Generalised Brownian Motion and Second Quantisation

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Abstract

A new approach to the generalised Brownian motion introduced by M. Bożejko and R. Speicher is described, based on symmetry rather than deformation. The symmetrisation principle is provided by Joyal's notions of tensorial and combinatorial species. Any such species V gives rise to an endofunctor $\mathcal{F}_V$ of the category of Hilbert spaces with contractions mapping a Hilbert space $\mathcal{H}$ to a symmetric Hilbert space $\mathcal{F}_V(\mathcal{H})$ with the same symmetry as the species V. A general framework for annihilation and creation operators on these spaces is developed and shown to give vacuum expectations as prescribed by Bożejko and Speicher. The existence of the second quantisation as functor from Hilbert spaces to von Neumann algebras with completely positive maps is investigated. For a certain one parameter interpolation between the classical and the free Brownian motion it is shown that the "field algebras" $\Gamma(\mathcal{K})$ are type II_1 factors when $\mathcal{K}$ is infinite dimensional.

1 Introduction

In non-commutative probability theory one is interested in finding generalisations of classical probabilistic concepts such as independence and processes with independent stationary increments. Motivated by a central limit theorem result and by the analogy with classical Brownian motion, M. Bożejko and R. Speicher proposed in [3] a class of operator algebras called "generalised Brownian motions" and investigated an example of interpolation between the classical [18] and the free motion of Voiculescu [23]. A better known interpolation is provided by the "q-deformed commutation relations" [2, 4, 5, 6, 7, 8, 16, 24].

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Such an operator algebra is obtained by performing the GNS representation of the free tensor algebra $\mathcal{A}(\mathcal{K})$ over an arbitrary infinite dimensional real Hilbert space $\mathcal{K}$, with respect to a "Gaussian state" $\tilde{\rho}_{\mathbf{t}}$ defined by the following "pairing prescription":

$$\tilde{\rho}_{\mathbf{t}}(\omega(f_1)\dots\omega(f_n)) = \begin{cases} 0 & \text{if } n \text{ odd} \\ \sum\limits_{\mathcal{V}\in\mathcal{P}_2(n)} \mathbf{t}(\mathcal{V}) \prod\limits_{(k,l)\in\mathcal{V}} \langle f_k, f_l\rangle & \text{if } n \text{ even} \end{cases} \tag{1.1}$$

where $f_i \in \mathcal{K}, \omega(f_i) \in \mathcal{A}(\mathcal{K})$ and the sum runs over all pair partitions of the ordered set $\{1, 2, \dots, n\}$. The functional is uniquely determined by the complex valued function $\mathbf{t}$ on pair partitions. Classical Brownian motion is obtained by taking $\mathcal{K} = \mathrm{L}^2(\mathbb{R}_+)$ and $B_s := \omega(\mathbf{1}_{[0,s)})$ with the constant function $\mathbf{t}(\mathcal{V}) = 1$ on all pair partitions; the free Brownian motion [23] requires $\mathbf{t}$ to be 0 on crossing partitions and 1 on non-crossing partitions.

If one considers complex Hilbert spaces, the analogue of a Gaussian state is called a Fock state. We show that the GNS representation of the free algebra $\mathcal{C}(\mathcal{H})$ of creation and annihilation operators with respect to a Fock state $\rho_{\mathbf{t}}$ can be described in a functorial way inspired by the notions of tensorial species of Joyal [12, 13]: the representation space has the form

$$\mathcal{F}_{\mathbf{t}}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \frac{1}{n!} V_n \otimes_s \mathcal{H}^{\otimes n} \tag{1.2}$$

where V_n are Hilbert spaces carrying unitary representations of the symmetric groups $\mathrm{S}(n)$ and $\otimes_s$ means the subspace of the tensor product containing vectors which are invariant under the double action of $\mathrm{S}(n)$. The creation operators have the expression:

$$a^*_{\mathbf{t}}(h)\ v \otimes_s (h_0 \otimes \dots \otimes h_{n-1}) = (j_n v) \otimes_s (h_0 \otimes \dots \otimes h_{n-1} \otimes h_n) \tag{1.3}$$

where $j_n : V_n \to V_{n+1}$ is an operator which intertwines the action of $\mathrm{S}(n)$ and $\mathrm{S}(n+1)$.

In Section 3 we connect these Fock representations with positive functionals on a certain algebraic object $\mathcal{BP}_2(\infty)$ which we call the *-semigroup of "broken pair partitions". The elements of this *-semigroup can be described graphically as segments located between two vertical lines which cut through the graphical representation of a pair partition. In particular, the pair partitions are elements of $\mathcal{BP}_2(\infty)$. We show that if $\rho_{\mathbf{t}}$ is a Fock state then the function $\mathbf{t}$ has a natural extension to a positive functional $\hat{\mathbf{t}}$ on $\mathcal{BP}_2(\infty)$. The GNS-like representation with respect to $\hat{\mathbf{t}}$ provides the combinatorial data $(V_n, j_n)_{n=0}^{\infty}$ associated to $\rho_{\mathbf{t}}$.

The representation of $\mathcal{A}(\mathcal{K})$ with respect to a Gaussian state $\tilde{\rho}_{\mathbf{t}}$ is a *-algebra generated by "fields" $\omega_{\mathbf{t}}(f)$. Monomials of such fields can be seen as moments, with the corresponding cumulants being a generalisation of the Wick products known from the q-deformed Brownian motion [2]. Using generalised Wick products we prove that any Gaussian state $\tilde{\rho}_{\mathbf{t}}$ extends to a Fock state $\rho_{\mathbf{t}}$ on the algebra of creation and annihilation operators $\mathcal{C}(\mathcal{K}_{\mathbb{C}})$ (see section 4).

Second quantisation is a functor from the category of real Hilbert spaces with contractions to the category of (non-commutative) probability spaces. For any real Hilbert space $\mathcal{K}$ we consider first the *-algebra $\tilde{\Gamma}_{\mathbf{t}}(\mathcal{K})$ of generalised Wick products. In a standard manner we associate to it a von Neumann algebra $\Gamma_{\mathbf{t}}(\mathcal{K})$ and investigate the possibility of making $\Gamma_{\mathbf{t}}$ into a functor of second quantisation. A necessary condition is that the function $\mathbf{t}$ has the multiplicative property, a form of statistical independence. A sufficient condition is the separating property of the vacuum. By modifying the definition of the "field algebras" we obtain another second quantisation functor denoted $\Delta_{\mathbf{t}}$ this time for *all* multiplicative (positive definite) functions $\mathbf{t}$.

In the last section we develop a useful criterion, in terms of the spectrum of a characteristic contraction, for factoriality of the algebras $\Gamma_{\mathbf{t}}(\ell^2(\mathbb{Z}))$ in the case when the vacuum state $\rho_{\mathbf{t}}$ is tracial. We then apply it to a particular example of positive definite function $\mathbf{t}_q$ where $0 \leq q < 1$, which interpolates between the bosonic and free cases and has been introduced in [3] (see [10] for another proof of the positivity). We conclude that that $\Gamma_{\mathbf{t}}(\ell^2(\mathbb{Z}))$ is a type II_1 factor. Further generalisation of this criterion to factors of type III will be investigated in a forthcoming paper [11].

2 Definitions and description of the Fock representation

The generalised Brownian motions [3] are representations with respect to special *gaussian* states on free algebras over real Hilbert spaces. We start by giving all necessary definitions and subsequently we will analyse the structure of the *Fock representations* which are intimately connected with the generalised Brownian motion (see section 4) .

Definition 2.1 Let $\mathcal{K}$ be a real Hilbert space. The algebra $\mathcal{A}(\mathcal{K})$ is the free unital *-algebra with generators $\omega(h)$ for all $h \in \mathcal{K}$, divided by the relations:

$$\omega(af + bg) = a\omega(f) + b\omega(g), \qquad \omega(f) = \omega(f)^* \tag{2.1}$$

for all $f, g \in \mathcal{K}$ and $a, b \in \mathbb{R}$.

Definition 2.2 Let $\mathcal{H}$ be a complex Hilbert space. The algebra $\mathcal{C}(\mathcal{H})$ is the free unital *-algebra with generators $a(h)$ and $a^*(h)$ for all $h \in \mathcal{H}$, divided by the relations:

$$a^*(\lambda f + \mu g) = \lambda a^*(f) + \mu a^*(g), \qquad a^*(f) = a(f)^* \tag{2.2}$$

for all $f, g \in \mathcal{H}$ and $\lambda, \mu \in \mathbb{C}$.

We notice the existence of the canonical injection from $\mathcal{A}(\mathcal{K})$ to $\mathcal{C}(\mathcal{K}_{\mathbb{C}})$

$$\omega(h) \mapsto a(h) + a^*(h) \tag{2.3}$$

where $\mathcal{K}_{\mathbb{C}}$ is the complexification of the real Hilbert space $\mathcal{K}$. On the algebras defined above we would like to define positive linear functionals by certain pairing prescriptions for which we need some notions of pair partitions.

Definition 2.3 Let S be a finite ordered set. We denote by $\mathcal{P}_2(S)$ is the set of pair partitions of S, that is $\mathcal{V} \in \mathcal{P}_2(S)$ if $\mathcal{V}$ consists of $\frac{1}{2}n$ disjoint ordered pairs (l, r) with $l < r$ having S as their reunion. The set of all pair partitions is

$$\mathcal{P}_2(\infty) := \bigcup_{r=0}^{\infty} \mathcal{P}_2(2r). \tag{2.4}$$

Note that $\mathcal{P}_2(n) = \emptyset$ if n is odd. In this paper the symbol $\mathbf{t}$ will always stand for a function $\mathbf{t} : \mathcal{P}_2(\infty) \to \mathbb{C}$.

Definition 2.4 A *Fock state* on the algebra $\mathcal{C}(\mathcal{H})$ is a positive normalised linear functional $\rho_{\mathbf{t}} : \mathcal{C}(\mathcal{H}) \to \mathbb{C}$ of the form

$$\rho_{\mathbf{t}}(a^{\sharp_1}(f_1) \ldots a^{\sharp_n}(f_n)) = \sum_{\mathcal{V} \in \mathcal{P}_2(n)} \mathbf{t}(\mathcal{V}) \prod_{(k,l) \in \mathcal{V}} \langle f_k, f_l \rangle \cdot Q(\sharp_k, \sharp_l) \tag{2.5}$$

the symbols $\sharp_i$ standing for creation or annihilation and the two by two covariance matrix Q is given by

$$Q = \begin{pmatrix} \rho(a_i a_i) & \rho(a_i a_i^*) \\ \rho(a_i^* a_i) & \rho(a_i^* a_i^*) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

where $a_i = a(e_i)$ and e_i is an arbitrary normalized vector in $\mathcal{H}$. Note that the l.h.s. of (2.5) is zero for odd values of n.

Definition 2.5 A *Gaussian state* on $\mathcal{A}(\mathcal{K})$ is a positive normalised linear functional $\tilde{\rho}_{\mathbf{t}}$ with moments

$$\tilde{\rho}_{\mathbf{t}}(\omega(f_1) \ldots \omega(f_n)) = \sum_{\mathcal{V} \in \mathcal{P}_2(n)} \mathbf{t}(\mathcal{V}) \prod_{(k,l) \in \mathcal{V}} \langle f_k, f_l \rangle \tag{2.6}$$

Remark. The restriction of a Fock state $\rho_{\mathbf{t}}$ on $\mathcal{C}(\mathcal{K}_{\mathbb{C}})$ to the subalgebra $\mathcal{A}(\mathcal{K})$ is the Gaussian state $\tilde{\rho}_{\mathbf{t}}$. If $\rho_{\mathbf{t}}$ is a Fock state for all choices of $\mathcal{K}$ then we call the function

$$\mathbf{t} : \mathcal{P}_2(\infty) \to \mathbb{C}$$

positive definite.

The GNS representations associated to pairs $(\mathcal{C}(\mathcal{H}), \rho_{\mathbf{t}})$ have been studied in a number of cases. One obtains a representation $\pi_{\mathbf{t}}$ of $\mathcal{C}(\mathcal{H})$ as *-algebra of creation and annihilation operators acting on a Hilbert space $\mathcal{F}_{\mathbf{t}}(\mathcal{H})$ which has a Fock-type structure

$$\mathcal{F}_{\mathbf{t}}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

with $\mathcal{H}_n$ being a (symmetric) subspace of $\mathcal{H}^{\otimes n}$ in the case of bosonic or fermionic algebras [18], the full tensor product in models of free probablity [23], a deformation of it in the case of q-deformations [2, 4, 5, 6, 7, 8, 16, 24], or even "larger" spaces containing more copies of $\mathcal{H}^{\otimes n}$ with a deformed inner product in the case of another deformation depending on a parameter $-1 \leq q \leq 1$ constructed in [3]. The action of the creation operators is $a^*(f)\Omega_{\mathbf{t}} = f \in \mathcal{H}$,

$$a^*(f)f_1 \otimes \ldots \otimes f_n = f \otimes f_1 \otimes \ldots \otimes f_n$$

while that of the annihilation operator is less transparent, depending on the inner product on $\mathcal{H}_n$. Proving the positivity of this inner product is in general nontrivial.

