Geometrically induced spectrum in curved leaky wires

P. Exner[a,b] and T. Ichinose[c]

a) Department of Theoretical Physics, Nuclear Physics Institute, Academy of Sciences, 25068 Řež, Czech Republic
b) Doppler Institute, Czech Technical University, Břehová 7, 11519 Prague, Czech Republic
c) Department of Mathematics, Faculty of Science, Kanazawa University, Kanazawa 920-1192, Japan
exner@ujf.cas.cz, ichinose@kappa.s.kanazawa-u.ac.jp

We study measure perturbations of the Laplacian in $L^2(\mathbb{R}^2)$ supported by an infinite curve Γ in the plane which is asymptotically straight in a suitable sense. We show that if Γ is not a straight line, such a "leaky quantum wire" has at least one bound state below the threshold of the essential spectrum.

1 Introduction

The aim of the present paper is to elucidate some geometrically induced spectral properties for the Laplacian in $L^2(\mathbb{R}^2)$ perturbed by a negative multiple of the Dirac measure of an infinite curve Γ in the plane.

This problem has at least two motivations. On the physics side we note that quantum mechanics of electrons confined to narrow tubelike regions has attracted a considerable interest, because such systems represent a natural model for semiconductor "quantum wires". In some examples the region in question is a strip or tube with hard walls – see, *e.g.*, [DE] and references therein – while other treatments assume even stronger localization to a curve or a graph – a rich bibliography to such models can be found in [KS]. Various interesting spectral effects were found in such a setting related to the geometry and topology of the underlying restricted configuration space. One of them, of a relevance for the present paper, is the existence of curvature-induced bound states in Dirichlet tubes observed for the first time more than a decade ago [EŠ].

On the other hand, the said models are certainly idealized as far as the nature of the confinement is concerned. In actual quantum wires, the electrons are trapped due to interfaces between two different semiconductor materials which represents a finite potential jump. Hence if two parts of a quantum wire are close to each other, a quantum tunneling is possible between them. The idealization thus makes an important difference, because without it one expects the spectral properties to be determined by the *global* geometry of the wire. At the same time, it is not *a priori* clear whether effects like the curvature-induced binding mentioned above will persist if a tunneling is allowed, because the techniques used to demonstrate them make essential use of the strict spatial localization.

Here we address the last question in the weak-coupling setting when the confinement is realized transversally by an attractive δ interaction [AGHH]. We will show that if such a confining interaction is supported by a non-straight curve which is, however, straight asymptotically in the sense which we make precise below, the corresponding Hamiltonian has a nontrivial discrete spectrum. This is our main result expressed by Theorem 5.2. Moreover, we will show in Theorem 4.1 that such Hamiltonians can be approximated in the norm-resolvent sense by a family of Schrödinger operators with regular potentials of the form of a bounded and infinitely stretched "ditch". Consequently, the approximating operators exhibit bound states too provided the ditch is squeezed enough.

On the other hand, the technique we employ to demonstrate these results may represent some mathematical interest. It is basically the Birman-Schwinger formalism in the form extended to measure-perturbed Laplacians in [BEKŠ]. In the present case, however, we deal with the situation where the operator appearing in the BS-kernel is not compact. Our treatment shows that one can nevertheless get a useful information, if the operator in question decomposes into a sum of two parts, of which one is an operator with a known spectrum and the other is its compact perturbation.

2 Generalized Schrödinger operators

The Hamiltonians we are going to study are generalized Schrödinger operators with a singular interaction supported by a zero-measure set. Let us first recall several facts about such operators. They are borrowed from the paper [BEKŠ] and we specify them to our present purpose by assuming the configuration space dimension $d = 2$ and the coupling "strength" constant on the interaction support.

Consider a positive Radon measure m on $\mathbb{R}^2$ and a number $\alpha > 0$ such that

$$(1+\alpha)\int_{\mathbb{R}^2} |\psi(x)|^2\,\mathrm{d}m(x) \le a\int_{\mathbb{R}^2} |\nabla\psi(x)|^2\,\mathrm{d}x + b\int_{\mathbb{R}^2} |\psi(x)|^2\,\mathrm{d}x \tag{2.1}$$

holds for all $\psi \in \mathcal{S}(\mathbb{R}^2)$ and some $a < 1$ and b. The map I_m defined by $I_m\psi = \psi$ on $\mathcal{S}(\mathbb{R}^2)$ extends by density uniquely to

$$I_m :\; W_{1,2}(\mathbb{R}^2) \;\to\; L^2(m) := L^2(\mathbb{R}^2, m)\;; \tag{2.2}$$

for the sake of brevity we employ the same symbol for a continuous function and the corresponding equivalence classes in both $L^2(\mathbb{R}^2)$ and $L^2(m)$. The inequality (2.1) extends to $W_{1,2}(\mathbb{R}^2)$ with ψ replaced by $I_m\psi$ at the l.h.s.

The operators we are interested in are introduced by means of the following quadratic form,

$$\mathcal{E}_{-\alpha m}(\psi,\phi) := \int_{\mathbb{R}^2} \overline{\nabla\psi(x)}\nabla\phi(x)\,\mathrm{d}x - \alpha\int_{\mathbb{R}^2} (I_m\bar{\psi})(x)(I_m\phi)(x)\,\mathrm{d}m(x)\,, \tag{2.3}$$

with the domain $W_{1,2}(\mathbb{R}^2)$. It is straightforward to see [BEKŠ] that under the condition (2.1) this form is closed and below bounded, with $C_0^\infty(\mathbb{R}^2)$ as a core, and consequently, it is associated with a unique self-adjoint operator denoted as $H_{-\alpha m}$. The condition (2.1) is satisfied, in particular, if the measure m belongs to the generalized Kato class

$$\lim_{\epsilon\to 0}\,\sup_{x\in\mathbb{R}^2}\int_{B(x,\epsilon)} |\ln|x-y||\,\mathrm{d}m(x) = 0\,, \tag{2.4}$$

where $B(x,\epsilon)$ is the ball of radius ϵ and center x. Moreover, any positive number can be in this case chosen as a.

