructions and ideas related to this problem in the context of the applications (i)–(v) mentioned above, and work out some simple examples.

In [**95**] R. Schrader used this idea to construct, from a *complementary series representation* of $SL(2n, \mathbb{C})$, a unitary representation of the group $SU(n, n) \times SU(n, n)$. In that paper the similarities to the *Yang-Baxter* relation were also discussed, a theme that we will leave out in this exposition. What was missing in R. Schrader's paper was the *identification* of the resulting representations and a general procedure *how* to construct those representations. We will see that we can do this for all simple Lie groups where the associated Riemannian symmetric space G/K is a tube domain, and the that the duality works between complementary series representations and highest weight representations.

In general this problem can be formulated in terms of *c-duality* of Lie groups and the analytic continuation of unitary representations from one real form to another. (See [**33**] for these terms.) The representations that show up in the case of semisimple groups are generalized principal series representations on the one side and highest weight representations on the other. The symmetric spaces are the causal symmetric spaces, and the duality is between non-compactly causal symmetric spaces and compactly causal symmetric spaces. The latter correspond bijectively to real forms of bounded symmetric domains. Therefore both the geometry and the representations are closely related to the work of Harish-Chandra on bounded symmetric domains and the holomorphic discrete series [**21, 22, 23**]. But the ideas are also related to the work of I. Segal and S. Paneitz through the notion of causality and invariant cones, [**96, 86, 87**]. A more complete exposure can be found in the joint paper by the coauthors: *Unitary Representations of Lie Groups with Reflection Symmetry*, [**46**].

There are other interesting and related questions, problems, and directions. In particular we would like to mention the analytic continuation of the H-invariant character of the highest weight representations, and the reproducing kernel of the Hardy space realization of this representation to a spherical distribution character (spherical function) of the generalized principal series representation that we started with. This connects the representations that show up in the duality on the level of distribution characters. For this we refer to [**74, 54**].

The paper is organized as follows. The list also includes some sources for additional background references:

1. Some Spectral Considerations Related to Reflection Positivity

1.1. Unitary One-Parameter Groups and Path Space Integrals. The term *reflection positivity* is from quantum field theory (QFT) where it refers to a certain reflection in the time-variable. As we explained in the introduction this reflection is also the one which makes the analytic continuation between the (Newtonian) group of rigid motions and the Poincaré group of relativity. This is the approach to QFT of Osterwalder and Schrader [**84, 85**]. The approach implies a change of the inner product, and the new Hilbert space which carries a *unitary* representation $\tilde{\pi}$ of the Poincaré group P_4 results from the corresponding "old" one by passing to a subspace where the positivity (the so-called Osterwalder-Schrader positivity) is satisfied. An energy operator may then be associated with this "new" unitary representation $\tilde{\pi}$ of P_4. This representation $\tilde{\pi}$ is "physical" in that the corresponding energy is positive. The basic connection between the two groups may further be understood from the corresponding quadratic forms on space-time (x,t), $x=(x_1,x_2,x_3)$, $(x,t)\mapsto \|x\|^2+t^2$ with $\|x\|^2=x_1^2+x_2^2+x_3^2$. The analytic continuation $t\mapsto it$, $i=\sqrt{-1}$, turns this into the form $\|x\|^2-t^2$ of relativity. This same philosophy may also be used in an analytic continuation argument connecting Feynman measure with the Wiener measure on path space. This is important since the Wiener measure seems to be the most efficient way of making precise the Feynman measure, which involves infinite "renormalizations" if given a literal interpretation. We refer to [**17**] and [**67**] for more details on this point.

For the convenience of the reader we include here a simple instance of reflection positivity for a path space measure which will be needed later: Let $\mathcal{D}=\mathcal{D}(\mathbb{R})=C_c^\infty(\mathbb{R})$ denote the usual test functions on $\mathbb{R}$, and the dual space $\mathcal{D}'=\mathcal{D}'(\mathbb{R})$ of distributions. (We use the notation $q(f)$, $q\in\mathcal{D}'$, $f\in\mathcal{D}$.) Let $H_0=-\frac{1}{2}\triangle+\frac{1}{2}q^2-\frac{1}{2}$ be the harmonic oscillator Hamiltonian, and form $\hat{H}=H_0-E_0$ picking E_0 such that $\hat{H}\geq 0$ and $\hat{H}\Omega=0$ for a ground state vector Ω. Then by [**17**] there is a unique path-space measure ϕ_0 on $\mathcal{D}'$ such that

$$\int q(t)\,d\phi_0(q)=0,$$
$$\int q(t_1)\,q(t_2)\,d\phi_0(q)=\frac{1}{2}e^{-|t_1-t_2|}.$$

One rigorous interpretation (see [**67, 68**]) is to view $(q(t))$ here as a stochastic process, i.e., a family of random variables indexed by t. Further, for each $t>0$, and for each "even + linear" real potential $V(q)$, there is a unique measure μ_t on $\mathcal{D}'$ such that

$$d\mu_t=Z_t^{-1}\exp\left(-\int_{-t/2}^{t/2}V(q(s))\,ds\right)\,d\phi_0(q)$$

with

$$Z_t = \int \exp\left(-\int_{-t/2}^{t/2} V(q(s))\, ds\right) d\phi_0(q).$$

The consideration leading to the relation between the measures $d\phi_0$ and $d\mu_t$ is the Trotter approximation for the (analytically continued) semigroup,

$$e^{-it\left(-\frac{1}{2}\triangle+V\right)} = \lim_{n\to\infty}\left(e^{(it/2n)\triangle}e^{-(it/n)V}\right)^n.$$

See [**67**] for more details on this point. When the operator $(\cdots)^n$ on the right-hand side is computed, we find the integral kernel

$$K^{(n)}(x_0, x_n, t) = \frac{1}{N_n}\int e^{i\mathcal{S}(x_0,\dots,x_n,t)}\, dx_1 \cdots dx_{n-1}$$

with

$$N_n = \left(\frac{2\pi it}{n}\right)^{3n/2}$$

and

$$\mathcal{S}(x_0,\dots,x_n,t) = \frac{1}{2}\sum_{j=1}^{n}|x_{j-1}-x_j|^2\left(\frac{t}{n}\right)^{-1} - \sum_{j=1}^{n} V(x_j)\left(\frac{t}{n}\right).$$

The heuristic motivation for ϕ_0 and μ_t is then the "action" $\mathcal{S}$ approximating

$$\mathcal{S}(q) = \frac{1}{2}\int_0^t (\dot{q}(s))^2\, ds - \int_0^t V(q(s))\, ds$$

via

$$q(t_j) = x_j, \qquad \triangle t_j = \frac{t}{n},$$

and

$$\dot{q}(t_j) \sim \frac{x_j - x_{j-1}}{t/n}$$

If $-t/2 < t_1 \le t_2 \le \cdots \le t_n < t/2$, then

$$\begin{aligned}(1.1)\quad &\left\langle \Omega \,\middle|\, A_1 e^{-(t_2-t_1)\hat{H}} A_2 e^{-(t_3-t_2)\hat{H}} A_3 \cdots A_n \Omega \right\rangle \\ &\qquad = \lim_{t\to\infty}\int \prod_{k=1}^{n} A_k(q(t_k))\, d\mu_t(q(\,\cdot\,)).\end{aligned}$$

(The reader can find more details in [**17**].) Using further Minlos' theorem (see, e.g., [**17**] or [**92**]), it can be shown that there is a measure μ on $\mathcal{D}'$ such that

$$(1.2)\qquad \lim_{t\to\infty}\int e^{iq(f)}\, d\mu_t(q) = \int e^{iq(f)}\, d\mu(q) =: S(f)$$

for all $f \in \mathcal{D}$. Since μ is a positive measure, we have

$$(1.3)\qquad \sum_k \sum_l \bar{c}_k c_l S\left(f_k - \bar{f}_l\right) \ge 0$$

for all $c_1, \dots, c_n \in \mathbb{C}$, and all $f_1, \dots, f_n \in \mathcal{D}$. When combining (1.1) and (1.2), we note that this limit-measure μ then accounts for the time-ordered n-point functions which occur on the left-hand side in formula (1.1). This observation will be further used in the analysis of a corresponding stationary stochastic process $\mathbf{X}_t$, $\mathbf{X}_t(q) =$

$q(t)$, which we proceed to analyze and review. But, more importantly, it can be checked from the construction that we also have the following reflection positivity: Let $(\theta f)(s) := f(-s)$, $f \in \mathcal{D}$, $s \in \mathbb{R}$, and set

$$\mathcal{D}_+ = \{f \in \mathcal{D} \mid f \text{ real valued, } f(s) = 0 \text{ for } s < 0\} .$$

Then

$$\sum_k \sum_l \bar{c}_k c_l S(\theta(f_k) - f_l) \geq 0 \tag{1.4}$$

for all $c_1, \dots, c_n \in \mathbb{C}$, and all $f_1, \dots, f_n \in \mathcal{D}_+$. (A small technical point: In the use of Minlos' theorem it is often more convenient to use the duality $\mathcal{S} \leftrightarrow \mathcal{S}'$ of tempered distributions, as opposed to the $\mathcal{D} \leftrightarrow \mathcal{D}'$ duality mentioned above. The reason for this is that the Hermite functions are in $\mathcal{S}$ but not in $\mathcal{D}$.) These concepts are covered in detail in [**84, 85**].

While our work here was centered on the role of reflection positivity for the structure of representations of non-compact Lie groups G, the basic concepts are important even in the case when $G = \mathbb{R}$ (the real line). In that case, the data is as follows:

- $\mathbf{H}$: a complex Hilbert space (compare with (1.3))
- $\mathbf{K}_0$: a closed subspace (compare with $\mathcal{D}_+$)
- $\{U(t) \mid t \in \mathbb{R}\}$: a unitary one-parameter group acting on $\mathbf{H}$
- $J \colon \mathbf{H} \to \mathbf{H}$: a period-2 unitary operator (compare with $f \mapsto \theta(f)$) satisfying

$$JU(t) = U(-t)J, \tag{1.5}$$

$$P_0 J P_0 \geq 0 \qquad \text{(compare with (1.4))}, \tag{1.6}$$

$$P_0 U(t) P_0 = U(t) P_0 \qquad \text{for all } t \geq 0, \tag{1.7}$$

  where P_0 is the orthogonal projection of $\mathbf{H}$ onto $\mathbf{K}_0$.
- Define $\mathbf{N} := \{k_0 \in \mathbf{K}_0 \mid \langle k_0 \mid J k_0 \rangle = 0\}$.

To illustrate that the axiom system (1.5)–(1.7) is very restrictive, we note that it is *not* satisfied for the usual translation group on $\mathbb{R}$. Specifically, let $\mathbf{H} = L^2(\mathbb{R})$, and $U(t) f(x) = f(x - t)$, $f \in L^2(\mathbb{R})$, $x, t \in \mathbb{R}$. Since the subspaces $\mathbf{K}_0$ for which (1.7) hold are known by Beurling's theorem, see [**29**] and [**59**], we see that we cannot have all three (1.5)–(1.7) in this example, unless $\mathbf{K}_0 = \mathbf{N}$. If (1.7) holds, then either (case 1) $\mathbf{K}_0$ is invariant under all $U(t)$, $t \in \mathbb{R}$, and then it consists of all functions whose Fourier transform is supported on some measurable subset of $\mathbb{R}$ (depending on $\mathbf{K}_0$), or else (case 2) it consists of the transforms of functions in $qH^2(\mathbb{R})$ where q is a unitary function and $H^2(\mathbb{R})$ is the usual Hardy space. Hence we may assume that J is given by

$$(Jf)\widehat{\ }(\xi) = \hat{f}(-\xi), \qquad \xi \in \mathbb{R},$$

where $\hat{f}$ is the Fourier transform. It is then easy to check in case 1 that we cannot have (1.6), and that in case 2, $\mathbf{N} = \mathbf{K}_0$.

The simplest instance of (1.5)–(1.7) arises in quantum field theory and for certain stochastic processes; see, e.g., [**1**, Lecture 4], [**47**]. Let $\{\mathbf{X}_t \mid t \in \mathbb{R}\}$ be a stationary stochastic process on a probability space (Ω, P) which is *symmetric* in the sense that $\mathbf{X}_{-t}$ has the same distribution as does $\mathbf{X}_t$. In that case $\mathbf{H} = L^2(\Omega, P)$ and $\mathbf{K}_0$ may be taken to be the subspace in $L^2(\Omega, P)$ generated by the functions which are measurable with respect to the σ-field generated by $\{\mathbf{X}_t \mid t \geq 0\}$, and P_0 may be taken to be the corresponding conditional expectation. Since the process is stationary, it generates a unitary one-parameter group in $L^2(\Omega, P)$, and the choice of P_0 makes it clear that (1.5) and (1.7) will be satisfied. Condition (1.6) is an extra condition, which is called Osterwalder-Schrader positivity (or O-S positivity, for short), although the O-S positivity concept was first formulated in a different context; see, e.g., [**84, 85**, Axioms 1–2].

Both in the case of unitary one-parameter groups and in the general context of representations of Lie groups, there is an operator-theoretic step used in passing from $\mathbf{H}$ to the new Hilbert space. It underlies two technical points involved in the construction: *positivity* and *norm-estimates*. It is given in Section 4 and referred to as the *Basic Lemma*. More details are in [**46**].

An application of our Basic Lemma, Section 4, produces a contraction W,

$$\begin{tikzcd} & \mathbf{K}_0/\mathbf{N} \arrow[dr, "\text{completion}"] & \\ \mathbf{K}_0 \arrow[ur, "\text{quotient}"] \arrow[rr, "W"'] & & (\mathbf{K}_0/\mathbf{N})^{\sim} \end{tikzcd} \tag{1.8}$$

such that $\{U(t) \mid t \geq 0\}$ is realized in $\hat{\mathbf{H}}_+ := (\mathbf{K}_0/\mathbf{N})^{\sim}$ as a contractive and selfadjoint semigroup $\{\hat{U}(t) \mid t \geq 0\}$, and

$$WU(t)|_{\mathbf{K}_0} = \hat{U}(t)W, \qquad t \geq 0. \tag{1.9}$$

Since this induced semigroup is contractive and selfadjoint, it can be shown to have the form

$$\hat{U}(t) = e^{-tH}, \qquad t \geq 0 \tag{1.10}$$

for a selfadjoint operator H, $H \geq 0$, H acting in $\hat{\mathbf{H}}_+$. The semigroup $\hat{U}(t)$ in (1.10) is constructed here by a procedure which is also applicable to a large class of non-compact Lie groups. In the present case, the idea is simple: From the axioms (1.5)–(1.7), we get

$$\langle k_0 \mid JU(t)k_0 \rangle \leq \langle k_0 \mid Jk_0 \rangle, \qquad \text{for } k_0 \in \mathbf{K}_0 \text{ and for all } t \geq 0. \tag{1.11}$$

Hence $\mathbf{N} := \{k_0 \in \mathbf{K}_0 \mid \langle k_0 \mid Jk_0 \rangle = 0\}$ is invariant under $\{U(t) \mid t \geq 0\}$ which then passes to the quotient $\mathbf{K}_0/\mathbf{N}$ as a contraction semigroup on the completed Hilbert space. The selfadjointness of the induced semigroup follows from the identity

$$\langle k_1 \mid JU(t)k_2 \rangle = \langle U(t)k_1 \mid Jk_2 \rangle, \tag{1.12}$$

valid for all $k_1, k_2 \in \mathbf{K}_0$, and $t \geq 0$. The proof of (1.12) in turn is immediate from axiom (1.5).

When applied to the stochastic process example, we get a concrete realization of this semigroup $\hat{U}(t) = e^{-tH}$ on $\hat{\mathbf{H}}_+$. In pure operator-theoretic terms, what results is the following data:

(i) $\mathbf{K}_0$ and $\hat{\mathbf{H}}_+$, two Hilbert spaces;
(ii) $\{V(t)\}_{t\geq 0}$, a semigroup of isometries in $\mathbf{K}_0$;
(iii) $H \geq 0$, a selfadjoint, generally unbounded operator, in $\hat{\mathbf{H}}_+$;
(iv) $W\colon \mathbf{K}_0 \to \hat{\mathbf{H}}_+$, a contractive linear operator which has the following properties: $\ker(W) = 0$; and $\ker(W^*) = 0$, i.e., the range of W is dense, which is to say, $W(\mathbf{K}_0)$ is dense in $\hat{\mathbf{H}}_+$ relative to the norm of $\hat{\mathbf{H}}_+$;
(v) W intertwines the semigroups in the respective spaces, i.e.,

$$e^{-tH}W = WV(t), \qquad \text{for all } t \geq 0.$$

Note that the properties listed in (iv) for W imply that the polar decomposition $W = W_0 (W^*W)^{1/2}$ has the partial isometric factor $W_0 \colon \mathbf{K}_0 \to \hat{\mathbf{H}}_+$ a unitary isomorphism, but, of course, W_0 will *not* intertwine e^{-tH} and $\{V(t)\}_{t\geq 0}$, i.e., the relation in (v) does not pass to the polar decomposition.

It is this realization which we call the O-S construction. Since the constant function $\mathbb{1} \in L^2(\Omega, P)$ is cyclic for the $L^\infty(\Omega)$-multiplication algebra acting on $L^2(\Omega, P)$, we get $\Omega := W(\mathbb{1}) \in \hat{\mathbf{H}}_+$ satisfying a similar cyclicity property in $\hat{\mathbf{H}}_+$ and also $\hat{U}(t)\Omega = \Omega$ for all $t \geq 0$, or equivalently, Ω is in the domain of H, and $H\Omega = 0$.

If $L^2(\Omega, P)$, $\{U(t)\}$ are constructed from a stationary stochastic process $\mathbf{X}_t$ as described, then we may get (1.6) satisfied if

$$\int_\Omega f_1 \circ \mathbf{X}_{t_1}\, f_2 \circ \mathbf{X}_{t_2} \cdots f_n \circ \mathbf{X}_{t_n}\, dP \geq 0 \qquad \text{for all } f_1, \dots, f_n \in C_c(\mathbb{R}), \quad \text{and all } t_1, \dots, t_n \in \mathbb{R} \text{ such that } -\infty < t_1 \leq t_2 \leq \cdots \leq t_n < \infty. \tag{1.13}$$

Moreover, condition (1.13) is alternately denoted the Osterwalder-Schrader positivity condition. It can be obtained if instead we start with a semigroup $(\hat{U}(t), \hat{\mathbf{H}})$ and a representation $f \mapsto \pi(f)$ of $C_c(\mathbb{R})$ on $\hat{\mathbf{H}}$ such that, for a vector $v \in \hat{\mathbf{H}}$, we have

$$\langle v \,|\, \pi(f_1)\, \hat{U}(t_2 - t_1)\, \pi(f_2)\, \hat{U}(t_3 - t_2) \cdots \hat{U}(t_n - t_{n-1})\, \pi(f_n)\, v \rangle \quad \text{for all } f_1, \dots, f_n \in C_c(\mathbb{R}), \text{ and } -\infty < t_1 \leq t_2 \leq \cdots \leq t_n < \infty. \tag{1.14}$$

Starting with one of the two, (1.13) or (1.14), the other can be constructed such that the expressions in (1.13) and (1.14) are equal. This is the content of the O-S construction in the formulation of E. Nelson and others [**68, 48**].

To understand (1.9) better, it is useful to compare it to the Nagy dilation of a semigroup; see [**101**]. The Nagy theorem states that every contraction semigroup $(\hat{U}(t), \hat{\mathbf{H}}, t \geq 0)$ admits a representation $(U(t), \mathbf{H}, t \in \mathbb{R})$ where $\hat{\mathbf{H}} \subset \mathbf{H}$, and $U(t)$ is a one-parameter unitary group in $\mathbf{H}$ which satisfies

$$P_{\hat{\mathbf{H}}} U(t)\,|_{\hat{\mathbf{H}}} = \hat{U}(t), \qquad t \geq 0. \tag{1.15}$$

While there are some generalizations of (1.15) to (non-abelian) Lie groups, e.g., [**45**], the relation (1.9) is the focus of the present paper, wherein we show that it characterizes a class of "physical" representations of semisimple non-compact Lie groups. In (1.15), $P_{\hat{\mathbf{H}}}$ denotes the orthogonal projection of $\mathbf{H}$ onto $\hat{\mathbf{H}}$. Hence (1.15) may be rephrased in terms of an isometry $V \colon \hat{\mathbf{H}} \to \mathbf{H}$, and we get

$$V^* U(t)\, V = \hat{U}(t), \qquad t \geq 0. \tag{1.16}$$

In case the stochastic process $(\mathbf{X}_t, t \in \mathbb{R})$ is given on path space $\mathbf{X}_t(\omega) = \omega(t)$, Arveson found (in [**1**, Proposition 4.6]) a reformulation of (1.9) much in the spirit of (1.16), but Arveson produces a simultaneous "dilation" of a given semigroup *and* a representation of $C_c(\mathbb{R})$. First consider the following two representations of $C_c(\mathbb{R})$:

- σ representing $C_c(\mathbb{R})$ on $L^2(\Omega, P)$ given by

$$\sigma(f)\, F(\omega) = f(\omega(0))\, F(\omega), \tag{1.17}$$

  for $f \in C_c(\mathbb{R})$, $F \in L^2(\Omega, P)$, and

- M representing $C_c(\mathbb{R})$ on $\hat{\mathbf{H}}$ by

$$(M_f h)(t) = f(t)\, h(t)\,, \tag{1.18}$$

for $f \in C_c(\mathbb{R})$, $h \in \hat{\mathbf{H}}$.

Let v be the vector in (1.14). Arveson then shows that the linear mapping

$$M_f v \longmapsto \sigma(f)\, \mathbb{1} \tag{1.19}$$

extends uniquely to an isometry

$$V : L^2(\mathbb{R}) \longrightarrow L^2(\Omega, P)$$

which satisfies

$$V^* U(t_1)\, \sigma(f_1)\, U(t_2)\, \sigma(f_2) \cdots U(t_n)\, \sigma(f_n)\, V = \hat{U}(t_1)\, M_{f_1} \hat{U}(t_2)\, M_{f_2} \cdots \hat{U}(t_n)\, M_{f_n} \tag{1.20}$$

for all $n = 1, 2, \ldots$, $t_i \in \mathbb{R}$, $t_i \geq 0$, and all $f_i \in C_c(\mathbb{R})$.

The space $L^2(\mathbb{R})$ is thereby identified as a closed subspace in $L^2(\Omega, P)$. It is the time-zero subspace, and is "much smaller" than the subspace generated by the σ-algebra of $(\mathbf{X}_t, t \geq 0)$. There is a selfadjoint semigroup induced from both of the subspaces. In general, it is the $t = 0$ semigroup which has the Markoff property in the sense of Nelson [**68**, Theorem 1].

1.2. The $(ax + b)$-Group. The general case breaks down into the analysis of the solvable case, and the semisimple case. Therefore it is appropriate to start with the 2-dimensional solvable case, that is the $(ax + b)$-group. We may realize this group G as the affine transformations $ax + b$ of $\mathbb{R}$, $a \in \mathbb{R}_+$, $b \in \mathbb{R}$. Hence up to a trivial scale, the Lie algebra is determined by the relation $[A, B] = B$. Let $\tau(a, b) = (a, -b)$. A unitary representation of G on a Hilbert space $\mathbf{H}$ is therefore specified by two unitary one-parameter groups $U_A(s)$, $U_B(t)$, $s, t \in \mathbb{R}$, satisfying

$$U_A(s)\, U_B(t)\, U_A(-s) = U_B(e^s \cdot t)\,. \tag{1.21}$$

The spectra of the two groups form subsets of $\mathbb{R}$, and (1.21) shows that the spectrum $\Lambda(B) = \operatorname{spec}(U_B)$ is invariant under positive dilations, that is

$$\mathbb{R}_+ \cdot \Lambda(B) = \Lambda(B)\,. \tag{1.22}$$

But (1.22) implies that either $\Lambda(B) = \mathbb{R}_+$, $\Lambda(B) = \mathbb{R}_-$, or $\Lambda(B) = \mathbb{R}$.

Based on these considerations, we have the following:

THEOREM 1.1. *There are no nontrivial reflection symmetries for infinite dimensional unitary representations of the $(ax + b)$ group.*

PROOF (SKETCH). It is enough to exclude the reflection which sends $A \mapsto A$, and $B \mapsto -B$. We must show that in every $\mathbf{K}_0 \subset \mathbf{H}$ which is invariant under $\{U_A(s) \mid s \in \mathbb{R}\}$ and under $\{U_B(t) \mid t \geq 0\}$, and for every $(J, \mathbf{K}_0)$ such that (1.6) holds, and $JU_A(s) = U_A(s)\, J$ and $JU_B(t) = U_B(-t)\, J$, we must have $\hat{\mathbf{H}}$ $(= (\mathbf{K}_0/\mathbf{N})^{\sim})$ the trivial zero-dimensional space.

Recalling the new norm in $\hat{\mathbf{H}}$, $f \mapsto \langle f \mid Jf \rangle = \|f\|^2_{\hat{\mathbf{H}}}$, and

$$\mathbf{N} = \{f \in \mathbf{K}_0 \mid \langle f \mid Jf \rangle = 0\}\,,$$

we conclude that the induced transformations $\hat{U}_A$ and $\hat{U}_B$ on $\hat{\mathbf{H}}$ satisfy:

(i) $\hat{U}_A$ is a unitary one-parameter group on $\hat{\mathbf{H}}$ and its spectrum is a subset of $\Lambda(A)$;

(ii) $\{\hat{U}_B(t) \mid t \geq 0\}$ is a contraction semigroup on $\hat{\mathbf{H}}$ satisfying $\hat{U}_B(t)^* = \hat{U}_B(t)$; and

(iii) $\hat{U}_A(s)\,\hat{U}_B(t)\,\hat{U}_A(-s) = \hat{U}_B(e^s \cdot t)$ for all $s \in \mathbb{R}$, and $t \geq 0$.

Combining (ii)–(iii), and using a theorem of [**88**], we may assume that the selfadjoint generator H_B of $U_B(t)$ $(= e^{itH_B})$ and J have the representation

$$H_B = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{1.23}$$

relative to $\mathbf{H} = \begin{pmatrix} \mathbf{H}_+ \\ \mathbf{H}_- \end{pmatrix}$ for closed subspaces $\mathbf{H}_\pm$. In this representation it is possible to check all the candidates for $\mathbf{K}_0 \subset \mathbf{H}$ with the stated properties; see also Sections 3 and 13 below. The presence of a nontrivial $\mathbf{K}_0$ leads to (1.23) and the conclusion that $\Lambda(B) = \mathbb{R}$. But, of the nontrivial candidates for $\mathbf{K}_0$, none can satisfy the added axioms

$$P_0 J P_0 \geq 0$$

and

$$P_0 U_B(t) P_0 = U_B(t) P_0, \qquad \text{for all } t \geq 0.$$

We postpone further details to a future paper on semidirect products. □

We refer to Section 13 for more detailed discussion on the $(ax+b)$-group.

1.3. The Hilbert Transform. In a recent paper [**97**], Segal obtains a positive energy representation of the Poincaré group P on a Hilbert space of "complex" spinors. The construction is a renormalization of the usual Klein-Gordon inner product, by use of an operator J much like the one described above. In Fourier variables, it is $J\colon \phi(k) \mapsto i\theta(k)\,\phi(k)$, where $k = (k_0, k_1, k_2, k_3)$ denotes a point in momentum space, and

$$\theta(k) = \begin{cases} +1 & \text{if } k_0 \geq 0, \\ -1 & \text{if } k_0 < 0. \end{cases}$$

Hence J appears as the Hilbert transform with respect to time. One of the corollaries of our results in Section 7 is that, even in the general case of reflection positivity for semisimple Lie groups, the appropriate J must indeed be a generalized Hilbert transform, and the construction is tied to the complementary series.

1.4. One-Parameter Groups. In addition to the examples from path-space measures, there are those which arise directly from the theory of representations of reductive Lie groups. While this applies completely generally, the $SL(2,\mathbb{R}) \simeq SU(1,1)$ case is worked out in detail in Section 9 below. When the representations there are restricted to a suitable one-parameter subgroup in $SL(2,\mathbb{R})$, we arrive at the following basic setup for the axiom system (1.5)–(1.7).

Let $0 < s < 1$ be given. Let $\mathbf{H} := \mathbf{H}(s)$ be the Hilbert space given by the norm squared

$$\|f\|^2 := \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{f(x)} f(y)\, |x-y|^{s-1}\, dx\, dy < \infty\,.$$

With s fixed, let $\mathbf{K}_0 \subset \mathbf{H}$ be the subspace of functions in $\mathbf{H}$ which have compact support in $(-1,1)$. Let $\{U(t)\}_{t\in\mathbb{R}}$ be the unitary one-parameter group given by

$$U_s(t) f(x) = e^{(s+1)t} f\left(e^{2t}x\right). \tag{1.24}$$

It follows from representation theory (see Section 9) that, for every $s \in (0,1)$, the spectrum of the group $U_s(\,\cdot\,)$ in (1.24) is continuous and is all of $\mathbb{R}$.

Let $J\colon \mathbf{H} \to \mathbf{H}$ be given by

$$Jf(x) = |x|^{-s-1} f(1/x).$$

Then an elementary calculation shows that the axioms (1.5)–(1.7) are satisfied when P_0 denotes the projection onto $\mathbf{K}_0$. We will then form the completion of $\mathbf{K}_0$ relative to the norm $\|\cdot\|_{\widehat{\mathbf{H}_+(s)}}$ given by

$$\|f\|^2_{\widehat{\mathbf{H}_+(s)}} = \langle f \mid Jf\rangle_{\mathbf{H}(s)}, \tag{1.25}$$

for $f \in \mathbf{K}_0$. While we are completing a space of functions, it turns out that the elements in the completion are generally distributions which may not be functions.

We stress this example since it is the simplest instance when the completion (1.8) is made explicit as a concrete space of distributions. It follows from the representation-theoretic setup (Section 9) that

$$\hat{\mathbf{H}}_+ = (\mathbf{K}_0/\mathbf{N})^\sim$$

is then the Hilbert space of distributions obtained from completion of measurable functions on $(-1,1)$ completed relative to the new norm squared from (1.25), viz.,

$$\int_{-1}^{1}\int_{-1}^{1} \overline{f(x)} f(y) |1-xy|^{s-1}\, dx\, dy < \infty. \tag{1.26}$$

The fact that $\widehat{\mathbf{H}_+(s)}$, for $0 < s < 1$, is a space of distributions is significant for the spectral theory. The Dirac delta "function" δ is in $\widehat{\mathbf{H}_+(s)}$ and is a ground state vector. When the unitary one-parameter group (1.24) is passed to $\widehat{\mathbf{H}_+(s)}$, we get (for each s) the positive selfadjoint generator $H = H_s$ from (1.10), and we check that $H\delta = (1-s)\,\delta$ with $1-s=$the bottom of the spectrum of H. It is true in fact that the full spectrum of H in $\widehat{\mathbf{H}_+(s)}$ is $\{2n+1-s \mid n = 0,1,2,\dots\}$, and that the spectrum is simple.

