Sternberg Theorem for Equivariant Hamiltonian Vector Fields

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Introduction

The well known Sternberg Theorem (see [1]) asserts that if two local smooth vector fields are formally conjugate at a hyperbolic singularity then they are smoothly conjugate. This result reduces local classification and normalization problems to the formal ones.

In the last years there is a splash of interest to systems with symmetries and anti-symmetries (*reversible systems*) (for references see [2]). This activity faces, in particular, a similar "equivariant" problem: could one provide a conjugacy (smooth, formal) of two symmetric or anti-symmetric vector fields via a transformation keeping the property? The linear and the formal aspects of the problem were considered in [3, 4]. In [5] the authors proved a related version of Smooth Conjugacy Sternberg Theorem.

A similar question arised for hamiltonian systems: is it possible to conjugate two hamiltonian vector fields via a *canonical* coordinate change? The affirmative answer was given in [6, 7, 8].

These two results are in the same streamline and lead to the next very natural setting: given two hamiltonian (anti-)symmetric vector fields can one choose a conjugacy which preserves both symmetric and symplectic structures? The aim of the present paper is to prove a hamiltonian equivariant version of Sternberg Theorem (Theorem 2.1 below).

We use the so-called *deformation method* or *homotopy trick* going back to J.Mather (see [9, 10, 8]). This method reduces a nonlinear problem on equivalence of two vector fields (or more generally, of two local mappings near a common singularity) to that on solving a linear functional equation (*cohomology equation*).

1 Definitions

By a *local diffeomorphism* (*local vector field*) we mean in what follows either a germ of diffeomorphisms (vector fields) at a point or a representative of this germ defined in a neighborhood of the point. Sometimes, we omit for briefness the word "local".

Let $\mathfrak{G}$ be a group of local diffeomorphisms $\Phi : (\mathbb{R}^{2d}, 0) \to (\mathbb{R}^{2d}, 0)$ and $\sigma : \mathfrak{G} \to \mathbb{R}$ be a *multiplicative character* of $\mathfrak{G}$, *i.e.*, a homomorphism into the multiplicative group $\mathbb{R}^*$.

Definition 1.1. *A local vector field ξ is said to be $(\mathfrak{G}, \sigma)$-equivariant if*

$$U_*\xi = \sigma(U)\xi \quad (U \in \mathfrak{G}).$$

In this definition, $U_*\xi$ denotes a usual action of the local diffeomorphism U on the vector field ξ:

$$(U_*\xi)(x) = DU(U^{-1}x)\xi(U^{-1}x),$$

where $DU(x)$ is the Jacobi matrix.

A local transformation G keeps the property of ξ to be $(\mathfrak{G}, \sigma)$-equivariant if it commutes with every element $U \in \mathfrak{G}$, *i.e.*, $UG = GU$. Such a local diffeomorphism is said to be $\mathfrak{G}$-*equivariant.*

Let $\mathbb{R}^{2d}$ be provided with the *symplectic structure*, *i.e.*, with a *closed non-degenerate 2-form* ω. The vector field ξ is said to be *symplectic* if $L_\xi\omega = 0$. The *hamiltonian* F, $i_\xi(\omega) = dF$, is a function determined up to a constant. Each function may serve as a hamiltonian of some symplectic vector field.

Let ω be a symplectic form on $\mathbb{R}^{2d}$. According to a theorem due to Darboux, there exist coordinates $(u, v) = (u_1, \ldots, u_d, v_1, \ldots, v_d)$ on $\mathbb{R}^{2d}$ such that

$$\omega(u, v) = du_1 \wedge dv_1 + \ldots + du_d \wedge dv_d.$$

This form is called *standard.* The changes of variables which respect this form, *i.e.*,

$$G^*\omega = \omega$$

are called *canonical.* By $G^*\omega$ we denote a usual action of the local diffeomorphism G on the symplectic form ω.