For operators of the described type the generalized Birman-Schwinger principle is valid. If k^2 belongs to the resolvent set of $H_{-\alpha m}$ we put $R^k_{-\alpha m} :=$

$(H_{-\alpha m}-k^2)^{-1}$. The free resolvent R_0^k is defined for $\mathrm{Im}\, k>0$ as an integral operator with the kernel

$$G_k(x-y)=\frac{i}{4}\, H_0^{(1)}(k|x-y|)\,. \tag{2.5}$$

Next we need embedding operators associated with R_0^k. Let μ,ν be arbitrary positive Radon measures on $\mathbb{R}^2$ with $\mu(x)=\nu(x)=0$ for any $x\in\mathbb{R}^2$. By $R_{\nu,\mu}^k$ we denote the integral operator from $L^2(\mu):=L^2(\mathbb{R}^2,\mu)$ to $L^2(\nu)$ with the kernel G_k, *i.e.*

$$R_{\nu,\mu}^k\phi=G_k*\phi\mu$$

holds ν-a.e. for all $\phi\in D(R_{\nu,\mu}^k)\subset L^2(\mu)$. In our case the two measures will be the m introduced above and the Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}^2$ in different combinations. With this notation one can express the generalized BS principle as follows [BEKŠ]:

Proposition 2.1 *(i) There is a $\kappa_0>0$ such that the operator $I-\alpha R_{m,m}^{i\kappa}$ on $L^2(m)$ has a bounded inverse for any $\kappa\ge\kappa_0$.*
(ii) Let $\mathrm{Im}\, k>0$. Suppose that $I-\alpha R_{m,m}^k$ is invertible and the operator

$$R^k:=R_0^k+\alpha R_{\mathrm{d}x,m}^k[I-\alpha R_{m,m}^k]^{-1}R_{m,\mathrm{d}x}^k$$

from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ is everywhere defined. Then k^2 belongs to $\rho(H_{-\alpha m})$ and $(H_{-\alpha m}-k^2)^{-1}=R^k$.
(iii) $\dim\ker(H_{-\alpha m}-k^2)=\dim\ker(I-\alpha R_{m,m}^k)$ for any k with $\mathrm{Im}\, k>0$.

3 Formulation of the problem

After this preliminary we will specify a class of operators which we discuss in the following, where the measure m will be the Dirac measure supported by a curve. Suppose that $\tilde\gamma:\mathbb{R}\to\mathbb{R}^2$ is a *continuous, piecewise C^1 smooth* function; its graph is a curve denoted as Γ. We can define its arc length,

$$s[\xi_1,\xi_2]:=\int_{\xi_1}^{\xi_2}\sqrt{\dot{\tilde\gamma}_1^2+\dot{\tilde\gamma}_2^2}\,\mathrm{d}\xi\,,$$

which is the natural parametrization of Γ: for a fixed ξ_1, $s[\xi_1,\cdot]$ is strictly increasing and piecewise smooth, so there is a unique inverse function ξ :

$\mathbb{R} \to \mathbb{R}$ with the same properties, and we can define $\gamma := \tilde{\gamma} \circ \xi$. In what follows we characterize the curve Γ always by the function γ. Since γ maps continuously into $\mathbb{R}^2$, we have

$$|\gamma(s) - \gamma(s')| \le |s - s'| \tag{3.1}$$

for any $s, s' \in \mathbb{R}$. In addition, we shall assume:

(a1) there is $c \in (0,1)$ such that $|\gamma(s) - \gamma(s')| \ge c|s - s'|$. In particular, Γ has no cusps and self-intersections, and its possible asymptotes are not parallel to each other.

(a2) Γ is asymptotically straight in the following sense: there are positive d, μ, and $\omega \in (0,1)$ such that the inequality

$$1 - \frac{|\gamma(s) - \gamma(s')|}{|s - s'|} \le d \left[1 + |s + s'|^{2\mu}\right]^{-1/2} \tag{3.2}$$

holds true in the sector $S_\omega := \left\{(s, s') :\ \omega < \frac{s}{s'} < \omega^{-1}\right\}$.

The operator we are interested in is a generalized Schrödinger operator with the interaction localized at the curve which can be formally written as

$$H_{\alpha,\gamma} = -\Delta - \alpha\delta(x - \Gamma)\,. \tag{3.3}$$

This definition can be given meaning if we identify $H_{\alpha,\gamma}$ with $H_{-\alpha m}$ of the preceding section, where m is the Dirac measure on Γ, or more exactly,

$$m :\ m(M) = \ell_1(M \cap \Gamma) \tag{3.4}$$

for any Borel $M \subset \mathbb{R}^2$, where ℓ_1 is the one-dimensional Hausdorff measure; for a piecewise smooth curve it is given, of course, by the arc length.

One has to make sure, of course, that the measure (3.4) satisfies the condition (2.1). This follows from Thm. 4.1 of [BEKŠ] if γ is continuous, piecewise C^1, and satisfies the assumption *(a1)*. Consequently, we may employ Proposition 2.1 for investigation of the resolvent of $H_{\alpha,\gamma}$.

4 Leaky wires as weakly coupled waveguides

Before proceeding further we want to show that the operators (3.3) can be regarded as weak-coupling approximation to a class of Schrödinger operators.