In [10] we have followed a different, more combinatorial approach to the study of the representations $\pi_{\mathbf{t}}(\mathcal{C}(\mathcal{H}))$ for various examples of positive definite functions $\mathbf{t}$. We give here a brief description of our construction. The representation space is denoted by $\mathcal{F}_V(\mathcal{H})$ and has certain symmetry properties encoded by a sequence $(V_n)_{n=0}^{\infty}$ of (not necessarily finite dimensional) Hilbert spaces such that each V_n carries a unitary representation of the symmetric group $\mathrm{S}(n)$

$$\mathrm{S}(n) \ni \pi \mapsto U(\pi) \in \mathcal{U}(V_n). \tag{2.7}$$

In concrete examples we have realised V_n as $\ell^2(F[n])$ where $F[\ ]$ is a *species of structures* [1, 12, 13], i.e., a functor from the category of finite sets with bijections as morphisms to the category of finite sets with maps as morphisms. For each finite set A, the rule F prescribes a finite set $F[A]$ whose elemens are called F-structures over the set A. Moreover for any bijection $\sigma : A \to B$ there is a map $F[\sigma] : F[A] \to F[B]$ such that $F[\sigma \circ \tau] = F[\sigma] \circ F[\tau]$ and $F[\mathrm{id}_A] = \mathrm{id}_{F[A]}$. In particular for $n := \{0, 1, \ldots n-1\}$ there is an action of the symmetric group $\mathrm{S}(n)$ on the set of structures:

$$\forall \pi \in \mathrm{S}(n), \qquad F[\pi] : F[n] \to F[n]$$

which gives a unitary representation $U(\cdot)$ of $\mathrm{S}(n)$ on $V_n := \ell^2(F[n])$. Simple examples are such species as sets, ordered sequences, trees, graphs, etc.
We define

$$\mathcal{F}_V(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \frac{1}{n!} V_n \otimes_s \mathcal{H}^{\otimes n} \tag{2.8}$$

where $V_n \otimes_s \mathcal{H}^{\otimes n}$ is the subspace of $V_n \otimes \mathcal{H}^{\otimes n}$ spanned by the vectors ψ invariant under the action of $\mathrm{S}(n)$:

$$\psi = (U(\pi) \otimes \tilde{U}(\pi))\psi, \qquad \text{for all } \pi \in \mathrm{S}(n)$$

with $\tilde{U}(\pi) \in \mathcal{U}(\mathcal{H}^{\otimes n})$,

$$\tilde{U}(\pi) : h_0 \otimes \ldots \otimes h_{n-1} \mapsto h_{\pi^{-1}(0)} \otimes \ldots \otimes h_{\pi^{-1}(n-1)}, \tag{2.9}$$

the factor $\frac{1}{n!}$ referring to the inner product. The symmetric Hilbert space $\mathcal{F}_V(\mathcal{H})$ is spanned by linear combinations of vectors of the form:

$$v \otimes_s h_0 \otimes \ldots \otimes h_{n-1} := \frac{1}{n!} \sum_{\pi \in \mathrm{S}(n)} U(\pi) v \otimes \tilde{U}(\pi) h_0 \otimes \ldots \otimes h_{n-1}. \tag{2.10}$$

The creation and annihilation operators are defined with the help of a sequence of densely defined linear maps $(j_n)_{n=0}^{\infty}$ with $j_n : V_n \to V_{n+1}$ satisfying the intertwining relations

$$j_n \cdot U(\pi) = U(\iota_n(\pi)) \cdot j_n, \qquad \forall \pi \in \mathrm{S}(n) \tag{2.11}$$

with $\iota_n : \mathrm{S}(n) \to \mathrm{S}(n+1)$ being the canonical embedding associated to the inclusion of sets

$$n := \{0, 1, \ldots, n-1\} \hookrightarrow n+1 := \{0, 1, \ldots, n\}. \tag{2.12}$$

In the examples using species of structures the map $j_n : \ell^2(F[n]) \to \ell^2(F[n+1])$ is constructed by giving the matrix elements $j_n(s,t) := \langle \delta_t, V_n \delta_s \rangle$ which can be seen as "transition coefficients" between $s \in F[n]$ and $t \in F[n+1]$. For example [10] if the species $F[\cdot]$ is that of rooted trees one can choose $j_n(s,t) = 1$ if the tree s is obtained by removing the leaf with label n from the tree t; otherwise we choose $j_n(s,t) = 0$. Notice that there is no canonical manner of defining j_n but certain species of structures offer rather natural definitions, for example the species of sets, ordered sequences, rooted trees, oriented graphs, sequences of non empty sets, etc [10].
Let $h \in \mathcal{H}$; the creation operator $a^*_{V,j}(h)$ has the action:

$$a^*_{V,j}(h) v \otimes_s (h_0 \otimes \ldots \otimes h_{n-1}) := (j_n v) \otimes_s (h_0 \otimes \ldots \otimes h_{n-1} \otimes h). \tag{2.13}$$

The annihilation operator $a_{V,j}(h)$ is the adjoint of $a^*_{V,j}(h)$. Its action on the $n+1$-th level is given by the restriction of the operator

$$\begin{aligned} \tilde{a}_{V,j}(h) : V_{n+1} \otimes \mathcal{H}^{\otimes n+1} &\to V_n \otimes \mathcal{H}^{\otimes n} \\ v \otimes (h_0 \otimes \ldots \otimes h_n) &\mapsto \langle h, h_n \rangle j_n^* v \otimes (h_0 \otimes \ldots \otimes h_{n-1}). \end{aligned} \tag{2.14}$$

to the subspace $V_{n+1} \otimes_s \mathcal{H}^{\otimes n+1}$. Note that due to condition (2.11) the operators $a^*_{V,j}(h), a_{V,j}(h)$ are well defined. Let us denote by $\mathcal{C}_{V,j}(\mathcal{H})$ the *-algebra generated by all operators $a^*_{V,j}(h), a_{V,j}(h)$ and by $\Omega_V \in V_0$ the normalised vacuum vector in $\mathcal{F}_V(\mathcal{H})$. The following theorem is a generalisation of Proposition 5.1 in [10]:

Theorem 2.6 *Let $(\mathcal{F}_V(\mathcal{H}), \mathcal{C}_{V,j}(\mathcal{H}), \Omega_V)$ be a representation of $\mathcal{C}(\mathcal{H})$ as described above, then the state $\rho_{V,j}(\cdot) = \langle \Omega_V, \cdot \Omega_V \rangle$ is a Fock state, i.e. there exists a positive definite function $\mathbf{t}$ on pair partitions depending on $(V_n, j_n)_{n=0}^{\infty}$ such that $\rho_{V,j} = \rho_{\mathbf{t}}$.*

Sketch of the proof. Let $A \in \mathcal{B}(\mathcal{H})$. On $\mathcal{F}_V(\mathcal{H})$ we define the operator

$$\mathrm{d}\Gamma_V(A) : v_n \otimes_s f_0 \otimes \ldots \otimes f_{n-1} \mapsto \sum_{k=0}^{n-1} v_n \otimes_s f_0 \otimes \ldots \otimes Af_k \otimes \ldots \otimes f_{n-1} \tag{2.15}$$

for $v_n \in V_n, f_i \in \mathcal{H}$. Then the following commutation relations hold:

$$[a_{V,j}(f), \mathrm{d}\Gamma_V(A)] = a_{V,j}(A^* f). \tag{2.16}$$

In particular by choosing an orthonormal basis $\{e_i\}_{i\in I}$ in $\mathcal{H}$ and denoting $a_i^\sharp := a_{V,j}^\sharp(e_i)$ we obtain for all $i_k \neq i_0$

$$[\mathrm{d}\Gamma_V(|e_{i_0}\rangle\langle e_i|), a_{i_k}^{\sharp_k}] = \delta_{i_k,i} \cdot \delta_{\sharp_k,*} \cdot a_{i_0}^*. \tag{2.17}$$

Let $\psi = \left(\prod_{k=1}^n a_{i_k}^{\sharp_k}\right) \Omega_V$. Then $a_{i_0}\psi = 0$ if $i_0 \neq i_k$ for all $k = 1, \ldots, n$. By using (2.16), it follows that

$$a_i\psi = [a_{i_0}, \mathrm{d}\Gamma(|e_{i_0}\rangle\langle e_i|)]\psi = a_{i_0}\mathrm{d}\Gamma(|e_{i_0}\rangle\langle e_i|)\psi. \tag{2.18}$$

We then apply (2.17) repeatedly to obtain

$$a_i\left(\prod_{k=1}^n a_{i_k}^{\sharp_k}\right)\Omega_V = \sum_{k=1}^n \delta_{i,i_k} \cdot \delta_{\sharp_k,*} \cdot a_{i_0}\left(\prod_{p=1}^{k-1} a_{i_p}^{\sharp_p}\right) \cdot a_{i_0}^* \cdot \left(\prod_{q=k+1}^n a_{i_q}^{\sharp_q}\right)\Omega_V. \tag{2.19}$$

The vacuum expectation of a monomial $\prod_{k=1}^n a_{i_k}^{\sharp_k}$ can be different from zero only if the number of creators is equal to the number of annihilators, $a_{i_1}^{\sharp_1}$ is an annihilator and $a_{i_n}^{\sharp_n}$ a creator. We will therefore assume that this is the case. We put the monomial in the form $a_{i_1}\prod_{k=2}^n a_{i_k}^{\sharp_k}$ and apply (2.19). We obtain a sum over all pairs $(a_{i_1}, a_{i_k}^*)$ of the same color ($i_1 = i_k$) and replace i_1 by a new color i_0. We pass now to the next annihilator in each term of the sum and repeat the procedure, the new color which we add this time being different from all the colors used previously. After $\frac{n}{2}$ steps we obtain a sum containing all possible pairings of annihilators and creators of the same color in $\prod_{k=1}^n a_{i_k}^{\sharp_k}$:

$$\rho_{V,j}(\prod_{k=1}^n a_{i_k}^{\sharp_k}) = \sum_{\mathcal{V}\in\mathcal{P}_2(n)} \prod_{k,l\in\mathcal{V}} \delta_{i_k,i_l} \cdot Q(\sharp_k, \sharp_l) \cdot t(\mathcal{V}) \tag{2.20}$$

with $t(\mathcal{V}) := \rho_{V,j}(\prod_{k=1}^n a_{j_k}^{\sharp_k})$, where the indices $j_k, \sharp_k$ satisfy the following conditions: if $k \neq l$ then $j_k = j_l$ if and only if $(k,l) \in \mathcal{V}$, in which case $a_{j_k}^{\sharp_k}$ is annihilator and $a_{j_l}^{\sharp_l}$ is creator.

□

We prove now that the converse is also true.

Theorem 2.7 *Let* $\mathbf{t}$ *be a positive definite function on pair partitions. Then for any complex Hilbert space* $\mathcal{H}$ *the GNS-representation of* $(\mathcal{C}(\mathcal{H}), \rho_{\mathbf{t}})$ *is unitarily equivalent to* $(\mathcal{F}_V(\mathcal{H}), \mathcal{C}_{V,j}(\mathcal{H}), \Omega_V)$ *for a sequence* $(V_n, j_n)_{n=0}^{\infty}$ *dependent only up to unitary equivalence on* $\mathbf{t}$.

Proof. We first consider $\mathcal{H} := \ell^2(\mathbb{N}^*)$ with the orthonormal basis $(e_i)_{i=1}^{\infty}$. We split the proof in 3 steps.
1. Identify the spaces V_n and the maps j_n.
Let $(\mathcal{F}_{\mathbf{t}}(\mathcal{H}), \mathcal{C}_{\mathbf{t}}(\mathcal{H}), \Omega_{\mathbf{t}})$ be the triple obtained from the GNS-construction. Let V_n be the closure of the subspace of $\mathcal{F}_{\mathbf{t}}(\mathcal{H})$ spanned by vectors of the form $v_n := (\prod_{k=1}^{2p+n} a_{\mathbf{t}}^{\sharp_k}(e_{i_k}))\Omega_{\mathbf{t}}$ for which the following conditions hold:

(i) in the sequence $(a_{\mathbf{t}}^{\sharp_k}(e_{i_k}))_{k=1}^{2p+n}$ each creation operator $a_{\mathbf{t}}^*(e_j)$ appears exactly once for $1 \leq j \leq n$;

(ii) the rest of the sequence contains p creation operators $(a_{\mathbf{t}}^*(e_{l_q}))_{q=1}^{p}$ and p annihilation operators $(a_{\mathbf{t}}(e_{l_q}))_{q=1}^{p}$ for p vectors $(e_{l_q})_{q=1}^{p}$ different among each other and with $l_q \notin \{1, \ldots, n\}$ for all $1 \leq q \leq p$. The vector v_n does not depend in fact on the colours $(l_q)_{q=1}^{p}$ but only on the positions of the creation and annihilation operators in the monomial. Thus when necessary we can consider $l_q > N$ for all $1 \leq q \leq n$ and some fixed big enough $N \in \mathbb{N}$.
The map j_n is defined as the restriction of $a_{\mathbf{t}}^*(e_{n+1})$ to V_n:

$$j_n \prod_{k=1}^{2p+n} a_{\mathbf{t}}^{\sharp_k}(e_{i_k})\Omega_{\mathbf{t}} = a_{\mathbf{t}}^*(e_{n+1}) \prod_{k=1}^{2p+n} a_{\mathbf{t}}^{\sharp_k}(e_{i_k})\Omega_{\mathbf{t}}.$$

Obviously, the image of j_n lies in V_{n+1}.
The state $\rho_{\mathbf{t}}$ is invariant under unitary transformations $U \in \mathcal{U}(\mathcal{H})$:

$$\rho_{\mathbf{t}}(\prod_{k=1}^{n} a^{\sharp_k}(e_{i_k})) = \rho_{\mathbf{t}}(\prod_{k=1}^{n} a^{\sharp_k}(Ue_{i_k})).$$

Thus

$$\mathcal{F}_{\mathbf{t}}(U) : \prod_{k=1}^{n} a_{\mathbf{t}}^{\sharp_k}(e_{i_k})\Omega_{\mathbf{t}} \mapsto \prod_{k=1}^{n} a_{\mathbf{t}}^{\sharp_k}(Ue_{i_k})\Omega_{\mathbf{t}} \tag{2.21}$$

is unitary and $\mathcal{F}_{\mathbf{t}}(U_1)\mathcal{F}_{\mathbf{t}}(U_2) = \mathcal{F}_{\mathbf{t}}(U_1U_2)$ for two unitaries U_1, U_2. The action on the algebra of creation and annihilation operators is

$$\mathcal{F}_{\mathbf{t}}(U)a_{\mathbf{t}}^{\sharp}(f)\mathcal{F}_{\mathbf{t}}(U^*) = a_{\mathbf{t}}^{\sharp}(Uf). \tag{2.22}$$