Definition 1.2. *A symplectic form ω is said to be $(\mathfrak{G}, \sigma)$-equivariant if*

$$U^*\omega = \sigma(U)\omega \quad (U \in \mathfrak{G}).$$

In what follows, we assume that $\mathfrak{G}$ is *a compact group of linear operators in* $\mathbb{R}^{2d}$ and σ *is continuous.* Without loss of generality $\mathfrak{G}$ is supposed to be a closed subgroup of the group $O(2d, \mathbb{R})$ of all orthogonal transformaions. Thus, every element $U \in \mathfrak{G}$ keeps the standard inner product in $\mathbb{R}^{2d}$. And besides, the continuity of σ implies

$$\sigma(U) = \pm 1 \quad (U \in \mathfrak{G}).$$

2 Main result

Recall that two local vector fields ξ and η are said to be C^k conjugate if there exists a C^k diffeomorphism G such that

$$G_*\xi = \eta.$$

Take a $(\mathfrak{G},\sigma)$-equivariant symplectic form ω and a local $(\mathfrak{G},\sigma)$-equivariant symplectic C^∞ vector field ξ, $\xi(0) = 0$, $D\xi(0) = \Lambda$. Choose canonical coordinates bringing ω to the standard form. Let the subspaces $E_u = \{v = 0\}$ and $E_v = \{u = 0\}$ be invariant with respect to the phase flow of ξ.

Suppose Λ has no pure imaginary eigenvalues, *i.e.*, ξ has a hyperbolic singularity at the origin. It is known that $\Lambda = IS$, where S is a symplectic linear operator and

$$I = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}.$$

According to Williamson's theorem (see [11]), there exists a canonical linear transformation which brings $\Lambda = IS$ to the form

$$\begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix}$$

where all the eigenvalues of the matrix R have positive real parts.

Denote by δ and Δ the minimum and the maximum of the real parts of the eigenvalues of the matrix R. Given an integer k put

$$\rho(k,\Lambda) = k\frac{\Delta}{\delta} + k + 1.$$

Theorem 2.1. *Let η be a local $(\mathfrak{G},\sigma)$-equivariant symplectic C^K vector field, $\eta(0) = 0$, $K \geq \rho(k,\Lambda)$, and all derivatives of the difference $\xi - \eta$ up to the order K vanish at the origin. Then there is a local $\mathfrak{G}$-equivariant canonical C^k diffeomorphism G which conjugates ξ and η, i.e. $G_*\xi = \eta$.*

In particular, if $K = \infty$ then $k = \infty$ as well.

This theorem is the main result of the paper. Its proof is given in Sections 3 – 4.

3 Deformation method.

As we have mentioned above, the deformation method reduces a conjugacy problem to that of solving a multidimensional linear equation.

Given vector fields ξ and η consider the *deformation* $h(x,\varepsilon) \equiv \varepsilon\eta(x) + (1-\varepsilon)\xi(x)$ and the vector field

$$\Xi \equiv h(x,\varepsilon)\frac{\partial}{\partial x} + 0\cdot\frac{\partial}{\partial \varepsilon}$$

on $\mathbb{R}^{n+1}$. Take another vector field

$$\Phi(x,\varepsilon) \;\equiv\; \varphi(x,\varepsilon)\frac{\partial}{\partial x} \;+\; 1\cdot\frac{\partial}{\partial\varepsilon}$$

Let

$$Q^t(x,\varepsilon) \;=\; (q^t(x,\varepsilon),\varepsilon), \quad P^t(x,\eta) \;=\; (p^t(x,\varepsilon),\varepsilon \;+\; t)$$

be the pase flows of Ξ and Φ respectively. Put $G(x) \;=\; p^1(x,0)$.

Lemma 3.1. *Let the vector fields Φ and Ξ commute, i.e.*

$$[\,\Phi,\Xi\,] \;=\; 0. \tag{1}$$

Then G conjugates ξ and η, i.e., $G_*\xi \;=\; \eta$.