Let Γ be again an infinite planar curve described by the function γ. Now we shall make a stronger assumption, namely that γ is C^2. Then we can define the (signed) curvature $k(s) := (\gamma_1'\gamma_2'' - \gamma_2'\gamma_1'')(s)$; we shall assume that it is bounded, $|k(s)| < c_+$ for some $c_+ > 0$ and all $s \in \mathbb{R}$. We employ the conventional symbol believing that the context will never allow to mix the curvature with the momentum variable. On the other hand, we will not impose the requirements *(a1), (a2)*. It is sufficient to assume that Γ has neither self-intersections nor "near-intersections", *i.e.*, that there is a $c_- > 0$ such that $|\gamma(s)-\gamma(s')| \ge c_-$ for any s, s' with $|s-s'| \ge c_-$.

Under these assumptions we are able to define in the vicinity of Γ a locally orthogonal system of coordinates: a point is characterized by the pair (s,u), where u is the (signed) distance from Γ measured along the appropriate normal $n(s)$ and s is the arc-length coordinate of the point of Γ where the normal is taken. It is easy to see that the curvilinear coordinates are well defined and unique in the strip neighbourhood of the curve, $\Sigma_\epsilon := \{x(s,u) : (s,u) \in \Sigma_\epsilon^0\}$, where

$$x(s,u) := \gamma(s) + n(s)u \tag{4.1}$$

and $\Sigma_\epsilon^0 := \{(s,u) : \; s \in \mathbb{R}, |u| < \epsilon\}$ is the straightened strip, as long as the condition $2\epsilon < c_-$ is valid. If there is no danger of misunderstanding, we shall write simply x instead of $x(s,u)$.

With these prerequisites we are able to construct the mentioned family of Schrödinger operators. Given $W \in L^\infty((-1,1))$, we define for all $\epsilon < \frac{1}{2}c_-$ the transversally scaled potential,

$$V_\epsilon(x) := \left\{ \begin{array}{ccc} 0 & \dots & x \notin \Sigma_\epsilon \\ -\frac{1}{\epsilon} W\left(\frac{u}{\epsilon}\right) & \dots & x \in \Sigma_\epsilon \end{array} \right. \tag{4.2}$$

and put

$$H_\epsilon(W,\gamma) := -\Delta + V_\epsilon \,. \tag{4.3}$$

The operators $H_\epsilon(W,\gamma)$ are obviously self-adjoint on $D(-\Delta) = W_{2,2}(\mathbb{R}^2)$ and the corresponding resolvent can be expressed in the Birman-Schwinger way,

$$\begin{aligned} \left(H_\epsilon(W,\gamma) - k^2\right)^{-1} &= \left(-\Delta - k^2\right)^{-1} \\ &- \left(-\Delta - k^2\right)^{-1} V_\epsilon^{1/2} \left[I + |V_\epsilon|^{1/2} \left(-\Delta - k^2\right)^{-1} V_\epsilon^{1/2}\right]^{-1} |V_\epsilon|^{1/2} \left(-\Delta - k^2\right)^{-1} \end{aligned} \tag{4.4}$$

for any $k^2 \in \rho\left(H_\epsilon(W,\gamma)\right) \cap \rho(-\Delta)$, where we have used the usual convention, $V_\epsilon^{1/2} := |V_\epsilon|^{1/2} \mathrm{sgn}\,(V_\epsilon)$.

Then we have the following approximation result the proof of which is given in the appendix:

Theorem 4.1 *With the stated assumptions,* $H_\epsilon(W,\Gamma) \to H_{\alpha,\gamma}$ *as* $\epsilon \to 0$*, where* $\alpha = \int_{-1}^{1} W(t)\,\mathrm{d}t$*, in the norm-resolvent sense.*

5 Curvature-induced discrete spectrum

Let us return now to the spectral analysis of the operator $H_{\alpha,\gamma}$. If Γ is a straight line corresponding to $\gamma_0(s) = as + b$ for some $a, b \in \mathbb{R}^2$ with $|a| = 1$, we can separate variables and show that

$$\sigma(H_{\alpha,\gamma_0}) = \left[-\frac{1}{4}\alpha^2, \infty\right) \tag{5.1}$$

is purely absolutely continuous. The aim of the present section is to show that for a non-straight Γ of the class specified in Sec. 3, $\sigma(H_{\alpha,\gamma})$ has a nonempty discrete component. Let us start with the essential spectrum.

Proposition 5.1 *Let* $\alpha > 0$ *and suppose that* $\gamma : \mathbb{R} \to \mathbb{R}^2$ *is a continuous, piecewise* C^1 *function satisfying (a1), (a2); then* $\sigma_{ess}(H_{\alpha,\gamma}) = \left[-\frac{1}{4}\alpha^2, \infty\right)$.

Proof: We shall show in a while that $\sigma(\mathcal{R}^\kappa_{\alpha,\gamma_0}) = [0, \alpha/2\kappa]$ holds for $\mathcal{R}^\kappa_{\alpha,\gamma} := \alpha R^{i\kappa}_{m,m}$ referring to $\gamma = \gamma_0$. In view of Lemma 5.4 below the same interval is contained in the spectrum of $\mathcal{R}^\kappa_{\alpha,\gamma}$, and thus by Proposition 2.1 no point of the interval $\left(-\frac{1}{4}\alpha^2, 0\right)$ belongs to the resolvent set of the operator $H_{\alpha,\gamma}$. Consequently, $\sigma_{ess}(H_{\alpha,\gamma}) \supset \left[-\frac{1}{4}\alpha^2, 0\right]$. By the same compact-perturbation argument we find that apart of a discrete set corresponding to eigenvalues of a finite multiplicity, the points $-\kappa^2$ with $\kappa > \frac{1}{2}\alpha$ belong to $\rho\left(H_{\alpha,\gamma}\right)$, so the interval $(-\infty, -\frac{1}{4}\alpha^2)$ is not contained in the essential spectrum.

It remains to deal with the positive halfline. First we notice that for any $R > 0$ one can find a disc $B_R \subset \mathbb{R}^2$ of radius R which does not intersect with Γ, for otherwise we may take a family of such discs centered at the points $(3n_1R, 0)$ and $(0, 3n_2R)$ with $n_1, n_2 \in \mathbb{Z}$, and any curve intersecting with all of them would violate the assumption *(a2)*.