Considering unitaries which act by permuting the basis vectors $\{e_1, \ldots, e_n\}$ and leave all the others invariant we obtain a unitary representation of $\mathrm{S}(n)$ on V_n. The intertwining property (2.11) follows immediately from the definition of j_n. Having the "combinatorial data" (V_n, j_n), we can construct the triple $(\mathcal{F}_V(\mathcal{H}), \mathcal{C}_{V,j}(\mathcal{H}), \Omega_V)$ according to equations (2.8, 2.13, 2.14). Similarly to $\mathcal{F}_{\mathbf{t}}(U)$ we have the unitary

$$\begin{aligned} \mathcal{F}_V(U) : \mathcal{F}_V(\mathcal{H}) &\rightarrow \mathcal{F}_V(\mathcal{H}) \\ v \otimes_s (h_0 \otimes \ldots h_{n-1}) &\mapsto v \otimes_s (Uh_0 \otimes \ldots \otimes Uh_{n-1}) \end{aligned} \tag{2.23}$$

for $U \in \mathcal{U}(\mathcal{H}), v \in V_n$. We call $F_V(U)$ the *second quantisation* of U at the Hilbert space level. Its action on operators is:

$$\mathcal{F}_V(U) a^\sharp_{V,j}(f) \mathcal{F}_V(U^*) = a^\sharp_{V,j}(Uf). \tag{2.24}$$

Analogously to V_n we define for any finite subset $\{i_1, \dots, i_n\} \subset \mathbb{N}$ the linear subspace $V(i_1, \dots, i_n)$ of $\mathcal{F}_{\mathbf{t}}(\mathcal{H})$ spanned by applying to the vacuum $\Omega_{\mathbf{t}}$ monomials $\prod_{k=1}^{2p+n} a^{\sharp_k}_{\mathbf{t}}(e_{j_k})$ for which the colours $(j_k)_{k=1}^{2p+n}$ satisfy conditions similar to i), ii) but now with $\{i_1, \dots, i_n\}$ instead of $\{1, \dots, n\}$. For a unitary U which permutes the basis vectors, $Ue_i = e_{u(i)}$ we get

$$\mathcal{F}_{\mathbf{t}}(U) V(i_1, \dots, i_n) = V(u(i_1), \dots, u(i_n)). \tag{2.25}$$

One can check by calculating inner products that any two such spaces are either orthogonal or coincide. Similarly, we define the following subspaces of $\mathcal{F}_V(\mathcal{H})$

$$\tilde{V}(i_1, \dots, i_n) := \overline{\mathrm{lin}\{v \otimes_s (e_{i_1} \otimes \dots \otimes e_{i_n}) : \quad v \in V_n\}} \tag{2.26}$$

which are also orthogonal for different sets of "colours" $\{i_1, \dots, i_n\}$.
2. We proceed by proving the equality of the states $\rho_{\mathbf{t}}$ and $\rho_{V,j}$.
As $\rho_{V,j}$ is a Fock state by Theorem 2.6, we need only verify that the positive definite function $\mathbf{t}$ we have started with and the one derived from $\rho_{V,j}$ coincide. By definition there is an isometry

$$\begin{aligned} T_n : V_n &\to \mathcal{F}_{V,j}(\mathcal{H}) \\ v &\mapsto v \otimes_s (e_1 \otimes \dots \otimes e_n). \end{aligned} \tag{2.27}$$

Furthermore for any unitary $U \in \mathcal{U}(\mathcal{H})$ which permutes the basis vectors such that $Ue_k = e_{i_k}$, the operator

$$T(i_1, \dots, i_n) : V(i_1, \dots, i_n) \to \tilde{V}(i_1, \dots, i_n)$$

defined by

$$T(i_1, \dots, i_n) := \mathcal{F}_V(U) T_n \mathcal{F}_{\mathbf{t}}(U^*) \tag{2.28}$$

depends only on the set $\{i_1, \dots, i_n\}$. Finally, the definitions of $j_n, a^\sharp_{V,j}(f)$ amounts to the fact that the following diagram commutes

$$\begin{array}{ccc} V_n & \xrightarrow{T_n} & \tilde{V}_n \\ {\scriptstyle a^*_{\mathbf{t}}(e_{n+1})}\downarrow & & \downarrow{\scriptstyle a^*_{V,j}(e_{n+1})} \\ V_{n+1} & \xrightarrow{T_{n+1}} & \tilde{V}_{n+1} \end{array} \tag{2.29}$$

and by acting from the left and from the right with the appropriate second quantisation operators and using (2.28, 2.22, 2.24) we obtain

$$\begin{array}{ccc} V(i_1, \dots, i_n) & \xrightarrow{T(i_1, \dots, i_n)} & \tilde{V}(i_1, \dots, i_n) \\ {\scriptstyle a^*_{\mathbf{t}}(e_{i_{n+1}})}\downarrow & & \downarrow{\scriptstyle a^*_{V,j}(e_{i_{n+1}})} \\ V(i_1, \dots, i_{n+1}) & \xrightarrow{T(i_1, \dots, i_{n+1})} & \tilde{V}(i_1, \dots, i_{n+1}) \end{array} \tag{2.30}$$

with a similar diagram for the annihilation operators. This is sufficient for proving the equality $\rho_{\mathbf{t}}(\prod_{k=1}^{2n} a_{\mathbf{t}}^{\sharp_k}(e_{i_k})) = \rho_{V,j}(\prod_{k=1}^{2n} a_{V,j}^{\sharp_k}(e_{i_k}))$ for monomials containing n pairs of creation and annihilation operators of n different colours.
3. Finally we prove that $\Omega_{V,j}$ is cyclic vector for $\mathcal{C}_{V,j}(\mathcal{H})$.
The space $\mathcal{F}_V(\mathcal{H})$ has a decomposition with respect to occupation numbers

$$\mathcal{F}_V(\mathcal{H}) = \bigoplus_{\{n_1,\dots,n_k\}} \mathcal{F}_V(n_1,\dots,n_k)$$

with

$$\mathcal{F}_V(n_1,\dots,n_k) = \overline{\mathrm{lin}\{v \otimes_s (\underbrace{e_1 \otimes \dots \otimes e_1}_{n_1} \otimes \dots \otimes \underbrace{e_k \otimes \dots \otimes e_k}_{n_k}, v \in V_{n_1+\dots+n_k}\}}. \tag{2.31}$$

We recall that $\tilde{V_n} = \mathcal{F}_V(\underbrace{1,\dots,1}_{n})$ is spanned by linear combinations of vectors of the form

$$\prod_{k=1}^{2p+n} a_{V,j}^{\sharp_k}(e_{i_k})\Omega_V = v \otimes_s (e_1 \otimes \dots \otimes e_n)$$

with monomials satisfying the conditions i) and ii). By replacing the creation operators $(a^*(e_k))_{k=1}^n$ appearing in the monomial, with the sequence containing n_i times the creator $a^*(e_i)$ for $i \in \{1,\dots p\}$ and $\sum_{i=1}^p n_i = n$ we obtain a set of vectors which are dense in $\mathcal{F}_V(n_1,\dots,n_p)$ and this completes the proof of the cyclicity of the vacuum. Putting together 1., 2. and 3. we conclude that the representations $(\mathcal{F}_{\mathbf{t}}(\mathcal{H}), \mathcal{C}_{\mathbf{t}}(\mathcal{H}), \Omega_{\mathbf{t}})$ and $(\mathcal{F}_V(\mathcal{H}), \mathcal{C}_{V,j}(\mathcal{H}), \Omega_V)$ are unitarily equivalent for infinite dimensional $\mathcal{H}$. The case $\mathcal{H}$ finite dimensional follows by restriction of the previous representations to the appropriate subspaces.

□

3 The *-semigroup of broken pair partitions

The content of the last two theorems can be summarised by the following fact: there exist a bijective correspondence between positive definite functions on pair partitions $\mathbf{t}$, and "combinatorial data" $(V_n, j_n)_{n=0}^{\infty}$. This suggests that the positivity of $\mathbf{t}$ can be characterised in a simpler way by regarding $\mathbf{t}$ as a positive functional on an algebraic object containing $\mathcal{P}_2(\infty)$ as a subset. Theorem 1 of [3] shows that a positive definite function on pair partitions $\mathbf{t}$ restricts to positive definite functions on the symmetric groups $\mathrm{S}(n)$ for all $n \in \mathbb{N}$ through the embedding

$$\mathrm{S}(n) \ni \tau \mapsto \mathcal{V}_\tau \in \mathcal{P}_2(n) \tag{3.1}$$

given by

$$\mathcal{V}_\tau := \{(i, 2n+1-\tau(i)) : \ i = 1,\dots,2n\}. \tag{3.2}$$

However $\mathbf{t}$ is not determined completely by its restriction and thus one would like to find another algebraic object which completely encodes the positivity requirement. We will show that this is the *-semigroup of *broken pair partitions* which we denote by $\mathcal{BP}_2(\infty)$ and will be described below. Pictorially, the elements of the semigroup are segments obtained by sectioning pair partitions with vertical lines.

Definition 3.1 *Let X be an arbitrary finite ordered set and (L, P, R) a disjoint partition of X. We consider all the triples $(\mathcal{V}, f_l, f_r)$ where $\mathcal{V} \in \mathcal{P}_2(P)$ and*

$$f_l : L \to \{1, \ldots, |L|\}, \qquad f_r : R \to \{1, \ldots, |R|\} \tag{3.3}$$

are bijections. Any order preserving bijection $\alpha : X \to Y$ induces an obvious map

$$(\mathcal{V}, f_l, f_r) \to (\alpha \circ \mathcal{V}, f_l \circ \alpha^{-1}, f_r \circ \alpha^{-1}) \tag{3.4}$$

where $\alpha \circ \mathcal{V} := \{(\alpha(a), \alpha(b)) : (a, b) \in \mathcal{V}\}$. This defines an equivalence relation; an element d of $\mathcal{BP}_2(\infty)$ is an equivalence class of triples $(\mathcal{V}, f_l, f_r)$ under this equivalence relation.

We have the following pictorial representation: an element d is given by a diagram containing a sequence of $l + r + 2n$ points displayed horizontally with $2n$ of them connected into n pairs, l points are connected with other l points vertically ordered on the left side (left legs) and r points are connected with r points vertically ordered on the right (right legs). An example is given in Figure 1. In this case we have $X = \{1, \ldots, 5\}$, $\mathcal{V} = \{(1, 4)\}$, the left legs are connecting the points labeled 2 and 5 on the horizontal to the the points on the left side which are ordered vertically and labeled by 1 and 2. Similarly for the right legs. Usually we will label the ordered set of horizontal points will be of the form $\{n, n + 1, \ldots n + m\}$.

The product of two diagrams is calculated by drawing the diagrams next to each other and joining the right legs of the left diagram with the left legs of the right diagram which are situated at the same level on the vertical. Figure 2 illustrates an example.

More formally if $d_i = (\mathcal{V}_i, f_{l,i}, f_{r,i})$ for $i = 1, 2$ with the notations from Definition 3.1, then $d_1 \cdot d_2 = (\mathcal{V}, f_l, f_r)$ with

$$\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \{(f_{r,1}^{-1}(i), f_{l,2}^{-1}(i)) : i \leq \min(|R_1|, |L_2|)\}, \tag{3.5}$$

f_l is defined on the disjoint union $L_1 + (L_2 \setminus f_{l,2}^{-1}(\{1, \ldots, \min(|R_1|, |L_2|)\})$ by

$$\begin{cases} f_l(a) = f_{l,1}(a) & \text{for } a \in L_1 \\ f_l(b) = f_{l,2}(b) + |L_1| & \text{for } b \in L_2 \setminus f_{l,2}^{-1}(\{1, \ldots, \min(|R_1|, |L_2|)\} \end{cases}$$

and similarly for f_r. The product does not depend on the chosen representatives for d_i in their equivlence class and is associative. The diagrams with no legs are the pair partitions, thus $\mathcal{P}_2(\infty) \subset \mathcal{BP}_2(\infty)$.

The involution is given by mirror reflection (see Figure 3). If $d = (\mathcal{V}, f_l, f_r)$ then $d^* = (\mathcal{V}^*, f_r, f_l)$ with the underlying set X^* obtained by reversing the order on X and

$$\mathcal{V}^* := \{(b, a) : (a, b) \in \mathcal{V}\} \tag{3.6}$$

is the adjoint of $\mathcal{V}$. It is easy to check that

$$(d_1 \cdot d_2)^* = d_2^* \cdot d_1^*.$$

Let $\mathbf{t}$ be a linear functional on pair partitions. We extend it to a function $\hat{\mathbf{t}}$ on $\mathcal{BP}_2(\infty)$ defined as

$$\hat{\mathbf{t}}(d) = \begin{cases} \mathbf{t}(d) & \text{if } d \in \mathcal{P}_2(\infty) \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Theorem 3.2 *The function* $\mathbf{t}$ *on pair partitions is positive definite if and only if* $\hat{\mathbf{t}}$ *is postive on the* $^*-$*semigroup* $\mathcal{BP}_2(\infty)$.

Proof. The main ideas are already present in the proof of Proposition 2.7. A GNS-type of construction associates to the pair $(\mathcal{BP}_2(\infty), \hat{\mathbf{t}})$ a cyclic representation $\chi_{\mathbf{t}}$ of $\mathcal{BP}_2(\infty)$ on a Hilbert space V with cyclic vector $\xi \in V$. We have $\langle \xi, \chi_{\mathbf{t}}(d)\xi \rangle = \hat{\mathbf{t}}(d)$. We denote by $\mathcal{BP}_2^{(n,0)}$ the set of diagrams with n left legs and no right legs. Then using

$$\langle \chi_{\mathbf{t}}(d_1)\xi, \chi_{\mathbf{t}}(d_2)\xi \rangle_V = \hat{\mathbf{t}}(d_1^* \cdot d_2) \tag{3.8}$$

we obtain:
1. the representation space V is of the form

$$V = \bigoplus_{n=0}^{\infty} V_n \qquad \text{where} \qquad V_n = \overline{\text{lin}\{\chi_{\mathbf{t}}(d)\xi : d \in \mathcal{BP}_2^{(n,0)}\}} \tag{3.9}$$

2. on $\mathcal{BP}_2^{(n,0)}$ there is an obvious action of $\mathrm{S}(n)$ by permutations of the positions of the left ends of the legs. Figure 4 shows the action of the transposition $\tau_{1,2}$. This induces a unitary representation of $\mathrm{S}(n)$ on V_n as

$$\tau(d_1)^* \cdot \tau(d_2) = d_1^* \cdot d_2 \tag{3.10}$$

for all $d_1, d_2 \in \mathcal{BP}_2^{(n,0)}$ and $\tau \in \mathrm{S}(n)$.
3. let $d_0 \in \mathcal{BP}_2^{(1,0)}$ be the "left hook" (the diagram with no pairs). Then $j := \chi_{\mathbf{t}}(d_0)$ is an operator on V whose restriction j_n to V_n maps it into V_{n+1} and satisfies the intertwining condition (2.11) with respect ot the representations of the symmetric groups on V_n and V_{n+1}.
Using the data (V_n, j_n) we construct the triple $(\mathcal{F}_V(\mathcal{H}), \mathcal{C}_{V,j}(\mathcal{H}), \Omega_V)$. According to Proposition 2.6 there exists a positive definite function on pair partitions $\mathbf{t}'$ such that $\rho_{V,j} = \rho_{\mathbf{t}'}$. We have to prove that $\mathbf{t}$, which is the restriction of $\hat{\mathbf{t}}$ to $\mathcal{P}_2(\infty)$ coincides with $\mathbf{t}'$.