Proof. Note that the phase flows of ξ and η are, respectively, $q_0^t(x) \;\equiv\; q^t(x,0)$ and $q_1^t(x) \;\equiv\; q^t(x,1)$. Since the phase flows P^t and Q^t commute then

$$Q^{t_2}(P^{t_1}(x,\varepsilon)) = (q^{t_2}(p^{t_1}(x,\varepsilon),\varepsilon+t_1),\varepsilon+t_1) = (p^{t_1}(q^{t_2}(x,\varepsilon),\varepsilon),\varepsilon+t_1) = P^{t_1}(Q^{t_2}(x,\varepsilon)),$$

or

$$q^{t_2}(p^{t_1}(x,\varepsilon),\varepsilon+t_1) = p^{t_1}(q^{t_2}(x,\varepsilon),\varepsilon).$$

For $\varepsilon=0$, $t_1=1$, $t_2=t$ this gives

$$q^t(p^1(x,0),1) = p^1(q^t(x,0),0).$$

For $G(x)=p^1(x,0)$ we get

$$q^t(G(x),1) = G(q^t(x,0)),$$

or

$$q_1^t(G(x)) = G(q_0^t(x)),$$

i.e., G is a conjugacy between ξ and η. □

Condition (1) is equivalent to the equation

$$D_1\varphi(x,\varepsilon)\cdot h(x,\varepsilon) \;=\; D_1h(x,\varepsilon)\cdot\varphi(x,\varepsilon) \;+\; \eta(x) \;-\; \xi(x), \tag{2}$$

where

$$D_1 \;\equiv\; \frac{\partial}{\partial x}.$$

This equation is called the *cohomology equation.*

Now let us turn to the case where the vector fields ξ and η are symplectic. In order to find a canonical conjugacy we make use of the relationship between vector fields and differential forms as suggested in [7].

Let ω be a symplectic form and ξ and η be vector fields which preserve ω. Fix ε and put $h_\varepsilon(x) \;=\; h(x,\varepsilon)$, $\varphi_\varepsilon(x) \;=\; \varphi(x,\varepsilon)$. It is not hard to verify that

$$i_{[h_\varepsilon,\varphi_\varepsilon]}(\omega) = d(i_{h_\varepsilon}(i_{\varphi_\varepsilon}(\omega))),$$

therefore (2) implies that

$$d(i_{h_\varepsilon}(i_{\varphi_\varepsilon}(\omega))) = i_{\eta-\xi}(\omega).$$

Let X and Y be the hamiltonians of the vector fields ξ and η. Denote by H_ε the hamiltonian of the vector field φ_ε we are seeking. Then

$$d(i_{h_\varepsilon}(dH_\varepsilon)) = dY - dX.$$

Taking into account the equality

$$L_{h_\varepsilon} H_\varepsilon = i_{\xi_\varepsilon}(dH_\varepsilon) + di_{h_\varepsilon}(H_\varepsilon)$$

we get

$$dL_{h_\varepsilon} H_\varepsilon = di_{h_\varepsilon}(dH_\varepsilon),$$

whence

$$d(L_{h_\varepsilon} H_\varepsilon - Y + X) = 0.$$

It suffices to find the hamiltonian H_ε from the equation

$$L_{h_\varepsilon} H_\varepsilon = Y - X$$

or

$$D_1 H(x,\varepsilon) \cdot h(x,\varepsilon) = Y - X \tag{3}$$

which is called the *cohomology equation for hamiltonians.*

Thus, in order to prove that two vector fields which preserve the symplectic form are conjugate via a canonical diffeomorphism we have to solve the cohomology equation (3), then find the corresponding vector field φ_ε and integrate the system $\dot{x} = \varphi_\varepsilon(x),\ \dot{\varepsilon} = 1$.

4 $\mathfrak{G}$-equivariant conjugacy

First of all, note that solvability of equation (3) under the assumption of Theorem 2.1 was proved in [8] (for $k < \infty$) and [5] (for $k = \infty$). Prove that the solution can be choosed $(\mathfrak{G}, \sigma)$-equivariant.