It can be shown that $\widehat{\mathbf{H}_+(s)}$, for $0 < s < 1$, from (1.25)–(1.26) in fact are associated with measures on $\mathbb{R}$ as follows. The assertion is that, for every $f \in \widehat{\mathbf{H}_+(s)}$,

$$C_c(-1,1) \ni \varphi \longmapsto \langle f \mid \varphi\rangle_{\widehat{\mathbf{H}_+(s)}} \in \mathbb{C}$$

extends to a Radon measure,

$$\mu_f(\varphi) = \int_{\mathbb{R}} \varphi(x)\, d\mu_f(x),$$

on $\mathbb{R}$. To see this, use the estimates

$$\left|\langle f \mid \varphi\rangle_{\widehat{\mathbf{H}_+(s)}}\right| \le \|f\|_{\widehat{\mathbf{H}_+(s)}} \|\varphi\|_{\widehat{\mathbf{H}_+(s)}}, \tag{1.27}$$

and

$$\|\varphi\|_{\widehat{\mathbf{H}_+(s)}} \leq \text{const} \times \sup_{x\in(-1,1)} |\varphi(x)| \,, \qquad \forall\, \varphi \in C_c(-1,1)\,. \tag{1.28}$$

We will study the Hilbert spaces $\mathbf{H}(s)$ and $\widehat{\mathbf{H}_+(s)}$ further in Section 9 below, where we show among other things that $\{(d/dx)^n\,\delta \mid n = 0,1,2,\dots\}$ forms an orthogonal basis in the reflection Hilbert space $\widehat{\mathbf{H}_+(s)}$ for all $s \in (0,1)$. This is a way to turn the Taylor expansion into an orthogonal decomposition.

2. The General Setting

The setup is a connected Lie group G with an nontrivial involution, that is a symmetry, $\tau\colon G \to G$. The differential $\tau\colon \mathfrak{g} \to \mathfrak{g}$ is then an involution on the Lie algebra $\mathfrak{g}$ of G. Let $H := G^\tau = \{x \in G \mid \tau(x) = x\}$ and $\mathfrak{h} := \{X \in \mathfrak{g} \mid \tau(X) = X\}$. Then $\mathfrak{h}$ is the Lie algebra of H. Let $\mathfrak{q} := \{X \in \mathfrak{g} \mid \tau(X) = -X\}$. Then $\mathfrak{q}$ is a vector space isomorphic to the tangent space $T_{x_o}(G/H)$, where $x_o = eH \in G/H$.

The c-dual of $(\mathfrak{g}, \tau)$ is defined to be the Lie algebra

$$\mathfrak{g}^c := \mathfrak{h} + i\mathfrak{q} \subset \mathfrak{g}_{\mathbb{C}}$$

with the involution $\tau^c := \tau|\mathfrak{g}^c$. We let G^c be the simply connected Lie group with the Lie algebra $\mathfrak{g}^c$. Then τ integrates to an involution $\tau^c\colon G^c \to G^c$, and $H^c := (G^c)^{\tau^c}$ is connected, [**70**]. Here are few examples of triples (G, G^c, τ):

1. Compact Lie groups: Let G be a compact Lie group. Then G^c is a simply connected reductive Lie group with Cartan involution τ^c, and every connected simply connected reductive Lie group can be constructed in this way.

2. The group case: Let H be a Lie group and let $G = H \times H$. We identify H with the diagonal $d(H) = \{(a,a) \in G \mid a \in H\}$. Define $\tau(a,b) := (b,a)$. Then $G^\tau = H$. Furthermore the map

$$G/H \ni (a,b)H \longmapsto ab^{-1} \in H$$

is a diffeomorphism, intertwining the canonical action of G on G/H and the action $(a,b)\cdot x = axb^{-1}$ on H. Denote the simply connected complex Lie group with the Lie algebra $\mathfrak{h}_{\mathbb{C}}$ by $H_{\mathbb{C}}$. Then $G^c = H_{\mathbb{C}}$, and τ^c is the conjugation with respect to the real form $\mathfrak{h}$. Hence the analysis on Lie groups and their complexification is a special case of symmetric space duality. The case $H_{\mathbb{C}} = SL(2n,\mathbb{C})$ and $H = SU(n,n)$ was treated by R. Schrader in [**95**].

3. Semidirect product with abelian normal subgroup: Let $G = QH$ with Q and H connected, Q normal subgroup of G and $Q \cap H = \{e\}$. Then we can define an involution on G by

$$\tau(qh) := q^{-1}h\,.$$

Thus $\mathfrak{q}$ is the Lie algebra of Q. Define $\Phi\colon \mathfrak{g} \to \mathfrak{g}^c$ by

$$\Phi(X+Y) := iX + Y\,, \quad X \in \mathfrak{q}\,,\ Y \in \mathfrak{h}\,.$$

Then Φ is a Lie algebra isomorphism. Thus G^c is the simply connected covering group of G. A special case hereof is the $(ax+b)$-group and the Heisenberg group, where H is also abelian.

4. The $(ax+b)$-group: The $(ax+b)$-group is the group of transformations $x \mapsto ax + b$, $\;a > 0$, $b \in \mathbb{R}$. We take $Q = \mathbb{R}$ and $H = \mathbb{R}^+$. Then τ is given by $\tau(a,b) = (a,-b)$.

5. The Heisenberg group: Let H_n be the $(2n+1)$-dimensional Heisenberg group. We write

$$H_n = \left\{ h(\mathbf{x},\mathbf{y},z) = \begin{pmatrix} 1 & \mathbf{x}^t & z \\ 0 & I_{n-1} & \mathbf{y} \\ 0 & 0 & 1 \end{pmatrix} \; \mathbf{x},\mathbf{y} \in \mathbb{R}^n,\, z \in \mathbb{R} \right\} \simeq \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$$

with multiplication given by

$$h(\mathbf{x},\mathbf{y},z)h(\mathbf{x}',\mathbf{y}',z') = h(\mathbf{x}+\mathbf{x}',\mathbf{y}+\mathbf{y}',z+z'+(\mathbf{x}\mid \mathbf{y}')) .$$

In particular $h(\mathbf{x},\mathbf{y},z)^{-1} = h(-\mathbf{x},-\mathbf{y},(\mathbf{x}\mid\mathbf{y})-z)$. In this case we take $H = \{h(\mathbf{x},0,0)\} \simeq \mathbb{R}^n$, which is abelian, and $Q = \{h(0,\mathbf{y},z) \in H_n \mid \mathbf{y} \in \mathbb{R}^n,\, z \in \mathbb{R}\} \simeq \mathbb{R}^{n+1}$. The involution is given by

$$\tau(h(\mathbf{x},\mathbf{y},z)) := h(\mathbf{x},-\mathbf{y},-z) .$$

Starting from the pair (G,τ) and a unitary representation $(\pi,\mathbf{H}(\pi))$ of G we need a compactible involution on the Hilbert space $\mathbf{H}(\pi)$, that is a unitary linear map $J\colon \mathbf{H}(\pi)\to\mathbf{H}(\pi)$ intertwining π and $\pi\circ\tau$. Thus

$$J\pi(g) = \pi(\tau(g))J \qquad \forall g \in G .$$

We will also need a semigroup S such that $H \subset S$ or at least $H \subset \overline{S}$ where the bar denotes topological closure. We shall consider closed subspaces $\mathbf{K}_0 \subset \mathbf{H}(\pi)$, where $\mathbf{H}(\pi)$ is the Hilbert space of π, such that $\mathbf{K}_0$ is invariant under $\pi(S^o)$. Let $J\colon \mathbf{H}(\pi)\to\mathbf{H}(\pi)$ be a unitary intertwining operator for π and $\pi\circ\tau$ such that $J^2=\mathrm{id}$. We assume that $\mathbf{K}_0$ may be chosen such that $\|v\|_J^2 := \langle v \mid Jv\rangle \geq 0$ for all $v\in\mathbf{K}_0$. We will always assume our inner product conjugate linear in the first argument. We form, in the usual way, the Hilbert space $\mathbf{K} = (\mathbf{K}_0/\mathbf{N})^\sim$ by dividing out with $\mathbf{N} = \{v\in\mathbf{K}_0 \mid \langle v\mid Jv\rangle = 0\}$ and completing in the norm $\|\cdot\|_J$. (This is of course a variation of the Gelfand-Naimark-Segal (GNS) construction.) With the properties of $(G,\pi,\mathbf{H}(\pi),\mathbf{K})$ as stated, we show, using the Lüscher-Mack theorem, that the simply connected Lie group G^c with Lie algebra $\mathfrak{g}^c = \mathfrak{h}\oplus i\mathfrak{q}$ carries a *unitary* representation π^c on $\mathbf{K}$ such that $\{\pi^c(h\exp(iY)) \mid h\in H, Y\in C^o\}$ is obtained from π by passing the corresponding operators $\pi(h\exp Y)$ to the quotient $\mathbf{K}_0/\mathbf{N}$. To see that this leads to a unitary representation π_c of G^c we use a basic result of Lüscher and Mack [**60**] and in a more general context one of Jorgensen [**41**]. In fact, when $Y\in C$, the selfadjoint operator $d\pi(Y)$ on $\mathbf{K}$ has spectrum contained in $(-\infty,0]$. As in Lemma 4.5, we show that in the case where C extends to an G^c invariant regular cone in $i\mathfrak{g}^c = i\mathfrak{h}\oplus\mathfrak{q}$ and π^c is injective, then each π^c (as a unitary representation of G^c) must be a direct integral of highest-weight representations of G^c. The examples show that one can relax the condition in different ways, that is one can avoid using the Lüscher-Mack theorem by instead constructing local representations and using only cones that are neither generating nor H-invariant.

Assume now that G is a semidirect product of H and N with N normal and abelian. Define $\tau\colon G\to G$ by $\tau(hn) = hn^{-1}$. Let $\pi\in\hat{H}$ (the unitary dual) and extend π to a unitary representation of G by setting $\pi(hn)=\pi(h)$. In this case, G^c is locally isomorphic to G, and π gives rise to a unitary representation π^c of G^c by the formula $d\pi^c(X) = d\pi(X)$, $X\in\mathfrak{h}$, and $d\pi^c|_{i\mathfrak{q}} = 0$. A special case of this is the 3-dimensional Heisenberg group, and the $(ax+b)$-group. In Sections 12 and 13, we show that, if we induce instead a character of the subgroup N to G, then we have $(\mathbf{K}_0/\mathbf{N})^\sim = \{0\}$.

Our approach to the general representation correspondence $\pi \mapsto \pi^c$ is related to the integrability problem for representations of Lie groups (see [**44**]); but the present positivity viewpoint comes from Osterwalder-Schrader positivity; see [**84, 85**]. In addition the following other papers are relevant in this connection: [**15, 41, 42, 50, 91, 95**].

3. Preliminaries

The setting for the paper is a general Lie group G with a nontrivial involutive automorphism τ.

DEFINITION 3.1. A unitary representation π acting on a Hilbert space $\mathbf{H}(\pi)$ is said to be *reflection symmetric* if there is a unitary operator $J\colon \mathbf{H}(\pi) \to \mathbf{H}(\pi)$ such that

R1) $J^2 = \mathrm{id}$;

R2) $J\pi(g) = \pi(\tau(g))J\,, \quad g \in G$.

If (R1) holds, then π and $\pi\circ\tau$ are equivalent. Furthermore, generally from (R2) we have $J^2\pi(g) = \pi(g)J^2$. Thus, if π is irreducible, then we can always renormalize J such that (R1) holds. Let $H = G^\tau = \{g \in G \mid \tau(g) = g\}$ and let $\mathfrak{h}$ be the Lie algebra of H. Then $\mathfrak{h} = \{X \in \mathfrak{g} \mid \tau(X) = X\}$. Define $\mathfrak{q} = \{Y \in \mathfrak{g} \mid \tau(Y) = -Y\}$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$ and $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$.

DEFINITION 3.2. A closed convex cone $C \subset \mathfrak{q}$ is *hyperbolic* if $C^o \neq \varnothing$ and if $\operatorname{ad} X$ is semisimple with real eigenvalues for every $X \in C^o$.

We will assume the following for (G, π, τ, J):

PR1) π is reflection symmetric with reflection J;

PR2) there is an H-invariant hyperbolic cone $C \subset \mathfrak{q}$ such that $S(C) = H \exp C$ is a closed semigroup and $S(C)^o = H \exp C^o$ is diffeomorphic to $H \times C^o$;

PR3) there is a subspace $0 \neq \mathbf{K}_0 \subset \mathbf{H}(\pi)$ invariant under $S(C)$ satisfying the positivity condition

$$\langle v \,|\, v\rangle_J := \langle v \,|\, J(v)\rangle \geq 0, \quad \forall v \in \mathbf{K}_0\,.$$

REMARK 3.3. In (PR3) we can always assume that $\mathbf{K}_0$ is closed, as the invariance and the positivity pass over to the closure. In (PR2) it is only necessary to assume that $\mathbf{K}_0$ is invariant under $\exp C$, as one can always replace $\mathbf{K}_0$ by $\overline{\langle \pi(H)\mathbf{K}_0\rangle}$, the closed space generated by $\pi(H)\mathbf{K}_0$, which is $S(C)$-invariant, as C is H-invariant. For the exact conditions on the cone for (PR2) to hold see the original paper by J. Lawson [**58**], or the monograph [**31**, pp. 194 ff.].

In some of the examples we will replace (PR2)and (PR3) by the following: weaker conditions

PR2′) C is (merely) some nontrivial cone in $\mathfrak{q}$.

PR3′) There is a subspace $0 \neq \mathbf{K}_0 \subset \mathbf{H}(\pi)$ invariant under H and $\exp C$ satisfying the positivity condition from (PR3).

(See Section 12 for further details.)

Since the operators $\{\pi(h) \mid h \in H\}$ commute with J, they clearly pass to the quotient by

$$\mathbf{N} := \{v \in \mathbf{K}_0 \mid \langle v \,|\, Jv \rangle = 0\}$$

and implement unitary operators on $\mathbf{K} := (\mathbf{K}_0/\mathbf{N})^{\sim}$ relative to the inner product induced by

$$\langle u \,|\, v \rangle_J := \langle u \,|\, J(v) \rangle \; . \tag{3.1}$$

which will be denoted by the same symbol. Hence we shall be concerned with passing the operators $\{\pi(\exp Y) \mid Y \in C\}$ to the quotient $\mathbf{K}_0/\mathbf{N}$, and for this we need a basic Lemma.

In general, when $(\mathbf{K}_0, J)$ is given, satisfying the positivity axiom, then the corresponding composite quotient mapping

$$\mathbf{K}_0 \longrightarrow \mathbf{K}_0/\mathbf{N} \hookrightarrow (\mathbf{K}_0/\mathbf{N})^{\sim} =: \mathbf{K}$$

is *contractive* relative to the respective Hilbert norms. The resulting (contractive) mapping will be denoted β. An operator γ on $\mathbf{H}$ which leaves $\mathbf{K}_0$ invariant is said to *induce* the operator $\tilde{\gamma}$ on $\mathbf{K}$ if $\beta \circ \gamma = \tilde{\gamma} \circ \beta$ holds on $\mathbf{K}_0$. In general, an induced operation $\gamma \mapsto \tilde{\gamma}$ may not exist; and, if it does, $\tilde{\gamma}$ may fail to be bounded, even if γ is bounded.

This above-mentioned operator-theoretic formulation of reflection positivity has applications to the Feynman-Kac formula in mathematical physics, and there is a considerable literature on that subject, with work by E. Nelson [**67, 68**], A. Klein and L.J. Landau [**47, 48, 49**], B. Simon, and W.B. Arveson [**1**]. Since we shall not use path space measures here, we will omit those applications, and instead refer the reader to the survey paper [**1**] (lecture 4) by W.B. Arveson. In addition to mathematical physics, our motivation also derives from recent papers on non-commutative harmonic analysis which explore analytic continuation of the underlying representations; see, e.g., [**34, 54, 63, 72, 73, 80**].

4. A Basic Lemma

LEMMA 4.1. 1) *Let J be a period-2 unitary operator on a Hilbert space $\mathbf{H}$, and let $\mathbf{K}_0 \subset \mathbf{H}$ be a closed subspace such that $\langle v \,|\, J(v) \rangle \geq 0$, $v \in \mathbf{K}_0$. Let γ be an invertible operator on $\mathbf{H}$ such that $J\gamma = \gamma^{-1}J$ and which leaves $\mathbf{K}_0$ invariant and has $(\gamma^{-1})^*\gamma$ bounded on $\mathbf{H}$. Then γ induces a bounded operator $\tilde{\gamma}$ on $\mathbf{K} = (\mathbf{K}_0/\mathbf{N})^{\sim}$, where $\mathbf{N} = \{v \in \mathbf{K}_0 \mid \langle v \,|\, Jv \rangle = 0\}$, and the norm of $\tilde{\gamma}$ relative to the J-inner product in $\mathbf{K}$ satisfies*

$$\|\tilde{\gamma}\| \leq \|(\gamma^{-1})^*\gamma\|_{sp}^{1/2} \; , \tag{4.1}$$

where $\|\cdot\|_{sp}$ is the spectral radius.

2) *If we have a semigroup S of operators on $\mathbf{H}$ satisfying the conditions in* (1), *then*

$$(\gamma_1\gamma_2)^{\sim} = \tilde{\gamma}_1\tilde{\gamma}_2 \, , \quad \gamma_1, \gamma_2 \in S \, . \tag{4.2}$$

PROOF. For $v \in \mathbf{K}_0$, $v \neq 0$, we have

$$\begin{aligned}
\|\gamma(v)\|_J^2 &= \langle \gamma(v) \,|\, J\gamma(v) \rangle \\
&= \big\langle \gamma(v) \,\big|\, \gamma^{-1} J(v) \big\rangle \\
&= \big\langle (\gamma^{-1})^* \gamma(v) \,\big|\, J(v) \big\rangle \\
&= \big\langle (\gamma^{-1})^* \gamma(v) \,\big|\, v \big\rangle_J \\
&\leq \|(\gamma^{-1})^* \gamma(v)\|_J \|v\|_J \\
&\leq \|((\gamma^{-1})^* \gamma)^2(v)\|_J^{1/2} \|v\|_J^{1+1/2} \\
&\ \vdots \\
&\leq \|((\gamma^{-1})^* \gamma)^{2^n}(v)\|_J^{1/2^n} \|v\|_J^{1+1/2+\cdots+1/2^n} \\
&\leq \Big(\|((\gamma^{-1})^* \gamma)^{2^n}\| \|v\| \Big)^{1/2^n} \|v\|_J^2 .
\end{aligned}$$

Since $\lim_{n\to\infty} \|((\gamma^{-1})^*\gamma)^{2^n}\|^{1/2^n} = \|(\gamma^{-1})^*\gamma\|_{sp}$, and $\lim_{n\to\infty} \|v\|^{1/2^n} = 1$, the result follows.

By this we get

$$\langle \gamma(v) \,|\, J\gamma(v) \rangle \leq \|(\gamma^{-1})^*\gamma\|_{sp} \, \langle v \,|\, J(v) \rangle$$

which shows that $\gamma(\mathbf{N}) \subset \mathbf{N}$, whence γ passes to a bounded operator on the quotient $\mathbf{K}_0/\mathbf{N}$ and then also on $\mathbf{K}$ satisfying the estimate stated in (1). If both the operators in (4.2) leave $\mathbf{N}$ invariant, so does $\gamma_1\gamma_2$ and the operator induced by $\gamma_1\gamma_2$ is $\tilde{\gamma_1}\tilde{\gamma_2}$ as stated. □

COROLLARY 4.2. *Let the notation be as above and assume that γ is unitary on $\mathbf{H}$. Then the constant on the right in* (4.1) *is one. Hence $\tilde{\gamma}$ is a contraction on $\mathbf{K}$.*

To understand the assumptions on the space $\mathbf{K}_0$, that is positivity and invariance, we include the following which is based on an idea of R.S. Phillips [**89**].

PROPOSITION 4.3. *Let $\mathbf{H}$ be a Hilbert space and let J be a period-2 unitary operator on $\mathbf{H}$. Let S be a commutative semigroup of unitary operators on $\mathbf{H}$ such that $S = S_+S_-$ with $S_+ = \{\gamma \in S \mid J\gamma = \gamma J\}$ and $S_- = \{\gamma \in S \mid J\gamma = \gamma^{-1}J\}$. Then $\mathbf{H}$ possesses a maximal positive and invariant subspace, that is a subspace $\mathbf{K}_0$ such that $\langle v \,|\, J(v) \rangle \geq 0$, $v \in \mathbf{K}_0$, and $\gamma\mathbf{K}_0 \subset \mathbf{K}_0$, $\gamma \in S$.*

PROOF. The basic idea is contained in [**89**, pp. 386 ff.]. □

REMARK 4.4. A nice application is to the case $\mathbf{H} = L^2(X, m)$ where X is a Stone space. There is a m-a.e.-defined automorphism $\theta \colon X \to X$ such that

$$J(f) = f \circ \theta, \qquad f \in L^2(X, m)$$

and S is represented by multiplication operators on $L^2(X, m)$. By [**89**] we know that there are clopen subsets $A, B \subset X$ such that with $M_0 := \{x \in X \mid \theta(x) = x\}$ and $M_1 = X \setminus M_0$ we have $A, B \subset M_1$, $A \cap B = \varnothing$, $A \cup B = M_1$ and $\theta(A) = B$. Let $\mathbf{K}_0 := L^2(M_0 \cup A)$. Then $\mathbf{K}_0$ is a maximal positive and invariant subspace.

LEMMA 4.5. *If $M_0 \subset X$ is of measure zero, then the space $\mathbf{K}$ is trivial, that is $\langle f \,|\, J(f) \rangle = 0$ for all $f \in \mathbf{K}_0$.*

REMARK 4.6. Assume that we have (PR1) and (PR2). By [**55**] there is an abelian subspace $\mathfrak{a} \subset \mathfrak{q}$ such that $C^o = \mathrm{Ad}(H)(C^o \cap \mathfrak{a})$. Let $S_A = \exp(C^o \cap \mathfrak{a})$. Then S_A is an abelian semigroup, so one can use Proposition 4.3 to construct a maximal positive and invariant subspace for S_A. But in general we can not expect this space to be invariant under S.

We read off from the basic Lemma the following Proposition:

PROPOSITION 4.7. *Let π be a unitary representation of the group G. Assume that $(\tau, J, C, \mathbf{K}_0)$ satisfies the conditions* (PR1), (PR2′), *and* (PR3′). *If $Y \in C$, then $\pi(\exp Y)$ induces a contractive selfadjoint operator $\tilde{\pi}(\exp Y)$ on $\mathbf{K}$.*

PROOF. If $Y \in C$, then $\pi(\exp Y)\mathbf{K}_0 \subset \mathbf{K}_0$, and $\pi(\exp Y)$ is unitary on $\mathbf{H}(\pi)$. Thus

$$\begin{aligned} \langle \pi(\exp Y)u \mid J(v)\rangle &= \langle u \mid \pi(\exp(-Y))J(v)\rangle \\ &= \langle u \mid J(\pi(\exp Y)v)\rangle \ , \end{aligned}$$

proving that $\pi(\exp Y)$ is selfadjoint in the J-inner product. Since $\pi(\exp Y)$ is unitary on $\mathbf{H}(\pi)$

$$\|\pi(\exp Y)\| = \|\pi(\exp Y)\|_{sp} = 1\,,$$

and the contractivity property follows. □

LEMMA 4.8. *Let π be a unitary representation of G such that $(\tau, J, C, \mathbf{K}_0)$ satisfies the conditions* (PR1), (PR2′), *and* (PR3′). *Then for $Y \in C$ there is a selfadjoint operator $d\tilde{\pi}(Y)$ in $\mathbf{K} = (\mathbf{K}_0/\mathbf{N})^{\sim}$, with spectrum contained in $(-\infty, 0]$, such that*

$$\tilde{\pi}(\exp(tY)) = e^{t\, d\tilde{\pi}(Y)}, \quad t \in \mathbb{R}_+$$

is a contractive semigroup on $\mathbf{K}$. Furthermore the following hold:

1) $t \mapsto e^{t\, d\tilde{\pi}(Y)}$ *extends to a continuous map $z \mapsto e^{z\, d\tilde{\pi}(Y)}$ on $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0\}$ holomorphic on the open right half-plane, and such that $e^{(z+w)\, d\tilde{\pi}(Y)} = e^{z\, d\tilde{\pi}(Y)} e^{w\, d\tilde{\pi}(Y)}$.*
2) *There exists a one-parameter group of unitary operators*

$$\tilde{\pi}\left(\exp(itY)\right) := e^{it\, d\tilde{\pi}(Y)}, \quad t \in \mathbb{R}$$

on $\mathbf{K}$.

PROOF. The last statement follows by the spectral theorem. By construction $\{\tilde{\pi}(\exp(tY)) \mid t \in \mathbb{R}_+\}$ is a semigroup of selfadjoint contractive operators on $\mathbf{K}$. The existence of the operators $d\tilde{\pi}(Y)$ as stated then follows from a general result in operator theory; see, e.g., [**14**] or [**49**]. □

COROLLARY 4.9. *Let the situation be as in the last corollary. If $Y \in C \cap -C$ then $e^{t\, d\tilde{\pi}(Y)} = \mathrm{id}$ for all $t \in \mathbb{R}_+$. In particular $d\tilde{\pi}(Y) = 0$ for every $Y \in C \cap -C$.*

PROOF. This follows as the spectrum of $d\tilde{\pi}(Y)$ and $d\tilde{\pi}(-Y)$ is contained in $(-\infty, 0]$. □

We remark here that we have introduced the map $d\pi$ without using the space of *smooth vectors* for the representation π. Let us recall that a vector $\mathbf{v} \in \mathbf{H}(\pi)$ is called *smooth* if the map

$$\mathbb{R} \ni t \longmapsto \hat{\mathbf{v}}(t) := \pi(\exp tX)\mathbf{v} \in \mathbf{H}(\pi)$$

is smooth for all $X \in \mathfrak{g}$. The vector is *analytic* if the above map is analytic. We denote by $\mathbf{H}^\infty(\pi)$ the space of smooth vectors and by $\mathbf{H}^\omega(\pi)$ the space of analytic vectors. Both $\mathbf{H}^\omega(\pi)$ and $\mathbf{H}^\infty(\pi)$ are G-invariant dense subspaces of $\mathbf{H}(\pi)$. We define a representation of $\mathfrak{g}$ on $\mathbf{H}^\infty(\pi)$ by

$$d\pi(X)\mathbf{v} = \lim_{t\to 0} \frac{\pi(\exp tX)\mathbf{v} - \mathbf{v}}{t}\,.$$

Recall that if π is infinite-dimensional, then $d\pi$ is a representation of $\mathfrak{g}$ by unbounded operator on $\mathbf{H}(\pi)$, but the analytic vectors and the C^∞-vectors form dense domains for $d\pi$; see [**66, 90, 103**].

The operator $d\tilde{\pi}(X)$ in the above statements is an extension of the operator $d\tilde{\pi}(X)$ on the space of smooth vectors. This allows us to use the same notation for those two objects. We extend this representation to $\mathfrak{g}_\mathbb{C}$ by complex linearity, $d\pi(X+iY) = d\pi(X) + i\, d\pi(Y)$, $X, Y \in \mathfrak{g}$. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. Then $d\pi$ extends to a representation on $U(\mathfrak{g})$, again denoted by $d\pi$. The space $\mathbf{H}^\infty(\pi)$ is a topological vector space in a natural way, cf. [**103**]. Furthermore $\mathbf{H}^\infty(\pi)$ is invariant under G and $U(\mathfrak{g})$. As $\pi(g)\pi(\exp(tX))\mathbf{v} = \pi(\exp(t\,\mathrm{Ad}(g)X))\pi(g)\mathbf{v}$, we get

$$\pi(g)\, d\pi(X)\mathbf{v} = d\pi(\mathrm{Ad}(g)X)\pi(g)\mathbf{v}, \qquad \mathbf{v} \in \mathbf{H}^\infty(\pi)$$

for all $g \in G$ and all $X \in \mathfrak{g}$. Define $Z^* = -\sigma(Z)$, $Z \in \mathfrak{g}_\mathbb{C}$, where σ is the conjugation $X + iY \mapsto X - iY$, $X, Y \in \mathfrak{g}$. Then a simple calculation shows that for the densely defined operator $\pi(Z)$, $Z \in \mathfrak{g}_\mathbb{C}$, we have $\pi(Z)^* = \pi(Z^*)$ on $\mathbf{H}^\infty(\pi)$.

When (R1–2) and (PR1–3) hold, and $Y \in C$, we showed in Lemma 4.8 that the operator $\tilde{\pi}\,(Y)$ is selfadjoint in $\mathbf{K} = (\mathbf{K}_0/\mathbf{N})^\sim$ with spectrum in $[0,\infty)$. Once $\tilde{\pi}$ is identified as a unitary representation of G^c, then $\tilde{\pi}\,(iY)$ is automatically a selfadjoint operator in $\mathbf{K}$ by [**69**], but semiboundedness of the corresponding spectrum of $\tilde{\pi}\,(Y)$ only holds for $Y \in C$. Yet if $\tilde{\pi}$ is obtained, as in Lemma 4.8, from a unitary representation π of G acting on $\mathbf{H}$, then the spectrum of $\pi\,(Y)$ is contained in the purely imaginary axis $i\mathbb{R}$, and yet $\tilde{\pi}\,(Y)$ has spectrum in $[0,\infty) \subset \mathbb{R}$. The explanation is that the Hilbert spaces $\mathbf{H}$ and $\mathbf{K}$ are different for the two representations π and $\tilde{\pi}$.

5. Holomorphic Representations

The unitary representations that show up in the duality are direct integrals of highest weight G^c-modules. Those representations can also be viewed as *holomorphic representations* of a semigroup related to an extension of the H-invariant cone C that we started with. We will therefore give a short overview over this theory, while referring to the forthcoming monograph [**65**] for more details.

The theory of highest weight modules and holomorphic representations will always be related to the name of Harish-Chandra because of his fundamental work on bounded symmetric domains and the holomorphic discrete series, [**21, 22, 23**]. Later Gelfand and Gindikin in [**16**] proposed a new approach for studying the Plancherel formula for semisimple Lie group G. Their idea was to consider functions in $\mathbf{L}^2(G)$ as the sum of boundary values of holomorphic functions defined on domains in $G_\mathbb{C}$. The first deep results in this direction are due to Ol'shanskii [**80**] and Stanton [**98**], who realized the holomorphic discrete series of the group G in a Hardy space of a local tube domain containing G in the boundary. The generalization to noncompactly causal symmetric spaces was carried out in [**33, 34, 75, 76**].

This program was carried out for solvable groups in [**32**] and for general groups in [**53, 63**].