Any pair partition $\mathcal{V}$ can be written in a "standard form" (see Figure 5):

$$\mathcal{V} = (d_0^*)^{p_m} \cdot \pi_{m-1}(\ldots \pi_2(d_0^{k_2} \cdot (d_0^*)^{p_1} \cdot \pi_1(d_0^{k_1}))) \tag{3.11}$$

where the permutations π_i are uniquely defined by the requirement that any two lines connecting two pairs in the associated graphic intersect minimally and at the rightmost possible position.
Let $\prod_{k=1}^{2n} a_{V,j}^{\sharp_k}(e_{i_k})$ be a monomial containing n creation operators and n annihilation operators such that by pairing creators with annihilators of the same colour on their right side, we generate a pair partition $\mathcal{V}$. The definitions (2.13), (2.14) of the creation and annihilation operators give their expressions in terms of the operator j, j^* and the unitary representations of the permutation groups on the spaces V_n. By using the intertwining property (2.11) we can pass all permutations to the left of the j-terms and obtain:

$$\begin{aligned}
\mathbf{t}'(\mathcal{V}) &= \left\langle \Omega_V, \prod_{k=1}^{2n} a_{V,j}^{\sharp_k}(e_{i_k})\Omega_V \right\rangle \\
&= \left\langle \xi, (j^*)^{p_m} \cdot U(\pi_{m-1}) \ldots U(\pi_2) \cdot j^{k_2} \cdot (j^*)^{p_1} \cdot U(\pi_1) \cdot j^{k_1}\xi \right\rangle_V \\
&= \langle \xi, \chi_{\mathbf{t}}(\mathcal{V})\xi \rangle_V = \hat{\mathbf{t}}(\mathcal{V})
\end{aligned}$$

Conversely, starting from a positive definite function $\mathbf{t}$ we construct the representation $(V, \chi_{\mathbf{t}}(\mathcal{BP}_2(\infty)), \xi)$ through applying Theorem 2.7 and thus $\hat{\mathbf{t}}$ is positive on $\mathcal{BP}_2(\infty)$.

□

4 Generalised Wick products

As argued in the introduction, the representations of the "field algebras" $\mathcal{A}(\mathcal{K})$ with respect to Gaussian states $\tilde{\rho}_{\mathbf{t}}$ give rise to (noncommutative) processes called generalised Brownian motions [3] for $\mathcal{K}$ (infinite dimensional) real Hilbert space. In all known examples such representations appear as restrictions to the subalgebra $\mathcal{A}(\mathcal{K})$ of Fock representations of the algebra of creation and annihilation operators $\mathcal{C}(\mathcal{K}_{\mathbb{C}})$ with respect to the state $\rho_{\mathbf{t}}$. We will prove that this is always the case, thus answering a question put in [3].
Let

$$\mathbf{t} : \mathcal{P}_2(\infty) \to \mathbb{C} \tag{4.1}$$

be such that $\tilde{\rho}_{\mathbf{t}}$ is a Gaussian state on $\mathcal{A}(\mathcal{K})$ for $\mathcal{K}$ infinite dimensional Hilbert space. Let $(\tilde{\mathcal{F}}_{\mathbf{t}}(\mathcal{K}), \tilde{\pi}_{\mathbf{t}}(\mathcal{A}(\mathcal{K})), \tilde{\Omega}_{\mathbf{t}})$ be the GNS-triple associated to $(\mathcal{A}(\mathcal{K}), \tilde{\rho}_{\mathbf{t}})$. The *-algebra $\tilde{\pi}_{\mathbf{t}}(\mathcal{A}(\mathcal{K}))$ is generated by the operators $\omega_{\mathbf{t}}(f) := \tilde{\pi}_{\mathbf{t}}(\omega(f))$ for all $f \in \mathcal{K}$.

Lemma 4.1 *The operators $\omega_{\mathbf{t}}(f)$ are essentially selfadjoint.*

Proof. By definition $\omega_{\mathbf{t}}(f)$ is symmetric as operator defined on the dense domain $D := \tilde{\pi}_{\mathbf{t}}(\mathcal{A}(\mathcal{K}))\tilde{\Omega}_{\mathbf{t}}$. By Theorem VIII.3 in [17], $\omega_{\mathbf{t}}(f)$ is essentially selfadjoint if and only if $\mathrm{Ran}(\omega_{\mathbf{t}}(f) \pm i)$ is dense in $\tilde{\mathcal{F}}_{\mathbf{t}}(\mathcal{K})$. Let $B \in \mathcal{A}(\mathcal{K})$ such that for all $A \in \mathcal{A}(\mathcal{K})$ we have

$$\langle (\omega_{\mathbf{t}}(f) + i)\pi_{\mathbf{t}}(A)\Omega_{\mathbf{t}}, \pi_{\mathbf{t}}(B)\Omega_{\mathbf{t}} \rangle = 0 \tag{4.2}$$

By choosing $A = \omega_{\mathbf{t}}(f)B$ we get $\rho_{\mathbf{t}}(B^*\omega(f)B) = 0$ and $\rho_{\mathbf{t}}(B^*\omega(f)^2B) = 0$. Choose now $A = B$, then $\rho_{\mathbf{t}}(B^*B) = 0$ thus $\pi_{\mathbf{t}}(B)\Omega_{\mathbf{t}} = 0$.

□

In analogy to (2.21) for any orthogonal operator $O \in \mathcal{O}(\mathcal{K})$ there exists a unitary

$$\tilde{\mathcal{F}}_{\mathbf{t}}(O) : \prod_{k=1}^{n} \omega_{\mathbf{t}}(f_k)\tilde{\Omega}_{\mathbf{t}} \to \prod_{k=1}^{n} \omega_{\mathbf{t}}(Of_k)\tilde{\Omega}_{\mathbf{t}} \tag{4.3}$$

and $\tilde{\mathcal{F}}_{\mathbf{t}}(O_1)\tilde{\mathcal{F}}_{\mathbf{t}}(O_2) = \tilde{\mathcal{F}}_{\mathbf{t}}(O_1 \cdot O_2)$ for $O_1, O_2 \in \mathcal{O}(\mathcal{K})$. This induces an action on the *-algebra $\tilde{\pi}_{\mathbf{t}}(\mathcal{A}(\mathcal{K}))$:

$$\tilde{\Gamma}_{\mathbf{t}}(O) : X \mapsto \tilde{\mathcal{F}}_{\mathbf{t}}(O)X\tilde{\mathcal{F}}_{\mathbf{t}}(O^*). \tag{4.4}$$

Certain operators play a similar role to that of the Wick products in quantum field theory [18, 21] or for the q-deformed Brownian motion [2, 4].

Definition 4.2 Let $\{P, F\}$ be a partition of the ordered set $\{1, \ldots, 2p+n\}$ with $|P| = 2p$ and $|F| = n$. Let $\mathcal{V} = \{(l_1, r_1), \ldots, (l_p, r_p)\} \in \mathcal{P}_2(P)$ and $\mathbf{f} : F \to \mathcal{K}$. For every $\mathcal{V}' = \{(l'_1, r'_1), \ldots, (l'_{p'}, r'_{p'})\} \in \mathcal{P}_2(P')$ with $P' \subset F$ we introduce

$$\eta_{\mathbf{f}}(\mathcal{V}') = \prod_{i=1}^{p'} \langle \mathbf{f}(l'_i), \mathbf{f}(r'_i) \rangle . \tag{4.5}$$

The *generalised Wick product* associated to $(\mathcal{V}, \mathbf{f})$ is the operator $\Psi(\mathcal{V}, \mathbf{f})$ determined recursively by

$$\begin{aligned} \Psi(\mathcal{V}, \mathbf{f}) \quad &+ \sum_{\emptyset \neq P' \subset F} \; \sum_{\mathcal{V}' \in \mathcal{P}_2(P')} \eta_{\mathbf{f}}(\mathcal{V}') \cdot \Psi(\mathcal{V} \cup \mathcal{V}', \mathbf{f} \restriction_{F \setminus P'}) = M(\mathcal{V}, \mathbf{f}) \\ M(\mathcal{V}, \mathbf{f}) \quad &:= \operatorname*{w-lim}_{n \to \infty} \prod_{k=1}^{2p+n} \omega_{\mathbf{t}}(f_{k,n}) \end{aligned} \tag{4.6}$$

where $f_{k,n} := \mathbf{f}(k)$ for $k \in F$ and $f_{l_i,n} = f_{r_i,n} = e_{np+i}$ for $i = 1, \ldots, p$ with $(e_l)_{l \in \mathbb{N}}$ a set of normalised vectors, orthogonal to each other and to the vectors $(\mathbf{f}(k))_{k=1}^{n}$.

Remarks. 1) The right side of the last equation needs some clarifications. All operators appearing in (4.6) are defined on the domain D. The operator $M(\mathcal{V}, \mathbf{f})$ is defined on D by its matrix elements. If $\psi_i = \prod_{a=1}^{m_i} \omega_{\mathbf{t}}(g_a^{(i)})\tilde{\Omega}_{\mathbf{t}}$ for $i = 1, 2$ are vectors in D then from the definition of the Gaussian state follows immediately that

$$\lim_{n\to\infty} \left\langle \psi_1, \prod_{k=1}^{2p+n} \omega_{\mathbf{t}}(f_{k,n})\psi_2 \right\rangle = \left\langle \psi_1, \prod_{k=1}^{2p+n} \omega_{\mathbf{t}}(f_{k,n_0})\psi_2 \right\rangle \tag{4.7}$$

for $n_0 \in \mathbb{N}$ large enough such that the subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ of $\mathcal{K}$ spanned by $(e_{n_0+j})_{j\geq 0}$ respectively by $(g_a^{(i)})_{a=1}^{m_i}$ for $i = 1, 2$ are orthogonal to each other. The limit does not depend on the choice of the vectors $(e_i)_{i\in\mathbb{N}}$ (as long as they are normal and orthogonal to each other) but depends only on their positions in the monomial which are determined by the pair partition $\mathcal{V}$. Thus $M(\mathcal{V}, \mathbf{f})$ is well defined.
2) If the vectors $(\mathbf{f}(k))_{k=1}^{n}$ are orthogonal on each other then $\eta_{\mathbf{f}}(\mathcal{V}') = 0$, thus $\Psi(\mathcal{V}, \mathbf{f}) = M(\mathcal{V}, \mathbf{f})$.
3) The dense domain D is spanned by the vectors of the form $\Psi(\mathcal{V}, \mathbf{f})\tilde{\Omega}_{\mathbf{t}}$. Indeed let $\psi = \prod_{k=1}^{n} \omega_{\mathbf{t}}(\mathbf{f}(k))\tilde{\Omega}_{\mathbf{t}}$; then

$$\psi = \Psi(\emptyset, \mathbf{f})\tilde{\Omega}_{\mathbf{t}} + \sum_{\emptyset\neq P'\subset F} \sum_{d'\in\mathcal{P}_2(P')} \eta_f(\mathcal{V}') \cdot \Psi(\mathcal{V}', \mathbf{f}\restriction_{F\setminus P'})\tilde{\Omega}_{\mathbf{t}} \tag{4.8}$$

with $F = \{1, \ldots, n\}$.
4) The choice for $\{1, \ldots, 2p + n\}$ as the underlying ordered set is not essential. It is useful to think of $\Psi(\mathcal{V}, \mathbf{f})$ in terms of an arbitrary underlying finite ordered set X, where $\mathcal{V} \in \mathcal{P}_2(A)$, $A \subset X$, $\mathbf{f} : X \setminus A \to \mathcal{K}$. For example we can consider the set $X = \{0\}$ and $\mathbf{f}(0) = h$, then $\Psi(\emptyset, \mathbf{f}) = \omega_{\mathbf{t}}(h)$.
The relation between $M(\mathcal{V}, \mathbf{f})$ and $\Psi(\mathcal{V}, \mathbf{f})$ is similar to the one between moments and cumulants.

Lemma 4.3 *Let $\Psi(\mathcal{V}, \mathbf{f})$, $M(\mathcal{V}, \mathbf{f})$ be as in Definition 4.2. The equations (4.6) can be inverted into:*

$$\Psi(\mathcal{V}, \mathbf{f}) = M(\mathcal{V}, \mathbf{f}) + \sum_{\emptyset\neq P'\subset F} \sum_{\mathcal{V}'\in\mathcal{P}_2(P')} (-1)^{\frac{|P'|}{2}} \eta_{\mathbf{f}}(\mathcal{V}') \cdot M(\mathcal{V} \cup \mathcal{V}', \mathbf{f}\restriction_{F\setminus P'}). \tag{4.9}$$

Proof. Direct application of the definition.

□

Let X be an ordered set. Let $\{P, F\}$ be a partition of X into disjoint sets and consider a pair $(\mathcal{V} \in \mathcal{P}_2(P), \mathbf{f} : F \to \mathcal{K})$. Then for X^* as underlying set we define the pair $(\mathcal{V}^*, \mathbf{f}^*)$ where $\mathcal{V}^* \in \mathcal{P}_2(X^*)$ contains the same pairs as $\mathcal{V}$ but with the reversed order and $\mathbf{f}^* = \mathbf{f}$.