Let ω be a $(\mathfrak{G}, \sigma)$-equivariant symplectic form. Take a matrix $U \in \mathfrak{G}$,

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are $d \times d$ matrices. Since ω is in the standard form then (see, for example, [12] Proposition 5.5.6) A^tC and B^tD are symmetric and $A^tD - C^tB = \sigma(U)E$. This gives, in particular, that

$$U^{-1} = \sigma(U) \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}.$$

Lemma 4.1. *Let ξ be a symplectic vector field and its hamiltonian F is $(\mathfrak{G}, \sigma)$-equivariant. Then ξ is $\mathfrak{G}$-equivariant.*

Proof. Let F be a $(\mathfrak{G},\sigma)$-equivariant hamiltonian. Denote

$$\xi(u,v)=\sum_{i=1}^{d}\left(\eta_i(u,v)\frac{\partial}{\partial u_i}+\zeta_i(u,v)\frac{\partial}{\partial v_i}\right).$$

By tansition formulas,

$$\eta_i(u,v)=\frac{\partial F(u,v)}{\partial v_i},\qquad \zeta_i(u,v)=-\frac{\partial F(u,v)}{\partial u_i}\qquad (i=1,\dots,d).$$

Fix a matrix $U\in\mathfrak{G}$. Since $F(u,v)=\sigma(U)F(Au+Bv,Cu+Dv)$ then

$$\eta(u,v)=\sigma(U)(D^t\eta(Au+Bv,Cu+Dv)-B^t\zeta(Au+Bv,Cu+Dv)),$$

$$\zeta(u,v)=\sigma(U)(-C^t\eta(Au+Bv,Cu+Dv)+A^t\zeta(Au+Bv,Cu+Dv)).$$

Thus we conclude that

$$\xi\;=\;U^{-1}\,\xi\circ U,$$

i.e., ξ is $\mathfrak{G}$-equivariant. □

Lemma 4.2. *If a symplectic vector field ξ with the hamiltonian F is $(\mathfrak{G},\sigma)$-equivariant then F is $\mathfrak{G}$-equivariant.*

Proof. Let ξ be a $(\mathfrak{G},\sigma)$-equivariant symplectic vector field. Fix $U\in\mathfrak{G}$. Since ω is $\mathfrak{G}$-equivariant then

$$\sigma(U)U^{-1}=\begin{pmatrix} D^t & -B^t\\ -C^t & A^t\end{pmatrix}.$$

Therefore,

$$\xi(x)=\begin{pmatrix} D^t & -B^t\\ -C^t & A^t\end{pmatrix}\xi(Ux).$$

Put $\bar u=Au+Bv$, $\bar v=Cu+Dv$. Then

$$\begin{aligned}F(u,v)&=\sum_{i=1}^{d}\int_0^1[\eta^i(ut,vt)v_i-\zeta^i(ut,vt)u_i]dt=\sum_{i=1}^{d}\int_0^1[\eta^i(\bar ut,\bar vt)\bar v_i-\zeta^i(\bar ut,\bar vt)\bar u_i]dt\\&=F(Au+Bv,Cu+Dv).\end{aligned}$$

Thus, F is $\mathfrak{G}$-equivariant. □

Lemma 4.3. *If H is a C^k solution of equation* (3) *then for every $U\in\mathfrak{G}$ the function $\tilde H$*

$$\tilde H(x,\varepsilon)\equiv\sigma(U)H(Ux,\varepsilon),$$

is a solution of equation (3) *as well.*

Proof. Put $\tilde H(x,\varepsilon)\equiv\sigma(U)H(Ux,\varepsilon)$. By virtue of $(\mathfrak{G},\sigma)$-equivariance of ξ and η, the following equalities are true:

$$\begin{aligned}Uh(x,\varepsilon)&=\sigma(U)h(Ux,\varepsilon);\\(Y-X)(x,\varepsilon)&=(Y-X)(Ux,\varepsilon).\end{aligned}$$

Hence,

$$\begin{aligned} D_1\tilde{H}(x,\varepsilon)h(x,\varepsilon) &= \sigma(U)D_1H(Ux,\varepsilon)Uh(x,\varepsilon) \\ &= \sigma(U)D_1H(Ux,\varepsilon)\sigma(U)h(Ux,\varepsilon) \\ &= D_1H(Ux,\varepsilon)h(Ux,\varepsilon) \\ &= (Y-X)(Ux,\varepsilon) \\ &= (Y-X)(x,\varepsilon). \end{aligned}$$