Let $G_{\mathbb{C}}$ be a complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and let $\mathfrak{g}$ be a real form of $\mathfrak{g}_{\mathbb{C}}$. We assume for simplicity that G, the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$, is closed in $G_{\mathbb{C}}$. Let C be a regular G-invariant cone in $\mathfrak{g}$ such that the set $S(C) = G \exp iC$ is a closed semigroup in $G_{\mathbb{C}}$. Moreover, we assume that the map

$$G \times C \ni (a, X) \longmapsto a \exp iX \in S(C)$$

is a homeomorphism and even a diffeomorphism when restricted to $G \times C^o$. Finally, we assume that there exists a real automorphism σ of $G_{\mathbb{C}}$ whose differential is the complex conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$, that is $\sigma(X + iY) = X - iY$ for $X, Y \in \mathfrak{g}$. We notice that in this case $G_{\mathbb{C}}/G$ is a symmetric space and that the corresponding subspace $\mathfrak{q}$ is just $i\mathfrak{g}$. Those hypotheses are also satisfied for Hermitian Lie groups and also for some solvable Lie groups; cf. [**32**]. Define

$$W(\pi) := \{X \in \mathfrak{g} \mid \forall u \in \mathbf{H}^\infty(\pi) \,:\, \langle i\, d\pi(X)u \,|\, u\rangle \leq 0\}\,. \tag{5.1}$$

Thus $W(\pi)$ is the set of elements of $\mathfrak{g}$ for which $d\pi(iX)$ is *negative*. The elements of $W(\pi)$ are called *negative elements* for the representation π.

LEMMA 5.1. *$W(\pi)$ is a closed G-invariant convex cone in $\mathfrak{g}$.*

DEFINITION 5.2. Let W be a G-invariant cone in $\mathfrak{g}$. We denote the set of all unitary representations π of G with $W \subset W(\pi)$ by $\mathcal{A}(W)$. A unitary representation π is called *W-admissible* if $\pi \in \mathcal{A}(W)$.

The representations in $\mathcal{A}(W)$ will be studied in detail in Section 8 below. We show in Theorem 8.4 that a $\rho \in \mathcal{A}(W)$ which is irreducible is in fact a *highest weight representation*, and the corresponding K^c-weights are determined. The representations are then identified as discrete summands in $\mathbf{L}^2\,(G^c)$.

Let S be a semigroup with unit and let $\sharp\colon S \to S$ be a bijective involutive antihomomorphism, that is

$$(ab)^\sharp = b^\sharp a^\sharp \qquad \text{and} \qquad a^{\sharp\sharp} = a$$

We call $\sharp$ an *involution* on the semigroup S, and we call the pair $(S, \sharp)$ a *semigroup with involution* or an *involutive semigroup*. For us the important examples are the semigroups of the form $S(C) = H \exp C$ with $\gamma^\sharp = \tau(\gamma^{-1})$. Another class of examples consists of the *contractive semigroups* on a Hilbert space $\mathbf{H}$. Let $S(\mathbf{H}) = \{T \in B(\mathbf{H}) \mid \|T\| \leq 1\}$. Denote by T^* the adjoint of T with respect to the inner product on $\mathbf{H}$. Then $(S, *)$ is a semigroup with involution.

DEFINITION 5.3. Let $(S, \sharp)$ be a topological semigroup with involution: then a semigroup homomorphism $\rho\colon S \to S(\mathbf{H})$ is called a contractive representation of $(S, \sharp)$ if $\rho(g^\sharp) = \rho(g)^*$ and ρ is continuous with respect to the weak operator topology of $S(\mathbf{H})$. A contractive representation is called irreducible if there is no closed nontrivial subspace of $\mathbf{H}$ invariant under $\rho(S)$.

DEFINITION 5.4. Let ρ be a contractive representation of the semigroup $S(W) = G \exp iW \subset G_{\mathbb{C}}$. Then ρ is holomorphic if the function $\rho\colon S(C)^o \to B(\mathbf{V})$ is holomorphic.

The following lemma shows that, if a unitary representation of the group G extends to a holomorphic representation of $S(C)$, then this extension is unique.

LEMMA 5.5. *If $f\colon S(W) \to S(\mathbf{H})$ is continuous and $f|_{S(W)^o}$ is holomorphic such that $f|_G = 0$, then $f = 0$.*

To construct a holomorphic extension ρ of a representation π we have to assume that $\pi \in \mathcal{A}(W)$. Then for any $X \in W$, the operator $i\, d\pi(X)$ generates a self adjoint contraction semigroup which we denote by

$$T_X(t) = e^{ti\, d\pi(X)}\,.$$

For $s = g \exp iX \in S(C)$ we define

$$\rho(s) := \rho(g) T_X(1) \tag{5.2}$$

THEOREM 5.6. *ρ is a contractive and holomorphic representation of the semigroup $S(W)$. In particular, every representation $\pi \in \mathcal{A}(W)$ extends uniquely to a holomorphic representation of $S(W)$ which is uniquely determined by π.*

We will usually denote the holomorphic extension of the representation π by the same letter. For the converse of Theorem 5.6, we remark the following simple fact: Let $(S, \sharp)$ be a semigroup with involution and let ρ be a contractive representation of S. Let

$$G(S) := \{s \in S \mid s^\sharp s = s s^\sharp = 1\}$$

Then $G(S)$ is a closed subgroup of S and $\pi := \rho|_{G(S)}$ is a unitary representation of $G(S)$. Obviously,

$$G \subset G(S(W))\,.$$

Thus every holomorphic representation of $S(W)$ defines a unique unitary representation of G by restriction.

THEOREM 5.7. *Let ρ be a holomorphic representation of $S(W)$. Then $\rho|_G \in \mathcal{A}(W)$ and the ρ agrees with the extension of $\rho|_G$ to $S(W)$.*

Two representations ρ and π of the semigroup $S(W)$ are said to (be unitarily) equivalent if there exists a unitary isomorphism $U\colon \mathbf{H}(\rho) \to \mathbf{H}(\pi)$ such that

$$U\rho(s) = \pi(s)U \quad \forall s \in S(W)$$

In particular, two contractive representations ρ and π of $S(W)$ are equivalent if and only if $\rho|_G$ and $\pi|_G$ are unitarily equivalent. We call a holomorphic contractive representation ρ of $S(W)$ *W-admissible* if $\rho|_G \in \mathcal{A}(W)$ and write $\rho \in \mathcal{A}(W)$.

We denote by $\widehat{S(W)}$ the set of equivalence classes of irreducible holomorphic representations of $S(W)$.

We say that a representation ρ is bounded if $\|\rho(s)\| \leq 1$ for all $s \in S(W)$. Note that this depends only on the unitary equivalence class of ρ. We denote by $\widehat{S(W)}_b$ the subset in $\widehat{S(W)}$ of bounded representations. Let ρ and π be holomorphic representations of $S(W)$. Define a representation of $S(W)$ in $\mathbf{H}(\rho)\hat{\otimes}\mathbf{H}(\pi)$ by

$$[\rho \otimes \pi](s) := \rho(s) \otimes \pi(s)$$

Then $\rho \otimes \pi \in \mathcal{A}(W)$. We denote the representation $s \mapsto \mathrm{id}$ by ι.

THEOREM 5.8 (Neeb, Ol'shanskii [**33, 63, 65**]). *Let ρ be a holomorphic representation of $S(W)_b$. Then there exists a Borel measure μ on $\widehat{S(W)}$ supported on*

$\widehat{S(W)}_b$ *and a direct integral of representations*

$$\left(\int_{\widehat{S(W)}_b}^{\oplus} \rho_\omega \, d\mu(\omega), \int_{\widehat{S(W)}_b}^{\oplus} \mathbf{H}(\omega) \, d\mu(\omega)\right)$$

such that:

(i) *The representation* ρ *is equivalent to* $\int_{\widehat{S(W)}_b}^{\oplus} \rho_\omega \, d\mu(\omega)$.

(ii) *There exists a subset* $N \subset \widehat{S(W)}_b$ *such that* $\mu(N) = 0$ *and if* $\omega \in \widehat{S(W)}_b \setminus N$, *then* ρ_ω *is equivalent to* $(\pi_\omega \otimes \iota, \mathbf{H}(\omega)\hat{\otimes}\mathbf{L}(\omega))$, *where* $\pi_\omega \in \omega$ *and* $\mathbf{L}(\omega)$ *is a Hilbert space.*

(iii) *If* $\omega \in \widehat{S(W)}_b$ *then set* $n(\omega) := \dim \mathbf{L}(\omega)$. *Then* n *is a* μ*-measurable function from* $\widehat{S(W)}_b$ *to the extended positive axis* $[0, \infty]$ *which is called the multiplicity function.*

PROOF. See [**65**], Theorem XI.6.13. ☐

6. The Lüscher-Mack Theorem

We use reference [**31**] for the Lüscher-Mack Theorem, but [**15**], [**18**], [**41**], [**42**], [**44**], [**50**], [**60**], and [**95**] should also be mentioned in this connection. We have two ways of making the connection between the unitary representations of G and those of G^c: one is based on the Lüscher-Mack principle, and the other on the notion of local representations from Jorgensen's papers [**41**] and [**42**].

Let π, C, $\mathbf{H}(\pi)$, J and $\mathbf{K}_0$ be as before. We have proved that the operators

$$\{\pi(h \exp(Y)) \mid h \in H, Y \in C\}$$

pass to the space $\mathbf{K} = (\mathbf{K}_0/\mathbf{N})^{\sim}$ such that $\tilde{\pi}(h)$ is unitary on $\mathbf{K}$, and $\tilde{\pi}(\exp Y)$ is contractive and selfadjoint on $\mathbf{K}$. As a result we arrive at selfadjoint operators $d\tilde{\pi}(Y)$ with spectrum in $(-\infty, 0]$ such that for $Y \in C$, $\tilde{\pi}(\exp Y) = e^{d\tilde{\pi}(Y)}$ on $\mathbf{K}$. As a consequence of that we notice that

$$t \longmapsto e^{t\, d\tilde{\pi}(Y)}$$

extends to a continuous map on $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0\}$ holomorphic on the open right half plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. Furthermore,

$$e^{(z+w)\, d\tilde{\pi}(Y)} = e^{z\, d\tilde{\pi}(Y)} e^{w\, d\tilde{\pi}(Y)} .$$

As $\mathbf{K}$ is a unitary H-module we know that the H-analytic vectors $\mathbf{K}^\omega(H)$ are dense in $\mathbf{K}$. Thus $\mathbf{K}_{oo} := S(C^o)\mathbf{K}^\omega(H)$ is dense in $\mathbf{K}$. We notice that for $u \in \mathbf{K}_{oo}$ and $X \in C^o$ the function $t \mapsto \tilde{\pi}(\exp tX)u$ extends to a holomorphic function on an open neighborhood of the right half-plane. This and the Campbell-Hausdorff formula are among the main tools used in proving the following Theorem of Lüscher and Mack [**60**]. We refer to [**31**, p. 292] for the proof. Our present use of Lie theory, cones, and semigroups will follow standard conventions (see, e.g., [**11, 26, 58, 103, 106**]): the exponential mapping from the Lie algebra $\mathfrak{g}$ to G is denoted exp, the adjoint representation of $\mathfrak{g}$, ad, and that of G is denoted Ad.

THEOREM 6.1 (Lüscher-Mack [**60**]). *Let* ρ *be a strongly continuous contractive representation of* $S(C)$ *on the Hilbert space* $\mathbf{H}$ *such that* $\rho(s)^* = \rho(\tau(s)^{-1})$. *Let* G^c *be the connected, simply connected Lie group with Lie algebra* $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}$. *Then there exists a continuous unitary representation* $\rho^c \colon G^c \to \mathrm{U}(\mathbf{H})$, *extending* ρ, *such that for the differentiated representations* $d\rho$ *and* $d\rho^c$ *we have:*

1) $d\rho^c(X) = d\rho(X) \quad \forall X \in \mathfrak{h}$.
2) $d\rho^c(iY) = i\, d\rho(Y) \quad \forall Y \in C$.

We apply this to our situation to get the following theorem:

THEOREM 6.2. *Assume that* $(\pi, C, \mathbf{H}, J)$ *satisfies* (PR1)–(PR3). *Then the following hold:*

1) *$S(C)$ acts via $s \mapsto \tilde{\pi}(s)$ by contractions on $\mathbf{K}$.*
2) *Let G^c be the simply connected Lie group with Lie algebra $\mathfrak{g}^c$. Then there exists a unitary representation $\tilde{\pi}^c$ of G^c such that $d\tilde{\pi}^c(X) = d\tilde{\pi}(X)$ for $X \in \mathfrak{h}$ and $i\, d\tilde{\pi}^c(Y) = d\tilde{\pi}(iY)$ for $Y \in C$.*
3) *The representation $\tilde{\pi}^c$ is irreducible if and only if $\tilde{\pi}$ is irreducible.*

PROOF. (1) and (2) follow by the Lüscher-Mack theorem and Proposition 4.7, as the resulting representation of S is obviously continuous.

(3) Let $\mathbf{L}$ be a G^c-invariant subspace in $\mathbf{K}$. Then $\mathbf{L}$ is $\tilde{\pi}(H)$ invariant. Let $Y \in C^o$, $u \in \mathbf{L}^\omega$ and $v \in \mathbf{L}^\perp$. Define $f\colon \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0\} \to \mathbb{C}$ by

$$f(z) := \left\langle v \,\middle|\, e^{z d\tilde{\pi}(Y)} u \right\rangle_J .$$

Then f is holomorphic in $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$, and $f(it) = 0$ for every (real) t. Thus f is identically zero. In particular $f(t) = 0$ for every $t > 0$. Thus

$$0 = \left\langle v \,\middle|\, e^{t d\tilde{\pi}(Y)} u \right\rangle_J = \langle v \mid \tilde{\pi}(\exp tY) u \rangle_J .$$

As $S^o = H \exp C^o$ it follows that $\tilde{\pi}(S^o)(\mathbf{L}^\omega) \subset (\mathbf{L}^\perp)^\perp = \mathbf{L}$. By continuity we get $\tilde{\pi}(S)\mathbf{L} \subset \mathbf{L}$. Thus $\mathbf{K}$ is reducible as an S-module.

The other direction follows in exactly the same way. $\square$

We notice now that $-iC \subset W(\tilde{\pi}^c)$. Thus $W(\tilde{\pi}^c)$ is non-trivial and contain the $-\tau$-stable and G-invariant cone generated by $-iC$, i.e. $-i \cdot \overline{\mathrm{conv}\{\mathrm{Ad}(G)C\}} \subset W(\tilde{\pi}^c)$. But in general $W(\tilde{\pi}^c)$ is neither generating nor pointed. It even does not have to be $-\tau$-invariant. In fact, the Lie algebra of the $(ax+b)$-group, and the Heisenberg group, do not have *any* pointed, generating, invariant cones.

LEMMA 6.3. $W(\tilde{\pi}^c) \cap -W(\tilde{\pi}^c) = \ker(\tilde{\pi}^c)$.

PROOF. This is obvious from the spectral theorem. $\square$

LEMMA 6.4. *$\mathfrak{g}_1^c := W(\tilde{\pi}^c) - W(\tilde{\pi}^c)$ is an ideal in $\mathfrak{g}^c$. Furthermore, $[\mathfrak{q}, \mathfrak{q}] \oplus i\mathfrak{q} \subset \mathfrak{g}_1^c$.*

PROOF. Let $X \in \mathfrak{g}^c$. Then, as $W(\tilde{\pi}^c)$ is invariant by construction, we conclude that

$$e^{t\,\mathrm{ad}(X)}\left(W(\tilde{\pi}^c) - W(\tilde{\pi}^c)\right) \subset W(\tilde{\pi}^c) - W(\tilde{\pi}^c),\ t \in \mathbb{R}.$$

By differentiation at $t = 0$, it follows that $[X, \mathfrak{g}_1^c] \subset \mathfrak{g}_1^c$. This shows that $\mathfrak{g}_1^c$ is an ideal in $\mathfrak{g}^c$. The last part follows as C is generating (in $\mathfrak{q}$). $\square$

It is not clear if $\mathfrak{g}_1^c$ is τ-stable. To get a τ-stable subalgebra one can replace $W(\tilde{\pi}^c)$ by the cone generated by $-\mathrm{Ad}(G)C \subset W(\tilde{\pi}^c)$ or by the maximal G- and $-\tau$-stable cone $W(\tilde{\pi}^c) \cap (-\tau(W(\tilde{\pi}^c)))$ in $W(\pi^c)$.

We have now the following important consequence of the Neeb-Ol'shanskii theorem:

THEOREM 6.5. *Let the analytic subgroup G_1^c of G^c corresponding to $\mathfrak{g}_1^c$ be as described, and let $W(\tilde{\pi}^c)$ be the corresponding module. Then $\tilde{\pi}^c|_{G_1^c}$ is a direct integral of irreducible representations in $\mathcal{A}(W)$.*

7. Bounded Symmetric Domains

We have seen that the representations π^c that we can produce using the duality are direct integrals of holomorphic representations of suitable subsemigroups of $G_{\mathbb{C}}^c$ (or a subgroup). Those on the other hand only exist if there is a G^c invariant cone in $\mathfrak{g}$. We will discuss the case of simple Lie group G^c in some detail here. We refer to [**46**], chapter 2, and the references therein for proofs. In the duality $G \leftrightarrow G^c$ it will be the group G^c that has holomorphic representations. Therefore we will start using the notation G^c for Hermitian groups.

THEOREM 7.1 (Kostant). *Suppose that $\mathbf{V}$ is a finite-dimensional real vector space. Let L be a connected reductive subgroup of $GL(\mathbf{V})$ acting irreducibly on $\mathbf{V}$. Let $G^c = L'$ be the commutator subgroup of L. Further let K^c be a maximal compact subgroup of G^c. Then the following properties are equivalent:*

(i) *There exists a regular L-invariant closed cone in $\mathbf{V}$.*
(ii) *The G^c-module $\mathbf{V}$ is spherical.*

Let $C \subset \mathbf{V}$ be a regular L-invariant cone. Then $\mathbf{V}$ is spherical as a G^c-module. Let K^c be a maximal compact subgroup of G^c. A K^c-invariant vector u_{K^c} can be constructed in the following way: Let $u \in C^o$, the interior of C, be arbitrary. Define

$$u_{K^c} = \int_{K^c} k \cdot u \, dk \, .$$

Then $u_{K^c} \in C^o$ is K^c-invariant. Suppose that the group L acts on $\mathbf{V}$. Let $\mathrm{Cone}_L(\mathbf{V})$ denote the set of regular L-invariant cones in $\mathbf{V}$.

THEOREM 7.2 (Vinberg). *Let L, G^c, and $\mathbf{V}$ be as in the theorem of Kostant. Then the following properties are equivalent:*

(i) $\mathrm{Cone}_L(\mathbf{V}) \neq \varnothing$*;*
(ii) *The G^c-module $\mathbf{V}$ is spherical;*
(iii) *There exists a ray in $\mathbf{V}$ through 0 which is invariant with respect to some minimal parabolic subgroup P of G^c.*

If those conditions hold, every invariant pointed cone in $\mathbf{V}$ is regular.

For the next theorem, see [**86, 87**].

THEOREM 7.3 (Paneitz, Vinberg). *Let G^c be a connected semisimple Lie group and $(\mathbf{V}, \pi)$ a real finite-dimensional irreducible G^c-module such that $\mathrm{Cone}_{G^c}(\mathbf{V}) \neq \varnothing$. Let θ be a Cartan involution on G^c. Choose an inner product on $\mathbf{V}$ such that $\pi(x)^* = \pi(\theta(x)^{-1})$ for all $x \in G^c$. Then there exists a unique up to multiplication by (-1) invariant cone $C_{\min} \in \mathrm{Cone}_{G^c}(\mathbf{V})$ given by*

$$C_{\min} = \mathrm{conv}(\pi(G^c)u) \cup \{0\} = \overline{\mathrm{conv}\{\pi(G^c)(\mathbb{R}^+ v_{K^c})\}} \, ,$$

where u is a highest weight vector, v_{K^c} is a nonzero K^c-fixed vector unique up to scalar multiple, and $(u, v_{K^c}) > 0$. The unique (up to multiplication by (-1)) maximal cone is given by

$$C_{\max} = C_{\min}^* := \{w \in \mathbf{V} \mid \forall v \in C_{\min} \, : \, (w, v) \geq 0\} \, .$$

Assume now that G^c is a connected simple Lie group. Then G^c acts on $\mathfrak{g}^c$ by the adjoint action. Let $K^c \subset G^c$ be a maximal almost compact subgroup. Then by Kostant's Theorem we have $\mathrm{Cone}_{G^c}(\mathfrak{g}^c) \neq \varnothing$ if and only if there exists a $Z^0 \in \mathfrak{g}^c$ which is invariant under $\mathrm{Ad}(K^c)$. Let $\mathfrak{k}^c$ be the Lie algebra of K^c. Then $[\mathfrak{k}^c, Z^0] = 0$. Hence $\mathbb{R}Z^0 + \mathfrak{k}^c \neq \mathfrak{g}^c$ is a Lie algebra containing $\mathfrak{k}^c$. But $\mathfrak{k}^c$ is maximal in $\mathfrak{g}^c$. Hence $Z^0 \in \mathfrak{k}^c$. Similarly it follows that $\mathfrak{z}_{\mathfrak{g}^c}(Z^0) = \mathfrak{k}^c$. Finally the Theorem of Paneitz and Vinberg implies that $\mathrm{Cone}_{G^c}(\mathfrak{g}^c) \neq \varnothing$ if and only if the center of $\mathfrak{k}^c$ is one dimensional. In that case we can normalize the element Z^0 such that $\mathrm{ad}(Z^0)$ has eigenvalues $0, i, -i$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}^c$ containing Z^0. Then $\mathfrak{k}^c \subset \mathfrak{z}_{\mathfrak{g}^c}(Z^0) \subset \mathfrak{k}^c$. Hence $\mathfrak{t}$ is contained in $\mathfrak{k}^c$. For $\alpha \in \mathfrak{t}_{\mathbb{C}}^*$ let

$$\mathfrak{g}^c_{\mathbb{C}\alpha} := \{X \in \mathfrak{g}^c_{\mathbb{C}} \mid \forall Z \in \mathfrak{t}_{\mathbb{C}} \,:\, [Z, X] = \alpha(Z)X\} \,.$$

It is well known that $\dim \mathfrak{g}^c_{\mathbb{C}\alpha} \leq 1$ for all $\alpha \neq 0$ and

$$\mathfrak{g}^c_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^c_{\mathbb{C}\alpha}$$

where $\Delta = \{\alpha \in \mathfrak{t}^*_{\mathbb{C}} \setminus \{0\} \mid \mathfrak{g}^c_{\mathbb{C}\alpha} \neq \{0\}\}$. We notice that $\alpha(\mathfrak{t}) \subset i\mathbb{R}$ for all $\alpha \in \Delta$ as $\mathfrak{t} \subset \mathfrak{k}$ and $ad(X)$ is skew-symmetric for all $X \in \mathfrak{k}$. Let $\theta \colon \mathfrak{g}^c \to \mathfrak{g}^c$ be the Cartan involution corresponding to $\mathfrak{k}^c$. We do denote the corresponding involution $\theta \otimes 1$ on $\mathfrak{g}^c_{\mathbb{C}}$ and the integrated involution on G^c by the same letter. Then $\mathfrak{g}^c = \mathfrak{k}^c \oplus \mathfrak{p}^c$ where $\mathfrak{p}^c = \{X \in \mathfrak{g}^c \mid \theta(X) = -X\}$. Thus $\mathfrak{g}^c_{\mathbb{C}} = \mathfrak{k}^c_{\mathbb{C}} \oplus \mathfrak{p}^c_{\mathbb{C}}$. Let $\Delta_c = \{\alpha \in \Delta \mid \alpha(Z^0) = 0\}$ and $\Delta_p = \{\alpha \in \Delta \mid \alpha(Z^0) = \pm i\}$. As $\mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(Z^0) = \mathfrak{k}_{\mathbb{C}}$ we get

$$\Delta_c = \{\alpha \mid \mathfrak{g}^c_{\mathbb{C}\alpha} \subset \mathfrak{k}^c_{\mathbb{C}}\} \qquad \text{and} \qquad \Delta_p = \{\alpha \mid \mathfrak{g}^c_{\mathbb{C}\alpha} \subset \mathfrak{p}^c_{\mathbb{C}}\} \,.$$

Choose a positive system Δ^+ in Δ such that $\Delta_p^+ := \{\alpha \in \Delta_p \mid \alpha(Z^0) = i\} \subset \Delta^+$. Then $\Delta^+ = \Delta_c^+ \cup \Delta_p^+$ and Δ_c^+ is a positive system in Δ_c. For $\Gamma \subset \Delta$ let $\mathfrak{g}^c_{\mathbb{C}}(\Gamma) := \bigoplus_{\alpha \in \Gamma} \mathfrak{g}^c_{\mathbb{C}\alpha}$. Then

$$\mathfrak{p}^+ := \{X \in \mathfrak{g}^c_{\mathbb{C}} \mid [Z^0, X] = iX\} = \mathfrak{g}^c_{\mathbb{C}}(\Delta_p^+)\,;$$
$$\mathfrak{p}^- := \{X \in \mathfrak{g}^c_{\mathbb{C}} \mid [Z^0, X] = -iX\} = \mathfrak{g}^c_{\mathbb{C}}(-\Delta_p^+)\,.$$

Furthermore $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are abelian subalgebras with $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$. Let $P^\pm := \exp(\mathfrak{p}^\pm)$ and $K^c_{\mathbb{C}} = \exp(\mathfrak{k}^c_{\mathbb{C}})$. Both P^+ and P^- are simply connected closed abelian subgroups of $G^c_{\mathbb{C}}$. Hence $\exp \colon \mathfrak{p}^\pm \to P^\pm$ is a diffeomorphism. Let

$$\zeta = (\exp|_{\mathfrak{p}^+})^{-1} \colon P^+ \longrightarrow \mathfrak{p}^+ \,. \tag{7.1}$$

The set $P^+K^c_{\mathbb{C}}P^-$ is open and dense in $G^c_{\mathbb{C}}$, $G^c \subset P^+K^c_{\mathbb{C}}P^-$, $G^cK^c_{\mathbb{C}}P^-$ is open in $G^c_{\mathbb{C}}$, and $G^c \cap K^c_{\mathbb{C}}P^- = K^c$. Thus G^c/K^c is holomorphically equivalent to an open submanifold D of the complex flag manifold $X_{\mathbb{C}} = G^c_{\mathbb{C}}/K^c_{\mathbb{C}}P^-$. Furthermore the map $pK^c_{\mathbb{C}}P^- \mapsto \zeta(p)$ induces a biholomorphic map—also denoted by ζ—of G^c/K^c onto a bounded symmetric domain $\Omega_{\mathbb{C}} \subset \mathfrak{p}^- \simeq \mathbb{C}^{\dim(G^c/K^c)}$.

For $x \in P^+K^c_{\mathbb{C}}P^-$ we can write in a unique way

$$x = p(x)k_{\mathbb{C}}(x)q(x) \tag{7.2}$$

with $p(x) \in P^+$, $k_{\mathbb{C}}(x) \in K^c_{\mathbb{C}}$ and $q(x) \in P^-$. For $g \in G^c_{\mathbb{C}}$ and $Z \in \mathfrak{p}^+$ we introduce the following notations when ever they make sense:

$$g \cdot Z = \zeta(p(g\exp(Z)) \in \mathfrak{p}^+$$
$$j(g, Z) = k_{\mathbb{C}}(g \exp Z) \in K^c_{\mathbb{C}} \,.$$

If $Z \in \Omega_{\mathbb{C}}$ and $g \in G^c$ then $g \cdot Z$ is defined and $g \cdot Z \in \Omega_{\mathbb{C}}$. Furthermore $(g, Z) \mapsto g \cdot Z$ defines an action of G^c on $\Omega_{\mathbb{C}}$ such that $\zeta \colon G^c/K^c \to \Omega_{\mathbb{C}}$ is a G^c-map. The map j is *the universal automorphic factor* and it satisfies the following:

$$\begin{aligned} j(k, Z) &= k, \\ j(p, Z) &= 1, \\ j(ab, Z) &= j(a, b \cdot Z) j(b, Z), \end{aligned} \tag{7.3}$$

if $k \in K^c_{\mathbb{C}}$, $Z \in \mathfrak{p}^+$, $p \in P^+$, and $a, b \in G^c_{\mathbb{C}}$ are such that the expressions above are defined.

Define

$$S(\Omega_{\mathbb{C}}) := \left\{ \gamma \in G^c_{\mathbb{C}} \mid \gamma^{-1} \cdot \Omega_{\mathbb{C}} \subset \Omega_{\mathbb{C}} \right\}$$

and

$$S(\Omega_{\mathbb{C}})^o := \left\{ \gamma \in G^c_{\mathbb{C}} \mid \gamma^{-1} \cdot \overline{\Omega_{\mathbb{C}}} \subset \Omega_{\mathbb{C}} \right\}$$

where $\overline{\{\Omega_{\mathbb{C}}\}}$ stands for the topological closure of $\Omega_{\mathbb{C}}$ in $\mathfrak{p}^+$. Then $S(\Omega_{\mathbb{C}})$ is a closed semigroup in $\mathfrak{g}^c_{\mathbb{C}}$ of the form

$$S(\Omega_{\mathbb{C}}) = G^c \exp(iC_{\max})$$

where $C_{\max}$ is the maximal G^c-invariant cone in $\mathfrak{g}^c_{\mathbb{C}}$ containing $-Z^0$. Furthermore $S(\Omega_{\mathbb{C}})^o$ is the topological interior of $S(\Omega_{\mathbb{C}})$ and

$$S(\Omega_{\mathbb{C}})^o = S(C^o_{\max}) = G^c \exp(iC^o_{\max})\,.$$

We refer to [**33**] or [**31**] for all of this.

8. Highest Weight Modules

Our notion of reflection for unitary representations leads to the class of representations in $\mathcal{A}(W)$ of Definition 5.2, and in the present section we analyze these representations more closely. The analysis is based in large part on [**33**], and involves results of (among others) M. Davidson and R. Fabec [**6**], K.-H. Neeb [**63, 64, 65**], Harish-Chandra [**24, 25**], Ol'shanskii [**80**], R.J. Stanton [**98**], Wallach [**103**], and H. Rossi and M. Vergne [**102**].