Lemma 4.4 *With the above notations the following relation holds:*

$$\Psi(\mathcal{V}, \mathbf{f})^* = \Psi(\mathcal{V}^*, \mathbf{f}^*). \tag{4.10}$$

Proof. Apply Lemma 4.3 and use $M(\mathcal{V},\mathbf{f})^* = M(\mathcal{V}^*,\mathbf{f}^*)$ which follows directly from Definition 4.2.

$\square$

For two ordered sets X and Y we define their concatenation $X+Y$ as the disjoint union with the original order on X and Y and with $x<y$ for any $x\in X, y\in Y$. If $\mathbf{f}_X : X\to\mathcal{K}$ and $\mathbf{f}_Y : Y\to\mathcal{K}$ then we denote by $\mathbf{f}_X\oplus\mathbf{f}_Y$ the function on $X+Y$ which restricts to $\mathbf{f}_X$ and $\mathbf{f}_Y$ on X respectively Y. Finally if $|X|=|Y|=m$ we identify the subset of $\mathcal{P}_2(X+Y)$:

$$\mathcal{P}_2(X,Y) := \{\{(x_1,y_1),\ldots(x_m,y_m)\} \;:\; x_i\in X, y_i\in Y, i=1,\ldots,m\} \tag{4.11}$$

Lemma 4.5 *Let (P_i,F_i) be a disjoint partition of X_i and $\mathcal{V}_i\in\mathcal{P}_2(P_i)$, $\mathbf{f}_i : F_i\to\mathcal{K}$ for $i=1,2$. Then*

$$\left\langle \Psi(\mathcal{V}_1,\mathbf{f}_1)\tilde{\Omega}_{\mathbf{t}},\ \Psi(\mathcal{V}_2,\mathbf{f}_2)\tilde{\Omega}_{\mathbf{t}}\right\rangle = \sum_{\mathcal{V}\in\mathcal{P}_2(F_1^*,F_2)} \eta_{\mathbf{f}_1^*\oplus\mathbf{f}_2}(\mathcal{V})\cdot\mathbf{t}(\mathcal{V}_1^*\cup\mathcal{V}_2\cup\mathcal{V}) \tag{4.12}$$

with the convention $\eta_{\mathbf{f}_1^*\oplus\mathbf{f}_2}(\mathcal{V})=1$ for $F_1=F_2=\emptyset$.

Proof. From Definitions 2.5, 4.2 it follows that

$$\left\langle M(\mathcal{V}_1,\mathbf{f}_1)\tilde{\Omega}_{\mathbf{t}},\ M(\mathcal{V}_2,\mathbf{f}_2)\tilde{\Omega}_{\mathbf{t}}\right\rangle = \sum_{\mathcal{V}\in\mathcal{P}_2(F_1^*+F_2)} \eta_{\mathbf{f}_1^*\oplus\mathbf{f}_2}(\mathcal{V})\cdot\mathbf{t}(\mathcal{V}_1^*\cup\mathcal{V}_2\cup\mathcal{V}). \tag{4.13}$$

We apply Lemma 4.3 and obtain:

$$\begin{aligned}\left\langle \Psi(\mathcal{V}_1,\mathbf{f}_1)\tilde{\Omega}_{\mathbf{t}},\ \Psi(\mathcal{V}_2,\mathbf{f}_2)\tilde{\Omega}_{\mathbf{t}}\right\rangle &= \sum_{\mathcal{V}_1',\mathcal{V}_2'}(-1)^{\frac{|P_1'|+|P_2'|}{2}}\cdot\eta_{\mathbf{f}_1^*\oplus\mathbf{f}_2}(\mathcal{V}_1'^*\cup\mathcal{V}_2')\cdot\\ &\cdot\left\langle M(\mathcal{V}_1\cup\mathcal{V}_1',\mathbf{f}_1\restriction_{F_1\setminus P_1'})\tilde{\Omega}_{\mathbf{t}},\ M(\mathcal{V}_2\cup\mathcal{V}_2',\mathbf{f}_2\restriction_{F_2\setminus P_2'})\tilde{\Omega}_{\mathbf{t}}\right\rangle\end{aligned}$$

where the sum runs over all $\mathcal{V}_i'\in\mathcal{P}_2(P_i'), P_i'\subset F_i$ for $i=1,2$. Substituting in the last expression the result from equation (4.13) it becomes:

$$\sum_{\mathcal{V}_1',\mathcal{V}_2'}\sum_{\mathcal{V}}(-1)^{\frac{|P_1'|+|P_2'|}{2}}\cdot\eta_{\mathbf{f}_1^*\oplus\mathbf{f}_2}(\mathcal{V}_1'^*\cup\mathcal{V}_2'\cup\mathcal{V})\cdot\mathbf{t}((\mathcal{V}_1\cup\mathcal{V}_1')^*\cup\mathcal{V}_2\cup\mathcal{V}_2'\cup\mathcal{V}) \tag{4.14}$$

with the second sum running over all $\mathcal{V}\in\mathcal{P}_2((F_1\setminus P_1')^*+(F_2\setminus P_2'))$. We make the notation $\tilde{\mathcal{V}} := \mathcal{V}_1'^*\cup\mathcal{V}_2'\cup\mathcal{V}$ and by grouping together all terms containing $\tilde{\mathcal{V}}$ the initial expression looks like:

$$\left\langle \Psi(\mathcal{V}_1,\mathbf{f}_1)\tilde{\Omega}_{\mathbf{t}},\ \Psi(\mathcal{V}_2,\mathbf{f}_2)\tilde{\Omega}_{\mathbf{t}}\right\rangle = \sum_{\tilde{\mathcal{V}}} m(\tilde{\mathcal{V}})\cdot\eta_{\mathbf{f}_1^*\oplus\mathbf{f}_2}(\tilde{\mathcal{V}})\cdot\mathbf{t}(\mathcal{V}_1^*\cup\tilde{\mathcal{V}}\cup\mathcal{V}_2) \tag{4.15}$$

where the symbol $m(\tilde{\mathcal{V}})$ stands for total contribution from the terms of the form $(-1)^{\frac{|P_1'|+|P_2'|}{2}}$. We calculate now $m(\tilde{\mathcal{V}})$:

$$m(\tilde{\mathcal{V}}) = \sum_{\mathcal{V}_1',\mathcal{V}_2',\mathcal{V}}(-1)^{|\mathcal{V}_1'|+|\mathcal{V}_2'|}, \tag{4.16}$$

this sum running over all $\mathcal{V} \in \mathcal{P}_2((F_1 \setminus P_1')^* + (F_2 \setminus P_2'))$, $\mathcal{V}_i' \in \mathcal{P}_2(P_i')$, $P_i' \subset F_i$ for $i = 1, 2$ with the constraint $\tilde{\mathcal{V}} = \mathcal{V}_1'^* \cup \mathcal{V}_2 \cup \mathcal{V}$.
Suppose that $\tilde{\mathcal{V}} \in \mathcal{P}_2(F_1^*, F_2)$, then $\mathcal{V}_1' = \mathcal{V}_2' = \emptyset$ and $m(\tilde{\mathcal{V}}) = 1$. Otherwise $\tilde{\mathcal{V}}$ can be written in a unique way as

$$\tilde{\mathcal{V}} = \tilde{\mathcal{V}}_1^* \cup \tilde{\mathcal{V}}_c \cup \tilde{\mathcal{V}}_2 \tag{4.17}$$

where $\tilde{\mathcal{V}}_i \in \mathcal{P}_2(\tilde{P}_i)$, $\emptyset \neq \tilde{P}_i \subset X_i$ for $i = 1, 2$ and $\tilde{\mathcal{V}}_c \in \mathcal{P}_2((X_1 \setminus \tilde{P}_1)^*, X_2 \setminus \tilde{P}_2)$. Then one has the inclusions $\mathcal{V}_i' \subset \tilde{\mathcal{V}}_i$ for $i = 1, 2$ and $\mathcal{V}_c \subset \mathcal{V}$. The calculation of $m(\tilde{\mathcal{V}})$ reduces then to

$$m(\tilde{\mathcal{V}}) = \sum_{\mathcal{V}_1' \subset \tilde{\mathcal{V}}_1, \mathcal{V}_2' \subset \tilde{\mathcal{V}}_2} (-1)^{|\mathcal{V}_1'| + |\mathcal{V}_2'|} = (1 - 1)^{|\tilde{\mathcal{V}}_1| + |\tilde{\mathcal{V}}_2|} = 0. \tag{4.18}$$

In conclusion

$$\left\langle \Psi(\mathcal{V}_1, \mathbf{f}_1)\tilde{\Omega}_{\mathbf{t}},\ \Psi(\mathcal{V}_2, \mathbf{f}_2)\tilde{\Omega}_{\mathbf{t}} \right\rangle = \sum_{\tilde{\mathcal{V}} \in \mathcal{P}_2(F_1^*, F_2)} \eta_{\mathbf{f}_1^* \oplus \mathbf{f}_2}(\tilde{\mathcal{V}}) \cdot \mathbf{t}(\mathcal{V}_1^* \cup \mathcal{V}_2 \cup \tilde{\mathcal{V}}) \tag{4.19}$$

□

A similar result holds for algebras of creation and annihilation operators. Suppose that $\mathbf{t}$ is a function (not neccesarily positive definite) on pair partitions. Let $P, F, \mathcal{V}, \mathbf{f}$ be as in Definition 4.2 and define in the representation space $\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}})$ the vectors

$$\psi(\mathcal{V}, \mathbf{f}) = \prod_{k=1}^{2p+n} a_{\mathbf{t}}^{\sharp_k}(f_k))\Omega_{\mathbf{t}} \tag{4.20}$$

with $a^{\sharp_k}(f_k) = a^*(\mathbf{f}(k))$ for $k \in F$, $a^{\sharp_{l_i}}(f_{l_i}) = (a^{\sharp_{r_i}}(f_{r_i}))^* = a(g_i)$ for $i = 1 \ldots p$ with $(g_i)_{i=1,\ldots,p}$ a set of normalised vectors, orthogonal to each other and to the vectors $(\mathbf{f}(k))_{k=1}^n$.

Lemma 4.6 *Let* $\mathbf{t}$ *be a function on pair partitions. Then*

$$\langle \psi(\mathcal{V}_1, \mathbf{f}_1),\ \psi(\mathcal{V}_2, \mathbf{f}_2) \rangle = \sum_{\mathcal{V} \in \mathcal{P}_2(F_1^*, F_2)} \eta_{\mathbf{f}_1^* \oplus \mathbf{f}_2}(\mathcal{V}) \cdot \mathbf{t}(\mathcal{V}_1^* \cup \mathcal{V}_2 \cup \mathcal{V}) \tag{4.21}$$

Proof. The equation follows then directly from Definition 2.4.

□

Now we are ready for the main result of this section.

Theorem 4.7 *Let* $\mathbf{t}$ *be a function on pair partitions. If* $\tilde{\rho}_{\mathbf{t}}$ *is a Gaussian state on* $\mathcal{A}(\mathcal{K})$ *for any real Hilbert space* $\mathcal{K}$ *then* $\rho_{\mathbf{t}}$ *is a Fock state on* $\mathcal{C}(\mathcal{K}_{\mathbb{C}})$.

Proof. Suppose that $\rho_{\mathbf{t}}$ is not a Fock state. Then in the representation space $\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}})$ there exists a vector of the form

$$\psi = \sum_{a=1}^{m} c_a \cdot \psi(\mathcal{V}_a, \mathbf{f}_a) \tag{4.22}$$

with all $\mathbf{f}_a$ taking values in the real subspace $\mathcal{K}$ of $\mathcal{K}_{\mathbb{C}}$ and $c_a \in \mathbb{C}$, such that $\langle \psi, \psi \rangle < 0$. But from lemmas 4.5 and 4.6 it results that $\| \sum_{a=1}^{m} c_a \cdot \Psi(\mathcal{V}_a, \mathbf{f}_a) \tilde{\Omega}_{\mathbf{t}} \|^2 < 0$ which is a contradiction. Thus $\rho_{\mathbf{t}}$ is a positive functional and $\mathbf{t}$ is a positive definite function on pair partitions.

□

5 Second Quantisation

This section is dedicated to the description of functorial properties of the generalised Brownian motion which go by the name of second quantisation and appear at two different levels depending on the categories with which we work. Let $\mathcal{H}$, $\mathcal{H}'$ be Hilbert spaces and T a contraction from $\mathcal{H}$ to $\mathcal{H}'$. Define the second quantisation of T at the Hilbert space level by

$$\begin{aligned} \mathcal{F}_{\mathbf{t}}(T) : \mathcal{F}_{\mathbf{t}}(\mathcal{H}) &\rightarrow \mathcal{F}_{\mathbf{t}}(\mathcal{H}') \\ v \otimes_s h_0 \otimes \ldots \otimes \mathcal{H}_{n-1} &\mapsto v \otimes_s T h_0 \otimes \ldots \otimes T h_{n-1} \end{aligned} \tag{5.1}$$

for all $v \in V_n$, $h_i \in \mathcal{H}$ when $n \geq 1$, and equal to the identity on V_0. Clearly $\mathcal{F}_{\mathbf{t}}(T)$ is a contraction, satisfies the equation $\mathcal{F}_{\mathbf{t}}(T_1) \cdot \mathcal{F}_{\mathbf{t}}(T_2) = \mathcal{F}_{\mathbf{t}}(T_1 \cdot T_2)$ and for T unitary it coincides with the operator defined in the equations (2.23) and (2.24).

Definition 5.1 We call $\mathcal{F}_{\mathbf{t}}$ the functor of *second quantisation at the Hilbert space level* .

There is however a more interesting notion of second quantisation.