Let $\phi \in C_0^\infty([0,2))$ with $\phi(r) \geq 0$ and $\int_{\mathbb{R}^2} \phi(|x|)^2 dx = 1$. Given $n \in \mathbb{Z}_0$ and $p, x_n \in \mathbb{R}^2$, we define

$$\psi_n(x; p, x_n) := \frac{1}{n}\phi\left(\frac{1}{n}|x - x_n|\right) \mathrm{e}^{ipx}\,.$$

The functions ψ_n are normalized and easily seen to provide for an appropriate sequence $\{x_n\} \subset \mathbb{R}^2$ with $|x_n| \to \infty$ a Weyl sequence of the free Hamiltonian H_0 corresponding to the point $|p|^2$ of its essential spectrum. Choosing now the sequence $\{x_n\}$ in such a way that the discs $B_{2n}(x_n)$ are mutually disjoint and do not intersect with Γ, we have $H_{\alpha,\gamma}\psi_n(\cdot;p,x_n) = H_0\psi_n(\cdot;p,x_n)$. In this way, we have constructed a Weyl sequence to $H_{\alpha,\gamma}$ for any point of $[0,\infty)$ concluding thus the proof. ∎

Now we can state our main result:

Theorem 5.2 *Adopt the assumptions of the previous proposition. If the inequality (3.1) is sharp for some $s, s' \in \mathbb{R}$, then $H_{\alpha,\gamma}$ has at least one isolated eigenvalue below $-\frac{1}{4}\alpha^2$.*

Proof: By Proposition 2.1 we look for solutions of the equation $\mathcal{R}^{\kappa}_{\alpha,\gamma}\psi = \psi$, where $\mathcal{R}^{\kappa}_{\alpha,\gamma} := \alpha R^{i\kappa}_{m,m}$ is an integral operator on $L^2(\mathbb{R})$ with the kernel

$$\mathcal{R}^{\kappa}_{\alpha,\gamma}(s,s') = \frac{\alpha}{2\pi}\, K_0\left(\kappa|\gamma(s)-\gamma(s')|\right)\ ;$$

here K_0 is the Macdonald function; recall that $K_0(z) = \frac{\pi i}{2} H_0^{(1)}(iz)$. The idea is to compare this operator with $\mathcal{R}^{\kappa}_{\alpha,\gamma_0}$ having the kernel in which $|\gamma(s)-\gamma(s')|$ is replaced by $|s-s'|$.

The Fourier transformation takes $K_0(\kappa x)$ to $(\pi/2)^{1/2}(p^2+\kappa^2)^{-1/2}$. The well known relation $f(-i\nabla)\psi = (2\pi)^{-1/2}(\mathcal{F}^{-1}f)*\psi$ then shows that $\mathcal{R}^{\kappa}_{\alpha,\gamma_0}$ is unitarily equivalent to the operator of multiplication by $\frac{1}{2}\alpha(p^2+\kappa^2)^{-1/2}$ on $L^2(\mathbb{R})$. Consequently, it is absolutely continuous and its spectrum is $[0,\alpha/2\kappa]$ in correspondence with (5.1).

We can obtain the spectrum of H_{α,γ_0} directly, of course, as pointed out above. Now we we want to know how the spectrum of $\mathcal{R}^{\kappa}_{\alpha,\gamma_0}$ changes under the perturbation $\mathcal{D}_\kappa := \mathcal{R}^{\kappa}_{\alpha,\gamma} - \mathcal{R}^{\kappa}_{\alpha,\gamma_0}$. Notice that

$$\mathcal{D}_\kappa(s,s') := \frac{\alpha}{2\pi}\Big(K_0\left(\kappa|\gamma(s)-\gamma(s')|\right) - K_0\left(\kappa|s-s'|\right)\Big) \ge 0 \qquad (5.2)$$

holds for the kernel of $\mathcal{D}_\kappa$ in view of (3.1) and the monotonicity of K_0.

Lemma 5.3 $\sup\sigma\left(\mathcal{R}^{\kappa}_{\alpha,\gamma}\right) > \frac{\alpha}{2\kappa}$ *if Γ is not straight.*

Proof: It is sufficient to find a real-valued $\psi \in \mathcal{S}(\mathbb{R})$ such that

$$\left(\psi, \mathcal{R}^{\kappa}_{\alpha,\gamma}\psi\right) - \frac{\alpha}{2\kappa}\|\psi\|^2 > 0\,,$$

which is equivalent to

$$\frac{2\kappa}{\alpha}\int_{\mathbb{R}^2}\mathcal{D}_\kappa(s,s')\,\psi(s)\psi(s')\,\mathrm{d}s\,\mathrm{d}s' \\ +\frac{\kappa}{\pi}\int_{\mathbb{R}^2}K_0\left(\kappa|s-s'|\right)\psi(s)\psi(s')\,\mathrm{d}s\,\mathrm{d}s' - \int_{\mathbb{R}}\psi(s)^2\mathrm{d}s > 0\,.$$

Using the above observation together with the Parseval relation we can rewrite the last two terms at the r.h.s. as

$$\int_{\mathbb{R}}\frac{\kappa}{\sqrt{p^2+\kappa^2}}\,|\hat\psi(p)|^2\,\mathrm{d}p - \int_{\mathbb{R}}|\hat\psi(p)|^2\,\mathrm{d}p\,.$$

Choosing

$$\psi(s)=\sqrt[4]{\frac{2\lambda^2}{\pi}}\;e^{-\lambda^2 s^2}\,,$$

we find by a direct computation that two terms equal

$$-\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(1-\frac{\kappa}{\sqrt{u^2\lambda^2+\kappa^2}}\right)e^{-u^2/2}\,\mathrm{d}u = -\frac{1}{\sqrt{2\pi}}\frac{\lambda^2}{2\kappa^2}\int_{\mathbb{R}}u^2\,e^{-u^2/2}\,\mathrm{d}u+\mathcal{O}(\lambda^3)\,.$$