We have seen that the interesting representations are those in $\mathcal{A}(W)$ where W is an invariant cone in $\mathfrak{g}^c$. It turns out that the irreducible representations in $\mathcal{A}(W)$ are *highest weight representations.* A $(\mathfrak{g}^c, K^c)$-module is a complex vector space $\mathbf{V}$ such that

1) $\mathbf{V}$ is a $\mathfrak{g}^c$-module.
2) $\mathbf{V}$ carries a representation of K^c, and the span of $K^c \cdot v$ is finite-dimensional for every $v \in \mathbf{V}$.
3) For $v \in \mathbf{V}$ and $X \in \mathfrak{k}^c$ we have

$$X \cdot v = \lim_{t \to 0} \frac{\exp(tX) \cdot v - v}{t}.$$

4) For $Y \in \mathfrak{g}^c$ and $k \in K^c$ the following holds for every $v \in \mathbf{V}$:

$$k \cdot (X \cdot v) = (\mathrm{Ad}(k)X) \cdot [k \cdot v]\,.$$

Note that (3) makes sense, as $K^c \cdot v$ is contained in a finite dimensional vector space and this space contains a unique Hausdorff topology as a topological vector space. The $(\mathfrak{g}^c, K^c)$-module is called *admissible* if the multiplicity of every irreducible representation of K^c in $\mathbf{V}$ is finite. If $(\pi, \mathbf{V})$ is an irreducible unitary representation of G^c, then the space of K^c-finite elements in $\mathbf{V}$, denoted by $\mathbf{V}_{K^c}$, is an admissible $(\mathfrak{g}^c, K^c)$-module.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}^c$ and $\mathfrak{g}^c$ as in the last section.

DEFINITION 8.1. Let $\mathbf{V}$ be a $(\mathfrak{g}^c, K^c)$-module. Then $\mathbf{V}$ is a highest-weight module if there exists a nonzero element $v \in \mathbf{V}$ and a $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ such that

1) $X \cdot v = \lambda(X)v$ for all $X \in \mathfrak{t}$.
2) There exists a positive system Δ^+ in Δ such that $\mathfrak{g}^c_{\mathbb{C}}(\Delta^+) \cdot v = 0$.
3) $\mathbf{V} = U(\mathfrak{g}^c) \cdot v$.

The element v is called a primitive element of weight λ.

Let $W \in \mathrm{Cone}_{G^c}(\mathfrak{g}^c)$ and $\pi \in \mathcal{A}(W)$ irreducible. We assume that $-Z^o \in W^o$. Then $\mathbf{V}_{K^c}$ is an irreducible admissible $(\mathfrak{g}^c, K^c)$-module, and

$$\mathbf{V}_{K^c} = \bigoplus_{\lambda \in \mathfrak{t}_{\mathbb{C}}^*} \mathbf{V}_{K^c}(\lambda)$$

where $\mathbf{V}_{K^c}(\lambda) = \mathbf{V}_{K^c}(\lambda, \mathfrak{t}_{\mathbb{C}})$. Let $v \in \mathbf{V}_{K^c}(\lambda)$ be nonzero. Let $\alpha \in \Delta_p^+$ and let $X \in \mathfrak{p}_\alpha^+ \setminus \{0\}$. Then

$$X^k \cdot v \in \mathbf{V}_{K^c}(\lambda + k\alpha).$$

In particular,

$$-iZ^0 \cdot (X^k \cdot v) = [-i\lambda(Z^0) + k]v.$$

This yields the following lemma.

LEMMA 8.2. *Let the notation be as above. Then the following holds:*

(i) $-i\lambda(Z^0) \leq 0$.
(ii) *There exists a* λ *such that* $\mathfrak{p}^+ \cdot \mathbf{V}_{K^c}(\lambda) = \{0\}$.

LEMMA 8.3. *Let* $\mathbf{W}^\lambda$ *be the* K^c*-module generated by* $\mathbf{V}_{K^c}(\lambda)$. *Then* $\mathbf{W}^\lambda$ *is irreducible,* $\mathbf{V}_{K^c} = U(\mathfrak{p}^-)\mathbf{W}^\lambda$ *and the multiplicity of* $\mathbf{W}^\lambda$ *in* $\mathbf{V}_{K^c}$ *is one.*

Let $\alpha \in \Delta_p^+$ then there exists a unique element $H_\alpha \in i\mathfrak{t} \cap [\mathfrak{g}_{\mathbb{C}\alpha}, \mathfrak{g}_{\mathbb{C}\alpha}]$ such that $\alpha(H_\alpha) = 2$. Let μ be the highest weight of $\mathbf{W}^\lambda$ with respect to Δ_c^+ and let v^λ be a nonzero highest weight vector. Then v^λ is a primitive element with respect to the positive system $\Delta_c^+ \cup \Delta_p^+$.

THEOREM 8.4. *Let* $\rho \in \mathcal{A}(W)$ *be irreducible. Then the corresponding* $(\mathfrak{g}^c, K^c)$*-module is a highest-weight module and equals* $U(\mathfrak{p}^-)\mathbf{W}^\lambda$. *In particular, every weight of* $\mathbf{V}_{K^c}$ *is of the form*

$$\nu - \sum_{\alpha \in \Delta(\mathfrak{p}^+, \mathfrak{t}_{\mathbb{C}})} n_\alpha \alpha\,.$$

Furthermore, $\langle \nu, H_\alpha \rangle \leq 0$ *for all* $\alpha \in \Delta_p^+$.

The K^c-representation π^λ on $\mathbf{W}^\lambda$ is called the *minimal* K^c-type of $\mathbf{V}$ and $\mathbf{V}_{K^c}$. The multiplicity of π^λ in $\mathbf{V}$ is one. We recall how to realize highest-weight modules in a space of holomorphic functions on G^c/K^c. We follow here the geometric construction by M. Davidson and R. Fabec [**6**]. For a more general approach, see [**64, 65**]. To explain the method we start with the example $G^c = SU(1,1) = \left\{\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha}\end{pmatrix} \,\middle|\, |\alpha|^2 - |\beta|^2 = 1\right\}$. We set $X = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}$ and $H := H_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$. Then $Z^0 = \frac{i}{2}H$ and $\mathfrak{p}^+ = \mathbb{C}X$, $\mathfrak{k}^c_{\mathbb{C}} = \mathbb{C}H$ and $\mathfrak{p}^- = \mathbb{C}Y$. We use this to identify those spaces with $\mathbb{C}$. Let $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,\mathbb{C})$. Then

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d\end{pmatrix} &= \begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix}\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1\end{pmatrix} \\ &= \begin{pmatrix} \gamma + \gamma^{-1}z & \gamma^{-1}z \\ \gamma^{-1}y & \gamma^{-1}\end{pmatrix}.\end{aligned}$$

Hence $P^+K^c_{\mathbb{C}}P^- = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \,\middle|\, d \neq 0\right\}$ and if $x = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in P^+K^c_{\mathbb{C}}P^-$, then

$$p(x) = b/d\,, \quad k_{\mathbb{C}}(x) = d^{-1} \quad \text{and} \quad q(x) = c/d\,.$$

Thus

$$\zeta(xK^c_{\mathbb{C}}P^-) = b/d \tag{8.1}$$

$$x \cdot z = \frac{az+b}{cz+d} \tag{8.2}$$

$$j(x,z) = (cz+d)^{-1}\,. \tag{8.3}$$

To identify $\Omega_{\mathbb{C}}$ we notice that on $SU(1,1)$ we have $\zeta(x) = \beta/\bar{\alpha}$. Hence $G^c/K^c \simeq D = \{z \in \mathbb{C} \mid |z| < 1\}$. The finite-dimensional holomorphic representations of $K^c_{\mathbb{C}}$ are the characters

$$\chi_n(\exp ziH) = e^{inz}\,.$$

In particular, $d\chi_n(Z^0) = in/2$ or

$$-i\,d\chi_n(Z^0) = \frac{n}{2}\,.$$

Let $(\pi, \mathbf{V})$ be a unitary highest-weight representation of $SU(1,1)$ and assume that $(\pi, \mathbf{V}) \in \mathcal{A}(W)$. Then $n \leq 0$ by Lemma 8.2 and Theorem 8.4. Let $\mathbf{V}(n)$ be the one-dimensional space of χ_n-isotropic vectors. Then

$$\mathbf{V}_{K^c} = \bigoplus_{k \in \mathbb{N}} \mathbf{V}(n-2k),$$

and the spaces $\mathbf{V}(m)$ and $\mathbf{V}(k)$ are orthogonal if $m \neq k$.

Let σ be the conjugation of $\mathfrak{sl}(2,\mathbb{C})$ with respect to $SU(1,1)$. Then σ is given by

$$\sigma\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) = \begin{pmatrix} -\bar{a} & \bar{c} \\ \bar{b} & -\bar{a}\end{pmatrix}$$

so that $\sigma(X) = Y$. Since $\pi(T)^* = -\pi(\sigma(T))$ for all $T \in \mathfrak{sl}(2,\mathbb{C})$ we get

$$\pi(Y)^* = \pi(-X).$$

Finally, it follows from $[Y,X]=-H$ that for $v\in V(n)$:

$$\begin{aligned}\|\pi(Y)^k v\|^2 &= \langle \pi(Y)^k v \,|\, \pi(Y)^k v\rangle\\ &= \langle \pi((-X)^k Y^k) v \,|\, v\rangle\end{aligned}$$

LEMMA 8.5. *Let the notation be as above. Then*

$$\begin{aligned}\pi(-X)^k\pi(Y)^k v &= (-1)^k k!\frac{\Gamma(n+1)}{\Gamma(n-k+1)}\,v\\ &= (-n)_k v\end{aligned}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$.

As

$$\sum_{k=0}^{\infty}(-n)_k\frac{|z^2|^k}{k!} = (1-|z|^2)^n,$$

(see [**19**]) converges if and only if $|z|<1$, it follows that

$$q_{(zX)}v := \sum_{k=0}^{\infty}\overline{z}^k\frac{Y^k v}{n!}$$

converges if and only if $zX\in\Omega_{\mathbb{C}}$.

Let now G^c be arbitrary. Let $\sigma\colon \mathfrak{g}^c_{\mathbb{C}}\to\mathfrak{g}^c_{\mathbb{C}}$ be the conjugation with respect to $\mathfrak{g}^c$. We use the notation from earlier in this section. Using the usual $\mathfrak{sl}(2,\mathbb{C})$ reduction, we get the following theorem.

THEOREM 8.6 (Davidson-Fabec). *Let $T\in\mathfrak{p}^+$. Define $q_T\colon \mathbf{W}^\lambda\to\mathbf{V}$ by the formula*

$$q_T v := \sum_{n=0}^{\infty}\frac{\sigma(T)^n v}{n!}\,.$$

1) *If $v\neq 0$, then the series that defines q_T converges in the Hilbert space $\mathbf{V}$ if and only if $T\in\Omega_{\mathbb{C}}$.*
2) *Let π_λ be the representation of K^c on $\mathbf{W}^\lambda$. Let*

$$J_\lambda(g,Z) := \pi_\lambda(j(g,Z))\,. \tag{8.4}$$

Then

$$\pi(g)v = q_{g\cdot 0}J_\lambda(g,0)^{*-1}v \tag{8.5}$$

for $g\in G^c$ and $v\in\mathbf{W}^\lambda$.

It follows that the span of the $q_Z\mathbf{W}^\lambda$ with $Z\in\Omega_{\mathbb{C}}$ is dense in $\mathbf{V}$, since $\mathbf{V}$ is assumed to be irreducible. Define $Q\colon \Omega_{\mathbb{C}}\times\Omega_{\mathbb{C}}\to GL(\mathbf{W}^\lambda)$ by

$$Q(W,Z) = q_W^* q_Z\,.$$

Then the following theorem holds.

THEOREM 8.7 (Davidson-Fabec). *Let the notation be as above. Then the following hold:*

(i) $Q(W,Z) = J_\lambda(\exp(-\sigma(W)),Z)^{*-1}$.
(ii) *$Q(W,Z)$ is holomorphic in the first variable and antiholomorphic in the second variable.*

(iii) $\langle v \,|\, Q(W,Z)u\rangle = \langle q_W v \,|\, q_Z u\rangle$ *for all* $u, v \in \mathbf{W}_\lambda$.
(iv) Q *is a positive-definite reproducing kernel.*
(v) $Q(g\cdot W, g\cdot Z) = J_\lambda(g,W)Q(W,Z)J_\lambda(g,Z)^*$.

For $Z \in \Omega_{\mathbb{C}}$ and $u \in \mathbf{W}^\lambda$, let $F_{Z,u}\colon \Omega_{\mathbb{C}} \to \mathbf{W}^\lambda$ be the holomorphic function

$$F_{Z,u}(W) := Q(W,Z)u \tag{8.6}$$

and define

$$\langle F_{T,w} \,|\, F_{Z,u}\rangle_Q := \langle w \,|\, Q(W,Z)u\rangle\,. \tag{8.7}$$

Let $\mathbf{H}(\Omega_{\mathbb{C}}, \mathbf{W}^\lambda)$ be the completion of the span of $\{F_{Z,u} \mid Z \in \Omega_{\mathbb{C}}, u \in \mathbf{W}^\lambda\}$ with respect to this inner product. Then $\mathbf{H}(\Omega_{\mathbb{C}}, \mathbf{W}^\lambda)$ is a Hilbert space consisting of $\mathbf{W}^\lambda$-valued holomorphic functions. Define a representation of G^c in $\mathbf{H}(\Omega_{\mathbb{C}}, \mathbf{W}^\lambda)$ by

$$(\rho(g)F)(W) := J_\lambda(g^{-1}, W)^{-1}F(g^{-1}\cdot W)\,. \tag{8.8}$$

Then ρ is a unitary representation of G^c in $\mathbf{H}(\Omega_{\mathbb{C}}, \mathbf{W}^\lambda)$ called the *geometric realization* of $(\pi, \mathbf{V})$.

THEOREM 8.8 (Davidson-Fabec). *The map* $q_Z v \mapsto F_{Z,v}$ *extends to a unitary intertwining operator* U *between* $(\pi, \mathbf{V})$ *and* $(\rho, \mathbf{H}(\Omega_{\mathbb{C}}, \mathbf{W}^\lambda))$. *It can be defined globally by*

$$[Uw](Z) = q_Z^* w, \quad w \in \mathbf{V},\ Z \in \Omega_{\mathbb{C}}\,.$$

As the theorem stands, it gives a geometric realization for every unitary highest-weight module. What it does not do is give a natural analytic construction of the inner product on $\mathbf{H}(\Omega_{\mathbb{C}}, \mathbf{W}^\lambda)$. This is known only for some special representations, e.g., the *holomorphic discrete series* of the group G^c [**23, 7, 35**] or symmetric spaces of Hermitian type [**75, 76**]. At this point we will only discuss the holomorphic discrete series, which was constructed by Harish-Chandra in [**23**], in particular Theorem 4 and Lemma 29. For that, let $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ and let μ denote the highest weight of the representation of K^c on $\mathbf{W}^\lambda$. For $f, g \in \mathbf{H}(\Omega_{\mathbb{C}}, \mathbf{W}^\lambda)$, let μ be the G^c-invariant measure on $\Omega_{\mathbb{C}}$ and

$$\langle g \,|\, f\rangle_\lambda := \int_{G^c/K^c} \langle g(Z) \,|\, Q(Z,Z)^{-1}f(Z)\rangle_{\mathbf{W}^\lambda}\, d\mu\,.$$

THEOREM 8.9 (Harish-Chandra [**24, 25**]). *Assume that*

$$\langle \mu + \rho, H_\alpha\rangle < 0 \qquad \textit{for all } \alpha \in \Delta_p^+.$$

Then $\langle g \,|\, f\rangle_\lambda$ *is finite for* $f, g \in \mathbf{H}(\Omega_{\mathbb{C}}, \mathbf{W}^\lambda)$ *and there exists a positive constant* c_λ *such that*

$$\langle g \,|\, f\rangle_Q = c_\lambda \langle g \,|\, f\rangle_\lambda\,.$$

Moreover, $(\rho, \mathbf{H}(\Omega_{\mathbb{C}}, \mathbf{W}_\lambda))$ *is unitarily equivalent to a discrete summand in* $\mathbf{L}^2(G^c)$.

The Theorem of Harish-Chandra relates some of the unitary highest weight modules to the discrete part of the Plancherel measure. It was shown by Ol'shanskii [**80**] and Stanton [**98**] that this "holomorphic" part of the discrete spectrum can be realized as a Hardy space of holomorphic functions on a local tube domain. Those results were generalized to symmetric spaces of *Hermitian type* (or *compactly causal symmetric spaces*) in a series of papers [**75, 76, 34, 79, 2**]

The last theorem shows in particular that the corresponding highest weight modules are unitary. It was shown by Wallach [**103**] and Rossi and Vergne [**102**] that those are not all the unitary highest weight modules. The problem is to decide for which representations of K^c the reproducing kernel $Q(Z,W)$ is positive definite. We refer to [**10, 38**] for the classification of unitary highest weight modules. We will from on now assume that the representation of K^c is a character χ_λ where $\lambda \in i\mathfrak{t}^*$ is trivial on $\mathfrak{t} \cap [\mathfrak{k},\mathfrak{k}]$. Choose a maximal set $\{\gamma_1,\dots,\gamma_r\}$ of long strongly orthogonal roots in Δ_p^+. This can be done by putting $r = \mathrm{rank}(G^c/K^c)$ and then choosing γ_r to be a maximal root in Δ_p^+, γ_{r-1} maximal in $\{\gamma \in \Delta_p^+ \mid \gamma$ strongly orthogonal to $\gamma_r\}$, etc. Let $H_j := H_{\gamma_j}$ and

$$\mathfrak{a} = i\bigoplus_j \mathbb{R}H_j \subset \mathfrak{t}. \tag{8.9}$$

By the theorem of Moore (see [**27**]) we know that the roots in Δ_p restricted to $\mathfrak{a}$, are given by $\pm\frac{1}{2}(\gamma_i+\gamma_j)$, $1 \le i \le j \le r$ and possibly $\frac{1}{2}\gamma_j$. The root spaces for γ_j are all one-dimensional and the root spaces $\mathfrak{g}_{\pm\frac{1}{2}(\gamma_i+\gamma_j)}$, $1 \le i < j \le r$, have all the common dimension d.

THEOREM 8.10 (Vergne-Rossi [**102**], Wallach [**105**]). *Assume that G^c is simple. Let $\lambda_0 \in \mathfrak{a}^*$ be such that $\langle \lambda_0, H_r\rangle = 1$. Let $\gamma = \langle \lambda_0, Z^0\rangle$ and let*

$$L_{\mathrm{pos}} := -\frac{\gamma(r-1)d}{2}\,.$$

For $\nu - \rho < L_{\mathrm{pos}}$ there exists a irreducible unitary highest weight representation $(\rho_\nu, \mathbf{K}_\nu)$ of G^c with one-dimensional minimal K^c-type $\nu-\rho$.

PROOF. By [**102**, pp. 41–42] (see also [**104**]), $(\rho_\nu, \mathbf{K}_\lambda)$ exists if $\langle \nu-\rho, H_r\rangle \le -\frac{(r-1)d}{2}$. But $\nu - \rho = \langle \nu-\rho, H_r\rangle\,\lambda_0$. Hence $\langle \nu-\rho, 2Z^0\rangle = \langle \nu-\rho, H_r\rangle \langle \lambda_0, 2Z^0\rangle = \gamma\,\langle \nu-\rho, H_r\rangle$. ☐

We will later specialize this to the case where G^c/K^c is a tube type domain, that is biholomorphically equivalent to $\mathbb{R}^n + i\Omega$, where $n = \dim_{\mathbb{C}} G^c/K^c$, and Ω is a self-dual regular cone in $\mathbb{R}^n$. If we assume G^c simple, then this is exactly the case if G^c is locally isomorphic to one of the groups: $SU(n,n)$, $SO^*(4n)$, $Sp(n,\mathbb{R})$, $SO_o(n,2)$ and $E_{7(-25)}$. In this case we have

$$Z^0 = \frac{i}{2}\sum_{j=1}^r H_j \tag{8.10}$$

and

$$\Delta_p^+ = \left\{\gamma_i, \frac{1}{2}(\gamma_k+\gamma_j) \;\middle|\; 1 \le i,j,k \le r,\ j \le k\right\} \tag{8.11}$$

$$\Delta_c^+ = \left\{\frac{1}{2}(\gamma_k-\gamma_j) \;\middle|\; 1 \le j \le k \le r\right\}\,. \tag{8.12}$$

In this case we have:

LEMMA 8.11. *Assume that G^c/K is a tube-type domain, then $\gamma = r$ and $\nu-\rho \le L_{\mathrm{pos}}$ if and only if $\nu \le r$.*

PROOF. If G^c/K^c is of Cayley type then $2Z^0 = i\sum_{j=1}^r H_j$ and $\gamma_j = \gamma_r - \sum n_\alpha \alpha$, $\alpha \in \Delta_c^+$, $n_\alpha \geq 0$. Thus $\langle \nu - \rho, Z^0 \rangle = r\,\langle \nu - \rho, H_r \rangle$. We also have (see [**79**])

$$\rho = \frac{1}{2}\left(1 + \frac{(r-1)d}{2}\right)(\gamma_1 + \cdots + \gamma_r)\,.$$

From this the theorem follows. $\square$

REMARK 8.12. Let us remind the reader that we have only described here the continuous part of the unitary spectrum. There are also finitely many discrete points, the so-called *Wallach set*, giving rise to unitary highest weight representations.

REMARK 8.13. Let $\sigma\colon \mathfrak{g}_{\mathbb{C}}^c \to \mathfrak{g}_{\mathbb{C}}^c$ be the conjugation with respect to $\mathfrak{g}^c$. Thus $\sigma(X+iY) = X - iY$, $X, Y \in \mathfrak{g}^c$. Then $\sigma(\mathfrak{p}^+) = \mathfrak{p}^-$ and $\sigma(\mathfrak{k}_{\mathbb{C}}^c) = \mathfrak{k}_{\mathbb{C}}^c$. In this section we viewed $Q(W,Z)$ as a function on $\Omega_{\mathbb{C}} \times \Omega_{\mathbb{C}}$, holomorphic in the first variable and antiholomorphic in the second variable. In many applications it is better to view Q as a function on $\Omega_{\mathbb{C}} \times \sigma(\Omega_{\mathbb{C}})$, holomorphic in both variables.

9. An Example: $SU(1,1)$

The simplest case of a non-trivial reflection positivity is the case $G = SL(2,\mathbb{R})$ and $G^c = SU(1,1)$. In this case

$$G^c/K^c = D := \{z \in \mathbb{C} \mid |z| < 1\}$$

and $SU(1,1)$ acts by

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \cdot z = \frac{az+b}{\bar{b}z + \bar{a}}\,.$$

Let $\sigma\colon D \to D$ be complex conjugation, $z \mapsto \bar{z}$ and let $\tau\colon SL(2,\mathbb{C}) \to SL(2,\mathbb{C})$ be the involution given by

$$\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}. \tag{9.1}$$

Then $\sigma(g \cdot 0) = \tau(g)\cdot 0$ for $g \in SU(1,1,)$. We have

$$H_{\mathbb{C}} = \left\{\begin{pmatrix} z & w \\ w & z \end{pmatrix} \,\middle|\, z^2 - w^2 = 1\right\}.$$

and

$$H = \pm\left\{h_t = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \,\middle|\, t \in \mathbb{R}\right\} = SU(1,1) \cap SL(2,\mathbb{R})\,, \tag{9.2}$$

$\underset{\sim}{H/\{\pm I\}} = (-1,1)$ and $G = SL(2,\mathbb{R})$. Knowing that the representations of $SU(1,1)$ that we can get are highest weight modules, we see by looking at the infinitesimal character of those representations, that we have to start with the complementary series representations of $G = SL(2,\mathbb{R})$. They are constructed in the following way. Let P be the parabolic subgroup

$$P := \left\{p(a,x) = \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a \in \mathbb{R}^*,\, x \in \mathbb{R}\right\} = G \cap K_{\mathbb{C}}^c P^+\,.$$

For $s \in \mathbb{C}$, let π_s be the representation of G acting by $[\pi_s(a)f](b) = f(a^{-1}b)$ on the space $\mathbf{H}_s$ of functions $f\colon G \to \mathbb{C}$,

$$f(gp(a,x)) = |a|^{-(s+1)} f(g)\,, \qquad \int_{SO(2)} |f(k)|^2\, dk < \infty\,,$$

and with inner product

$$\langle f \mid g\rangle = \int_{SO(2)} \overline{f(k)} g(k)\, dk\,,$$

that is π_s is the principal series representation of G with parameter s. A simple calculation shows that the pairing

$$\mathbf{H}_s \times \mathbf{H}_{-s} \ni (f,g) \longmapsto \int_{SO(2)} f(k)g(k)\, dk \in \mathbb{C} \tag{9.3}$$

is invariant under the group action. The representations π_s are unitary in the above Hilbert-space structure as long as $s \in i\mathbb{R}$. Let

$$\bar{N} = \left\{ \bar{n}_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

Then

$$\bar{n}_t p(\gamma, x) = \begin{pmatrix} \gamma & x \\ \gamma t & \gamma^{-1} + xt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Hence as we have seen before $\bar{N}P = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a \neq 0 \right\}$ and

$$\gamma = a,\ t = c/a\,. \tag{9.4}$$

In particular we have

$$\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \tanh(t) & 1 \end{pmatrix} \begin{pmatrix} \cosh(t) & 0 \\ 0 & 1/\cosh(t) \end{pmatrix} \begin{pmatrix} 1 & \tanh(t) \\ 0 & 1 \end{pmatrix}.$$

Thus $HP/P \simeq (-1,1)$, but we notice that this is not the realization in $\mathfrak{p}^+$ but in $\mathfrak{p}^-$. By (9.4) this is expressed in the action of G by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot_{\mathrm{opp}} z = \frac{dz + c}{bz + a}\,. \tag{9.5}$$

Notice that this is the usual action twisted by τ, that is

$$g \cdot_{\mathrm{opp}} z = \tau(g) \cdot z, \tag{9.6}$$

where $\cdot$ stands for the usual action. By identify $\bar{N}$ with $\mathbb{R}$ using $\bar{n}_t \mapsto t$, we can realize the principal series representations as acting on functions on $\mathbb{R}$ (compare to (9.5)):

$$\pi_s\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f(t) = |d - bt|^{-s-1} f\left(\frac{-c + at}{d - bt}\right) \tag{9.7}$$

$$\langle f \mid g \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} \overline{f(t)}\, g(t)(1+t^2)^{1+\mathrm{Re}(s)}\, dt \tag{9.8}$$

and the pairing in (9.3) is simply

$$\langle f \mid g \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} \overline{f(t)}\, g(t)\, dt\,. \tag{9.9}$$

For defining the complementary series we need the intertwining operator $A_s\colon \mathbf{H}_s \to \mathbf{H}_{-s}$ defined by

$$A_s(f)(g) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(gw\bar{n}_y)\,dy \tag{9.10}$$

for $\operatorname{Re} s \geq 0$ and then generally by analytic continuation. Here w is the Weyl group element $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\bar{n}_y = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$. In our realization of the representation on $\mathbb{R}$ we get

$$A_s f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)|x-y|^{s-1}\,dy\,. \tag{9.11}$$

By (9.9) the bilinear form

$$\langle f \mid A_s g\rangle = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(x)}g(y)|x-y|^{s-1}\,dx\,dy \tag{9.12}$$

is G-invariant and actually the representation π_s is unitary for $0 < s < 1$. Define

$$Jf(t) = |t|^{-s-1}\,f(1/t)$$

or on the group level $Jf(a) := f(\tau(a)w^{-1}) = f(\tau(aw))$. The map $J\colon \mathbf{H}_s \to \mathbf{H}_s$ intertwines π_s and $\pi_s \circ \tau \simeq \pi_s$, $J^2 = 1$, and

$$A_s(J(g))(x) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} g(y)|1-xy|^{s-1}\,dy\,.$$

Hence

$$\langle f \mid g\rangle_J = \langle f \mid A_s J g\rangle = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(x)}g(y)|1-xy|^{s-1}\,dy\,dx\,. \tag{9.13}$$

We recall now the reproducing kernel $Q(w,z)$ from Section 8, corresponding to the lowest K^c-type $s-1$. In our case

$$Q(w,z) = (1-w\bar{z})^{s-1}\,. \tag{9.14}$$

which by Theorem 8.11 is positive if and only if $s \leq 1$. It follows that $\langle \cdot \mid \cdot \rangle_J$ is positive definite on the space of functions supported on the H-orbit $(-1,1)$. Let $\mathbf{K}_0$ be the closure of $\mathcal{C}_c^\infty(-1,1)$. Notice that the above inner product is defined on $\mathcal{C}_c^\infty(-1,1)$ for every s as we only integrate over compact subsets of $(-1,1)$. As we are using the realization in $\mathfrak{p}^-$ we define the semigroup now by

$$S = S_-(\Omega) := \{\gamma \in SL(2,\mathbb{R}) \mid \gamma\cdot(-1,1) \subset (-1,1)\}\,. \tag{9.15}$$

Then S is a closed semigroup of the form $H = \exp(C)$ where C is the H-invariant cone generated by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Let us remark here, that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S$, then $d - bt > 0$ for $|t| < 1$. The semigroup S acts on $\mathbf{K}_0$ and by the Lüscher-Mack Theorem 6 we get an highest weight module for $\widetilde{SU(1,1)}$, which is irreducible as we will see in a moment.

We also know (see [**33**]) that $S = H\exp C$ is a closed semigroup and that $\gamma I \subset I$, and actually S is exactly the semigroup of elements in $SL(2,\mathbb{R})$ that act by contractions on I. Hence S acts on $\mathbf{K}$. By a theorem of Lüscher and Mack [**31, 60**], the representation of S on $\mathbf{K}$ extends to a representation of G^c, which in this case is the universal covering of $SU(1,1)$ that is locally isomorphic to $SL(2,\mathbb{R})$. According to Theorem 6.5 the resulting representation is a direct integral of highest weight representations. We notice that this defines a representation of $SL(2,\mathbb{R})$ if and only if certain integrality conditions hold; see [**44**]. The question then arises to identify this direct integral and construct an explicit intertwining operator into the corresponding space of holomorphic functions on D.

We notice first that the kernel $(y,x)\mapsto Q(y,x)=(1-yx)^{s-1}$ is the reproducing kernel of the irreducible highest weight representation given by

$$[\rho_s(g)f](z)=(-cz+a)^{s-1}f\left(\frac{dz-b}{-cz+a}\right)\,,\quad g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}. \tag{9.16}$$

In particular $Q(y,x)$ extends to a holomorphic function on $D\times\sigma(D)$ according to Remark 8.13. For $f\in C_c^\infty(-1,1)$ define

$$Uf(z):=\frac{1}{\pi}\int_{-1}^{1}f(u)(1-zu)^{s-1}\,du=\frac{1}{\pi}\int_{-1}^{1}f(u)Q(z,u)\,du\,. \tag{9.17}$$

By simple calculation, using (9.6), we get for $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in S_-(\Omega)$:

$$\begin{aligned}
U(\pi_s(\gamma)f)(z)&=\frac{1}{\pi}\int_{-1}^{1}(d-bt)^{-s-1}f\left(\frac{-c+at}{at-c}\right)(1-zt)^{s-1}\,dt\\
&=\frac{1}{\pi}\int_{-1}^{1}f(u)(cu+d)^{s-1}\left(1-z\frac{du+c}{bu+a}\right)^{s-1}du\\
&=\frac{1}{\pi}\int_{-1}^{1}f(u)\left(bu+a-dzu-cz\right)^{s-1}\,du\\
&=\frac{1}{\pi}\int_{-1}^{1}f(u)\left(-cz+a-(dz-b)u\right)^{s-1}\,du\\
&=(-cz+a)^{s-1}\frac{1}{\pi}\int_{-1}^{1}f(u)\left(1-\frac{dz-b}{-cz+a}\right)^{s-1}du\\
&=\rho_s(\gamma)Uf(z)\,,
\end{aligned}$$

where the respective representations are given by (9.7) and (9.16). Here the last equality follows from (8.3), (8.4) and (8.8). As ρ_s is irreducible it follows that either U is surjective or identically zero. Using that $Q(z,u)$ is the reproducing kernel for the representation ρ_s we get for f and g with compact support:

$$\begin{aligned}
\langle Uf\,|\,Ug\rangle&=\frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\overline{f(u)}g(v)\ \langle Q(\cdot,u)\,|\,Q(\cdot,v)\rangle\ dv\,du\\
&=\frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\overline{f(u)}g(v)Q(u,v)\,dv\,du\\
&=\langle f\,|\,g\rangle\ .
\end{aligned}$$

It follows that U is a unitary isomorphism.