Therefore, $\tilde{H}$ is a solution of the cohomology equation. □

It is well known that since $\mathfrak{G}$ is compact then one can find a Haar measure μ on $\mathfrak{G}$ which is right invariant with respect to $\mathfrak{G}$. We can suppose without loss of generality that $\mu(\mathfrak{G}) = 1$. Let H be a C^k solution of equation (3). Define a C^k function $\mathcal{H} : \mathbb{R}^{2d+1} \to \mathbb{R}$ by

$$\mathcal{H}(x,\varepsilon) = \int_{\mathfrak{G}} \sigma(U)H(Ux,\varepsilon)d\mu(U) \qquad (x \in \mathbb{R}^{2d},\ \varepsilon \in \mathbb{R}). \tag{4}$$

Lemma 4.4. *The function $\mathcal{H}$ is a $(\mathfrak{G},\sigma)$-equivariant solution of the cohomology equation* (3).

Proof. By linearity of the cohomology equation, it follows from Lemma 4.3 that $\mathcal{H}$ is a solution. Since μ is right invariant, for any $V \in \mathfrak{G}$ we have

$$\mathcal{H}(Vx,\varepsilon) = \int_{\mathfrak{G}} \sigma(U)H(UVx,\varepsilon)d\mu(U) =$$

$$= \sigma(V)\int_{\mathfrak{G}} \sigma(V)\sigma(U)H(UVx,\varepsilon)d\mu(U) = \sigma(V)\int_{\mathfrak{G}} \sigma(W)H(Wx,\varepsilon)d\mu(W) = \sigma(V)\mathcal{H}(x,\varepsilon).$$

Here we put $W = UV$. As a result we get

$$\mathcal{H}(Vx,\varepsilon) = \sigma(V)\mathcal{H}(x,\varepsilon),$$

i.e., $\mathcal{H}$ is $(\mathfrak{G},\sigma)$-equivariant. □

Proof of Theorem 2.1. Take a local C^k solution H of equation (3). Then the function $\mathcal{H}$ given by formula (4) is a local $(\mathfrak{G},\sigma)$-equivariant C^k solution of (3). Let $F^t(x,\varepsilon) \equiv (f^t(x,\varepsilon), \varepsilon + t)$ be the phase flow of the vector field

$$\Psi(x,\varepsilon) = \psi(x,\varepsilon)\frac{\partial}{\partial x} + 1\cdot\frac{\partial}{\partial \varepsilon},$$

where $\psi(\cdot,\varepsilon)$ is a symplectic vector field generated by the hamiltonian $\mathcal{H}(\cdot,\varepsilon)$. Put $G(x) = f^1(x,0)$. Since $[\Psi,\Xi] = 0$ then, by Lemma 3.1, the local C^k diffeomorphism G conjugates ξ and η. Show that G is $\mathfrak{G}$-equivariant and canonical. In fact, since G is a shift along the trajectories of the symplectic phase flow it presrves the symplectic structure. Note that the vector field $\psi(\cdot,\varepsilon)$ is $\mathfrak{G}$-equivariant. Choose $U \in \mathfrak{G}$ and denote $\tilde{f}^t(x,\varepsilon) \equiv U^{-1}f^t(Ux,\varepsilon)$. Then

$$\left.\frac{d\tilde{f}^t(x,\varepsilon)}{dt}\right|_{t=0} = U^{-1}\left.\frac{df^t(Ux,\varepsilon)}{dt}\right|_{t=0} = U^{-1}\psi(Ux,\varepsilon) = \psi(x,\varepsilon).$$

Hence $\tilde{F}^t(x,\varepsilon) \equiv (\tilde{f}^t(x,\varepsilon),\varepsilon + t)$ is also a phase flow of the vector field Ψ. It follows from Uniqueness Theorem that $\tilde{F}^t = F^t$ and $G(x) = U^{-1}G(Ux)$. We conclude that G is $\mathfrak{G}$-equivariant completing the proof of Theorem 2.1. $\square$

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