On the other hand, the inequality in (5.2) is sharp in an open subset of $\mathbb{R}^2$ if Γ is not straight, so the first term is

$$\sqrt{\frac{2}{\pi}}\,\lambda\int_{\mathbb{R}^2}\frac{2\kappa}{\alpha}\,\mathcal{D}_\kappa(s,s')\,e^{-\lambda^2(s^2+s'^2)}\,\mathrm{d}s\,\mathrm{d}s' \ge c\lambda$$

for some $c>0$ as $\lambda\to 0+$. Hence the above ψ is the sought trial function for λ small enough. ∎

Next we shall show that perturbation (5.2) is compact under the assumption *(a2)*, and thus it can change only the discrete spectrum of $\mathcal{R}^\kappa_{\alpha,\gamma}$.

Lemma 5.4 *$\mathcal{D}_\kappa$ is Hilbert-Schmidt if $\mu>\frac{1}{2}$.*

Proof: For the sake of brevity, we denote

$$\varrho\equiv\varrho(s,s') := \kappa|\gamma(s)-\gamma(s')|\,,\quad \sigma\equiv\sigma(s,s') := \kappa|s-s'|\,.$$

To estimate $K_0(\varrho)-K_0(\sigma)$ we use convexity of K_0 together with the relation $K_0'(z)=-K_1(z)$,

$$K_1(\sigma)(\sigma-\varrho)\le K_0(\varrho)-K_0(\sigma)\le \varrho K_1(\varrho)\frac{\sigma-\varrho}{\varrho}\,. \tag{5.3}$$

Hence the kernel of $\mathcal{D}_\kappa$ is bounded, because $\varrho \mapsto \varrho K_1(\varrho)$ is bounded in $(0,\infty)$ and the inequality $c\sigma \le \varrho \le \sigma$ yields

$$0 \le \frac{\sigma - \varrho}{\varrho} \le \frac{1-c}{c}\,. \tag{5.4}$$

Moreover, there is $c_1 > 0$ such that

$$\varrho K_1(\varrho) \le c_1\, e^{-\varrho/2} \le c_1\, e^{-c\sigma/2}\,, \tag{5.5}$$

and by *(a2)* we have

$$\frac{\sigma - \varrho}{\varrho} \le \frac{\sigma - \varrho}{c\sigma} \le \frac{d}{c}\,\left[1 + |s+s'|^{2\mu}\right]^{-1/2} \tag{5.6}$$

in the sector S_ω. Putting together the inequalities (5.3)–(5.6) we can estimate the Hilbert-Schmidt norm of the operator in question:

$$\begin{aligned} \left(\frac{2\kappa}{\alpha}\right)^2 \int_{\mathbb{R}^2} \mathcal{D}_\kappa(s,s')^2\, \mathrm{d}s\, \mathrm{d}s' &\le \left(\frac{1-c}{c}\right)^2 c_1^2 \int_{\mathbb{R}^2\setminus S_\omega} e^{-c\kappa|s-s'|}\, \mathrm{d}s\, \mathrm{d}s' \\ &+ \left(\frac{c_1 d}{c}\right)^2 \int_{S_\omega} \frac{e^{-c\kappa|s-s'|}}{1+|s+s'|^{2\mu}}\, \mathrm{d}s\, \mathrm{d}s' \\ &\le \left(2c_1 \frac{1-c}{c}\right)^2 \frac{1+\omega}{1-\omega} \int_0^\infty u\, e^{-\sqrt{2}c\kappa u}\, \mathrm{d}u + \left(\frac{c_1 d}{c}\right)^2 \int_{\mathbb{R}^2} \frac{e^{-c\kappa|s-s'|}}{1+|s+s'|^{2\mu}}\, \mathrm{d}s\, \mathrm{d}s', \end{aligned} \tag{5.7}$$

which is finite for $\mu > \frac{1}{2}$. ∎

Finally, we need the following continuity result.

Lemma 5.5 *With the above stated assumptions, the function* $\kappa \mapsto \mathcal{R}^\kappa_{\alpha,\gamma}$ *is operator-norm continuous and* $\mathcal{R}^\kappa_{\alpha,\gamma} \to 0$ *as* $\kappa \to \infty$.

Proof: Using the above established equivalence between $\mathcal{R}^\kappa_{\alpha,\gamma_0}$ and the multiplication by $\frac{1}{2}\,\alpha[p^2+\kappa^2]^{-1/2}$ we easily check the claim for the "free" operator, so it is sufficient to show that the perturbation $\mathcal{D}_\kappa$ has the same properties. The inequality

$$|(\mathcal{D}_\kappa - \mathcal{D}_{\kappa'})\,(s,s')|^2 \le 2\left[\mathcal{D}_\kappa(s,s')^2 + \mathcal{D}_{\kappa'}(s,s')^2\right] \le 4\mathcal{D}_{\kappa_0}(s,s')^2$$

valid for any $\kappa_0 \le \min(\kappa,\kappa')$ allows us to use the dominated convergence by which

$$\|\mathcal{D}_\kappa - \mathcal{D}_{\kappa'}\|_{\mathrm{HS}} \to 0 \qquad \text{as} \qquad \kappa' \to \kappa\,.$$