We can describe U in a different way using the representation ρ_s instead of the reproducing kernel. Let $\mathbb{1}$ be the constant function $z\mapsto 1$. Then

$$[\rho_s(g)\mathbb{1}](z)=J_s(g^{-1},z)=(-cz+a)^{s-1}\,. \tag{9.18}$$

We therefore get

(9.19)
$$\begin{aligned}\int_H f(h)\rho_s(h)\mathbb{1}(z)\,dh &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\cosh(t)^{-s-1}f(\tanh(t))\right](-\sinh(t)z+\cosh(t))^{s-1}\,dt\\ &= \frac{1}{\pi}\int_{-1}^{1} f(u)(1-uz)^{s-1}\,du\\ &= Uf(z)\,.\end{aligned}$$

We will meet the transform in (9.19) again in the generalization of the Bargmann transform in Section 11. That shows that the Bargmann transform introduced in [**78**] is closely related to the reflection positivity and the Osterwalder-Schrader duality.

In summary, we have the representation π_s from (9.7) acting on the Hilbert space $\widehat{\mathbf{H}_+(s)}$ of distributions obtained from completion with respect to

$$\int_{-1}^{1}\int_{-1}^{1}\overline{f(x)}\,f(y)\,(1-xy)^{s-1}\;dx\,dy\,,$$

and the unitarily equivalent representation ρ_s from (9.16). The operator U from (9.17) intertwines the two. Moreover U passes to the distributions on $(-1,1)$, in the completion $\widehat{\mathbf{H}_+(s)}$, and we have

$$U\left(\delta^{(n)}\right) = \frac{(s-1)\,(s-2)\cdots(s-n)}{\pi}\,z^n, \tag{9.20}$$

where, for $n=0,1,2,\dots$, $\delta^{(n)} = (d/dx)^n\,\delta$ are the derivatives of the Dirac "function", defined by

$$\left\langle f,\delta^{(n)}\right\rangle = \left\langle (-1)^n\,f^{(n)},\delta\right\rangle = (-1)^n\,f^{(n)}\,(0)\,, \tag{9.21}$$

where f is a test function. Furthermore, z^n are the monomials in the reproducing kernel Hilbert space $\mathbf{H}(s)$ corresponding to the complex kernel $(1-\bar{z}w)^{s-1}$. This Hilbert space consists of analytic functions $f(z) = \sum_{n=0}^{\infty} C_n z^n$, defined in $D = \{z\in\mathbb{C}\mid |z|<1\}$, and satisfying

$$\sum_{n=0}^{\infty}|C_n|^2\,\frac{1}{\left|\binom{s-1}{n}\right|}<\infty,$$

where the $\binom{s-1}{n}$ refers to the (fractional) binomial coefficients. This sum also defines the norm in $\mathbf{H}(s)$. For every $w\in D$, the function $u_w(z) := (1-\bar{w}z)^{s-1}$ is in $\mathbf{H}(s)$, and for the inner product, we have

$$\langle u_{w_1}\,|\,u_{w_2}\rangle_{\mathbf{H}(s)} = (1-\bar{w}_1 w_2)^{s-1}\;.$$

So $\mathbf{H}(s)$ is indeed a reproducing kernel Hilbert space, as it follows that the values $f(z)$, for $f\in\mathbf{H}(s)$ and $w\in D$, are given by the inner products

$$f(w) = \langle u_w\,|\,f\rangle_{\mathbf{H}(s)}\;.$$

Since $u_w(z) = \sum_{n=0}^{\infty}\binom{s-1}{n}\bar{w}^n z^n$, we conclude that the monomials z^n form an orthogonal basis in $\mathbf{H}(s)$, and it follows from (9.20) that the distributions $\delta^{(n)}$,

$n = 0, 1, 2, \ldots$, form an orthogonal basis in the Hilbert space $\widehat{\mathbf{H}_+(s)}$, and that

$$\left\|\delta^{(n)}\right\|^2_{\widehat{\mathbf{H}_+(s)}} = \frac{n!\,(1-s)\,(2-s)\cdots(n-s)}{\pi^2}.$$

10. Reflection Symmetry for Semisimple Symmetric Spaces

The main results in this section are Theorems 10.18 and 10.20. They are stated for non-compactly causal symmetric spaces, and the proofs are based on our Basic Lemma and the Lüscher-Mack theorem. At the end of the section we show that results from Jorgensen's paper [**41**] lead to an extension of the scope of the two theorems.

We now generalize the construction from the last section to a bigger class of semisimple symmetric pairs. We restrict ourself to the case of characters induced from a maximal parabolic subgroup, which leads to highest weight modules with one-dimensional lowest K^c-type. This is meant as a simplification and not as a limitation of our method. An additional source of inspiration for the present chapter is the following series of papers: [**66, 72, 77, 75, 76, 78, 46, 50, 84, 91, 95**].

Assume that $G^c/K^c = D \simeq \Omega_{\mathbb{C}} \subset \mathfrak{p}^+$ is a bounded symmetric domain with G^c simply connected and simple. Let θ^c be the Cartan involution on G^c corresponding to K^c. Let $\sigma\colon D \to D$ be a conjugation, that is a non-trivial order two antiholomorphic map. Those involutions were classified in [**36, 37**], see also [**33, 71, 72**]. Then σ defines an involution on the group $I_o(D)$, the connected component of holomorphic isometries of D, by

$$\tau(f)(Z) = \sigma\left(f(\sigma(Z))\right).$$

But $I_o(D)$ is locally isomorphic to G^c, see [**27**], Chapter VIII. Hence τ defines an involution on G^c and $\mathfrak{g}^c$. Let $H^c = G^{c\tau}$, and $\mathfrak{h} = \{X \in \mathfrak{g}^c \mid \tau(X) = X\}$ and $\mathfrak{q}^c = \{X \in \mathfrak{g}^c \mid \tau(X) = -X\}$. Then $\mathfrak{g}^c = \mathfrak{h} \oplus \mathfrak{q}^c$. We define

$$\mathfrak{g} := \mathfrak{h} \oplus i\mathfrak{q}^c$$

and $\mathfrak{q} = \{X \in \mathfrak{g} \mid \tau(X) = -X\} = i\mathfrak{q}^c$. Then $(\mathfrak{g}, \tau)$ is a symmetric pair. Let $G_{\mathbb{C}}$ be a simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and let $G \subset G_{\mathbb{C}}$ be the connect Lie group with Lie algebra $\mathfrak{g}$. Then τ integrates to an involution on G. Let $H = G^\tau = \{a \in G \mid \tau(a) = a\}$. Then G/H is a symmetric space. The involution θ^c integrates to an involution on G and $\theta := \tau\theta^c$ is a Cartan involution on G that commutes with τ. Let K be the corresponding maximal almost compact subgroup. Denote the corresponding Cartan decomposition as usually by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

As τ is antiholomorphic it follows that $\tau(Z^0) = -Z^0$, where Z^0 is a central element in $\mathfrak{k}^c$ with eigenvalues $0, i, -i$. Hence G^c/H^c is a symmetric space of *Hermitian type*, in the sense of [**75**]. Those spaces are now usually called *compactly causal symmetric spaces* because those are exactly the symmetric spaces such that $\mathfrak{q}$ contains a regular H-invariant cone C with $C^o \cap \mathfrak{k} \neq \varnothing$. The minimal cone is given by

$$C^c_{\min} = \mathbb{R}^+ \cdot \overline{\operatorname{conv}\left\{\operatorname{Ad}(H)Z^0\right\}}.$$

The dual spaces G/H are exactly the *non-compactly causal symmetric spaces*. Those are the symmetric spaces containing H-invariant regular cones C such that $C^o \cap \mathfrak{p} \neq \varnothing$. We use [**33**] as a standard reference to the causal symmetric spaces.

EXAMPLE 10.1 (Cayley type spaces). A special case of the above construction is when G^c/K^c is a tube type domain. Let $\mathbf{c}$ be a Cayley transform from the bounded realization of G^c/K^c to the unbounded realization. This can be done by choosing $\mathbf{c} = \mathrm{Ad}(\exp(\frac{\pi i}{2}Y^0))$ where $\mathrm{ad}(Y^0)$ has eigenvalues $0, 1, -1$. Then $\mathrm{Ad}(\mathbf{c})^4 = \mathrm{id}$ and $\tau = \mathrm{Ad}(\mathbf{c})^2(G^c) = G^c$. Hence τ is an involution on G^c. It is also well known that $\tau(Z^0) = -Z^0$. Hence τ defines a conjugation on D. The symmetric spaces G/H are the symmetric spaces of Cayley type. We have $Y^0 \in \mathfrak{h}$ is central and $\mathfrak{z}_{\mathfrak{g}}(Y^0) = \mathfrak{h}$. Furthermore $\mathrm{Ad}(\mathbf{c})$ is an isomorphism $\mathfrak{g}^c \simeq \mathfrak{g}$. The spaces that we get from this construction are locally isomorphic to one of the following symmetric spaces, where we denote by the subscript $+$ the group of elements having positive determinant: $Sp(n,\mathbb{R})/GL(n,\mathbb{R})_+$, $SU(n,n)/GL(n,\mathbb{C})_+$, $SO^*(4n)/SU^*(2n)\mathbb{R}_+$, $SO(2,k)/SO(1,k-1)\mathbb{R}_+$ and $E_{7(-25)}/E_{6(-26)}\mathbb{R}_+$.

EXAMPLE 10.2. Assume that H is a connected Lie group such that H/K_H, K_H a maximal compact subgroup of H, is a bounded symmetric domain. Let $G^c = H\times H$ and $D = H/H_K \times \overline{H/H_K}$, where the bar denotes opposite complex structure. Let $\tau(d,c) = (c,d)$. Then τ is a conjugation with fixed-point set the diagonal. The corresponding involution on G^c is $\tau(a,b) = (b,a)$. Thus $G^{c\tau} = \text{diagonal} \simeq H$. Identify $G^{c\tau}$ with H. Then $G/H \ni (a,b)H \mapsto ab^{-1} \in H$ is an isomorphism. In this case G is locally isomorphic to $H_{\mathbb{C}}$ and the involution τ on $\mathfrak{g}$ is the conjugation with respect to the real form $\mathfrak{h} \subset \mathfrak{g}$. Let H_1 be the corresponding analytic subgroup. Then H_1 is locally isomorphic to H and the symmetric space we are looking at is $G_{\mathbb{C}}/H_1$.

We will need the following facts. Let $X^0 = -iZ^0 \in \mathfrak{q}\cap\mathfrak{p}$. Then X^0 is $H\cap K$-invariant,

$$\mathfrak{z}_{\mathfrak{g}}(X^0) = \mathfrak{k}^c_{\mathbb{C}} \cap \mathfrak{g} = \mathfrak{k}\cap\mathfrak{h} \oplus \mathfrak{p}\cap\mathfrak{q}, \tag{10.1}$$

Let $\mathfrak{n} := \mathfrak{p}^+ \cap \mathfrak{g}$, $\bar{\mathfrak{n}} := \mathfrak{p}^- \cap \mathfrak{g}$, and $\mathfrak{p}_{\max} := (\mathfrak{k}^c_{\mathbb{C}} \oplus \mathfrak{p}^+) \cap \mathfrak{g}$. Then $\mathfrak{p}_{\max}$ is a maximal parabolic subgroup of $\mathfrak{g}$ of the form $\mathfrak{p}_{\max} = \mathfrak{m} \oplus \mathbb{R}X^0 \oplus \mathfrak{n}$, where $\mathfrak{m} = \{X \in \mathfrak{k}\cap\mathfrak{h} \oplus \mathfrak{p}\cap\mathfrak{q} \mid B(X,X^0) = 0\}$, B the Killing form on $\mathfrak{g}$. We have $H\cap K = Z(X^0)$. Let $A := \exp(\mathfrak{a})$, $N := \exp(\mathfrak{n})$, $\bar{N} := \exp(\bar{\mathfrak{n}})$. M_0 the analytic subgroup of G corresponding to $\mathfrak{m}$, and $M := (H\cap K)M_0$. Then M is a closed and τ-stable subgroup of G, $M\cap A = \{1\}$, $MA = Z_G(A)$, and $P_{\max} = N_G(\mathfrak{p}_{\max}) = MAN$. We have

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_{\max}\,.$$

Let $\Omega = \tau(\Omega_{\mathbb{C}}) \cap \mathfrak{g} \subset \bar{\mathfrak{n}}$. Then by [**11**]:

LEMMA 10.3. *$HP_{\min}$ is open in G and $HP_{\max} = \exp(\Omega)P_{\max} \subset \bar{N}P_{\max}$.*

Let $\mathfrak{a} = \mathbb{R}X^0$ and $A := \exp(\mathfrak{a})$. We need to fix the normalization of measures before we discuss the generalized principal series representations. Let the measure da on A be given by

$$\int_A f(a)\,da = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(a_t)\,dt, \quad a_t = \exp 2tX^0\,.$$

Then Fourier inversion holds without any additional constants. We fix the Lebesgue measure dX on $\bar{\mathfrak{n}}$ such that, for $d\bar{n} = \exp(dX)$, we then have

$$\int_{\bar{\mathfrak{n}}} a(\bar{n})^{-2\rho}\,d\bar{n} = 1\,.$$

Here $\rho(X) = \frac{1}{2}\operatorname{tr}(\operatorname{ad}(X))|_{\mathfrak{n}}$ as usual, and $a(g) \in A$, $g \in G$, is determined by $g \in KMa(g)N$. The Haar measure on compact groups will usually be normalized to have total measure one. The measure on N is $\theta(d\bar{n})$. We fix a Haar measure dm on M, and dg on G such that

$$\int_G f(g)\,dg = \int_K \int_M \int_A \int_N f(kman)a^{2\rho}\,dn\,da\,dm\,dk\,, \qquad f \in \mathcal{C}_c^\infty(G).$$

Then we can normalize the Haar measure dh on H such that for $f \in \mathcal{C}_c^\infty(G)$, $\operatorname{supp}(f) \subset HP_{\max}$, we have, see [**70**]:

$$\int_G f(g)\,dg = \int_H \int_M \int_A \int_N f(hman)a^{2\rho}\,dn\,da\,dm\,dh\,.$$

The invariant measure $d\dot{x}$ on G/H is then given by

$$\int_G f(x)\,dx = \int_{G/H} \int_H f(xh)\,dh\,d\dot{x}, \quad f \in \mathcal{C}_c(G)$$

and similarly for $K/H \cap K$.

LEMMA 10.4. *Let the measures be normalized as above. Then the following hold:*

1) *Let $f \in \mathcal{C}_c(\bar{N}MAN)$. Then*
$$\int_G f(g)\,dg = \int_{\bar{N}} \int_M \int_A \int_N f(\bar{n}man)a^{2\rho}\,d\bar{n}\,dm\,da\,dn\,.$$
2) *Let $f \in \mathcal{C}_c(\bar{N})$. For $y \in \bar{N}MAN$ write $y = \bar{n}(y)m_{\bar{N}}(y)a_{\bar{N}}(y)n_{\bar{N}}(y)$. Let $x \in G$. Then*
$$\int_{\bar{N}} f(\bar{n}(x\bar{n}))a_{\bar{N}}(x\bar{n})^{-2\rho}\,d\bar{n} = \int_{\bar{N}} f(\bar{n})\,d\bar{n}\,.$$
3) *Write, for $g \in G$, $g = k(g)m(g)a(g)n(g)$ according to $G = KMAN$. Let $h \in \mathcal{C}(K/H \cap K)$. Then*
$$\int_{K/H\cap K} h(\dot{k})\,d\dot{k} = \int_{\bar{N}} h(k(\bar{n})H \cap K)a(\bar{n})^{-2\rho}\,d\bar{n}\,.$$
4) *Let $h \in \mathcal{C}(K/H \cap K)$ and let $x \in G$. Then*
$$\int_{K/H\cap K} f(k(xk)H \cap K)a(xk)^{-2\rho}\,d\dot{k} = \int_{K/H\cap K} f(\dot{k})\,d\dot{k}$$
5) *Assume that* $\operatorname{supp}(f) \subset H/H \cap K \subset K/H \cap K$. *Then*
$$\int_{K/H\cap K} f(\dot{k})\,d\dot{k} = \int_{H/H\cap K} f(k(h)H \cap K)a(h)^{-2\rho}\,d\dot{h}\,.$$
6) *Let $f \in \mathcal{C}_c(\bar{N})$. Then*
$$\int_{\bar{N}} f(\bar{n})\,d\bar{n} = \int_{H/H\cap K} f(\bar{n}(h))a_{\bar{N}}(h)^{-2\rho}\,d\bar{n}\,.$$
7) *For $x \in HP_{\max}$ write $x = h(x)m_H(x)a_H(x)n_H(x)$ with $h(x) \in H$, $m_H(x) \in M$, $a_H(x) \in A$, and $n_H(x) \in N$. Let $f \in \mathcal{C}_c^\infty(H/H \cap K)$ and let $x \in G$ be such that $xHP_{\max} \subset HP_{\max}$. Then*
$$\int_{H/H\cap K} f(h(xh)H \cap K)a_H(xh)^{-2\rho}\,d\dot{h} = \int_{H/H\cap K} f(\dot{h})\,d\dot{h}$$

Identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}$ by $\mathfrak{a}_{\mathbb{C}}^* \ni \nu \mapsto 2\nu(X^0) \in \mathbb{C}$. Then ρ corresponds to $\dim \mathfrak{n}$. For $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, let $\mathcal{C}^\infty(\nu)$ be the space of $\mathcal{C}^\infty$-functions $f\colon G \to \mathbb{C}$ such that, for $a_t = \exp t(2X^0)$, we have

$$f(gma_t n) = e^{-(\nu+\rho)t} f(g) = a_t^{-(\nu+\rho)} f(g)\,.$$

Define an inner product on $\mathcal{C}^\infty(\nu)$ by

$$\langle f \,|\, g\rangle_\nu := \int_K \overline{f(k)} g(k)\, dk = \int_{K/H\cap K} \overline{f(k)} g(k)\, d\dot{k}\,.$$

Then $\mathcal{C}^\infty(\nu)$ becomes a pre-Hilbert space. We denote by $\mathbf{H}(\nu)$ the completion of $\mathcal{C}^\infty(\nu)$. Define $\pi(\nu)$ by

$$[\pi(\nu)(x)f](g) := f(x^{-1}g), \quad x, g \in G, \quad f \in \mathcal{C}^\infty(\nu)\,.$$

Then $\pi(\nu)(x)$ is bounded, so it extends to a bounded operator on $\mathbf{H}(\nu)$, which we denote by the same symbol. Furthermore $\pi(\nu)$ is a continuous representation of G which is unitary if and only if $\nu \in i\mathbb{R}$. By [**90**] we have $\mathbf{H}(\nu)^\infty = \mathcal{C}^\infty(\nu)$. We can realize $\mathbf{H}(\nu)$ as $\mathbf{L}^2(K/H\cap K)$ and as $\mathbf{L}^2(\bar{N}, a(\bar{n})^{2\,\mathrm{Re}(\nu)}\, d\bar{n})$ by restriction (see Lemma 10.7). In the first realization the representation $\pi(\nu)$ becomes

$$[\pi(\nu)(x)f](k) = a(x^{-1}k)^{-\nu-\rho} f(k(x^{-1}k))$$

and in the second

$$[\pi(\nu)(x)f](\bar{n}) = a_{\bar{N}}(x^{-1}\bar{n})^{-\nu-\rho} f(\bar{n}(x^{-1}\bar{n}))\,.$$

LEMMA 10.5. *The pairing*

$$\mathbf{H}(\nu) \times \mathbf{H}(-\bar{\nu}) \ni (f, g) \longmapsto \langle f \,|\, g\rangle_\nu := \int_K \overline{f(k)} g(k)\, dk = \int_{K/H\cap K} \overline{f(k)} g(k)\, d\dot{k}$$

is G-invariant, i.e.

$$\langle \pi(\nu)(x) f \,|\, g\rangle_\nu = \left\langle f \,\middle|\, \pi(-\bar{\nu})(x^{-1}) g\right\rangle_\nu\,.$$

REMARK 10.6. We notice that if ν is purely imaginary, that is $-\bar{\nu} = \nu$, the above shows that $(\pi(\nu), \mathbf{H}(\nu))$ is then unitary.

LEMMA 10.7. *Let the notation be as above.*

1) *On $\bar{N}$ the invariant pairing $\langle \cdot \,|\, \cdot\rangle_\nu$ is given by*

$$\langle f \,|\, g\rangle_\nu = \int_{\bar{N}} \overline{f(\bar{n})} g(\bar{n})\, d\bar{n}\,, \quad f \in \mathbf{H}(\nu),\ g \in \mathbf{H}(-\bar{\nu})\,.$$

2) *Let $\mathbf{H}_H(\nu)$ be the closure of $\{f \in \mathcal{C}^\infty(\nu) \mid \mathrm{supp}(f) \subset HP_{\max}\}$. Then $\mathbf{H}_H(\nu) \ni f \mapsto f|_H \in \mathbf{L}^2(H/H\cap K, a(h)^{2\,\mathrm{Re}(\nu)}\, d\dot{h})$ is an isometry.*
3) *Let $f \in \mathbf{H}(\nu)$, $g \in \mathbf{H}(-\bar{\nu})$ and assume that $\mathrm{supp}(fg) \subset HP_{\max}$. Then*

$$\langle f \,|\, g\rangle_\nu = \int_{H/H\cap K} \overline{f(h)} g(h)\, d\dot{h}\,.$$

Let us assume, from now on, that there exists an element $w \in K$ such that $\mathrm{Ad}(w)(X^0) = -X^0$. In particular such an element exists if -1 is in the Weyl group $W(\mathfrak{a}_q) := N_K(\mathfrak{a}_q)/Z_K(\mathfrak{a}_q)$, where $\mathfrak{a}_q \subset \mathfrak{p}\cap\mathfrak{q}$ is maximal abelian (and then maximal abelian in $\mathfrak{p}$ and $\mathfrak{q}$). This is always the case if G/H is a Cayley type space because G is then a Hermitian groups which implies that θ is an inner automorphism. The element w does also exists if G is a complexification of one of the groups $\mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{su}(n, n)$, $\mathfrak{so}^*(4n)$, $\mathfrak{so}(2, k)$ and $\mathfrak{e}_{7(-25)}$, see [**46**], Lemma 5.20.

Let us work out more explicitly the details for the representations of the Cayley-type spaces in order to compare the existence of $(\rho_\nu, \mathbf{K}_\nu)$ to the existence of the complementary series, see Lemma 10.8 below:

For "big" ν define the intertwining operator $A(\nu)\colon \mathbf{H}(\nu) \to \mathbf{H}(-\nu)$ by the converging integral

$$[A(\nu)f](x) := \int_N f(gnw)\, dn\,,$$

see [**51, 105**]. The map $\nu \mapsto A(\nu)$ has an analytic continuation to a meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$ intertwining $\pi(\nu)$ and $\pi(-\nu)$. Using Lemma 10.5 we define a new invariant bilinear form on $\mathcal{C}_c^\infty(\nu)$ by

$$\langle f \mid g \rangle_\nu := \langle f \mid A(\nu) g \rangle_\nu\,.$$

If there exists a (maximal) constant $R > 0$ such that the invariant bilinear form $\langle \cdot \mid \cdot \rangle_\nu$ is positive definite for $|\nu| < R$, then we complete $\mathcal{C}_c^\infty(\nu)$ with respect to this new inner product, but denote the resulting space by the same symbol $\mathbf{H}(\nu)$ as before. We call the resulting unitary representations *the complementary series.* Otherwise we set $R = 0$. We compare here the constant R for the Cayley type symmetric space, see [**81, 93, 94**], and the constant $L_{\mathrm{pos}} + \rho$ from Theorem 8.11. We notice that in all cases $R \le L_{\mathrm{pos}} + \rho$.

LEMMA 10.8. *For the Cayley-type symmetric spaces the constants R and $L_{\mathrm{pos}}+\rho$ are given by the following table:*

$\mathfrak{g}$	R	$L_{\mathrm{pos}} + \rho$
$\mathfrak{su}(2n+1, 2n+1)$	$2n+1$	$2n+1$
$\mathfrak{su}(2n, 2n)$	0	$2n$
$\mathfrak{so}^*(4n)$	n	$2n$
$\mathfrak{sp}(2n, \mathbb{R})$	n	$2n$
$\mathfrak{sp}(2n+1, \mathbb{R})$	0	$2n+1$
$\mathfrak{so}(4n+2, 2)$	2	2
$\mathfrak{so}(2n+1, 2)$	1	2
$\mathfrak{so}(4n, 2)$	0	2
$E_{7(-25)}$	3	3

LEMMA 10.9. *$w^{-1}\tau(\bar{N})w = \bar{N}$, and $\varphi\colon \bar{N} \ni \bar{n} \mapsto w^{-1}\tau(\bar{n})w \in \bar{N}$ is unimodular.*

PROOF. The first claim follows as $\mathrm{Ad}(w)$ and τ act by -1 on $\mathfrak{a}$, and thus map N onto $\bar{N}$, and $\bar{N}$ onto N. The second follows as we can realize φ^2 by conjugation by an element in $M \cap K$. □

LEMMA 10.10. *For $f \in \mathbf{H}(\nu)$ let $J(f)(x) := f(\tau(xw))$. Then the following properties hold:*

1) $J(f)(x) = f(\tau(x)w^{-1})$.
2) $J(f) \in \mathbf{H}(\nu)$ *and* $A(\nu)J = JA(\nu)$.
3) $J\colon \mathbf{H}(\nu) \to \mathbf{H}(\nu)$ *is a unitary isomorphism.*
4) $J^2 = \mathrm{id}$.
5) *For $x \in G$ we have* $J \circ \pi(\nu)(x) = \pi(\nu)(\tau(x)) \circ J$.

Notice that

$$[A(\nu)J](f)(x) := \int_{\bar{N}} f(\tau(x)\bar{n})\, d\bar{n} \tag{10.2}$$

for $\operatorname{Re}\lambda$ "big". By simple calculation we get:

LEMMA 10.11. *Assume that G/H is non-compactly causal. Then $A(\nu)J$ intertwines $\pi(\nu)$, and $\pi(-\nu)\circ\tau$ if $A(\nu)J$ has no pole at ν.*

Equation (10.2) and Lemma 10.11 show that even if each one of the operators $A(\nu)$ and J does not exist, the operator $A(\nu)J\colon \mathbf{H}(\nu)\to\mathbf{H}(-\nu)$, $[A(\nu)J]\circ\pi(\nu) = [\pi(-\nu)\circ\tau][A(\nu)J]$, will always exist. The next theorem shows that the intertwining operator $A(\nu)J$ is a convolution operator which kernel $y,x\mapsto a_{\bar{N}}(\tau(y)^{-1}x)^{\nu-\rho}$. The importance of that is that $\bar{N}MAN = (P^-K_{\mathbb{C}}^cP^+)\cap G$ and hence

$$a_{\bar{N}}(x)^{\nu-\rho} = k_{\mathbb{C}}(\tau(x))^{-\nu+\rho} \tag{10.3}$$

where $k_{\mathbb{C}}$ denotes the $K_{\mathbb{C}}^c$−projection from (7.2). Here we have used that $\tau(X^0) = -X^0$. The reflection positivity then reduces to the problem to determine those ν for which this kernel is positive semidefinite.

THEOREM 10.12.

(i) *Let $f\in\mathcal{C}^\infty(\nu)$. Then*

$$[A(\nu)J](f)(\bar{n}) = \int_{\bar{N}} f(x)a_{\bar{N}}(\tau(\bar{n})^{-1}x)^{\nu-\rho}\, dx\,.$$

(ii) *If $\operatorname{supp}(f)\subset HP_{\max}$, then for $h\in H$*

$$[A(\nu)J](f)(h) = \int_{H/H\cap K} f(x)a_{\bar{N}}(h^{-1}x)^{\nu-\rho}\, d\dot{x}\,.$$

PROOF. We may assume that ν is big enough such that the integral defining $A(\nu)$ converges. The general statement follows then by analytic continuation. We have

$$\begin{aligned}
[A(\nu)J]f(\bar{n}) &= \int_{\bar{N}} Jf(\bar{n}wx)\, dx\\
&= \int f(\tau(\bar{n})w^{-1}\tau(x)w)\, dx\\
&= \int f(\tau(\bar{n})x)\, dx\\
&= \int f(\bar{n}(\tau(\bar{n})x))a_{\bar{N}}(\tau(\bar{n})x)^{-(\nu+\rho)}\, dx\,.
\end{aligned}$$

Now $a_{\bar{N}}(\tau(\bar{n})x) = a_{\bar{N}}(\tau(\bar{n})^{-1}\bar{n}(\tau(\bar{n})x))^{-1}$. By Lemma 10.4 we get

$$\begin{aligned}
[A(\nu)J]f(\bar{n}) &= \int f(\bar{n}(\tau(\bar{n})x))a_{\bar{N}}(\tau(\bar{n})^{-1}\bar{n}(\tau(\bar{n})x))^{\nu-\rho}a_{\bar{N}}(\tau(\bar{n})x)^{-2\rho}\, dx\\
&= \int f(x)a_{\bar{N}}(\tau(\bar{n})^{-1}x)^{\nu-\rho}\, dx.
\end{aligned}$$

The second statement follows in the same way. Let $\mathcal{C}_c^\infty(\Omega,\nu)$ be the subspace of $\mathcal{C}^\infty(\nu)$ consisting of functions f with support in $\exp\Omega P_{\max}$, and with $\operatorname{supp}(f|_\Omega)$ compact. □

LEMMA 10.13. *Let $f,g\in\mathcal{C}_c^\infty(\Omega,\nu)$. Then the following holds:*

(i) $\langle f \mid g\rangle_J = \int_{\bar N}\int_{\bar N} \overline{f(x)} g(y) a_{\bar N}(\tau(x)^{-1}y)^{\nu-\rho}\, dx\, dy$.
(ii) $\langle f \mid g\rangle_J = \int_{H/H\cap K}\int_{H/H\cap K} \overline{f(h)} g(k) a_{\bar N}(h^{-1}k)^{\nu-\rho}\, dh\, dk$.