Definition 5.2 [14] i) The category of *non-commutative probability spaces* has as objects pairs $(\mathcal{A}, \rho_{\mathcal{A}})$ of von Neumann algebras and normal states and as morphisms between two objects $(\mathcal{A}, \rho_{\mathcal{A}})$ and $(\mathcal{B}, \rho_{\mathcal{B}})$ all completely positive maps $T : \mathcal{A} \rightarrow \mathcal{B}$ such that $T(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{A}}$ and $\rho_{\mathcal{B}}(Tx) = \rho_{\mathcal{A}}(x)$ for all $x \in \mathcal{A}$.
ii) A functor Γ from the category of (real) Hilbert spaces with contractions to the category of non-commutative probability spaces such that $\Gamma(\{0\}) = \mathbb{C}$ where $\{0\}$ stands for the zero dimensional Hilbert space is called *functor of white noise.*

From Lemmas 4.3 and 4.4 we know that the generalised Wick products $\Psi(\mathcal{V}, \mathbf{f})$ acting on $\mathcal{F}_{\mathbf{t}}(\mathcal{K})$ form a *-algebra of operators which contains $\pi_{\mathbf{t}}(\mathcal{A}(\mathcal{K}))$ and will be denoted by $\tilde{\Gamma}_{\mathbf{t}}(\mathcal{K})$. In general this is not an algebra of bounded operators. However in a standard manner we construct the von Neumann algebra $\Gamma_{\mathbf{t}}(\mathcal{K})$ generated by all the spectral projections of selfadjoint operators in $\tilde{\Gamma}_{\mathbf{t}}(\mathcal{K})$. Any $X \in \tilde{\Gamma}_{\mathbf{t}}(\mathcal{K})$ is affiliated to $\Gamma_{\mathbf{t}}(\mathcal{K})$. The pair $(\Gamma_{\mathbf{t}}(\mathcal{K}), \langle \Omega_{\mathbf{t}}, \cdot \Omega_{\mathbf{t}} \rangle)$ makes up a non-commutative probability space. Let us first note that Theorem 4.7 implies that the representations of $\tilde{\Gamma}_{\mathbf{t}}(\mathcal{K})$ on $\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}})$ and $\tilde{\mathcal{F}}_{\mathbf{t}}(\mathcal{K})$ are unitarily equivalent, thus:

Corollary 5.3 *The vacuum vector $\Omega_{\mathbf{t}}$ is cyclic for the field algebra $\Gamma_{\mathbf{t}}(\mathcal{K})$ for any real Hilbert space $\mathcal{K}$.*

Our task is to find for which functions $\mathbf{t}$, the map $\mathcal{K} \to \Gamma_{\mathbf{t}}(\mathcal{K})$ can be enriched with the morphisms

$$\Gamma_{\mathbf{t}}(T) : \Big(\Gamma_{\mathbf{t}}(\mathcal{K}), \langle \Omega_{\mathbf{t}}, \cdot \Omega_{\mathbf{t}} \rangle\Big) \to \Big(\Gamma_{\mathbf{t}}(\mathcal{K}'), \langle \Omega_{\mathbf{t}}, \cdot \Omega_{\mathbf{t}} \rangle\Big)$$

for all contractions $T : \mathcal{K} \to \mathcal{K}'$ such that $\Gamma_{\mathbf{t}}$ is a functor of white noise and moreover the following continuity requirement is satisfied: for any pair of Hilbert spaces $\mathcal{K}, \mathcal{K}'$, any $X \in \Gamma_{\mathbf{t}}(\mathcal{K})$ and any sequence of contractions $T_n : \mathcal{K} \to \mathcal{K}'$ converging weakly to T we have

$$\operatorname*{w-lim}_{n\to\infty} \Gamma_{\mathbf{t}}(T_n)(X) = \Gamma_{\mathbf{t}}(T)(X). \tag{5.2}$$

When this is true we call $\Gamma_{\mathbf{t}}$ a functor of *second quantisation at algebraic level* and the completely positive map $\Gamma_{\mathbf{t}}(T)$ the second quantisation of the contraction T. The existence of the second quantisation at algebraic level turns out to be connected to a property of the functions on pair partitions.

Definition 5.4 [3] A function $\mathbf{t}$ on pair partitions is called *multiplicative* if for all $k, l, n \in \mathbb{N}$ with $0 \leq k < l \leq n$ and all $\mathcal{V}_1 \in \mathcal{P}_2(\{1, \dots, k, l+1, \dots, n\})$ and $\mathcal{V}_2 \in \mathcal{P}_2(\{k+1, \dots, l\})$ we have

$$\mathbf{t}(\mathcal{V}_1 \cup \mathcal{V}_2) = \mathbf{t}(\mathcal{V}_1) \cdot \mathbf{t}(\mathcal{V}_2). \tag{5.3}$$

Proposition 5.5 *If Γ is a functor of second quantisation then $\Gamma(T)$ is an automorphism (injective *-homomorphism) if T is a unitary (isometry), and $\Gamma(P)$ is a conditional expectation if P is an orthogonal projection.*

Proof. For separating vacuum the proof has been given in [15]. Here we do not assume this property.
1. Let $O : \mathcal{K} \to \mathcal{K}'$ be an orthogonal operator and $X \in \Gamma(\mathcal{K})$. As $\Gamma(O^*)$ and $\Gamma(O)$ are completely positive we have the inequalities

$$\Gamma(O^*)(\Gamma(O)(X^*) \cdot \Gamma(O)(X)) \geq \Gamma(O^*O)(X^*) \cdot \Gamma(O^*O)(X) = X^*X \tag{5.4}$$

and

$$\Gamma(O)(X^*)\Gamma(O)(X) \leq \Gamma(O)(X^*X) \tag{5.5}$$

which by applying the positive operator $\Gamma(O^*)$ becomes

$$\Gamma(O^*)(\Gamma(O)(X^*) \cdot \Gamma(O)(X)) \leq \Gamma(O^*O)(X^*X) = X^*X \tag{5.6}$$

From (5.4, 5.6) we get $\Gamma(O)(X^*) \cdot \Gamma(O)(X) = \Gamma(O)(X^*X)$ and by reapeating the argument for $X+Y$ and $X+iY$ we obtain that $\Gamma(O)$ is a *-isomorphism.
2. Let $\mathcal{K}$ be a real Hilbert space and $I : \mathcal{K} \to \mathcal{K} \oplus \ell^2(\mathbb{Z})$ the natural isometry. Let S be the shift operator on $\ell^2(\mathbb{Z})$. The operator $O := \mathbf{1} \oplus S$ is orthogonal and $\text{w-lim}_{n\to\infty} O^n = P$ where P is the projection on $\mathcal{K}$. By the continuity assumption 5.2 we have then

$$\operatorname*{w-lim}_{n\to\infty} \Gamma(O^n)(X) = \Gamma(P)(X) \tag{5.7}$$

for all $X \in \Gamma(\mathcal{K} \oplus \ell^2(\mathbb{Z}))$. Let $\Gamma(\mathcal{K} \oplus \ell^2(\mathbb{Z}))^{\mathbb{Z}}$ be the subalgebra of $\Gamma(\mathcal{K} \oplus \ell^2(\mathbb{Z}))$ of operators invariant under the action of the group of automorphisms $(\Gamma(O^n))_{n\in\mathbb{Z}}$. Then from $O^n P = P$ we get $\Gamma(P)(X) \in \Gamma(\mathcal{K} \oplus \ell^2(\mathbb{Z}))^{\mathbb{Z}}$. For arbitrary $Y_1, Y_2 \in \Gamma(\mathcal{K} \oplus \ell^2(\mathbb{Z}))^{\mathbb{Z}}$ the following holds

$$\Gamma(P)(Y_1 X Y_2) = Y_1 \Gamma(P)(X) Y_2 \tag{5.8}$$

which means that $\Gamma(P)$ is a conditional expectation from $\Gamma(\mathcal{K} \oplus \ell^2(\mathbb{Z}))$ onto $\Gamma(\mathcal{K} \oplus \ell^2(\mathbb{Z}))^{\mathbb{Z}}$. We show now that $\Gamma(I)$ is an injective *-homomorphism. By a similar argument to that used in (5.4, 5.6) we have:

$$\Gamma(I^*)(\Gamma(I)(X) \cdot \Gamma(I)(Y)) = XY \tag{5.9}$$

for all $X, Y \in \Gamma(\mathcal{K})$. Now for any $Z \in \Gamma(\mathcal{K})$

$$\Gamma(I)(Z) = \Gamma(PI)(Z) \in \Gamma(\mathcal{K} \oplus \ell^2(\mathbb{Z}))^{\mathbb{Z}} \tag{5.10}$$

which together with (5.9) implies

$$\Gamma(I)(XY) = \Gamma(II^*)(\Gamma(I)(X) \cdot \Gamma(I)(Y)) = \Gamma(I)(X) \cdot \Gamma(I)(Y) \tag{5.11}$$

and thus $\Gamma(I)$ is an injective *-homomorphism.
3. Let $I : \mathcal{K} \to \mathcal{K}'$ be an isometry. We consider the natural isometries $I'_{\mathcal{K}} : \mathcal{K}' \to \mathcal{K}' \oplus \ell^2(\mathbb{Z})$ and $I_{\mathcal{K}} = I_{\mathcal{K}'} I$. From the previous argument we know that $\Gamma(I_{\mathcal{K}'}), \Gamma(I_{\mathcal{K}})$ are injective *-homomorphisms. Let $X, Y \in \Gamma(\mathcal{K})$. Then

$$\begin{aligned} \Gamma(I_{\mathcal{K}'})(\Gamma(I)(XY)) &= \Gamma(I_{\mathcal{K}})(XY) = \Gamma(I_{\mathcal{K}})(X) \cdot \Gamma(I_{\mathcal{K}})(Y) \\ &= \Gamma(I_{\mathcal{K}'})(\Gamma(I)(X) \cdot \Gamma(I)(Y)). \end{aligned} \tag{5.12}$$

As $\Gamma(I_{\mathcal{K}'})$ is injective *-homomorphism we obtain

$$\Gamma(I)(XY) = \Gamma(I)(X)\Gamma(I)(Y). \tag{5.13}$$

Thus $\Gamma(I)$ is a *-homomorphism. The injectivity follows from $I^* I = \mathbf{1}_{\mathcal{K}}$.
4. Using the previous step of the proof we see that $\Gamma(P)$ is a norm one projection from $\Gamma(\mathcal{K}')$ onto its von Neumann subalgebra $\Gamma(I)(\Gamma(\mathcal{K}))$. Thus $\Gamma(P)$ is a conditional expectation [22].

□

Corollary 5.6 *If Γ is a functor of second quantisation then for any infinite dimensional real Hilbert space $\mathcal{K}$ we have $\Gamma(\mathcal{K})^{\mathcal{O}(\mathcal{K})} = \mathbb{C}\mathbf{1}$.*

Proof. We can choose $\mathcal{K} = \ell^2(\mathbb{Z})$. Let S be the right shift. Then S^n converges weakly to 0 as $n \to \infty$. By the previous proposition we have that $\Gamma(\mathcal{K})^{\mathbb{Z}}$ is isomorphic with $\Gamma(I)(\{0\})$ where I is the trivial isometry from $\{0\}$ to $\ell^2(\mathbb{Z})$. Thus $\Gamma(\mathcal{K})^{\mathcal{O}(\mathcal{K})} = \mathbb{C}\mathbf{1}$.

□

Corollary 5.7 *Let* $\mathbf{t}$ *be a positive definite function on pair partitions and suppose that the functor of second quantisation* $\Gamma_{\mathbf{t}}$ *exists and* $\Gamma_{\mathbf{t}}(O)(\cdot) = \mathcal{F}_{\mathbf{t}}(O) \cdot \mathcal{F}_{\mathbf{t}}(O^*)$*. Then the function* $\mathbf{t}$ *is multiplicative.*

Proof. Let $\mathcal{K} = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ with the two right shifts S_1, S_2 acting separately on the two $\ell^2(\mathbb{Z})$. Let $\mathcal{V}_1 \cup \mathcal{V}_2$ be a pair partition as in Definition 5.4. For any pair partition $\mathcal{V}$ the operator $\Psi(\mathcal{V}, \emptyset)$ on $\mathcal{F}_{\mathbf{t}}(\mathcal{K})$ commutes with $\mathcal{F}_{\mathbf{t}}(O)$ for all $O \in \mathcal{O}(\mathcal{K})$ and is affiliated to $\Gamma_{\mathbf{t}}(\mathcal{K})$. By corollary 5.6 we have

$$\Psi(\mathcal{V}, \emptyset) = \langle \Omega_{\mathbf{t}}, \Psi(\mathcal{V}, \emptyset)\Omega_{\mathbf{t}} \rangle = \mathbf{t}(\mathcal{V})\mathbf{1}. \tag{5.14}$$

We consider the monomial of fields $M(\mathcal{V}_1 \cup \mathcal{V}_2)$ containing $|\mathcal{V}_1| + |\mathcal{V}_2|$ pairs of different colours arranged according to the pair partition $\mathcal{V}_1 \cup \mathcal{V}_2$ and such that the colours for the pairs in $\mathcal{V}_1$ belong to the first $\ell^2(\mathbb{Z})$ in $\mathcal{K}$, and those for the pairs in $\mathcal{V}_2$ belong to the second term. Then

$$\begin{aligned}\Psi(\mathcal{V}_1 \cup \mathcal{V}_2, \emptyset) &= \mathbf{t}(\mathcal{V}_1 \cup \mathcal{V}_2)\mathbf{1} = \operatorname*{w-lim}_{n,m\to\infty} \Gamma_{\mathbf{t}}(S_1^n S_2^m)(M(\mathcal{V}_1 \cup \mathcal{V}_2)) \\ &= \operatorname*{w-lim}_{m\to\infty} M(\mathcal{V}_1)\Psi(\mathcal{V}_2) = \Psi(\mathcal{V}_1, \emptyset) \cdot \mathbf{t}(\mathcal{V}_2) = \mathbf{t}(\mathcal{V}_1)\mathbf{t}(\mathcal{V}_2) \cdot \mathbf{1}. \end{aligned} \tag{5.15}$$

□

Lemma 5.8 *Let* $\psi(\mathcal{V}, \mathbf{f})$ *as defined in equation (4.20). Then*

$$\mathcal{F}_{\mathbf{t}}(T)\psi(\mathcal{V}, \mathbf{f}) = \psi(\mathcal{V}, T \circ \mathbf{f}). \tag{5.16}$$

Proof. We use the representation $\chi_{\mathbf{t}}$ of the *-semigroup of broken pair partitions $\mathcal{BP}_2(\infty)$ with respect to the state $\hat{t}$ (see equation 3.7). Let $\{F, P\}$ be a partition of $\{1, \dots, 2p+n\}$ and $\mathcal{V} \in \mathcal{P}_2(P)$, $\mathbf{f} : F \to \mathcal{H}$. Then using (4.20) and the equations (2.13, 2.14) we obtain

$$\psi(\mathcal{V}, \mathbf{f}) = \chi_{\mathbf{t}}(\tilde{\mathcal{V}})\xi \otimes_s \bigotimes_{k \in F} \mathbf{f}(k) \tag{5.17}$$

for $\tilde{\mathcal{V}} \in \mathcal{BP}_2^{n,0}$ the diagram with the set of pairs $\mathcal{V}$ and n legs to the left which do not intersect each other.