Finally, the estimate (5.7) shows at the same time that

$$\|\mathcal{D}_\kappa\|_{\mathrm{HS}} \to 0 \qquad \text{as} \qquad \kappa \to \infty\,,$$

which concludes the proof. ∎

Proof of Theorem 5.2, continued: By Lemma 5.3 $\sup\sigma\left(\mathcal{R}^{\kappa}_{\alpha,\gamma}\right) > \frac{\alpha}{2\kappa}$ holds whenever Γ is not straight. On the other hand, the essential spectrum of $\mathcal{R}^{\kappa}_{\alpha,\gamma_0}$ is by Lemma 5.4 preserved under the geometric perturbation, so $\mathcal{R}^{\kappa}_{\alpha,\gamma}$ has in $\left(\frac{\alpha}{2\kappa},\infty\right)$ just isolated eigenvalues; in combination with the previous result we infer that at least one such eigenvalue $\lambda_{\alpha,\gamma}(\kappa)$ of $\mathcal{R}^{\kappa}_{\alpha,\gamma}$ exists for any $\kappa > 0$. Finally, by Lemma 5.5 the function $\lambda_{\alpha,\gamma}(\cdot)$ is continuous and $\lim_{\kappa\to\infty}\lambda_{\alpha,\gamma}(\kappa) = 0$. Hence there is a point $\kappa_0 > \frac{1}{2}\alpha$ such that $\lambda_{\alpha,\gamma}(\kappa_0) = 1$, and therefore, recalling that $\mathcal{R}^{\kappa}_{\alpha,\gamma} = \alpha R^{i\kappa}_{m,m}$, we infer by Proposition 2.1 that $-\kappa_0^2$ is an eigenvalue of the operator $H_{\alpha,\gamma}$. ∎

Remark 5.6 One asks naturally how strong is the asymptotic restriction imposed by *(a2)*. To answer this question, suppose that γ is C^2 smooth. The Γ can be described – uniquely up to Euclidean transformations of the plane – by its signed curvature $k(s)$. Using the standard expression of γ in terms of k we can estimate

$$\begin{aligned}|\gamma(s)-\gamma(s')| &= \Bigg[\left(\int_{s'}^{s}\cos\left(\int_{s'}^{s_1}k(s_2)\,\mathrm{d}s_2\right)\mathrm{d}s_1\right)^2 \\ &\quad + \left(\int_{s'}^{s}\sin\left(\int_{s'}^{s_1}k(s_2)\,\mathrm{d}s_2\right)\mathrm{d}s_1\right)^2\Bigg]^{1/2} \\ &\geq \int_{s'}^{s}\cos\left(\int_{s'}^{s_1}k(s_2)\,\mathrm{d}s_2\right)\mathrm{d}s_1 \geq \int_{s'}^{s}\left[1-\frac{1}{2}\left(\int_{s'}^{s_1}k(s_2)\,\mathrm{d}s_2\right)^2\right]\mathrm{d}s_1\,,\end{aligned}$$

where we have assumed $s > s'$ without loss of generality; hence

$$1-\frac{|\gamma(s)-\gamma(s')|}{|s-s'|} \leq \frac{1}{2|s-s'|}\int_{s'}^{s}\left(\int_{s'}^{s_1}k(s_2)\,\mathrm{d}s_2\right)^2\mathrm{d}s_1\,.$$

Suppose that $|k(s)| \leq c_2|s|^{-\beta}$ for some $\beta > 0$, then the r.h.s. of the last inequality can be estimated by

$$\frac{1}{2|s-s'|}\,\frac{c_2^2}{|s|^{2\beta}}\int_{s'}^{s}(s_1-s')^2\mathrm{d}s_1 \;\leq\; \frac{c_2^2}{|s'|^{2\beta}}\,\frac{|s-s'|^2}{6} \;\leq\; \frac{c_2^2 s^2}{6|s'|^{2\beta}} \;\leq\; \frac{c_2^2}{6\omega^2}\,|s'|^{2-2\beta}\,.$$

Consequently, *(a2)* with $\mu > \frac{1}{2}$ holds for $\beta > \frac{5}{4}$. This is a slightly stronger restriction than for curved Dirichlet strips [DE] where $\beta > 1$ is sufficient.

Appendix

To prove Theorem 4.1 we have to show that (4.4) approximates the resolvent of the formal operator (3.3) which we have identified with $H_{-\alpha m}$. We will write the resolvents in question in a way similar to that used for the analogous purpose in [AGHH, Sec. I.3]. The first term at the r.h.s. of (4.4) is ϵ-independent and subtracts in the difference. The action of the second one on a vector $\psi \in L^2(\mathbb{R}^2)$ can be written as

$$
\begin{aligned}
-\iiint_{\mathbb{R}^2} & G_k(x-x')V_\epsilon^{1/2}(x')\left[I+|V_\epsilon|^{1/2}R_0^k V_\epsilon^{1/2}\right]^{-1}(x',x'')|V_\epsilon|^{1/2}(x'') \\
&\times G_k(x''-x''')\,\psi(x''')\,\mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}x''' \\
= \iint_\Sigma \int_{\mathbb{R}^2} & G_k\left(x-x(s',u')\right)\frac{1}{\epsilon}W^{1/2}\left(\frac{u'}{\epsilon}\right) \\
&\times \epsilon\left[I+|V_\epsilon|^{1/2}R_0^k V_\epsilon^{1/2}\right]^{-1}(s',u';s'',u'')\,\frac{1}{\epsilon}\left|W\left(\frac{u''}{\epsilon}\right)\right|^{1/2} \\
&\times G_k\left(x'''-x(s'',u'')\right)\,(1+u'k(s'))(1+u''k(s'')) \\
&\times \psi(x''')\,\mathrm{d}s'\,\mathrm{d}u'\,\mathrm{d}s''\,\mathrm{d}u''\,\mathrm{d}x''' \qquad \text{(A.1)}
\end{aligned}
$$

where $x(s,u)$ is given by (4.1) and $(1+uk(s))$ is the Jacobian of the transformation between the Cartesian and curvilinear coordinates. Changing the integration variables to $t' := u'/\epsilon$ and $t'' := u''/\epsilon$ we can rewrite the last expression as