Let $\sigma\colon \mathfrak{g}^c_{\mathbb{C}} \to \mathfrak{g}^c_{\mathbb{C}}$ be the conjugation with respect to $\mathfrak{g}^c$. Then $\sigma|_{\mathfrak{g}} = \tau|_{\mathfrak{g}}$. This, the fact that $\Omega = \sigma(\Omega_{\mathbb{C}}) \cap \mathfrak{g}$, and Theorem 8.7, part (i), implies that

$$(10.4)\quad (X, Y) \longmapsto a_{\bar N}(\tau(\exp(X))^{-1}\exp(Y))^{\nu-\rho} = Q(\sigma(Y), \sigma(X)) =: Q_\sigma(Y, X)\,,$$

where Q is the reproducing kernel of the representation ρ_ν, see (10.3). Notice the twist by σ that comes from the fact that we are using the realization of Ω inside $\mathfrak{p}^-$ whereas we have realized $\Omega_{\mathbb{C}}$ inside $\mathfrak{p}^+$. It follows that the integral kernel is positive definite. Hence $\langle \cdot \mid \cdot \rangle_J$ is positive definite on the space of functions supported in $HP_{\max}$. We let $\mathbf{K}_0$ be the completion of $\mathcal{C}^\infty_c(\Omega, \nu)$.

What is still needed for the application of the Lüscher-Mack Theorem is the invariant cone $C \subset \mathfrak{q}$ and the semigroup S. For that we choose $C_{\min}$ as the minimal invariant cone in $\mathfrak{q}$ containing X^0, that is $C_{\min}$ is generated by $\mathrm{Ad}(H)X^0$. We notice that

$$\mathrm{Ad}(\exp(tX^0))X = e^{-t}X\,, \qquad \forall X \in \bar{\mathfrak{n}}.$$

Hence $\mathrm{Ad}(\exp tX^0)$ acts by contractions on Ω if $t > 0$. Let

$$S(\Omega) := \{g \in G \mid gH \subset HP_{\max}\} = \{g \in G \mid g\cdot\Omega \subset \Omega\}\ .$$

Then $S(\Omega)$ is a closed semigroup invariant under $s \mapsto s^\sharp := \tau(s)^{-1}$. It follows by construction that $S(\Omega) \subset HP_{\max}$. We remark the following results:

Lemma 10.14. *Let $C = C_{\max}$ be the maximal pointed generating cone in $\mathfrak{q}$ containing X^0. Then the following hold:*
(i) *Let $t > 0$ and $Y \in \Omega$. Then $\exp tX^0 \in S$ and $\exp tX^0 \cdot Y = e^{-t}Y$.*
(ii) *$S(\Omega) = H\exp C_{\max}$.*

Proof. (1) is a simple calculation. For (2) see [**31**] and [**33**]. ☐

Corollary 10.15. *The semigroup $S(C_{\min}) = H\exp(C_{\min})$ acts by contractions on Ω.*

Lemma 10.16. *Let $s \in S(\Omega)$ and $f \in \mathcal{C}^\infty_c(\Omega, \nu)$. Then $\pi(\nu)(s)f \in \mathcal{C}^\infty_c(\Omega, \nu)$, that is $\mathcal{C}^\infty_c(\Omega)$ is $S(\Omega)$-invariant.*

Proof. Let $f \in \mathcal{C}^\infty_c(\Omega)$ and $s \in S$. Then $\pi(\nu)(s)f(x) = f(s^{-1}x) \neq 0$ only if $s^{-1}x \in \mathrm{supp}(f) \subset HP_{\max}$. Thus $\mathrm{supp}(\pi(\nu)(s)f) \subset s\,\mathrm{supp}(f) \subset sHP_{\max} \subset HP_{\max}$. ☐

We still assume that G^c is simple. Let $(\rho_\nu, \mathbf{K}_\nu)$ be as above. Let $\mathbb{1} \in \mathbf{K}_\nu(\lambda-\rho)$ be the constant function $Z \mapsto 1$. Then $\|\mathbb{1}\| = 1$. Let $H^c := (G^c)^\tau$. Then H^c is connected. Let $\tilde H$ be the universal covering of H^c and then also H_o. We notice that

$$H^c/H^c \cap K^c = H/H \cap K\,.$$

Denote the restriction of ρ_ν to H^c by $\rho_{\nu,H}$. We can lift $\rho_{\nu,H}$ to a representation of $\tilde H$ also denoted by $\rho_{\nu,H}$. We let $C = C_{\min}$ be the minimal H-invariant cone in $\mathfrak{q}$ generated by X^0. We denote by $\tilde C = \tilde C_{\min}$ the minimal G^c-invariant cone in $i\mathfrak{g}^c$ with $\tilde C \cap \mathfrak{q} = \mathrm{pr}_{\mathfrak{q}}(\tilde C) = C$, where $\mathrm{pr}_{\mathfrak{q}}\colon \mathfrak{g} \to \mathfrak{q}$ denotes the orthogonal projection (see [**33, 71**]). As $L_{\mathrm{pos}} \le 0$ it follows that ρ_λ extends to a holomorphic representation of the universal semigroup $\Gamma(G^c, \tilde C)$ corresponding to G^c and $\tilde C$ (see [**31, 56**]). Let

G_1^c be the analytic subgroup of $G_{\mathbb{C}}$ corresponding to the Lie algebra $\mathfrak{g}^c$. Let H_1 be the analytic subgroup of G_1^c corresponding to $\mathfrak{h}$. Then—as we are assuming that $G \subset G_{\mathbb{C}}$—we have $H_1 = H_o$. Let $\kappa\colon G^c \to G_1^c$ be the canonical projection and let $Z_H = \kappa^{-1}(Z_{G_1^c} \cap H_o)$. Then ρ_ν is trivial on Z_H as $\nu - \rho$ is trivial on $\exp([\mathfrak{k}^c, \mathfrak{k}^c]) \supset H^c \cap K^c$. Thus ρ_ν factors to G^c/Z_H, and to $\Gamma(G^c, \tilde{C})/Z_H$. Notice that $(G^c/Z_H)_o^\tau$ is isomorphic to H_o. Therefore we can view H_o as subgroup of G^c/Z_H, and $S_o(C) = H_o \exp C$ as a subsemigroup of $\Gamma(G^c, \tilde{C})/Z_H$. In particular $\tau_\nu(s)$ is defined for $s \in S_o(C)$. This allows us to write $\rho_\nu(h)$ or $\rho_{\nu,H}(h)$ for $h \in H_o$. Using (10.3) and Lemma 8.2 we get

$$a_{\bar{N}}(h)^{\nu-\rho} = \langle \mathbb{1}, \rho_{\nu,H}(h)\mathbb{1}\rangle \ .$$

In particular we get that $(h,k) \mapsto a_{\bar{N}}(h^{-1}k)^{\nu-\rho}$ is positive semidefinite if $\nu - \rho \leq L_{\mathrm{pos}}$.

Let us now consider the case $G = H_{\mathbb{C}}$ and $G^c = \tilde{H} \times \tilde{H}$. Denote the constant L_{pos} for $\tilde{H}$ by S_{pos} and denote, for $\mu \leq S_{\mathrm{pos}}$, the representation with lowest $\tilde{H} \cap \tilde{K}$-type μ by (τ_μ, L_μ). Let $\bar{\tau}_\mu$ be the conjugate representation. Recall that we view $\tilde{H}$ as a subset of G^c by the diagonal embedding

$$\tilde{H} \ni h \longmapsto (h,h) \in \Delta(G^c) := \{(x,x) \in G^c \mid x \in \tilde{H}\}\,.$$

The center of $\mathfrak{k}^c$ is two dimensional (over $\mathbb{R}$) and generated by $i(X^0, X^0)$ and $i(X^0, -X^0)$. We choose $Z^0 = i(X^0, -X^0)$. Then $\mathfrak{p}^+ = \mathfrak{n} \times \bar{\mathfrak{n}}$. Let $\mathbb{1}$ again be a lowest weight vector of norm one. Denote the corresponding vector in the conjugate Hilbert space by $\overline{\mathbb{1}}$. Then for $h \in \tilde{H}$:

$$\begin{aligned}\left\langle \mathbb{1} \otimes \overline{\mathbb{1}} \,\middle|\, \tau_\mu \otimes \bar{\tau}_\mu(h,h)\mathbb{1} \otimes \overline{\mathbb{1}}\right\rangle &= \langle \mathbb{1} \mid \tau_\lambda(h)\mathbb{1}\rangle\, \overline{\langle \mathbb{1} \mid \tau_\lambda(h)\mathbb{1}\rangle} \\ &= |\langle \mathbb{1} \mid \tau_\lambda(h)\mathbb{1}\rangle|^2 \\ &= a_{\bar{N}}(h)^{2\mu}\end{aligned}$$

Thus we define in this case $L_{\mathrm{pos}} := 2S_{\mathrm{pos}}$. As before we notice that $\tau_\nu \otimes \bar{\tau}_\nu(h,h)\mathbb{1} \otimes \overline{\mathbb{1}}$ is well defined on H. We now have a new proof that $\langle \cdot \mid \cdot \rangle$ is positive definite on Ω.

Lemma 10.17. *For $\nu - \rho \leq L_{\mathrm{pos}}$ there exists an unitary irreducible highest weight representation $(\rho_\nu, \mathbf{K}_\nu)$ of G^c and a lowest K^c-type vector $\mathbb{1}$ of norm one such that for every $h \in H$*

$$a_{\bar{N}}(h)^{\nu-\rho} = \langle \mathbb{1} \mid \tau_\lambda(h)\mathbb{1}\rangle \ .$$

Hence the kernel

$$(H \times H) \ni (h,k) \longmapsto a_{\bar{N}}(k^{-1}h)^{\nu-\rho} \in \mathbb{R}$$

is positive semidefinite. In particular $\langle \cdot \mid \cdot \rangle_J$ is positive semidefinite on $\mathbf{C}_c^\infty(\Omega, \nu)$ for $\nu - \rho \leq L_{\mathrm{pos}}$.

The Basic Lemma and the Lüscher-Mack Theorem, together with the above, now imply the following Theorem:

Theorem 10.18 (Reflection Symmetry for Complementary Series). *Let G/H be a non-compactly causal symmetric space such that there exists a $w \in K$ such that $\mathrm{Ad}(w)|_{\mathfrak{a}} = -1$. Let $\pi(\nu)$ be a complementary series such that $\nu \leq L_{\mathrm{pos}}$. Let C be the minimal H-invariant cone in $\mathfrak{q}$ such that $S(C)$ is contained in the contraction semigroup of $HP_{\max}$ in $G/P_{\max}$. Let Ω be the bounded realization of $H/H \cap K$ in*

$\bar{\mathfrak{n}}$. *Let* $J(f)(x) := f(\tau(x)w^{-1})$. *Let* $\mathbf{K}_0$ *be the closure of* $\mathcal{C}_c^\infty(\Omega,\nu)$ *in* $\mathbf{H}(\nu)$. *Then the following hold:*

1) $(G,\tau,\pi(\nu),C,J,\mathbf{K}_0)$ *satisfies the positivity conditions* (PR1)–(PR2).
2) $\pi(\nu)$ *defines a contractive representation* $\tilde{\pi}(\nu)$ *of* $S(C)$ *on* $\mathbf{K}$ *such that*
$$\tilde{\pi}(\nu)(\gamma)^* = \tilde{\pi}(\nu)(\tau(\gamma)^{-1}).$$
3) *There exists a unitary representation* $\tilde{\pi}^c$ *of* G^c *such that*
   i) $d\tilde{\pi}(\nu)^c(X) = d\tilde{\pi}(\nu)(X) \quad \forall X \in \mathfrak{h}$.
   ii) $d\tilde{\pi}(\nu)^c(iY) = i\, d\tilde{\pi}(\nu)(Y) \quad \forall Y \in C$.

We remark that this Theorem includes the results of R. Schrader for the complementary series of $SL(2n,\mathbb{C})$ [**95**]. In a moment we will show that actually $\tilde{\pi}(\nu)^c \simeq \rho_\nu$, where ρ_ν is the irreducible unitary highest weight representation of G^c such that
$$a(h)^{\nu-\rho} = \langle \mathbb{1} \,|\, \rho_\nu(h)\mathbb{1}\rangle$$
as before. From now on we assume that $\nu-\rho \le L_{\mathrm{pos}}$. We notice that the inner product $\langle\,\cdot\,|\,A(\nu)J(\,\cdot\,)\rangle$ makes sense independent of the existence of w. Let $\mathbf{K}_0$ be the completion of $\mathcal{C}_c^\infty(\Omega,\nu)$ in the norm $\langle\,\cdot\,|\,A(\nu)J(\,\cdot\,)\rangle$. Let $\mathbf{N}$ be the space of vectors of zero length and let $\mathbf{K}$ be the completion of $\mathbf{K}_0/\mathbf{N}$ in the induced norm. First of all we have to show that $\pi(\nu)(\gamma)$ passes to a continuous operator $\tilde{\pi}(\nu)(\gamma)$ on $\mathbf{K}$ such that $\tilde{\pi}(\nu)(\gamma)^* = \tilde{\pi}(\nu)(\tau(\gamma)^{-1})$. For that we recall that
$$H/H\cap K = H_o/H_o\cap K = \Omega \tag{10.5}$$
so we my replace the integration over H in $\langle f\,|\,A(\nu)Jf\rangle_\nu$ with integration over H_o. Motivated by (9.19) we define for $f\in\mathcal{C}_c^\infty(\Omega,\nu)$
$$U(f) = \rho_\nu(f)\mathbb{1} := \int_{H_o} f(h)\rho_\nu(h)\mathbb{1}\,dh \tag{10.6}$$
$$= \int_{H_o} a_{\bar{N}}(h)^{-\nu-\rho} f(h\cdot 0)J_\nu(h,\cdot)^{-1}\,dh. \tag{10.7}$$

LEMMA 10.19. *Assume that* $\nu-\rho\le L_{\mathrm{pos}}$. *Let* ρ_ν, $\mathbf{K}_\nu$ *and* $\mathbb{1}$ *be as specified in Lemma* 10.17 *and let* $f,g\in\mathcal{C}_c^\infty(\Omega,\nu)$ *and* $s\in S(C)$. *Then the following hold:*

1) $\langle f\,|\,[A(\nu)J](g)\rangle_\nu = \langle Uf\,|\,Ug\rangle$.
2) $U(\pi(\nu)(s)f) = \rho_\nu(s)U(f)$.
3) $\pi_\nu(s)$ *passes to a contractive operator* $\pi(\nu)(s)$ *on* $\mathbf{K}$ *such that* $\tilde{\pi}(\nu)(s)^* = \tilde{\pi}(\nu)(\tau(s)^{-1})$.

PROOF. (1) Let f and g be as above. Then
$$\begin{aligned}\langle f\,|\,[A(\nu)J](g)\rangle &= \int_{H_o/H_o\cap K}\int_{H_o/H_o\cap K} \overline{f(h)}g(k)a_{\bar{N}}(h^{-1}k)^{\nu-\rho}\,dh\,dk\\ &= \int_{H_o/H_o\cap K}\int_{H_o/H_o\cap K} \overline{f(h)}g(k)\left\langle \mathbb{1}\,\middle|\,\rho_\nu(h^{-1}k)\mathbb{1}\right\rangle\,dh\,dk\\ &= \int_{H_o/H_o\cap K}\int_{H_o/H_o\cap K} \overline{f(h)}g(k)\left\langle \rho_\nu(h)\mathbb{1}\,\middle|\,\rho_\nu(k)\mathbb{1}\right\rangle\,dh\,dk\\ &= \langle Uf\,|\,Ug\rangle\,.\end{aligned}$$
This proves (1).

(2) This follows from Lemma 10.4,7) and the following calculation:

$$\begin{aligned}U(\pi_\nu(s)f)\mathbb{1} &= \int f(s^{-1}h)\rho_\nu(h)\mathbb{1}\,dh \\ &= \int f(h(s^{-1}h))a_H(s^{-1}h)^{-(\nu+\rho)}\rho_\nu(h)\mathbb{1}\,dh \\ &= \int f(h(s^{-1}h))a_H(sh(s^{-1}))^{\nu-\rho}\rho_\nu(h)a_H(s^{-1}h)^{-2\rho}\mathbb{1}\,dh \\ &= \int f(h)\rho_\nu(sh)\mathbb{1}\,dh \\ &= \rho_\nu(s)U(f)\,,\end{aligned}$$

where we have used that

$$\rho_\nu(sh)\mathbb{1} = a_H(sh)^{\nu-\rho}\rho_\nu(h(sh))\mathbb{1}\,.$$

(3) By (1) and (2) we get:

$$\begin{aligned}\|\pi_\nu(s)f\|_J^2 &= \|\rho_\nu(s)U(f)\|^2 \\ &\le \|U(f)\|^2 \\ &= \langle f\,|\,[A(\nu)J]f\rangle_\nu \qquad (= \|f\|_J^2)\ .\end{aligned}$$

Thus $\pi_\nu(s)$ passes to a contractive operator on $\mathbf{K}$. That $\tilde{\pi}_\nu(s)^* = \tilde{\pi}_\nu(\tau(s)^{-1})$ follows from Lemma 10.11. □

THEOREM 10.20 (Identification Theorem). *Assume that G/H is non-compactly causal and that $\nu-\rho \le L_{\mathrm{pos}}$. Let ρ_ν, $\mathbf{K}_\nu$ and $\mathbb{1}\in\mathbf{K}_\nu$ be as in Lemma 10.17. Then the following hold:*

1) *There exists a continuous contractive representation $\pi(\nu)$ of $S_o(C)$ on $\mathbf{K}$ such that*

$$\pi(\nu)(s)^* = \pi(\nu)(\tau(s)^{-1})\,, \quad \forall s\in S_o(C)\,.$$

2) *There exists a unitary representation $\tilde{\pi}(\nu)^c$ of G^c such that*
   i) $d\tilde{\pi}(\nu)^c(X) = d\tilde{\pi}(\nu)(X) \quad \forall X\in\mathfrak{h}$.
   ii) $d\tilde{\pi}(\nu)^c(iY) = i\,d\tilde{\pi}(\nu)(Y) \quad \forall Y\in C$.
3) *The map*

$$\mathcal{C}_c^\infty(\Omega,\nu)\ni f\longmapsto U(f)\in\mathbf{K}_\nu$$

extends to an isometry $\mathbf{K}\simeq\mathbf{K}_\nu$ intertwining $\tilde{\pi}(\nu)^c$ and ρ_ν. In particular $\tilde{\pi}(\nu)^c$ is irreducible and isomorphic to ρ_ν.

PROOF. (1) follows from Lemma 10.19 as obviously $\tilde{\pi}(\nu)(sr) = \tilde{\pi}(\nu)(s)\tilde{\pi}_\nu(r)$.

(2) This follows now from the Theorem of Lüscher-Mack.

(3) By Lemma 10.19 we know that $f\mapsto U(f)$ defines an isometric $S_o(C)$-intertwining operator. Let $f\in\mathcal{C}_c^\infty(\Omega,\nu)$. Differentiation and the fact that ρ_ν is holomorphic gives

i) $U(d\tilde{\pi}(\nu)^c(X)f) = d\rho_\nu(X)U(f)\,, \quad \forall X\in\mathfrak{h}$.
ii) $U(i\,d\tilde{\pi}(\nu)^c(Y)f) = i\,d\rho_\nu(Y)U(f)\,, \quad \forall Y\in C$.

But those are exactly the relations that define $\tilde{\pi}(\nu)^c$. The fact that $\mathfrak{h}\oplus iC$ generates $\mathfrak{g}^c$ implies that $f\mapsto U(f)$ induces an $\mathfrak{g}^c$-intertwining operator intertwining $\tilde{\pi}(\nu)^c$ and ρ_ν. As both are also representations of G^c, it follows that this is an isometric G^c-map. In particular as it is an isometry by Lemma 10.19, part 1, it follows that the map U is an isomorphism. This proves the theorem. $\square$

We will now explain another view of the above results using local representations instead of the Lüscher-Mack Theorem. Let $\mathfrak{a}_p$ be a maximal abelian subspace of $\mathfrak{p}$ containing X^0. Then $\mathfrak{a}_p$ is contained in $\mathfrak{q}$. Let $\Delta(\mathfrak{g},\mathfrak{a})$ be the set of roots of $\mathfrak{a}$ in $\mathfrak{g}$. We choose a positive system such that $\Delta_+ = \{\alpha \mid \alpha(X^0) = 1\} \subset \Delta^+(\mathfrak{g},\mathfrak{a})$. Choose a maximal set of long strongly orthogonal roots $\gamma_1,\ldots,\gamma_r$, $r = rank(H/H\cap K)$. Choose $X_j \in \mathfrak{g}_{\gamma_j}$ such that with $X_{-j} = \tau(X_j)$ we have $[X_j, X_{-j}] = H_j := H_{\gamma_j}$. Then by Theorem 5.1.8 in [**33**] we have

$$\Omega = \mathrm{Ad}(H\cap K)\left\{\sum_{j=1}^r t_j X_{-j}\,\middle|\, -1 < t_j < 1,\, 1 \le j \le r\right\}.$$

For $R > 0$, let

$$B_R := \mathrm{Ad}(H\cap K)\left\{\sum_{j=1}^r t_j X_{-j}\,\middle|\, -R < t_j < R\right\}.$$

Then B_R is open in $\bar{\mathfrak{n}}$. Let $\beta\colon \mathbf{K}_0 \to (\mathbf{K}_0/\mathbf{N})^{\sim} = \mathbf{K}$ be the canonical map. Then β is a contraction ($\|\beta(f)\|_J^2 = \langle f \mid Jf\rangle = \langle f \mid f\rangle_J \le \|f\|^2$). For $U \subset \Omega$ open, let

$$\mathcal{C}_c^\infty(U,\nu) := \{f \in \mathcal{C}_c^\infty(\Omega,\nu) \mid \mathrm{supp}(f) \subset U\}$$

and $\mathbf{K}(U) := \beta(\mathcal{C}_c^\infty(U))$.

THEOREM 10.21. *Let $U \subset \Omega$ be open. Then $\mathbf{K}(U)$ is dense in $\mathbf{K}$.*

PROOF. Let $x \in U$. Then we can choose $h \in H$ such that $h\cdot x = 0$. As $\mathcal{C}_c^\infty(U,\nu) = h\cdot \mathcal{C}_c^\infty(h\cdot U,\nu)$ and H acts unitarily, it follows that we can assume that $0 \in U$. Let $R > 0$ be such that $B_R \subset U$. Then $\mathcal{C}_c^\infty(B_R,\nu) \subset \mathcal{C}_c^\infty(U,\nu)$. Hence we can assume that $U = B_R$. Let $g \in \mathcal{C}_c^\infty(U,\nu)^\perp$ and let $f \in \mathcal{C}_c^\infty(\Omega,\nu)$. We want to show that $\langle g \mid f\rangle_J = 0$. Choose $0 < L < 1$ such that $\mathrm{supp}(f) \subset B_L$. For $t \in \mathbb{R}$ and $a_t = \exp(2tX^0)$ we have $a_t \cdot B_L = B_{e^{-2t}L}$. Thus $\mathrm{supp}([\pi(\nu)(a_t)f]_\Omega) \subset B_{e^{-2t}L}$. Choose $0 < s_0$ such that $e^{-2t}L < R$ for every $t > s_0$. Then $\pi(\nu)(a_t)(f) \in \mathcal{C}_c^\infty(U,\nu)$ for every $t > s_0$. It follows that for $t > s_0$:

$$\begin{aligned}
0 &= \langle g \mid \pi(\nu)(a_t)f\rangle_J \\
&= \int_\Omega\int_\Omega \overline{g(x)}\,[\pi(\nu)(a_t)f](y) Q_\sigma(x,y)\,dx\,dy \\
&= e^{(\lambda+1)t}\int_\Omega\int_\Omega \overline{g(x)} f(e^{2t}y) Q_\sigma(x,y)\,dx\,dy \\
&= e^{(\lambda-1)t}\int_\Omega\int_\Omega \overline{g(x)} f(y) Q_\sigma(x,e^{-2t}y)\,dx\,dy\,.
\end{aligned}$$

By Lemma 8.7 we know that $z \mapsto Q(zX,Y)$ is holomorphic on $D = \{z \mid |z| < 1\}$. As g and f both have compact support it follows by (10.4) that

$$F(z) := \int_\Omega\int_\Omega \overline{g(x)} f(y) Q(x,zy)\,dx\,dy$$

is holomorphic on D. But $F(z) = 0$ for $0 < z < e^{-2s_0}$. Thus $F(z) = 0$ for every z. In particular

$$\langle g \mid \pi(\nu)(a_t)f\rangle_J = 0$$

for every $t > 0$. By continuity $\langle g \,|\, f \rangle_J = 0$. Thus $g = 0$. $\square$

Let us recall some basic facts from [**41**]. Let ρ be a local homomorphism of a neighborhood U of e in G into the space of linear operators on the Hilbert space $\mathbf{H}$ such that $\rho(g)$ is densely defined for $g \in U$. Furthermore $\rho|_{(U\cap H)}$ extends to a strongly continuous representation of H in $\mathbf{H}$. ρ is called a *local representation* if there exists a dense subspace $\mathbf{D} \subset \mathbf{H}$ such that the following hold:

LR1) $\forall g \in U$, $\mathbf{D} \subset \mathbf{D}(\rho(g))$, where $\mathbf{D}(\rho(g))$ is the domain of definition for $\rho(g)$.

LR2) If $g_1, g_2, g_1 g_2 \in U$ and $u \in \mathbf{D}$ then $\rho(g_2)u \in \mathbf{D}(\rho(g_1))$ and
$$\rho(g_1)[\rho(g_2)u] = \rho(g_1 g_2)u\,.$$

LR3) Let $Y \in \mathfrak{h}$ such that $\exp tY \in U$ for $0 \le t \le 1$. Then for every $u \in \mathbf{D}$
$$\lim_{t\to 0} \rho(\exp tY)u = u\,.$$

LR4) $\rho(Y)\mathbf{D} \subset \mathbf{D}$ for every $Y \in \mathfrak{h}$.

LR5) $\forall u \in \mathbf{D}\, \exists V_u$ an open 1-neighborhod in H such that $UV_u \subset U^2$ and $\rho(h)u \in \mathbf{D}$ for every $h \in V_u$.

LR6) For every $Y \in \mathfrak{q}$ and every $u \in \mathbf{D}$ the function
$$h \longmapsto \rho(\exp(\mathrm{Ad}(h)Y))u$$
is locally integrable on $\{h \in H \mid \exp(\mathrm{Ad}(h)Y) \in U\}$.

[**41**] now states that every local representation extends to a unitary representation of G^c. We now want to use Theorem 10.21 to construct a local representation of G. For that let $0 < R < 1$ and let $\mathbf{D} = \mathbf{K}(B_R(0))$. Let V be a symmetric open neighborhood of $1 \in G$ such that $V \cdot B_R(0) \subset \Omega$. Let U_1 be a convex symmetric neighborhood of 0 in $\mathfrak{g}$ such that with $U := \exp U_1$ we have $U^2 \subset V$. If $g \in U$ then obviously (LR1)–(LR3) are satisfied. (LR4) is satisfied as differentiation does not increase support. (LR6) is also clear as $u = \beta(f)$ with $f \in \mathcal{C}_c^\infty(U)$ and hence $\|\rho_c(\exp \mathrm{Ad}(h)Y)u\|$ is continuous as a function of h.

(LR5) Let $u = \beta(f) \in \mathbf{K}(B_R(0))$. Let $L = \mathrm{supp}(f) \subset B_R(0)$. Let V_u be such that $V_u^{-1} = V_u$, $V_u L \subset B_R(0)$, and $V_u \subset U$. Then $UV_u \subset U^2$ and $\tilde{\pi}(\nu)(h)u = \beta(\pi(\nu)(h)f)$ is defined and in D. This now implies that $\tilde{\pi}$ restricted to U is a local representation. Hence the existence of $\tilde{\pi}^c$ follows from [**41**]. We notice that this construction of $\tilde{\pi}^c$ does not use the full semigroup S but only H and $\exp \mathbb{R}_+ X^o$.

Remark 10.22. Here we have used reflection positivity to go from the generalized principal series to a highest weight module. Similar, but purely algebraic construction for the special case $G_\mathbb{C}$ and $G_\mathbb{C}^c = G \times G$ is due to T. J. Enright, [**9**].

11. The Segal-Bargmann Transform

We have seen that the reflection positivity and the Osterwalder-Schrader duality both correspond to transforming functions on the real form $\Omega \subset \Omega_\mathbb{C}$ to a reproducing Hilbert space of holomorphic functions on $\Omega_\mathbb{C}$. A classical example of a similar situation is given by the Segal-Bargmann transform of the Schrödinger representation realized on $\mathbf{L}^2(\mathbb{R}^n)$ to the Fock model realized in the space of holomorphic function on $\mathbb{C}^n$ with finite norm
$$\|F\|_\mathbf{F}^2 = \int_{\mathbb{C}^n} |F(z)|^2\, e^{-|z|^2}\, d\mu(z)$$

where $d\mu$ is the π^{-n} times the Lebesgue measure on $\mathbb{C}^n$. We normalize the Lebesgue measure on $\mathbb{R}^n$ by $(2\pi)^{-n/2}$ times the usual Lebesgue measure. We denote the resulting measure by dx. There are no further constants in the Fourier inversion formula in this normalization.

Our first observation is that $\mathbb{C}^n$ is a "complexification" of $\mathbb{R}^n$. Therefore the "restriction" map

$$R\colon \mathbf{F}(\mathbb{C}^n) \longrightarrow \mathbf{L}^2(\mathbb{R}^n), \qquad RF(x) := e^{-x^2/2}F(x)$$

is injective. It can be shown that R is continuous with dense image. Hence $R^*\colon \mathbf{L}^2(\mathbb{R}^n) \to \mathbf{F}(\mathbb{C}^n)$ is well defined and continuous.

The next observation is, that $\mathbf{F}(\mathbb{C}^n)$ is a reproducing Hilbert space with reproducing kernel $K(w,z) = K_z(w) = e^{w\overline{z}}$. Therefore

$$\begin{aligned} R^*f(z) &= \langle K_z \,|\, R^*f\rangle \\ &= \langle RK_z \,|\, f\rangle \\ &= \int f(x)e^{-x^2/2}e^{zx}\,dx \\ &= e^{z^2/2}\int f(x)e^{-(z-x)^2/2}\,dx \\ &= e^{z^2/2}H * f(z) \end{aligned} \tag{11.1}$$

where $H(z) = e^{-z^2/2}$.