□

Lemma 5.9 *Let* $P : \mathcal{K} \to \mathcal{K}'$ *be a coisometry i.e.* $PP^* = \mathbf{1}_{\mathcal{K}'}$ *and* $\mathbf{t}$ *a positive definite function on pair partitions. Then the map* $\tilde{\Gamma}_{\mathbf{t}}(P)$ *from* $\tilde{\Gamma}_{\mathbf{t}}(\mathcal{K})$ *to* $\tilde{\Gamma}_{\mathbf{t}}(\mathcal{K}')$

$$\tilde{\Gamma}_{\mathbf{t}}(P) : X \mapsto \mathcal{F}_{\mathbf{t}}(P)X\mathcal{F}_{\mathbf{t}}(P^*) \tag{5.18}$$

is well defined.

Proof. It is enough to show that

$$\mathcal{F}_{\mathbf{t}}(P)\Psi(\mathcal{V}, \mathbf{f})\mathcal{F}_{\mathbf{t}}(P^*) = \Psi(\mathcal{V}, P \circ \mathbf{f}). \tag{5.19}$$

For this we prove the equality of the matrix elements

$$\begin{aligned}
&\langle\Psi(\mathcal{V}_1,\mathbf{f}_1)\Omega_{\mathbf{t}},\ \mathcal{F}_{\mathbf{t}}(P)\Psi(\mathcal{V},\mathbf{f})\mathcal{F}_{\mathbf{t}}(P^*)\Psi(\mathcal{V}_2,\mathbf{f}_2)\Omega_{\mathbf{t}}\rangle = \\
&\langle\psi(\mathcal{V}_1,\mathbf{f}_1),\ \mathcal{F}_{\mathbf{t}}(P)\Psi(\mathcal{V},\mathbf{f})\mathcal{F}_{\mathbf{t}}(P^*)\psi(\mathcal{V}_2,\mathbf{f}_2)\rangle = \\
&\langle\psi(\mathcal{V}_1,P^*\circ\mathbf{f}_1),\ \Psi(\mathcal{V},\mathbf{f})\psi(\mathcal{V}_2,P^*\circ\mathbf{f}_2)\rangle = \\
&\langle\psi(\mathcal{V}_1,\mathbf{f}_1),\ \Psi(\mathcal{V},P\circ\mathbf{f})\psi(\mathcal{V}_2,\mathbf{f}_2)\rangle
\end{aligned} \tag{5.20}$$

□

From Lemma 5.9 we deduce that if $P:\mathcal{K}\to\mathcal{K}'$ is a coisometry then $\Gamma_{\mathbf{t}}(P)(\cdot):=\mathcal{F}_{\mathbf{t}}(P)\cdot\mathcal{F}_{\mathbf{t}}(P^*)$ is well defined from $\Gamma_{\mathbf{t}}(\mathcal{K})$ to $\Gamma_{\mathbf{t}}(\mathcal{K}')$ and completely positive.

Theorem 5.10 *Let* $\mathbf{t}$ *be a positive definite function on pair partitions such that the vacuum* $\Omega_{\mathbf{t}}\in\mathcal{F}_{\mathbf{t}}(\mathcal{K})$ *is separating vector for the von Neumann algebra* $\Gamma_{\mathbf{t}}(\mathcal{K})$ *for any real Hilbert space* $\mathcal{K}$*. Let* $T:\mathcal{K}_1\to\mathcal{K}_2$ *be a contraction between real Hilbert spaces. Then the second quantisation of* T *is the completely positive map* $\Gamma_{\mathbf{t}}(T)$ *from* $\Gamma_{\mathbf{t}}(\mathcal{K}_1)$ *to* $\Gamma_{\mathbf{t}}(\mathcal{K}_2)$ *such that*

$$\mathcal{F}_{\mathbf{t}}(T)X\Omega_{\mathbf{t}}=\Gamma_{\mathbf{t}}(T)(X)\Omega_{\mathbf{t}} \tag{5.21}$$

for $X\in\Gamma_{\mathbf{t}}(\mathcal{K}_1)$*. This defines the linear map* $\tilde{\Gamma}_{\mathbf{t}}(T)$ *from* $\tilde{\Gamma}_{\mathbf{t}}(\mathcal{K}_1)$ *to* $\tilde{\Gamma}_{\mathbf{t}}(\mathcal{K}_2)$*:*

$$\Gamma_{\mathbf{t}}(T):\Psi(\mathcal{V},\mathbf{f})\mapsto\Psi(\mathcal{V},T\circ\mathbf{f}). \tag{5.22}$$

Proof. The contraction T can be written as $T=PI$ where $I:\mathcal{K}_1\to\mathcal{K}'$ is an isometry and $P:\mathcal{K}'\to\mathcal{K}_2$ is a coisometry.
The faithfulness of the vacuum and the fact that the isometry $\mathcal{F}_{\mathbf{t}}(I)$ has the action $\mathcal{F}_{\mathbf{t}}(I)\Psi(\mathcal{V},\mathbf{f})\Omega_{\mathbf{t}}=\Psi(\mathcal{V},I\circ\mathbf{f})\Omega_{\mathbf{t}}$ implies that the map

$$\tilde{\Gamma}_{\mathbf{t}}(I):\Psi(\mathcal{V},\mathbf{f})\mapsto\Psi(\mathcal{V},I\circ\mathbf{f}) \tag{5.23}$$

is an injective *-homomorphism from $\tilde{\Gamma}_{\mathbf{t}}(\mathcal{K}_1)$ to $\tilde{\Gamma}_{\mathbf{t}}(\mathcal{K}')$. On the level of von Neumann algebras we obtain the homomorphism (and thus completely positive map) $\Gamma_{\mathbf{t}}(I)$ from $\Gamma_{\mathbf{t}}(\mathcal{K}_1)$ to $\Gamma_{\mathbf{t}}(\mathcal{K}_2)$.
By Lemma 5.9, $\Gamma_{\mathbf{t}}(P)$ is a conditional expectation and $\Gamma_{\mathbf{t}}(I)\Gamma_{\mathbf{t}}(P)$ depends only on $T=PI$:

$$\Gamma_{\mathbf{t}}(I)\Gamma_{\mathbf{t}}(P)(X)\Omega_{\mathbf{t}}=\mathcal{F}_{\mathbf{t}}(T)X\Omega_{\mathbf{t}}. \tag{5.24}$$

□

5.1 A second definition for the functor of second quantisation

Theorem 5.10 points out that a separating vacuum insures the existence of the injective *-homomorphism $\Gamma_{\mathbf{t}}(I)$ and thus of the functor of second quatisation $\Gamma_{\mathbf{t}}$. We will show here that this condition can be removed if one uses a different definition for the "field algebra" $\Gamma_{\mathbf{t}}(\mathcal{K})$. Let $\mathcal{K}$ be a real Hilbert space and define the von Neumann algebra

$$\Delta_{\mathbf{t}}(\mathcal{K}) := \Gamma_{\mathbf{t}}(\mathcal{K} \oplus \ell^2(\mathbb{Z}))^{\mathbb{Z}} \tag{5.25}$$

acting on $\mathcal{F}_{\mathbf{t}}(\mathcal{K} \oplus \ell^2(\mathbb{Z}))$. Similarly to $\Gamma_{\mathbf{t}}(\mathcal{K})$, this von Neumann algebra is generated by the spectral projections of the selfadjoint operators in the *-algebra $\tilde{\Delta}_{\mathbf{t}}(\mathcal{K})$ of generalised Wick products $\Psi(\mathcal{V}, \mathbf{f})$ with $\mathrm{Im}(\mathbf{f}) \subset \mathcal{K}$. The subrepresentation of $\Delta_{\mathbf{t}}(\mathcal{K})$ for which $\Omega_{\mathbf{t}}$ is cyclic, is unitarily equivalent to $\Gamma_{\mathbf{t}}(\mathcal{K})$.

Lemma 5.11 *Let $\mathcal{K}, \mathcal{K}'$ be real Hilbert spaces and $I : \mathcal{K} \to \mathcal{K}'$ an isometry. Then there exists an injective *-homomorphism $\Delta_{\mathbf{t}}(I)$ from $\Delta_{\mathbf{t}}(\mathcal{K})$ to $\Delta_{\mathbf{t}}(\mathcal{K}')$. The corresponding map $\tilde{\Delta}_{\mathbf{t}}(I)$ acts as follows*

$$\begin{aligned} \tilde{\Delta}_{\mathbf{t}}(I) \quad &: \quad \tilde{\Delta}_{\mathbf{t}}(\mathcal{K}) \to \tilde{\Delta}_{\mathbf{t}}(\mathcal{K}') \\ \tilde{\Delta}_{\mathbf{t}}(I) \quad &: \quad \Psi(\mathcal{V}, \mathbf{f}) \mapsto \Psi(\mathcal{V}, I \circ \mathbf{f}) \end{aligned} \tag{5.26}$$

where $\mathrm{Im}(\mathbf{f}) \subset \mathcal{K}$.

Proof. We have

$$\Delta_{\mathbf{t}}(\mathcal{K}) = \Gamma_{\mathbf{t}}(\mathcal{K} \oplus \ell^2(\mathbb{Z}))^{\mathbb{Z}} = \Gamma_{\mathbf{t}}(\mathcal{K} \oplus \ell^2(\mathbb{Z}))^{\mathcal{O}(\ell^2(\mathbb{Z}))} \tag{5.27}$$

Let $O_{\mathcal{K}} : \mathcal{K} \oplus \ell^2(\mathbb{Z}) \to \mathcal{K}' \oplus \ell^2(\mathbb{Z})$ be an orthogonal operator such that $O_{\mathcal{K}}(f \oplus 0) = I(f) \oplus 0$. Let us make the notation $\mathcal{K}_1 := (\mathcal{K}' \ominus I\mathcal{K}) \oplus \ell^2(\mathbb{Z})$. Then we have the injective *-homomorphism:

$$\begin{aligned} \Delta_{\mathbf{t}}(I) : \Gamma_{\mathbf{t}}(\mathcal{K} \oplus \ell^2(\mathbb{Z}))^{\mathcal{O}(\ell^2(\mathbb{Z}))} \quad &\to \quad \Gamma_{\mathbf{t}}(I\mathcal{K} \oplus \mathcal{K}_1)^{\mathcal{O}(\mathcal{K}_1)} \\ X \quad &\mapsto \quad \mathcal{F}_{\mathbf{t}}(O_{\mathcal{K}}) X \mathcal{F}_{\mathbf{t}}(O_{\mathcal{K}}^*). \end{aligned} \tag{5.28}$$

On the other hand $\Gamma_{\mathbf{t}}(I\mathcal{K} \oplus \mathcal{K}_1)^{\mathcal{O}(\mathcal{K}_1)}$ is a von Neumann subalgebra of $\Gamma_{\mathbf{t}}(I\mathcal{K} \oplus \mathcal{K}_1)^{\mathcal{O}(\ell^2(\mathbb{Z}))}$ which is the identical with $\Delta_{\mathbf{t}}(\mathcal{K}') = \Gamma_{\mathbf{t}}(\mathcal{K}' \oplus \ell^2(\mathbb{Z}))^{\mathbb{Z}}$. Thus we have shown the existence of the injective *-homomorphism $\Delta_{\mathbf{t}}(I)$. This does not depend on the choice of $O_{\mathcal{K}}$. The action on $\tilde{\Delta}_{\mathbf{t}}(\mathcal{K})$ is obvious.

□

Now let $P : \mathcal{K} \to \mathcal{K}'$ be a coisometry. We define the second quantisation of P (for the functor $\Delta_{\mathbf{t}}$) in a similar manner to $\Gamma_{\mathbf{t}}(P)$ (see 5.18) by the completely positive map:

$$\begin{aligned} \Delta_{\mathbf{t}}(P) : \Delta_{\mathbf{t}}(\mathcal{K}) \quad &\to \quad \Delta_{\mathbf{t}}(\mathcal{K}') \\ X \quad &\mapsto \quad \mathcal{F}_{\mathbf{t}}(P \oplus \mathbf{1}) X \mathcal{F}_{\mathbf{t}}(P^* \oplus \mathbf{1}). \end{aligned} \tag{5.29}$$

According to Lemma 5.9, on the *-algebra $\tilde{\Delta}_{\mathbf{t}}(\mathcal{K})$ we have:

$$\tilde{\Delta}_{\mathbf{t}}(P) : \Psi(\mathcal{V}, \mathbf{f}) \mapsto \Psi(\mathcal{V}, P \circ \mathbf{f}). \tag{5.30}$$

Finally let T be an arbitrary contraction. Independent of the choice of the decomposition $T = PI$ we have

$$\tilde{\Delta}_{\mathbf{t}}(T)(\Psi(\mathcal{V}, \mathbf{f})) := \tilde{\Delta}_{\mathbf{t}}(P)\tilde{\Delta}_{\mathbf{t}}(I)(\Psi(\mathcal{V}, \mathbf{f})) = \Psi(\mathcal{V}, T \circ \mathbf{f}) \tag{5.31}$$

which is implemented by a compression of the type $\mathcal{F}_{\mathbf{t}}(P') \cdot \mathcal{F}_{\mathbf{t}}(P'^*)$ where $P' = (P \oplus \mathbf{1})O_{\mathcal{K}}$. This leads us to the result of this subsection:

Theorem 5.12 *If $\mathbf{t}$ is positive definite and multiplicative then $\Delta_{\mathbf{t}}$ is a functor of second quantisation.*