$$
\begin{aligned}
\iint_\Sigma \int_{\mathbb{R}^2} & G_k\left(x-\gamma(s')-n(s')\epsilon t'\right)\,W^{1/2}\left(t'\right) \\
&\times \epsilon\left[I+|V_\epsilon|^{1/2}R_0^k V_\epsilon^{1/2}\right]^{-1}(s',\epsilon t';s'',\epsilon t'')\,\left|W\left(t''\right)\right|^{1/2} \\
&\times G_k\left(x'''-\gamma(s'')-n(s'')\epsilon t''\right)\,(1+\epsilon t'k(s'))(1+\epsilon t''k(s'')) \\
&\times \psi(x''')\,\mathrm{d}s'\,\mathrm{d}u'\,\mathrm{d}s''\,\mathrm{d}u''\,\mathrm{d}x'''\,.
\end{aligned}
$$

If $\||V_\epsilon|^{1/2}R_0^k V_\epsilon^{1/2}\| < 1$, the inverse can be written as a geometric series with the integral-operator kernel

$$
\begin{aligned}
&\epsilon\left[I+|V_\epsilon|^{1/2}R_0^k V_\epsilon^{1/2}\right]^{-1}(s',\epsilon t';s'',\epsilon t'') \\
&\quad = \delta(s'-s'')\,\delta(t'-t'') - |W(s',t')|^{1/2}G_k(s',\epsilon t';s'',\epsilon t'')W(s'',t'')^{1/2} + \dots
\end{aligned}
$$

Consequently, the operator given by (A.1) can be written as the product $B_\epsilon(I-C_\epsilon)^{-1}\tilde{B}_\epsilon$ of operators mapping $L^2(\mathbb{R}^2) \to L^2(\Sigma_1^0) \to L^2(\Sigma_1^0) \to L^2(\mathbb{R}^2)$, with the following kernels

$$\begin{aligned} B_\epsilon(x;s',t') &:= G_k\left(x-x(s',\epsilon t')\right)\,(1+\epsilon t' k(s'))W\left(t'\right)^{1/2}\,, \\ \tilde{B}_\epsilon(s,t;x') &:= |W\left(t\right)|^{1/2}(1+\epsilon t k(s))\,G_k\left(x'-x(s,\epsilon t)\right), \\ C_\epsilon(s,t;s',t') &:= |W\left(t\right)|^{1/2} G_k(x(s,\epsilon t)-x(s',\epsilon t'))W\left(t'\right)^{1/2}\,. \end{aligned}$$

We have $\|C_\epsilon\| \le \|W\|_\infty \|P_1 R_0^k P_1\| \le \|W\|_\infty |k|^{-2}$ for $k = i\kappa$ with $\kappa > 0$, where P_1 is the projection onto $L^2(\Sigma_1^0) \subset L^2(\mathbb{R}^2)$, hence $\|C_\epsilon\| \le \text{const} < 1$ holds for κ large enough uniformly w.r.t. ϵ, and the operator in question equals

$$B_\epsilon(I-C_\epsilon)^{-1}\tilde{B}_\epsilon = \sum_{j=0}^{\infty} B_\epsilon C_\epsilon^j \tilde{B}_\epsilon\,. \tag{A.2}$$

Let us now turn to the resolvent of $H_{\alpha,\gamma}$. Since the operator $I-\alpha R^k_{m,m}$ is by Proposition 2.1 boundedly invertible with for $k = i\kappa$ with κ large enough, we can again write its second terms as a geometric series. Furthermore, $\alpha = (W^{1/2}, |W|^{1/2})$ by assumption, so we have

$$\begin{aligned} \alpha R^k_{\mathrm{d}x,m} \sum_{j=0}^{\infty} \left(\alpha R^k_{m,m}\right)^j R^k_{m,\mathrm{d}x} &= R^k_{\mathrm{d}x,m}(W^{1/2}, |W|^{1/2}) R^k_{m,\mathrm{d}x} \\ &+ R^k_{\mathrm{d}x,m}(W^{1/2}, |W|^{1/2}) R^k_{m,m}(W^{1/2}, |W|^{1/2}) R^k_{m,\mathrm{d}x} + \dots \\ &= \sum_{j=0}^{\infty} B C^j \tilde{B}\,, \end{aligned} \tag{A.3}$$

where $B, C, \tilde{B}$ are operators between the same spaces as their indexed counterparts above given by their integral kernels:

$$\begin{aligned} B(x;s',t') &:= G_k\left(x-\gamma(s')\right)\,W\left(t'\right)^{1/2}\,, \\ \tilde{B}(s,t;x') &:= |W\left(t\right)|^{1/2}\,G_k\left(x'-\gamma(s)\right), \\ C(s,t;s',t') &:= |W\left(t\right)|^{1/2} G_k(\gamma(s)-\gamma(s'))W\left(t'\right)^{1/2}\,. \end{aligned}$$

Let us stress that while these operators depend on W, the expression (A.3) contains just the integral of the approximating potential, which is why the limit does not depend on a particular shape of W. The operator norm of the difference between (A.2) and (A.3) can be estimated by means of the telescopic trick,

$$\left\|B_\epsilon(I-C_\epsilon)^{-1}\tilde B_\epsilon - B(I-C)^{-1}\tilde B\right\| \le \sum_{n=0}^{\infty}\Big\{\|B_\epsilon - B\|\|C_\epsilon\|^n\|\tilde B_\epsilon\|$$

$$+\|B\|\sum_{\ell=0}^{n-1}\|C\|^\ell\|C_\epsilon - C\|\|C_\epsilon\|^{n-\ell-1}\|\tilde B_\epsilon\| + \|B\|\|C\|^n\|\tilde B_\epsilon - \tilde B\|\Big\},$$