The third observation is that this is just a special case of the *heat semigroup*

$$H_t(f)(y) = H_t * f(y) = t^{-n/2}\int f(x)e^{-(y-x)^2/2t}\,dx\,.$$

As the name indicate the family $\{H_t\}_{t>0}$ is a semigroup, that is

$$H_t * H_s = H_{t+s}. \tag{11.2}$$

This gives us the following form for RR^* and $|R^*| = \sqrt{RR^*}$:

$$\begin{aligned} RR^*h(x) &= e^{-x^2/2}R^*h(x) \\ &= H_1 * h(x)\,. \end{aligned} \tag{11.3}$$

Thus

$$\begin{aligned} \sqrt{RR^*}(h)(x) &= H_{1/2} * h(x) \\ &= 2^{n/2}\int h(y)e^{-(x-y)^2}\,dy\,. \end{aligned} \tag{11.4}$$

From this we derive the following expression for the unitary part of the polar decomposition of $R^* = B\sqrt{RR^*}$ of R^*:

$$\begin{aligned} Bh(z) &= R^*\,|R^*|^{-1}\,h(z) \\ &= e^{z^2/2}H_{1/2} * h(z) \\ &= 2^{n/2}\int h(x)e^{-x^2+2zx-z^2/2}\,dy \\ &= 2^{n/2}e^{-z^2/2}\int h(x)e^{-x^2+2xz}\,dx \end{aligned}$$

which is, up to a scaling factor, the usual Bargmann transform, see [**13**], p. 40. In particular it follows directly from our construction that the Bargmann transform is a unitary isomorphism.

This example shows that we can recover the Bargmann transform from the *restriction principle*, see [**78**], that is we have

- Manifolds $M \subset M_{\mathbb{C}}$ where $M_{\mathbb{C}}$ is a "complexification" of M;
- Groups $H \subset G$ such that H acts on M and G acts on $M_{\mathbb{C}}$;
- Measures μ and λ on M respectively $M_{\mathbb{C}}$ with the measure μ being H-invariant;
- A reproducing Hilbert space $\mathbf{F}(M_{\mathbb{C}}) \subset \mathbf{L}^2(M_{\mathbb{C}}, \lambda) \cap \mathcal{O}(M_{\mathbb{C}})$ with a representation π_G of G given by
$$[\pi_G(g)F](z) = m(g^{-1}, z)^{-1} F(g^{-1} \cdot z)$$
where m is a "multiplier".
- A function χ such that the "restriction" map $R(F)(x) := \chi(x)F(x)$, $x \in M$, from $\mathbf{F}(M_{\mathbb{C}}) \to \mathbf{L}^2(M, \mu)$ is a closed (or continuous) H-intertwining operator with dense image.

Denote the reproducing kernel of $\mathbf{F}(M_{\mathbb{C}})$ by $K(z,w) = K_w(z)$. Then $K_w \in \mathbf{F}(M_{\mathbb{C}})$ and $K(z,w) = \overline{K(w,z)}$. The map $R^* \colon L^2(M,\mu) \to \mathbf{F}(M_{\mathbb{C}})$ has the form
$$\begin{aligned} R^* h(w) &= \langle K_w \,|\, R^* h \rangle \\ &= \langle R K_w \,|\, h \rangle \\ &= \int_M h(m) \overline{\chi(m)} K(w,m)\, d\mu(m)\,. \end{aligned} \tag{11.5}$$
Hence
$$RR^* h(x) = \int_M h(m)\chi(x)\overline{\chi(m)}\, K(x,m)\, d\mu(m)\,. \tag{11.6}$$
Write $R^* = B\,|R^*|$ for the polar decomposition of R^*.

DEFINITION 11.1. The unitary isomorphism $B \colon \mathbf{L}^2(M,\mu) \to \mathbf{F}(M_{\mathbb{C}})$ is called the (generalized) Bargmann transform.

The natural setting that we are looking at now is $H/H\cap K \simeq \Omega \subset G^c/K^c = \Omega_{\mathbb{C}}$ and one of the highest weight modules as a generalization of the Fock spaces with the representation $[\rho_\nu(g)F](Z) = J_\lambda(g^{-1}, Z)^{-1} F(g^{-1} \cdot Z)$, see (8.8). Here λ corresponds to $\nu - \rho$ in our previous sections. Define
$$RF(x) = J_\lambda(x, 0)^{-1} F(x \cdot 0), \quad x \in H\,. \tag{11.7}$$
Using the multiplier relation $J_\lambda(h^{-1}x, 0) = J_\lambda(h^{-1}, x\cdot 0) J_\lambda(x, 0)$ which follows from (7.3) we get:
$$\begin{aligned} R\left[\rho_\lambda(h)F\right](x) &= J_\lambda(x,0)^{-1}\left[\rho_\lambda(h)F\right](x\cdot 0) \\ &= J_\lambda(x,0)^{-1} J_\lambda(h^{-1}, x\cdot 0)^{-1} F(h^{-1}\cdot(x\cdot 0)) \\ &= J_\lambda(h^{-1}x, 0)^{-1} F((h^{-1}x)\cdot 0) \\ &= [L(h)RF](x) \end{aligned}$$
where L stands for the left regular representation of H on $H/H\cap K$. Hence R is an intertwining operator. Let $\mathfrak{a}_p$, Δ and Δ^+ be as in the end of Section 10. By [**75, 76**] the following is known:

THEOREM 11.2. *Suppose that* $\langle \lambda + \rho, H_\alpha \rangle < 0$ *for all* $\alpha \in \Delta_+$. *Then* ρ_λ *is equivalent to a discrete summand in* $\mathbf{L}^2(G^c/H)$, *that is there exists an injective* G^c*-map* $T\colon \mathbf{H}(\rho_\lambda) \to \mathbf{L}^2(G^c/H)$.

We call the resulting discrete part of $\mathbf{L}^2(G^c/H)$ the *holomorphic discrete series* of G^c/H. We have (see [**78**]):

THEOREM 11.3. *Assume that* ρ_λ *is a holomorphic discrete series representation of* G^c/H *then the following holds:*

(i) *The restriction map is injective, closed and with dense image.*
(ii) *The generalized Bargmann transform* $B\colon \mathbf{L}^2(H/H\cap K) \to \mathbf{H}(\rho_\lambda)$ *is a* H*-isomorphism.*

Denote as before the reproducing kernel of $\mathbf{H}(\rho_\lambda)$ by $Q(W,Z)$. Then by (11.5) and Theorem 8.7:

$$R^*f(Z) = \int_{H/H\cap K} f(h\cdot 0)\cdot \overline{J_\lambda(h,0)^{-1}Q(h\cdot 0, Z)}\,dh$$
$$= \int_{H/H\cap K} f(h\cdot 0) J_\lambda(h^{-1},Z)\,dh\,. \tag{11.8}$$

If $Z = x\cdot 0$, $x\in H$, then $J_\lambda(h^{-1},Z) = J_\lambda(h^{-1}x,0)J(x,0)^{-1}$. Let $\Psi_\lambda(h) := J_\lambda(h,0)$ and view $f(h\cdot 0)$ as a right $H\cap K$-invariant function on H. Then RR^* is the convolution operator

$$RR^*f(x) = \int_H f(h)\Psi_\lambda(h^{-1}x)\,dh\,.$$

The problem that we face here is, that $\lambda \mapsto \Psi_\lambda *$ is *not* a semigroup of operators. Hence it becomes harder to find the square root of RR^*, but it can still be shown, that it is a convolution operator. The final remark in this section is the following connection between our map U that comes from the reflection positivity and R^*.

THEOREM 11.4. *Let* $f \in \mathcal{C}_c^\infty(\Omega,\nu)$. *Define a function* F *on* $H/H\cap K$ *by* $F(h) = a_{\bar N}(h)^{-\nu-\rho} f(h\cdot_{\mathrm{opp}} 0)$. *Then*

$$Uf(Z) = R^*(F)(Z)\,.$$

PROOF. This follows from (11.8) and (10.7) $\square$

12. The Heisenberg Group

A special case of the setup in Definition 3.1 above arises as follows: Let the group G, and $\tau \in \mathrm{Aut}_2(G)$ be as described there. Let $\mathbf{H}_\pm$ be two given complex Hilbert spaces, and $\pi_\pm \in \mathrm{Rep}(G,\mathbf{H}_\pm)$ a pair of unitary representations. Suppose $T\colon \mathbf{H}_- \to \mathbf{H}_+$ is a unitary operator such that $T\pi_- = (\pi_+\circ\tau)\,T$, or equivalently,

$$T\pi_-(g)f_- = \pi_+(\tau(g))\,Tf_- \tag{12.1}$$

for all $g\in G$, and all $f_-\in\mathbf{H}_-$. Form the direct sum $\mathbf{H} := \mathbf{H}_+\oplus\mathbf{H}_-$ with inner product

$$\langle f_+\oplus f_- \,|\, f'_+\oplus f'_-\rangle := \langle f_+\,|\,f'_+\rangle_+ + \langle f_-\,|\,f'_-\rangle_- \tag{12.2}$$

where the $\pm$ subscripts are put in to refer to the respective Hilbert spaces $\mathbf{H}_\pm$, and we may form $\pi := \pi_+\oplus\pi_-$ as a unitary representation on $\mathbf{H} = \mathbf{H}_+\oplus\mathbf{H}_-$ by

$$\pi(g)\,(f_+\oplus f_-) = \pi_+(g)f_+\oplus\pi_-(g)f_-\,,\quad g\in G,\ f_\pm\in\mathbf{H}_\pm\,.$$

Setting

$$J := \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}, \tag{12.3}$$

that is $J(f_+ \oplus f_-) = (Tf_-) \oplus (T^* f_+)$, it is then clear that properties (1)–(2) from Definition 3.1 will be satisfied for the pair (J, π). Formula (12.1) may be recovered by writing out the relation

$$J\pi = (\pi \circ \tau) J \tag{12.4}$$

in matrix form, specifically

$$\begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix} \begin{pmatrix} \pi_+(g) & 0 \\ 0 & \pi_-(g) \end{pmatrix} = \begin{pmatrix} \pi_+(\tau(g)) & 0 \\ 0 & \pi_-(\tau(g)) \end{pmatrix} \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}.$$

If, conversely, (12.4) is assumed for some unitary period-2 operator J on $\mathbf{H} = \mathbf{H}_+ \oplus \mathbf{H}_-$, and, if the two representations π_+ and π_- are *disjoint,* in the sense that no irreducible in one occurs in the other (or, equivalently, there is no nonzero intertwiner between them), then, in fact, (12.1) will follow from (12.4). The diagonal terms in (12.3) will be zero if (12.4) holds. This last implication is an application of Schur's lemma.

LEMMA 12.1. *Let $0 \neq \mathbf{K}_0$ be a closed linear subspace of $\mathbf{H} = \mathbf{H}_+ \oplus \mathbf{H}_-$ satisfying the positivity condition* (PR3) *in Definition* 3.2, *that is*

$$\langle v \,|\, Jv \rangle \geq 0\,, \quad \forall v \in \mathbf{K}_0 \tag{12.5}$$

where

$$J = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix} \tag{12.6}$$

is given from a fixed unitary isomorphism $T\colon \mathbf{H}_- \to \mathbf{H}_+$ as in (12.1). *For $v = f_+ \oplus f_- \in \mathbf{H} = \mathbf{H}_+ \oplus \mathbf{H}_-$, set $P_+ v := f_+$. The closure of the subspace $P_+\mathbf{K}_0$ in $\mathbf{H}_+$ will be denoted $\overline{P_+\mathbf{K}_0}$. Then the subspace*

$$\mathbf{G} = \left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in \mathbf{K}_0 \;\middle|\; f_- \in T^*\left(\overline{P_+\mathbf{K}_0}\right) \right\}$$

is the graph of a closed linear operator M with domain

$$\mathbf{D} = \left\{ f_+ \in \mathbf{H}_+ \;\middle|\; \exists f_- \in T^*\left(\overline{P_+\mathbf{K}_0}\right) \text{s.t.} \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in \mathbf{K}_0 \right\}; \tag{12.7}$$

and, moreover, the product operator $L := TM$ is dissipative on this domain, that is

$$\langle Lf_+ \,|\, f_+ \rangle_+ + \langle f_+ \,|\, Lf_+ \rangle_+ \geq 0 \tag{12.8}$$

holds for all $f_+ \in \mathbf{D}$.

PROOF. The details will only be sketched here, but the reader is referred to [**100**] and [**40**] for definitions and background literature. An important argument in the proof is the verification that, if a column vector of the form $\begin{pmatrix} 0 \\ f_- \end{pmatrix}$ is in $\mathbf{G}$, then f_- must necessarily be zero in $\mathbf{H}_-$. But using positivity, we have

$$\left|\langle u \,|\, Jv \rangle\right|^2 \leq \langle u \,|\, Ju \rangle \langle v \,|\, Jv \rangle\,, \quad \forall u, v \in \mathbf{K}_0\,. \tag{12.9}$$

Using this on the vectors $u = \begin{pmatrix} 0 \\ f_- \end{pmatrix}$ and $v = \begin{pmatrix} k_+ \\ k_- \end{pmatrix} \in \mathbf{K}_0$, we get

$$\left\langle \begin{pmatrix} 0 \\ f_- \end{pmatrix} \middle| \begin{pmatrix} Tk_- \\ T^* k_+ \end{pmatrix} \right\rangle = \langle f_- \, | \, T^* k_+ \rangle = 0 \,, \quad \forall k_+ = P_+ v \,.$$

But, since f_- is also in $T^* \left(\overline{P_+ \mathbf{K}_0} \right)$, we conclude that $f_- = 0$, proving that $\mathbf{G}$ is the graph of an operator M as specified. The dissipativity of the operator $L = TM$ is just a restatement of (PR3). □

The above result involves only the operator-theoretic information implied by the data in Definition 3.2, and, in the next lemma, we introduce the representations:

LEMMA 12.2. *Let the representations $\pi_\pm$ and the intertwiner T be given as specified before. Let $H = G^\tau$; and suppose we have a cone $C \subset \mathfrak{q}$ as specified in* (PR1), (PR2′) *and* (PR3′). *Assume further that*

PR4) $\mathbf{D}$ *is dense in* $\mathbf{H}_+$*;*

PR5) *The commutant of $\pi(H)$ is abelian.*

Then $L = TM$ is normal.

PROOF. Since T is a unitary isomorphism $\mathbf{H}_- \to \mathbf{H}_+$ we may make an identification and reduce the proof to the case where $\mathbf{H}_+ = \mathbf{H}_-$ and T is the identity operator. We then have

$$\pi_- = T^{-1} \left(\pi_+ \circ \tau \right) T = \pi_+ \circ \tau \,;$$

and if $h \in H$, then

$$\pi_-(h) = \pi_+ \left(\tau(h) \right) = \pi_+(h) \,;$$

while, if $\tau(g) = g^{-1}$, then

$$\pi_-(g) = \pi_+ \left(\tau(g) \right) = \pi_+ \left(g^{-1} \right) \,.$$

Using only the H part from (PR2′), we conclude that $\mathbf{K}_0$ is invariant under $\pi_+ \oplus \pi_+(H)$. If the projection $P_{\mathbf{K}_0}$ of $\mathbf{H}_+ \oplus \mathbf{H}_+$ onto $\mathbf{K}_0$ is written as an operator matrix $\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ with entries representing operators in $\mathbf{H}_+$, and satisfying

$$\begin{aligned} P_{11}^* &= P_{11} \,, \\ P_{22}^* &= P_{22} \,, \\ P_{12}^* &= P_{21} \,, \\ P_{ij} &= P_{i1} P_{1j} + P_{i2} P_{2j} \,, \end{aligned}$$

then it follows that

$$P_{ij} \pi_+(h) = \pi_+(h) P_{ij} \quad \forall i, j = 1, 2, \; \forall h \in H \,, \tag{12.10}$$

which puts each of the four operators P_{ij} in the commutant $\pi_+(H)'$ from (PR5). Using (PR4), we then conclude that L is a dissipative operator with $\mathbf{D}$ as dense domain, and that $\mathbf{K}_0$ is the graph of this operator. Using (PR5), and a theorem of Stone [**100**], we finally conclude that L is a normal operator, that is it can be represented as a multiplication operator with dense domain $\mathbf{D}$ in $\mathbf{H}_+$. □

We shall consider two cases below (the Heisenberg group, and the $(ax+b)$-group) when conditions (PR4)–(PR5) can be verified from the context of the representations. Suppose G has two abelian subgroups H, N, and the second N also a normal subgroup, such that $G = HN$ is a product representation in the sense of Mackey [**61**]. Define $\tau \in \mathrm{Aut}_2(G)$ by setting

$$\tau(h) = h\,, \quad \forall h \in H\,, \text{ and } \tau(n) = n^{-1}\,, \quad \forall n \in N\,. \tag{12.11}$$

The Heisenberg group is a copy of $\mathbb{R}^3$ represented as matrices $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$, or equivalently vectors $(a,b,c) \in \mathbb{R}^3$. Setting $H = \{(a,0,0) \mid a \in \mathbb{R}\}$ and

$$N = \{(0,b,c) \mid b,c \in \mathbb{R}\}\ , \tag{12.12}$$

we arrive at one example.

The $(ax+b)$-group is a copy of $\mathbb{R}^2$ represented as matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, $a = e^s$, $b \in \mathbb{R}$, $s \in \mathbb{R}$. Here we may take $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in \mathbb{R}_+ \right\}$ and

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathbb{R} \right\}\ , \tag{12.13}$$

and we have a second example of the Mackey factorization. Generally, if $G = HN$ is specified as described, we use the representations of G which are induced from one-dimensional representations of N. If G is the Heisenberg group, or the $(ax+b)$-group, we get all the infinite-dimensional irreducible representations of G by this induction (up to unitary equivalence, of course). For the Heisenberg group, the representations are indexed by $\hbar \in \mathbb{R} \setminus \{0\}$, $\hbar$ denoting Planck's constant. The representation $\pi_\hbar$ may be given in $\mathbf{H} = \mathbf{L}^2(\mathbb{R})$ by

$$\pi_\hbar(a,b,c)f(x) = e^{i\hbar(c+bx)}f(x+a)\,, \quad \forall f \in \mathbf{L}^2(\mathbb{R}),\ (a,b,c) \in G\,. \tag{12.14}$$

The Stone-von Neumann uniqueness theorem asserts that every unitary representation π of G satisfying

$$\pi(0,0,c) = e^{i\hbar c} I_{\mathbf{H}(\pi)} \quad (\hbar \neq 0)$$

is unitarily equivalent to a direct sum of copies of the representation $\pi_\hbar$ in (12.14).

The $(ax+b)$-group (in the form $\left\{ \begin{pmatrix} e^s & b \\ 0 & 1 \end{pmatrix} \,\middle|\, s,b \in \mathbb{R} \right\}$) has only two inequivalent unitary irreducible representations, and they may also be given in the same Hilbert space $\mathbf{L}^2(\mathbb{R})$ by

$$\pi_\pm \begin{pmatrix} e^s & b \\ 0 & 1 \end{pmatrix} f(x) = e^{\pm i e^x b} f(x+s)\,, \quad \forall f \in \mathbf{L}^2(\mathbb{R})\,. \tag{12.15}$$

There are many references for these standard facts from representation theory; see, e.g., [**43**].

LEMMA 12.3. *Let the group G have the form $G = HN$ for locally compact abelian subgroups H, N, with N normal, and $H \cap N = \{e\}$. Let χ be a one-dimensional unitary representation of N, and let $\pi = \mathrm{ind}_N^G(\chi)$ be the corresponding induced representation. Then the commutant of $\{\pi(H) \mid h \in H\}$ is an abelian von Neumann algebra: in other words, condition* (PR5) *in Lemma* 12.2 *is satisfied.*

PROOF. See, e.g., [**43**]. □

In the rest of the present section, we will treat the case of the *Heisenberg group,* and the $(ax+b)$*-group* will be the subject of the next section.

For both groups we get pairs of unitary representations $\pi_\pm$ arising from some $\tau \in \mathrm{Aut}_2(G)$ and described as in (12.4) above. But when the two representations π_+ and $\pi_- = \pi_+ \circ \tau$ are irreducible and disjoint, we will show that there are no spaces $\mathbf{K}_0$ satisfying (PR1), (PR2′), and (PR3) such that $\mathbf{K} = (\mathbf{K}_0/\mathbf{N})^\sim$ is nontrivial. Here (PR2) is replaced by

PR2′) C is a nontrivial cone in $\mathfrak{q}$.

Since for both groups, and common to all the representations, we noted that the Hilbert space $\mathbf{H}_+$ may be taken as $\mathbf{L}^2(\mathbb{R})$, we can have J from (12.6) represented in the form $J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Then the J-inner product on $\mathbf{H}_+ \oplus \mathbf{H}_- = \mathbf{L}^2(\mathbb{R}) \oplus \mathbf{L}^2(\mathbb{R}) \simeq \mathbf{L}^2(\mathbb{R}, \mathbb{C}^2)$ may be brought into the form

$$\left\langle \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \middle| \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \right\rangle_J = 2\,\mathrm{Re}\,\langle f_+ \mid f_- \rangle = 2\int_{-\infty}^{\infty} \mathrm{Re}\left(\overline{f_+(x)}f_-(x)\right)\,dx\,. \tag{12.16}$$

For the two examples, we introduce

$$N_+ = \{(0,b,c) \mid b,c \in \mathbb{R}_+\}$$

where N is defined in (12.12), but N_+ is not H-invariant. Alternatively, set

$$N_+ = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{R}_+ \right\}$$

for the alternative case where N is defined from (12.13), and note that this N_+ is H-invariant. In fact there are the following 4 invariant cones in $\mathfrak{q}$:

$$\begin{aligned} C_1^+ &= \{(0,0,t) \mid t \geq 0\} \\ C_1^- &= \{(0,0,t) \mid t \leq 0\} \\ C_2^+ &= \{(0,x,y) \mid x \in \mathbb{R},\ y \geq 0\} \\ C_2^- &= \{(0,x,y) \mid x \in \mathbb{R},\ y \leq 0\} \end{aligned}$$

Let π denote one of the representations of $G = HN$ from the discussion above (see formulas (12.14) and (12.15)) and let $\mathcal{D}$ be a closed subspace of $\mathbf{H} = \mathbf{L}^2(\mathbb{R})$ which is assumed invariant under $\pi(HN_+)$. Then it follows that the two spaces

$$\mathcal{D}_\infty := \bigvee \{\pi(n)\mathcal{D} \mid n \in N\} \tag{12.17}$$

$$\mathcal{D}_{-\infty} := \bigwedge \{\pi(n)\mathcal{D} \mid n \in N\} \tag{12.18}$$

are invariant under $\pi(G)$, where the symbols $\bigvee$ and $\bigwedge$ are used for the usual lattice operations on closed subspaces in $\mathbf{H}$. We leave the easy verification to the reader, but the issue is resumed in the next section. If P_∞, resp., $P_{-\infty}$, denotes the projection of $\mathbf{H}$ onto $\mathcal{D}_\infty$, resp., $\mathcal{D}_{-\infty}$, then we assert that both projections $P_{\pm\infty}$ are in the commutant of $\pi(G)$. So, if π is irreducible, then each P_∞, or $P_{-\infty}$, must be 0 or I. Since $\mathcal{D}_{-\infty} \subset \mathcal{D} \subset \mathcal{D}_\infty$ from the assumption, it follows that $P_\infty = I$ if $\mathcal{D} \neq \{0\}$.

LEMMA 12.4. *Let G be the Heisenberg group, and let the notation be as described above. Let π_+ be one of the representations $\pi_\hbar$ and let π_- be the corresponding $\pi_{-\hbar}$ representation. Let $0 \neq \mathbf{K}_0 \subset \mathbf{L}^2(\mathbb{R}) \oplus \mathbf{L}^2(\mathbb{R})$ be a closed subspace which is invariant under $(\pi_+ \oplus \pi_-)(HN_+)$. Then it follows that there are only the following possibilities for $\overline{P_+\mathbf{K}_0}$: $\{0\}$, $\mathbf{L}^2(\mathbb{R})$, or $A\mathcal{H}_+$ where $\mathcal{H}_+$ denotes the Hardy space in $\mathbf{L}^2(\mathbb{R})$ consisting of functions f with Fourier transform $\hat{f}$ supported in the half-line $[0, \infty)$, and where $A \in \mathbf{L}^\infty(\mathbb{R})$ is such that $|A(x)| = 1$ a.e. $x \in \mathbb{R}$. For the space $\overline{P_-\mathbf{K}_0}$, there are the possibilities: $\{0\}$, $\mathbf{L}^2(\mathbb{R})$, and $A\mathcal{H}_-$, where A is a (possibly different) unitary $\mathbf{L}^\infty$-function, and $\mathcal{H}_-$ denotes the negative Hardy space.*

PROOF. Immediate from the discussion, and the Beurling-Lax theorem classifying the closed subspaces in $\mathbf{L}^2(\mathbb{R})$ which are invariant under the multiplication operators, $f(x) \mapsto e^{iax} f(x)$, $a \in \mathbb{R}_+$. We refer to [**59**], or [**29**], for a review of the Beurling-Lax theorem. ☐

COROLLARY 12.5. *Let $\pi_\pm$ be the representations of the Heisenberg group, and suppose that the subspace $\mathbf{K}_0$ from Lemma* 12.4 *is chosen such that* (PR1)–(PR3) *in Definition* 3.2 *hold. Then* $(\mathbf{K}_0/\mathbf{N})^\sim = \{0\}$.

PROOF. Suppose there are unitary functions $A_\pm \in \mathbf{L}^\infty(\mathbb{R})$ such that $\overline{P_\pm\mathbf{K}_0} = A_\pm\mathcal{H}_\pm$. Then this would violate the Schwarz-estimate (12.9), and therefore condition (PR3). Using irreducibility of $\pi_+ = \pi_\hbar$ and of $\pi_- = \pi_+ \circ \tau = \pi_{-\hbar}$, we may reduce to considering the cases when one of the spaces $\overline{P_\pm\mathbf{K}_0}$ is $\mathbf{L}^2(\mathbb{R})$. By Lemma 12.2, we are then back to the case when $\mathbf{K}_0$ or $\mathbf{K}_0^{-1}$ is the graph of a densely defined normal and dissipative operator L, or L^{-1}, respectively. We will consider L only. The other case goes the same way. Since

$$(\pi_+ \oplus \pi_-)(0, b, 0)(f_+ \oplus f_-)(x) = e^{i\hbar bx} f_+(x) \oplus e^{-i\hbar bx} f_-(x) \tag{12.19}$$

it follows that L must anti-commute with the multiplication operator ix on $\mathbf{L}^2(\mathbb{R})$. For deriving this, we used assumption (PR3) at this point. We also showed in Lemma 12.2 that L must act as a multiplication operator on the Fourier-transform side. But the anti-commutativity is inconsistent with a known structure theorem in [**88**], specifically Corollary 3.3 in that paper. Hence there are unitary functions $A_\pm$ in $\mathbf{L}^\infty(\mathbb{R})$ such that $\overline{P_\pm\mathbf{K}_0} = A_\pm\mathcal{H}_\pm$. But this possibility is inconsistent with positivity in the form $\operatorname{Re}\langle f_+ \,|\, f_-\rangle \geq 0$, $\forall (f_+, f_-) \in \mathbf{K}_0$ (see (12.16)) if $(\mathbf{K}_0/\mathbf{N})^\sim \neq \{0\}$. To see this, note that $\mathbf{K}_0$ is invariant under the unitary operators (12.19) for $b \in \mathbb{R}_+$. The argument from Lemma 12.4, now applied to $\pi_+ \oplus \pi_-$, shows that the two subspaces

$$\mathbf{K}_0^\infty := \bigvee_{b\in\mathbb{R}} (\pi_+ \oplus \pi_-)(0, b, 0)\,\mathbf{K}_0$$

and

$$\mathbf{K}_0^{-\infty} := \bigwedge_{b\in\mathbb{R}} (\pi_+ \oplus \pi_-)(0, b, 0)\,\mathbf{K}_0$$

are both invariant under the whole group $(\pi_+ \oplus \pi_-)(G)$. But the commutant of this is 2-dimensional: the only projections in the commutant are represented as one of the following,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

relative to the decomposition $\mathbf{L}^2(\mathbb{R}) \oplus \mathbf{L}^2(\mathbb{R})$ of $\pi_+ \oplus \pi_-$. The above analysis of the anti-commutator rules out the cases $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$, and if $(\mathbf{K}_0/\mathbf{N})^\sim \neq \{0\}$, we are left with the cases $\mathbf{K}_0^\infty = \{0\}$ and $\mathbf{K}_0^\infty = \mathbf{L}^2(\mathbb{R}) \oplus \mathbf{L}^2(\mathbb{R})$. Recall, generally $\mathbf{K}_0^{-\infty} \subset \mathbf{K}_0 \subset \mathbf{K}_0^\infty$, as a starting point for the analysis. A final application of the Beurling-Lax theorem (as in [**59**]; see also [**8**]) to (12.19) then shows that there must be a pair of unitary functions $A_\pm$ in $\mathbf{L}^\infty(\mathbb{R})$ such that

$$\mathbf{K}_0 = A_+\mathcal{H}_+ \oplus A_-\mathcal{H}_- \tag{12.20}$$

where $\mathcal{H}_\pm$ are the two Hardy spaces given by having $\hat{f}$ supported in $[0,\infty)$, respectively, $(-\infty,0]$. The argument is now completed by noting that (12.20) is inconsistent with the positivity of $\mathbf{K}_0$ in (12.5); that is, we clearly do not have $\left\langle \begin{pmatrix} A_+h_+ \\ A_-h_- \end{pmatrix} \middle| J \begin{pmatrix} A_+h_+ \\ A_-h_- \end{pmatrix} \right\rangle = 2\,\mathrm{Re}\,\langle A_+h_+ \,|\, A_-h_- \rangle$ semidefinite, for all $h_+ \in \mathcal{H}_+$ and all $h_- \in \mathcal{H}_-$. This concludes the proof of the Corollary. $\square$

Remark 12.6. At the end of the above proof of Corollary 12.5, we arrived at the conclusion (12.20) for the subspace $\mathbf{K}_0$ under consideration. Motivated by this, we define a closed subspace $\mathbf{K}_0$ in a direct sum Hilbert space $\mathbf{H}_+ \oplus \mathbf{H}_-$ to be *uncorrelated* if there are closed subspaces $\mathbf{D}_\pm \subset \mathbf{H}_\pm$ in the respective summands such that

$$\mathbf{K}_0 = \mathbf{D}_+ \oplus \mathbf{D}_- \tag{12.21}$$

Contained in the corollary is then the assertion that every semigroup-invariant $\mathbf{K}_0$ in $\mathbf{L}^2(\mathbb{R}) \oplus \mathbf{L}^2(\mathbb{R})$ is uncorrelated, where the semigroup here is the subsemigroup S in the Heisenberg group G given by

$$S = \{(a,b,c) \mid b \in \mathbb{R}_+,\ a,c \in \mathbb{R}\}\ , \tag{12.22}$$

and the parameterization is the one from (12.12). We also had the representation π in the form $\pi_+ \oplus \pi_-$ where the respective summand representations $\pi_\pm$ of G are given by (12.14) relative to a pair $(\hbar, -\hbar)$, $\hbar \in \mathbb{R} \setminus \{0\}$ some fixed value of Planck's constant. In particular, it is assumed in Corollary 12.5 that each representation $\pi_\pm$ is *irreducible.* But for proving that some given semigroup-invariant $\mathbf{K}_0$ must be uncorrelated, this last condition can be relaxed considerably; and this turns out to be relevant for applications to Lax-Phillips scattering theory for the wave equation with obstacle scattering [**59**]. In that context, the spaces $\mathbf{D}_\pm$ will be outgoing, respectively, incoming subspaces; and the wave equation translates backwards, respectively forwards, according to the unitary one-parameter groups $\pi_-(0,b,0)$, respectively, $\pi_+(0,b,0)$, with $b \in \mathbb{R}$ representing the time-variable t for the unitary time-evolution one-parameter group which solves the wave equation under consideration. The unitary-equivalence identity (12.4) stated before Lemma 12.1 then implies equivalence of the wave-dynamics before, and after, the obstacle scattering.