6 An Example

In [3] and [10] it has been proved that for all $0 \leq q \leq 1$, the following function on pair partitions is positive definite:

$$\mathbf{t}_q(\mathcal{V}) = q^{|\mathcal{V}|-|\mathrm{B}(\mathcal{V})|} \tag{6.1}$$

where $|\mathrm{B}(\mathcal{V})|$ is the number of blocks of the pair partition $\mathcal{V}$. A block is a subpartition whose graphical representation is connected and does not intersect other pairs from the rest of the partition. The corresponding vacuum state $\rho_{\mathbf{t}_q}(\cdot) = \left\langle \Omega_{\mathbf{t}_q}, \cdot \Omega_{\mathbf{t}_q} \right\rangle$ is tracial for any von Neumann algebra $\Gamma_{\mathbf{t}_q}(\mathcal{K})$ associated to a real Hilbert space $\mathcal{K}$. Indeed for any pair partition $\mathcal{V}$ we have $\mathbf{t}_q(\mathcal{V}) = \left\langle \Omega_{\mathbf{t}_q}, M_{\mathcal{V}} \Omega_{\mathbf{t}_q} \right\rangle$ with $M_{\mathcal{V}}$ a monomial of fields containing $|\mathcal{V}|$ pairs of different colours arranged according to the pair partition $\mathcal{V}$. The trace property for the vacuum is equivalent with the invariance under circular permutations of the fields in the monomial $M_{\mathcal{V}}$ which is equivalent to the invariance of $\mathbf{t}_q$ under transformations described as follows:

$$\begin{aligned} \mathcal{P}_2(\{1,\ldots,2r\}) \ni \mathcal{V} \quad &\mapsto \quad \tilde{\mathcal{V}} \in \mathcal{P}_2(\{0,\ldots,2r-1\}) && (6.2)\\ \{p_1,\ldots,p_{r-1}\} \cup \{(l,2r)\} \quad &\mapsto \quad \{(0,l)\} \cup \{p_1,\ldots,p_{r-1}\}. && (6.3) \end{aligned}$$

Under such transformations the number of blocks remains unchanged thus $\mathbf{t}_q(\mathcal{V})$ is equal to $\mathbf{t}_q(\tilde{\mathcal{V}})$ and $\rho_{\mathbf{t}_q}$ is tracial. Thus the assumption of Theorem 5.10 is satisfied and we have second quantisation at algebraic level.
Next, we want to investigate the type of the von Neumann algebras $\Gamma_{\mathbf{t}_q}(\mathcal{K})$ for dim $\mathcal{K} = \infty$. Inspired by [2], we will find a sufficient condition for $\Gamma_{\mathbf{t}}(\mathcal{K})$ to be a type II_1 factor, and we will apply it to $\mathbf{t}_q$.
Let $\mathbf{t}$ be a multiplicative positive definite function such that $\rho_{\mathbf{t}}$ is trace state on $\Gamma_{\mathbf{t}}(\mathcal{K})$ for $\mathcal{K}$ infinite dimensional. Let I be the natural isometry from $\mathcal{K}$ to $\mathcal{K} \oplus \mathbb{R}$, and e_0 a unit vector in the orthogonal complement of its image. The function $\mathbf{t}$ being multiplicative implies that the map

$$\phi : \mathcal{F}_{\mathbf{t}}(\mathcal{K}) \to \mathcal{F}_{\mathbf{t}}(\mathcal{K} \oplus \mathbb{R}) \tag{6.4}$$

defined by $\phi = \omega_{\mathbf{t}}(e_0)\mathcal{F}_{\mathbf{t}}(I)$ is an isometry.

Definition 6.1 Let (P, L, R) be a disjoint partition of the ordered set $\{1,\ldots,2n+l+r\}$ and $d = (\mathcal{V}, f_l, f_r)$ an element of the $^*-$semigroup of broken pair partitions with $\mathcal{V} \in \mathcal{P}_2(P)$, $f_l : L \to \{1,\ldots l\}$ the left legs and $f_r : R \to \{1,\ldots r\}$ the right legs. We denote by $\underline{d} := (\underline{\mathcal{V}}, f_l, f_r)$ the element obtained by adding to $\mathcal{V}$ one pair which embraces all other pairs

$$\underline{\mathcal{V}} := \mathcal{V} \cup \{(0, 2n+l+r+1)\} \in \mathcal{P}_2(\{0\} \cup P \cup \{2n+l+r+1\}). \tag{6.5}$$

Then the map $\tilde{\Phi}(\cdot) := \phi^* \tilde{\Gamma}_{\mathbf{t}}(I)(\cdot)\phi$ has the following action on $\tilde{\Gamma}_{\mathbf{t}}(\mathcal{K})$:

$$\tilde{\Phi}(\Psi(\mathcal{V}, \mathbf{f})) = \Psi(\underline{\mathcal{V}}, \mathbf{f}) \tag{6.6}$$

which on the level of von Neumann algebras gives the completely positive contraction $\Phi(\cdot) := \phi^* \Gamma_{\mathbf{t}}(I)(\cdot)\phi$ from $\Gamma_{\mathbf{t}}(\mathcal{K})$ to itself. We fix an orthonormal basis $(e_n)_{n=1}^{\infty}$ in $\mathcal{K}$. Then by direct computation one can check that:

$$\Phi(X) = \operatorname*{w-lim}_{n\to\infty} \omega_{\mathbf{t}}(e_n) X \omega_{\mathbf{t}}(e_n). \tag{6.7}$$

Let now τ be a tracial normal state on $\Gamma_{\mathbf{t}}(\mathcal{K})$. Then using the fact that $\operatorname{w-lim}_{n\to\infty} \omega_{\mathbf{t}}(e_n)^2 = \mathbf{1}$ we get:

$$\tau(\Phi(X)) = \lim_{n\to\infty} \tau(\omega_{\mathbf{t}}(e_n) X \omega_{\mathbf{t}}(e_n)) = \lim_{n\to\infty} \tau(\omega_{\mathbf{t}}(e_n)^2 X) = \tau(X). \tag{6.8}$$

Suppose now that

$$\operatorname*{w-lim}_{k\to\infty} \Phi^k(X) = \rho_{\mathbf{t}}(X)\mathbf{1} \tag{6.9}$$

for all $X \in \Gamma_{\mathbf{t}}(\mathcal{K})$ which by the faithfulness of the vacuum state is equivalent to $\lim_{k\to\infty} \Phi^k(X)\Omega_{\mathbf{t}} = \rho_{\mathbf{t}}(X)\Omega_{\mathbf{t}}$. Then by equation (6.8) we conclude that $\rho_{\mathbf{t}}$ is the only trace state on $\Gamma_{\mathbf{t}}(\mathcal{K})$ which is thus a type II_1 factor. Let us take a closer look at the contraction

$$\Theta : X\Omega_{\mathbf{t}} \mapsto \Phi(X)\Omega_{\mathbf{t}}. \tag{6.10}$$

From equation (6.6) the operator Θ commutes with the orthogonal projectors on the spaces with definite "occupation numbers" $\mathcal{F}_{\mathbf{t}}(n_1, \ldots, n_k)$ (see 2.31). Thus

$$\Theta : v \otimes_s e(\underline{n}) \mapsto \theta(v) \otimes_s e(\underline{n}) \tag{6.11}$$

where

$$e(\underline{n}) := \underbrace{e_1 \otimes \ldots \otimes e_1}_{n_1 \text{times}} \otimes \ldots \otimes \underbrace{e_k \otimes \ldots \otimes e_k}_{n_k \text{times}}, \tag{6.12}$$

$v \in V_n$ and $\theta : V \to V$ is the linear operator defined by

$$\theta : \chi_{\mathbf{t}}(d)\xi \mapsto \chi_{\mathbf{t}}(\underline{d})\xi, \qquad (d \in \mathcal{BP}_2(\infty)). \tag{6.13}$$

Lemma 6.2 *Let* $\mathbf{t}$ *be a multiplicative positive definite function such that* $\rho_{\mathbf{t}}$ *is trace. Then the operator* $\theta : V \to V$ *defined by 6.13 is a selfadjoint contraction.*

Proof. Let $d_1, d_2 \in \mathcal{BP}_2^{(n,0)}$ be two diagrams with n left legs and no right legs. Then

$$\langle \chi_{\mathbf{t}}(d_1)\xi,\ \theta\ \chi_{\mathbf{t}}(d_2)\xi \rangle_V = \hat{\mathbf{t}}(d_1^* \cdot \underline{d_2}). \tag{6.14}$$

But if $\rho_{\mathbf{t}}$ is a trace then

$$\hat{\mathbf{t}}(d_1^* \cdot \underline{d_2}) = \hat{\mathbf{t}}(\underline{d_1}^* \cdot d_2) \tag{6.15}$$

which implies that

$$\langle \chi_{\mathbf{t}}(d_1)\xi,\ \theta\ \chi_{\mathbf{t}}(d_2)\xi\rangle_V = \langle\ \theta\ \chi_{\mathbf{t}}(d_1)\xi, \chi_{\mathbf{t}}(d_2)\xi\rangle_V\,. \tag{6.16}$$

Thus θ is a selfadjoint contraction.

□

Theorem 6.3 *If ξ is the only eigenvector of θ with eigenvalue 1 then $\Gamma_{\mathbf{t}}(\mathcal{K})$ is a II_1 factor for any infinite dimensional real Hilbert space $\mathcal{K}$.*

Proof. The operator θ is a selfadjoint contraction, thus

$$\mathrm{w}\!-\!\lim_{k\to\infty} \theta^k = P_\xi \tag{6.17}$$

where P_ξ is the projection on the subspace $\mathbb{C}\xi$. This implies 6.9 and thus $\Gamma_{\mathbf{t}}(\mathcal{K})$ is a II_1 factor .

□

Corollary 6.4 *Let $\mathcal{K}$ be an infinite dimensional real Hilbert space and $\mathbf{t}$ the function given by 6.1 with $0 \leq q < 1$. Then the von Neumann algebra $\Gamma_{\mathbf{t}}(\mathcal{K})$ is a type II_1 factor.*

Proof. Let $d_1, d_2 \in \mathcal{BP}_2^{(n,0)}$ be two diagrams with $n \geq 1$ left legs and no right legs. Then

$$\begin{aligned} \left\langle \chi_{\mathbf{t}}(d_1)\xi,\ \theta^2\ \chi_{\mathbf{t}}(d_2)\xi\right\rangle_V &= \hat{\mathbf{t}}(\underline{d_1}^* \cdot \underline{d_2}) = q \cdot \hat{\mathbf{t}}(d_1^* \cdot \underline{d_2}) \\ &= q \cdot \langle \chi_{\mathbf{t}}(d_1)\xi,\ \theta\ \chi_{\mathbf{t}}(d_2)\xi\rangle_V\,. \end{aligned} \tag{6.18}$$

where we have used the selfadjointness of θ in the first step and

$$\begin{aligned} &|B(\underline{d_1}^* \cdot \underline{d_2})| = |B(d_1^* \cdot \underline{d_2})|, \\ &|\underline{d_1}^* \cdot \underline{d_2}| = |d_1^* \cdot \underline{d_2}| + 1 \end{aligned}$$

in the second equality. Thus the restriction of θ to $V \ominus \mathbb{C}\xi$ has norm $q < 1$ and we can apply Theorem 6.3.

□

Remark. If $\rho_{\mathbf{t}}$ is a faithful, multiplicative but non-tracial state for $\Gamma_{\mathbf{t}}(\mathcal{K})$ then the operators $\phi, \Phi, \Theta, \theta$ can still be defined in the same way. If moreover, ξ is the only eigenvector with eigenvalue 1 of the operator θ, then by a similar argument it can be shown that the algebra $\Gamma_{\mathbf{t}}(\mathcal{K})$ is a factor. Indeed if X is an element in the center of $\Gamma_{\mathbf{t}}(\mathcal{K})$ then $\Phi(X) = \mathrm{w}\!-\!\lim_{k\to\infty} \omega_{\mathbf{t}}(e_n)^2 X = X$. which contradicts the assumption on θ. The factor must be infinite; otherwise by a limit argument it would result that $\rho_{\mathbf{t}}$ is tracial. Examples of such positive definite multplicative functions on pair partitions will be given in a forthcoming paper [11].

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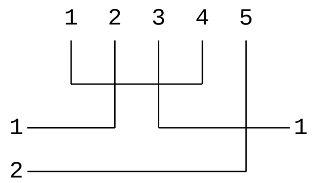

Figure 1: Diagram corresponding to an element of the semigroup

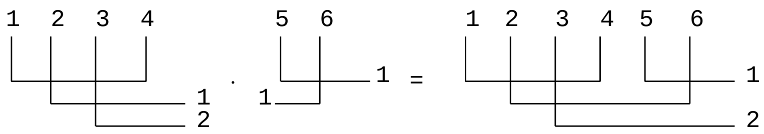

Figure 2: Product of two elements

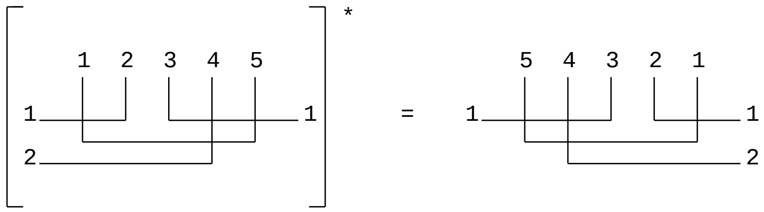

Figure 3: The adjoint

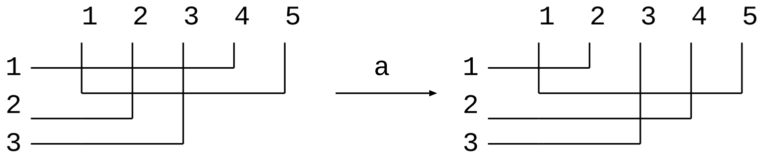

Figure 4: The action of a transposition

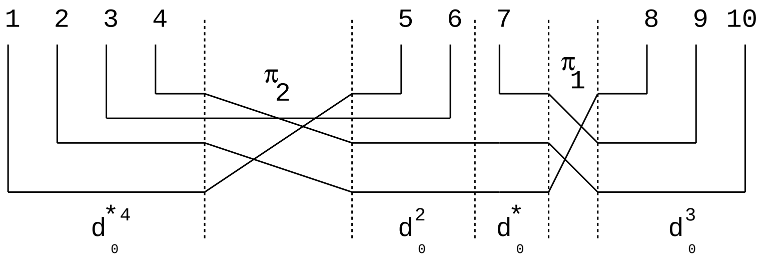

Figure 5: The standard form of a pair partition