where the second term at the r.h.s. is conventionally put to zero if $n=0$. As above, we have $\|R_0^k\| \le |k|^{-2}$ for $-ik=\kappa>0$, with $\|W^{1/2}\|\le\|W\|_\infty^{1/2}$ and $|1+\epsilon t k(s)| \le 1+\epsilon\|k\|_\infty < 1+\|k\|_\infty$, hence for k^2 large enough negative there is a positive $c_3<1$ such that

$$\max\{\|B\|,\|B_\epsilon\|,\|C\|,\|C_\epsilon\|,\|\tilde B\|,\|\tilde B_\epsilon\|\} \le c_3$$

holds for any $\epsilon\in(0,1)$. Consequently, the norm in question is estimated by

$$\Big\{\|B_\epsilon - B\| + \|\tilde B_\epsilon - \tilde B\|\Big\}\sum_n c_3^{n+1} + \|C_\epsilon - C\|\sum_n n\, c_3^{n+1},$$

so it is sufficient to investigate the three norms involved here. Consider the first one which we can estimate as follows,

$$\|B_\epsilon - B\| \le \|W\|_\infty^{1/2}\left\{(1+\|k\|_\infty)\left\|R^k_{\Sigma,\epsilon}-R^k_{\Sigma,0}\right\| + \epsilon\|k\|_\infty\left\|R^k_{\Sigma,0}\right\|\right\},$$

where $R^k_{\Sigma,\epsilon}, R^k_{\Sigma,0}$ are the resolvent factors in this expression, *i.e.*, integral operators $L^2(\Sigma_1^0)\to L^2(\mathbb{R}^2)$ with kernels $G_k\left(x-x(s',\epsilon t')\right)$ and $G_k\left(x-\gamma(s')\right)$, respectively. To show that $R^k_{\Sigma,\epsilon}\to R^k_{\Sigma,0}$ in the operator-norm topology, let us rewrite the kernel of the difference by means of the mean value theorem,

$$\begin{aligned} &G_k\left(x-x(s',\epsilon t')\right) - G_k\left(x-\gamma(s')\right)\\ &= \frac{1}{2\pi}\Big[K_0\left(\kappa|x-x(s',\epsilon t')|\right) - K_0\left(\kappa|x-\gamma(s')|\right)\Big]\\ &= -\frac{\epsilon t'}{2\pi}\int_0^1 K_1\left(\kappa|x-\gamma(s')-n(s')\epsilon t'\vartheta|\right)\kappa\left(\frac{d}{d\vartheta}\,\mathrm{dist}(x,\gamma(s')+n(s')\epsilon t'\vartheta)\right)\mathrm{d}\vartheta\,. \end{aligned}$$

Since the last factor does not exceed one in modulus, we have

$$\left|\left(R^k_{\Sigma,\epsilon}-R^k_{\Sigma,0}\right)(x,x(s',\epsilon t')\right| \le \frac{\epsilon\kappa|t'|}{2\pi}\int_0^1 K_1\left(\kappa|x-\gamma(s')-n(s')\epsilon t'\vartheta|\right)\mathrm{d}\vartheta\,. \tag{A.4}$$

This makes it possible to estimate the quantity

$$\begin{aligned} h_\infty &:= \sup_{x\in\mathbb{R}^2} \int_{\mathbb{R}} \mathrm{d}s' \int_{-1}^{1} \mathrm{d}t' \left| \left(R^k_{\Sigma,\epsilon} - R^k_{\Sigma,0} \right) (x, x(s', \epsilon t') \right| \\ &\le \frac{\epsilon\kappa}{2\pi} \sup_{x\in\mathbb{R}^2} \int_{\Sigma^0_1} K_1\left(\kappa|x - x(\sigma'))|\right) \mathrm{d}\sigma' \\ &\le \frac{\epsilon\kappa}{2\pi} \sup_{x\in\mathbb{R}^2} \int_{\mathbb{R}^2} K_1\left(\kappa|x - x')|\right) \mathrm{d}x' = \frac{\epsilon\kappa}{2\pi} \, \|K_1(\kappa|\cdot|)\|_{L^1(\mathbb{R}^2)} \,, \end{aligned}$$

where the r.h.s. is finite, because the function $K_1(\kappa|\cdot|)$ decays exponentially at large distances and has the integrable singularity $|\cdot|^{-1}$ at the origin. In the same way we find

$$h_1 := \sup_{x'\in\Sigma_1} \int_{\mathbb{R}^2} \left| \left(R^k_{\Sigma,\epsilon} - R^k_{\Sigma,0} \right) (x, x') \right| \mathrm{d}x \le \frac{\epsilon\kappa}{2\pi} \, \|K_1(\kappa|\cdot|)\|_{L^1(\mathbb{R}^2)} \,.$$

The norm under consideration can be the estimated by the corresponding Schur-Holmgren bound – see, *e.g.*, [Ka, Ex. III.3.2] – as

$$\left\| R^k_{\Sigma,\epsilon} - R^k_{\Sigma,0} \right\| \le (h_1 h_\infty)^{1/2} \le \frac{\epsilon\kappa}{2\pi} \, \|K_1(\kappa|\cdot|)\|_{L^1(\mathbb{R}^2)} \,,$$

so it tends to zero as $\epsilon \to 0$. Analogous estimates are valid for $\|\tilde{B}_\epsilon - \tilde{B}\|$ and $\|C_\epsilon - C\|$ which concludes the proof.

Remark: With our goal in mind we examined the situation when the approximating potential depends on the transverse variable only. If we replace it by $W \in L^\infty(\Sigma^0_1)$, the analogous argument shows that corresponding family (4.3) converges in the norm-resolvent sense to the operator $-\Delta + \alpha(s)\delta(x - \gamma(s))$ with $\alpha(s) := \int_{-1}^{1} W(s, u)\, \mathrm{d}u$, which is properly defined by a quadratic form similar to (2.3) – see [BEKŠ].

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