Before stating our next result, we call attention to the $(2n+1)$-dimensional Heisenberg group G_n in the form $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, in parameter form: $a, b \in \mathbb{R}^n$, $c \in \mathbb{R}$, and product rule

$$(a,b,c) \cdot (a',b',c') = (a+a', b+b', c+c'+a \cdot b')$$

where $a + a' = (a_1 + a'_1, \dots, a_n + a'_n)$ and $a \cdot b' = \sum_{j=1}^{n} a_j b'_j$. For every (fixed) $b \in \mathbb{R}^n \setminus \{0\}$, we then have a subsemigroup

$$S(b) = \{(a, \beta b, c) \mid \beta \in \mathbb{R}_+ ,\ a \in \mathbb{R}^n ,\ c \in \mathbb{R}\} \ ; \tag{12.23}$$

and we show in the next result that it is enough to have invariance under such a semigroup in G_n, just for a single direction, defined from some fixed $b \in \mathbb{R}^n \setminus \{0\}$.

THEOREM 12.7. *Let $\pi_\pm$ be unitary representations of the Heisenberg group G on respective Hilbert spaces $\mathbf{H}_\pm$, and let $T\colon \mathbf{H}_- \to \mathbf{H}_+$ be a unitary isomorphism which intertwines π_- and $\pi_+ \circ \tau$ as in* (12.1) *where*

$$\tau(a, b, c) = (a, -b, -c) , \quad \forall (a, b, c) \in G \simeq \mathbb{R}^{2n+1} . \tag{12.24}$$

Suppose there is $\hbar \in \mathbb{R} \setminus \{0\}$ such that

$$\pi_+(0, 0, c) = e^{i\hbar c} I_{\mathbf{H}_+} . \tag{12.25}$$

If $\mathbf{K}_0 \subset \mathbf{H}_+ \oplus \mathbf{H}_-$ is a closed subspace which is invariant under

$$\{(\pi_+ \oplus \pi_-)(a, \beta b, c) \mid a \in \mathbb{R}^n ,\ \beta \in \mathbb{R}_+ ,\ c \in \mathbb{R}\}$$

from (12.23), *$b \in \mathbb{R}^n \setminus \{0\}$, then we conclude that $\mathbf{K}_0$ must automatically be uncorrelated.*

PROOF. The group-law in the Heisenberg group yields the following commutator rule:

$$(a, 0, 0)(0, b, 0)(-a, 0, 0) = (0, b, a \cdot b)$$

for all $a, b \in \mathbb{R}^n$. We now apply $\pi = \pi_+ \oplus \pi_-$ to this, and evaluate on a general vector $f_+ \oplus f_- \in \mathbf{K}_0 \subset \mathbf{H}_+ \oplus \mathbf{H}_-$: abbreviating $\pi(a)$ for $\pi(a, 0, 0)$, and $\pi(b)$ for $\pi(0, b, 0)$, we get

$$\pi(a)\pi(\beta b)\pi(-a)(f_+ \oplus f_-) = e^{i\hbar\beta a \cdot b}\pi_+(\beta b) f_+ \oplus e^{-i\hbar\beta a\cdot b}\pi_-(\beta b) f_- \in \mathbf{K}_0$$

valid for all $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}_+$. Note, in (12.25), we are assuming that π_+ takes on some specific value $e^{i\hbar c}$ on the one-dimensional center. Since π_- is unitarily equivalent to $\pi_+ \circ \tau$ by assumption (see (12.25)), we conclude that

$$\pi_-(0, 0, c) = e^{-i\hbar c} I_{\mathbf{H}_-} , \quad \forall c \in \mathbb{R}.$$

The argument really only needs that the two representations $\pi_\pm$ define *different* characters on the center. (Clearly $\hbar \neq -\hbar$ since $\hbar \neq 0$.) Multiplying through first with $e^{-i\hbar\beta a\cdot b}$, and integrating the resulting term

$$\pi_+(\beta b) f_+ \oplus e^{-i2\hbar\beta a\cdot b}\pi_-(\beta b) f_- \in \mathbf{K}_0$$

in the a-variable, we get $\pi_+(\beta b) f_+ \oplus 0 \in \mathbf{K}_0$. The last conclusion is just using that $\mathbf{K}_0$ is a closed subspace. But we can do the same with the term

$$e^{i2\hbar\beta a\cdot b}\pi_+(\beta b) f_+ \oplus \pi_-(\beta b) f_- \in \mathbf{K}_0 ,$$

and we arrive at $0 \oplus \pi_-(\beta b) f_- \in \mathbf{K}_0$. Finally letting $\beta \to 0_+$, and using strong continuity, we get $f_+ \oplus 0$ and $0 \oplus f_-$ both in $\mathbf{K}_0$. Recalling that $f_\pm$ are general vectors in $P_\pm \mathbf{K}_0$, we conclude that $P_+\mathbf{K}_0 \oplus P_-\mathbf{K}_0 \subset \mathbf{K}_0$, and therefore $\overline{P_+\mathbf{K}_0} \oplus \overline{P_-\mathbf{K}_0} \subset \mathbf{K}_0$. Since the converse inclusion is obvious, we arrive at (12.21) with $\mathbf{D}_\pm = \overline{P_\pm \mathbf{K}_0}$. $\square$

The next result shows among other things that there are representations π of the Heisenberg group G_n (for each n) such that the reflected representation π^c of $G_n^c \simeq G_n$ (see Theorem 6.2) acts on a nonzero Hilbert space $\mathbf{H}^c = (\mathbf{K}_0/\mathbf{N})^{\sim}$. However, because of Lemma 6.3, $\pi^c(G_n^c)$ will automatically be an *abelian* group of operators on $\mathbf{H}^c$. To see this, note that the proof of Theorem 12.7 shows that π^c must act as the identity operator on $\mathbf{H}^c$ when restricted to the one-dimensional center in $G_n^c \simeq G_n$.

It will be convenient for us to read off this result from a more general context: we shall consider a general Lie group G, and we fix a right-invariant Haar measure on G.

A distribution F on the Lie group G will be said to be positive definite (PD) if

$$\int_G \int_G F(uv^{-1}) \overline{f(u)} f(v)\, du\, dv \geq 0 \tag{PD}$$

for all $f \in C_c^\infty(G)$; and we say that F is PD on some open subset $\Omega \subset G$ if this holds for all $f \in C_c^\infty(\Omega)$. The interpretation of the expression in (PD) is in the sense of distributions. But presently measurable functions F will serve as the prime examples.

We say that the distribution is reflection-positive (RP) on Ω ((RP$_\Omega$) for emphasis) if, for some period-2 automorphism τ of G, we have

$$F \circ \tau = F \tag{12.26}$$

and

$$\int_G \int_G F(\tau(u)v^{-1}) \overline{f(u)} f(v)\, du\, dv \geq 0 \tag{RP$_\Omega$}$$

for all $f \in C_c^\infty(\Omega)$.

We say that $x \in G$ is (RP$_\Omega$)-contractive if (RP$_\Omega$) holds, and for all $f \in C_c^\infty(\Omega)$

$$\begin{aligned} 0 &\leq \int_G \int_G F(\tau(u)v^{-1}) \overline{f(ux)} f(vx)\, du\, dv \\ &\leq \int_G \int_G F(\tau(u)v^{-1}) \overline{f(u)} f(v)\, du\, dv\,. \end{aligned}$$

Note that, since

$$\int_G \int_G F(\tau(u)v^{-1}) \overline{f(ux)} f(vx)\, du\, dv = \int_G \int_G F(\tau(u)\tau(x)^{-1}xv^{-1}) \overline{f(u)} f(v)\, du\, dv$$

it follows that every $x \in H$ is contractive; in fact, isometric. If instead we have $\tau(x) = x^{-1}$, then contractivity amounts to the estimate

$$\int_G \int_G F(\tau(u)x^2v^{-1}) \overline{f(u)} f(v)\, du\, dv \leq \int_G \int_G F(\tau(u)v^{-1}) \overline{f(u)} f(v)\, du\, dv$$

for all $f \in C_c^\infty(\Omega)$. Using the Basic Lemma one can also show that x acts by contractions.

The following result is useful, but an easy consequence of the definitions and standard techniques for positive definite distributions; see for example [**43, 91**].

THEOREM 12.8. *Let F be a distribution on a Lie group G with a period-2 automorphism τ, and suppose F is τ-invariant,* (PD) *holds on G, and* (RP$_\Omega$) *holds on some open, and semigroup-invariant, subset Ω in G. Then define*

$$(\pi(u)f)(v) := f(vu)\,, \quad \forall u, v \in G,\ \forall f \in C_c^\infty(G)\,;$$

and

$$Jf := f \circ \tau \, .$$

Let $\mathbf{H}(F)$ *be the Hilbert space obtained from the GNS construction, applied to* (PD), *with inner product on* $C_c^\infty(G)$ *given by*

$$\langle f \,|\, g \rangle := \int_G \int_G F(uv^{-1}) \overline{f(u)} g(v) \, du \, dv \, .$$

Then π *extends to a unitary representation of* G *on* $\mathbf{H}(F)$, *and* J *to a unitary operator, such that*

$$J\pi = (\pi \circ \tau) J \, .$$

If (RP_Ω) *further holds, as described, then* π *induces* (*via Theorem* 6.2) *a unitary representation* π^c *of* G^c *acting on the new Hilbert space* $\mathbf{H}^c$ *obtained from completing in the new inner product from* (RP_Ω), *and dividing out with the corresponding kernel.*

The simplest example of a function F on the Heisenberg group G_n satisfying (PD), but not (RP_Ω), for nontrivial Ω's, may be obtained from the Green's function for the sub-Laplacian on G_n; see [**99**, p. 599] for details.

If complex coordinates are introduced in G_n, the formula for F takes the following simple form: let $z \in \mathbb{C}^n$, $c \in \mathbb{R}$, and define

$$F(z,c) = \frac{1}{\left(|z|^4 + c^2 \right)^n} \, .$$

Then we adapt the product in G_n to the modified definition as follows:

$$(z,c) \cdot (z',c') = (z + z', c + c' + \langle z, z' \rangle) \quad \forall z, z' \in \mathbb{C}^n, \ \forall c, c' \in \mathbb{R},$$

where $\langle z, z' \rangle$ is the symplectic form

$$\langle z, z' \rangle := 2 \operatorname{Im}(z \cdot \bar{z}') \, .$$

The period-2 automorphism τ on G_n we take as

$$\tau(z,c) = (\bar{z}, -c)$$

with $\bar{z}$ denoting complex conjugation $(z_1, \dots, z_n) \mapsto (\bar{z}_1, \dots, \bar{z}_n)$.

The simplest example where both (PD) and (RP_Ω) hold on the Heisenberg group G_n is the following:

EXAMPLE 12.9. Let $\zeta = (\zeta_1, \dots \zeta_n) \in \mathbb{C}^n$, $\xi_j = \operatorname{Re} \zeta_j$, $\eta_j = \operatorname{Im} \zeta_j$, $j = 1, \dots, n$. Define

$$F(z,c) = \int_{\mathbb{R}^{2n}} \frac{e^{i \operatorname{Re}(z \cdot \bar{\zeta})}}{\prod_{j=1}^n (|\zeta_j|^2 + 1)} \, d\xi_1 \cdots d\xi_n \, d\eta_1 \cdots d\eta_n \, .$$

Let $\Omega := \{(z,c) \in G_n \mid z = (z_j)_{j=1}^n, \ \operatorname{Im} z_j > 0\}$. Then (PD) holds on G_n, and (RP_Ω) holds, referring to this Ω. Since the expression for $F(z,c)$ factors, the problem reduces to the $(n = 1)$ special case. There we have

$$F(z,c) = \int_{\mathbb{R}^2} \frac{e^{i(x\xi + y\eta)}}{\xi^2 + \eta^2 + 1} \, d\xi \, d\eta \, ;$$

and if $f \in C_c^\infty(\Omega)$ with $\Omega = \{(z,c) \mid y > 0\}$, then

$$\begin{aligned}\int_{G_1}\int_{G_1} & F(\tau(u)v^{-1})\overline{f(u)}f(v)\,du\,dv \\ &= \int_{\mathbb{R}^8} \frac{e^{i(x-x')\xi}\,e^{-i(y+y')\eta}}{\xi^2+\eta^2+1}\overline{f(x+iy,c)}f(x'+iy',c')\,d\xi\,d\eta\,dx\,dy\,dc\,dx'\,dy'\,dc'\,.\end{aligned}$$

Let $\tilde{f}$ denote the Fourier transform in the x-variable, keeping the last two variables (y,c) separate. Then the integral transforms as follows:

$$\int_{\mathbb{R}^5} \frac{e^{-(y+y')\sqrt{1+\xi^2}}}{\sqrt{1+\xi^2}}\overline{\tilde{f}(\xi,y,c)}\tilde{f}(\xi,y',c')\,d\xi\,dy\,dy'\,dc\,dc'\,.$$

Introducing the Laplace transform in the middle variable y, we then get (since f is supported in $y > 0$)

$$\int_0^\infty e^{-y\sqrt{1+\xi^2}}\tilde{f}(\xi,y,c)\,dy = \tilde{f}_\lambda(\xi,\sqrt{1+\xi^2},c)\,;$$

the combined integral reduces further:

$$\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\tilde{f}_\lambda(\xi,\sqrt{1+\xi^2},c)\,dc\right|^2\frac{d\xi}{1+\xi^2}$$

which is clearly positive; and we have demonstrated that (RP_Ω) holds. It is immediate that F is τ-invariant (see (12.26)), and also that it satisfies (PD) on G_n.

13. The $(ax+b)$-Group Revisited

We showed that in general we get a unitary representation π^c of the group G^c from an old one π of G, provided π satisfies the assumptions of reflection positivity. The construction as we saw uses a certain cone C and a semigroup $H\exp C$, which are part of the axiom system. What results is a new class of unitary representations π^c satisfying a certain spectrum condition (semi-bounded spectrum).

But, for the simplest non-trivial group G, this semi-boundedness turns out *not* to be satisfied in the general case. Nonetheless, we still have a reflection construction getting us from unitary representations π of the $(ax+b)$-group, such that $\pi\circ\tau \simeq \pi$ (unitary equivalence), to associated unitary representations π^c of the same group. The (up to conjugation) unique non-trivial period-2 automorphism τ of G, where G is the $(ax+b)$-group, is given by

$$\tau(a,b) = (a,-b)\,.$$

Recall that the G may be identified with the matrix-group

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a > 0, b \in \mathbb{R} \right\}$$

and (a,b) corresponds to the matrix $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. In this realization the Lie algebra of G has the basis

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,.$$

We have $\exp(tX) = (e^t,0)$ and $\exp(sY) = (1,s)$. Hence $\tau(X) = X$ and $\tau(Y) = -Y$. Thus $\mathfrak{h} = \mathbb{R}X$ and $\mathfrak{q} = \mathbb{R}Y$. We notice the commutator relation $[X,Y] = Y$. The possible H-invariant cones in $\mathfrak{q}$ are $\pm\{tY \mid t \geq 0\}$. It is known from Mackey's theory that G has two inequivalent, unitary, irreducible, infinite-dimensional representations $\pi_\pm$, and it is immediate that we have the unitary equivalence (see details below):

$$\pi_+ \circ \tau \simeq \pi_- \,. \tag{13.1}$$

Hence, if we set $\pi := \pi_+ \oplus \pi_-$, then $\pi \circ \tau \simeq \pi$, so we have the setup for the general theory. We show that π may be realized on $\mathbf{L}^2(\mathbb{R}) \oplus \mathbf{L}^2(\mathbb{R}) \simeq \mathbf{L}^2(\mathbb{R}, \mathbb{C}^2)$, and we find and classify the invariant positive subspaces $\mathbf{K}_0 \subset \mathbf{L}^2(\mathbb{R}, \mathbb{C}^2)$. To understand the interesting cases for the $(ax+b)$-group G, we need to relax the invariance condition: We shall *not* assume invariance of $\mathbf{K}_0$ under the semigroup $\{\pi(1,b) \mid b \geq 0\}$, but only under the infinitesimal unbounded generator $\pi(Y)$. With this, we still get the correspondence $\pi \mapsto \pi^c_{\mathbf{K}_0}$ as described above.

We use the above notation. We know from Mackey's theory [**61**] that there are two inequivalent irreducible infinite-dimensional representations of G, and we shall need them in the following alternative formulations: Let $\mathcal{L}_\pm$ denote the respective Hilbert space $\mathbf{L}^2(\mathbb{R}_\pm)$ with the multiplicative invariant measure $d\mu_\pm = dp/|p|$, $p \in \mathbb{R}_\pm$. Then the formula

$$f \longmapsto e^{ipb} f(pa) \tag{13.2}$$

for functions f on $\mathbb{R}$ restricts to two unitary irreducible representations, denoted by $\pi_\pm$ of G on the respective spaces $\mathcal{L}_\pm$. Let $Q(f)(p) := f(-p)$ denote the canonical mapping from $\mathcal{L}_+$ to $\mathcal{L}_-$, or equivalently from $\mathcal{L}_-$ to $\mathcal{L}_+$. Then we have for $g \in G$ (cf. (13.1)):

$$Q\pi_+(g) = \pi_-(\tau(g))Q \tag{13.3}$$

For the representation $\pi := \pi_+ \oplus \pi_-$ on $\mathbf{H} := \mathcal{L}_+ \oplus \mathcal{L}_-$ we therefore have

$$J\pi(g) = \pi(\tau(g))J, \quad g \in G\,, \tag{13.4}$$

where J is the unitary involutive operator on $\mathbf{H}$ given by

$$J = \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix} . \tag{13.5}$$

Instead of the above p-realization of π we will mainly use the following x-formalism. The map $t \mapsto \pm e^t$ defines an isomorphism $L_\pm \colon \mathcal{L}_\pm \to \mathbf{L}^2(\mathbb{R})$, where we use the additive Haar measure dx on $\mathbb{R}$. For $g = (e^s, b) \in G$ and $f \in \mathbf{L}^2(\mathbb{R})$, set

$$(\pi_\pm(g)f)(x) := e^{\pm i e^x b} f(x+s), \quad x \in \mathbb{R}\,. \tag{13.6}$$

A simple calculation shows that $L_\pm$ intertwines the old and new construction of $\pi_\pm$, excusing our abuse of notation. In this realization Q becomes simply the identity operator $Q(f)(x) = f(x)$. The involution $J \colon \mathbf{L}^2(\mathbb{R}, \mathbb{C}^2)$ is now simply given by

$$J(f_0, f_1) = (f_1, f_0) \tag{13.7}$$

or $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In this formulation the operator

$$L := \pi_\pm(\Delta_H - \Delta_Q) = \pi_\pm(X^2 - Y^2) \tag{13.8}$$

takes the form

$$L = \left(\frac{d}{dx}\right)^2 + e^{2x}\,, \tag{13.9}$$

but it is on $\mathbf{L}^2(\mathbb{R})$ and $-\infty < x < \infty$. This operator is known to have defect indices $(1,0)$ [**39, 69**], which means that it cannot be extended to a selfadjoint operator on $\mathbf{L}^2(\mathbb{R})$. Using a theorem from [**39, 92**] we can see this by comparing the quantum mechanical problem for a particle governed by $-L$ as a Schrödinger operator (that is a strongly repulsive force) with the corresponding classical one governed (on each energy surface) by

$$E_{\text{kin}} + E_{\text{pot}} = \left(\frac{dx}{dt}\right)^2 - e^{2x} = E\,.$$

The escape time for this particle to $x = \pm\infty$ is

$$t_\pm = \int_{\text{finite}}^{\pm\infty} \frac{dx}{\sqrt{E + e^{2x}}}\,, \tag{13.10}$$

that is t_∞ is finite, and $t_{-\infty} = \infty$. We elaborate on this point below. The nonzero defect vector for the quantum mechanical problem corresponds to a boundary condition at $x = \infty$ since this is the singularity which is reached in finite time.

The fact from [**39**] we use for the defect index assertion is this: The Schrödinger operator $H = -\left(\frac{d}{dx}\right)^2 + V(x)$ for a single particle has nonzero defect solutions $f_\pm \in \mathbf{L}^2(\mathbb{R})$ to $H^* f_\pm = \pm i f_\pm$ iff there are solutions $t \mapsto x(t)$ to the corresponding classical problem

$$E = \left(\frac{dx(t)}{dt}\right)^2 + V(x(t))$$

with finite travel-time to $x = +\infty$, respectively, $x = -\infty$. The respective (possibly infinite) travel-times are

$$t_{\pm\infty} = \int_{\text{finite}}^{\pm\infty} \frac{dx}{\sqrt{E - V(x)}}\,.$$

The correspondence principle states that one finite travel-time to $+\infty$ (say) yields a dimension in the associated defect space, and similarly for the other travel-time to $-\infty$.

In the x-formalism, (13.3) from above then simplifies to the following identity for operators on the *same* Hilbert space $\mathbf{L}^2(\mathbb{R})$ (carrying the two inequivalent representations π_+ and π_-):

$$\pi_+(g) = \pi_-(\tau(g)), \quad g \in G\,. \tag{13.11}$$

We realize the representation $\pi = \pi_+ \oplus \pi_-$ in the Hilbert space $\mathbf{H} = \mathbf{L}^2(\mathbb{R}) \oplus \mathbf{L}^2(\mathbb{R}) = \mathbf{L}^2(X_2)$ where $X_2 = 0 \times \mathbb{R} \cup 1 \times \mathbb{R}$. We may represent J by an automorphism $\theta\colon X_2 \to X_2$ (as illustrated in Proposition 4.3):

$$\theta(0,x) := (1,x) \quad \text{and} \quad \theta(1,y) = (0,y)\,, \quad x,y \in \mathbb{R}\,,$$

and

$$J(f)(\omega) = f(\theta(\omega))\,, \quad \omega \in X_2\,.$$

Notice that the subset

$$X_2^\theta = \{\omega \in X_2 \mid \theta(\omega) = \omega\}$$

is empty. Define for $f \in \mathbf{L}^2(X_2)$, $f_k(x) = f(k,x)$, $k = 0,1$, $x \in \mathbb{R}$. We have for $g = (e^s, b) \in G$:

$$(\pi(g)f)_0(x) = e^{ibe^x} f_0(x+s) = (\bar{\pi}_+(g)f_0)(x)$$

and

$$(\pi(g)f)_1(x) = e^{-ibe^x} f_1(x+s) = (\bar{\pi}_-(g)f_1)(x)\,.$$

PROPOSITION 13.1. *Let $\pi = \pi_+ \oplus \pi_-$ be the representation from* (13.1)–(13.4) *above of the $(ax+b)$-group G. Then the only choices of reflections $\mathbf{K}_0$ as in Remark* 3.3 *for the sub-semigroup $S = \{(a,b) \in G \mid b > 0\}$ will have $\mathbf{K} = (\mathbf{K}_0/\mathbf{N})^{\sim}$ equal to* 0.

LEMMA 13.2. *Let Q be the projection in $\mathbf{L}^2(\mathbb{R}) \oplus \mathbf{L}^2(\mathbb{R})$ onto a translation-invariant J-positive subspace. Then Q is represented by a measurable field of 2×2 complex matrices $\mathbb{R} \ni \xi \mapsto (Q_{ij}(\xi))_{ij=1}^2$ such that $|Q_{12}(\xi)|^2 = Q_{11}(\xi)Q_{22}(\xi)$ a.e. on $\mathbb{R}$, and $Q_{12}(\xi) + Q_{21}(\xi) \geq 0$ a.e.; and conversely.*

COROLLARY 13.3. *These relations imply the following for the matrix Q:*

1) *If $Q_{12}(\xi) = 0$ then we have the three possibilities:*

$$Q(\xi) = 0\,,$$

$$Q(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\,, \quad \text{and}$$

$$Q(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\,.$$

In all those cases, we have $Q(\xi)JQ(\xi) = 0$.

2) *If $Q_{12}(\xi) \neq 0$, then $0 < Q_{22}(\xi) = 1 - Q_{11}(\xi) < 1$. Let $\mu(\xi) = Q_{12}(\xi)/Q_{11}(\xi)$. Then by $\operatorname{Tr}(Q(\xi)JQ(\xi)) \geq 0$ we have $\operatorname{Re}\mu(\xi) \geq 0$ and*

$$Q(\xi) = \frac{1}{1 + |\mu(\xi)|^2} \begin{pmatrix} 1 & \mu(\xi) \\ \overline{\mu(\xi)} & |\mu(\xi)|^2 \end{pmatrix}\,. \tag{13.12}$$

With $\lambda = \bar{\mu}$ we get that the image of $Q(\xi)$ is given by

$$\left\{ u(\xi) \begin{pmatrix} 1 \\ \lambda(\xi) \end{pmatrix} \,\middle|\, u(\xi) \in \mathbb{C} \right\}\,.$$

Specifying to our situation, $f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \in \mathbf{K}_0$ if and only if

$$\hat{f}_1(\xi) = \lambda(\xi)\hat{f}_0(\xi)\,. \tag{13.13}$$

Since $Q(\xi)$ is a measurable field of projections, the function $\mathbb{R} \ni \xi \mapsto \lambda(\xi)$ must be measurable, but it may be unbounded. This also means that $P_{\mathbf{K}_0}$ is the projection onto the graph of the operator $T_0\colon f_0 \mapsto f_1$ where f_0 and f_1 are related as in (13.13), and the Fourier transform $\hat{}$ is in the $\mathbf{L}^2$-sense.

The following argument deals with the general case, avoiding the separation of the proof into the two cases (I) and (II): If vectors $v \in \mathbf{K}_0$ are expanded as $v = \begin{pmatrix} h \\ k \end{pmatrix}$, $h = Q_{11}h + Q_{12}k$, $k = Q_{21}h + Q_{22}k$, we can introduce $\mathcal{D} =$

$\left\{ h \in \mathbf{L}^2(\mathbb{R}) \;\middle|\; \exists k \in \mathbf{L}^2(\mathbb{R}) \text{ s.t. } \begin{pmatrix} h \\ k \end{pmatrix} \in \mathbf{N} \right\}$. If $b > 0$, then:

$$\begin{aligned} \pi_+(b)h &= Q_{11}\pi_+(b)h + Q_{12}\pi_+(-b)k\,, \\ \pi_+(-b)k &= Q_{21}\pi_+(b)h + Q_{22}\pi_+(-b)k\,, \end{aligned}$$

valid for any $\begin{pmatrix} h \\ k \end{pmatrix} \in \mathbf{K}_0$, and $b \in \mathbb{R}_+$. So it follows from Lemma 4.1 again that $\mathcal{D}$ is invariant under $\{\pi_+(b) \mid b > 0\}$, and also under the whole semigroup $\{\pi_+(g) \mid g \in S\}$ where π_+ is now denoting the corresponding irreducible representation of G on $\mathbf{L}^2(\mathbb{R})$. Let

$$\mathcal{D}_\infty := \bigvee_{b\in\mathbb{R}} \pi_+(b)\mathcal{D}\,, \tag{13.14}$$

$$\mathcal{D}_{-\infty} := \bigwedge_{b\in\mathbb{R}} \pi_+(b)\mathcal{D}\,, \tag{13.15}$$

where $\bigvee$ and $\bigwedge$ denote the lattice operations on closed subspaces in $\mathbf{L}^2(\mathbb{R})$, and

$$\left(\pi_+(b)f\right)(x) = e^{ibe^x} f(x)\,, \quad f \in \mathbf{L}^2(\mathbb{R}),\ b, x \in \mathbb{R}.$$

We may now apply the Lax-Phillips argument to the spaces $\mathcal{D}_{\pm\infty}$. If $(\mathbf{K}_0/\mathbf{N})^\sim$ should be $\neq \{0\}$, then $\mathcal{D} = \{0\}$ by the argument. Since we are assuming $(\mathbf{K}_0/\mathbf{N})^\sim \neq \{0\}$, we get $\mathcal{D} = \{0\}$, and as a consequence the following operator graph representation for $\mathbf{K}_0$: $(\mathbf{K}_0/\mathbf{N})^\sim = \beta\left(G(L)\right)$ where $G(L)$ is the graph of a closed operator L in $\mathbf{L}^2(\mathbb{R})$. Specifically, this means that the linear mapping $\mathbf{K}_0/\mathbf{N} \ni \begin{pmatrix} h \\ k \end{pmatrix} + \mathbf{N} \mapsto h$ is well-defined as a linear closed operator. This in turn means that $\mathbf{K}_0$ may be represented as the graph of a closable operator in $\mathbf{L}^2(\mathbb{R})$ as discussed in the first part of the proof. Hence such a representation could have been assumed at the outset.

Remark 13.4. In a recent paper on local quantum field theory [**4**], Borchers considers in his Theorem II.9 a representation π of the $(ax+b)$-group G on a Hilbert space $\mathbf{H}$ such that there is a *conjugate linear* J (that is a period-2 antiunitary) such that $J\pi J = \pi \circ \tau$ where τ is the period-2 automorphism of G given by $\tau(a,b) := (a,-b)$. In Borchers's example, the one-parameter subgroup $b \mapsto \pi(1,b)$ has semibounded spectrum, and there is a unit-vector $v_0 \in \mathbf{H}$ such that $\pi(1,b)v_0 = v_0$, $\forall b \in \mathbb{R}$. The vector v_0 is cyclic and separating for a von Neumann algebra M such that $\pi(1,b)M\pi(1,-b) \subset M$, $\forall b \in \mathbb{R}_+$. Let $a = e^t$, $t \in \mathbb{R}$. Then, in Borchers's construction, the other one-parameter subgroup $t \mapsto \pi(e^t, 0)$ is the modular group Δ^{it} associated with the cyclic and separating vector v_0 (from Tomita-Takesaki theory [**5**, vol. I]). Finally, J is the corresponding modular conjugation satisfying $JMJ = M'$ when M' is the commutant of M.

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