On systems with finite ergodic degree

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Abstract

In this paper we study the ergodic theory of a class of symbolic dynamical systems (Ω, T, μ) where $T : \Omega \to \Omega$ the the left shift transformation on $\Omega = \prod_0^\infty \{0,1\}$ and μ is a σ-finite T-invariant measure having the property that one can find a real number d so that $\mu(\tau^d) = \infty$ but $\mu(\tau^{d-\epsilon}) < \infty$ for all $\epsilon > 0$, where τ is the first passage time function in the reference state 1. In particular we shall consider invariant measures μ arising from a potential V which is uniformly continuous but not of summable variation. If $d > 0$ then μ can be normalized to give the unique non-atomic equilibrium probability measure of V for which we compute the (asymptotically) exact mixing rate, of order n^{-d}. We also establish the weak-Bernoulli property and a polynomial cluster property (decay of correlations) for observables of polynomial variation. If instead $d \le 0$ then μ is an infinite measure with scaling rate of order n^d. Moreover, the analytic properties of the weighted dynamical zeta function and those of the Fourier transform of correlation functions are shown to be related to one another via the spectral properties of an operator-valued power series which naturally arises from a standard inducing procedure. A detailed control of the singular behaviour of these functions in the vicinity of their non-polar singularity at $z = 1$ is achieved through an approximation scheme which uses generating functions of a suitable renewal process.

In the perspective of differentiable dynamics, these are statements about the unique absolutely continuous invariant measure of a class of piecewise smooth interval maps with an indifferent fixed point.

1 Introduction

It is well known that for a subshifts of finite type and equilibrium measure associated to a Hölder continuous potential V one has exponential decay of correlations. In fact, Hölder continuity of the potential implies that $\mathrm{var}_n V$ decays exponentially fast so that the corresponding Ruelle-Perron-Frobenius transfer operator has a spectral gap when acting on the Banach space of Hölder continuous functions (see [Ba1], [Bo], [PP],[Ru4]). Moreover, in this case both the weighted dynamical zeta function and the Fourier transform of the correlation function extend meromorphically to some complex

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domain where their poles are in correspondence with the isolated eigenvalues of the transfer operator

(see [Ba2], [Hay], [Pol1], [Ru3]). If $\mathrm{var}_n V$ decays at a sub-exponential rate one does not expect a spectral gap any more and the determination of the rate of mixing becomes a challenging problem. In [Pol2] this problem has been adressed for potentials of summable variation, for which it is known [Wal1] that there is only one equilibrium state. In this paper we study this and related questions for a class of potentials which are not even of summable variation, but have an induced version which is Hölder continuous. In this case, there is a σ-finite invariant measure which is either infinite or can be normalized to give a (non-unique) equilibrium state, depending on the value of a parameter that we identify as the *ergodic degree*. Roughly speaking, this parameter controls in a continuous fashion the number of finite moments possessed by the first passage time function in a given reference set.

The main motivation for this study comes from an attempt to understand the ergodic properties of some class of non-uniformly hyperbolic dynamical systems, the simplest example being that of smooth interval maps which are expanding everywhere but at an indifferent fixed point (see [Th], [PS], [LSV], [Yo], [Yu], [MRTVV]).

The paper is organized as follows. In Section 2 we give some preliminaries on the inducing procedure and the main assumptions are settled down. In Section 3 we introduce an operator-valued power series $\mathcal{M}_z$ which will play an essential role in the sequel and study its spectral properties when acting on a Banach space of locally Hölder continuous functions. An algebraic relation between $\mathcal{M}_z$ and the transfer operator $\mathcal{L}$ of the original system is established in Section 4. This relation is then used to construct the σ-finite invariant measure mentioned above and to discuss some of its properties, depending of its ergodic degree d. Moreover, the logarithm of the leading eigenvalue of $\mathcal{M}_z$, for z in a complex open neighbourhood of $(0, 1]$, is interpreted as a pressure function $P(z)$ for a suitable 'grand canonical' potential, and in Section 5 we examine its behaviour, showing in particular how the number of its derivatives at $z = 1$ is related to the ergodic degree d.

An approximation scheme based on a renewal Markov chain is introduced in Section 6 and the asymptotic behaviour when $z \uparrow 1$ of the operator valued function $(1-\mathcal{M}_z)^{-1}$ acting on suitable test functions is determined in terms of the generating function of this renewal process. In Section 7 we study the Fourier transform of a 'renewal density' sequence yielding the probability to observe a return in the reference set after n iterates, and we show that it is asymptotically equivalent to the corresponding renewal sequence for the Markov approximation. Thereafter, this is extended to any Borel set where the first passage time function in the above reference set is bounded, thus determining the (asymptotically) exact scaling rate when the measure is infinite ($d \leq 0$) and the (asymptotically) exact mixing rate when it is finite ($d > 0$). In the latter case, we also establish a weak-Bernoulli property and a polynomial decay of correlations for test functions of polynomial variation (Section 8). In Section 9 we study dynamical zeta functions. We first establish an algebraic relation between zeta functions of the original and the induced system which is the counterpart of the operators relation mentioned above. We then show that the singular behaviour of the dynamical zeta function is characterized by a non-polar singularity at $z = 1$ which can be inspected in terms of the generating function of the approximating renewal process. Finally, in Section 10 we apply the preceeding results to the symbolic description of the dynamics of interval maps with indifferent fixed points. In this simple situation we have a nice control of the asymptotic behaviour of the first passage function which allows to partially sharpen the general results. Although we believe that the approach described here can be carried out for more general examples of non-uniformly hyperbolic systems, we leave the actual investigation of this point to be discussed elsewhere.

2 Inducing on shift spaces

We shall consider the following situation. Let $\Omega = \prod_0^\infty \{0,1\}$ be the set of all one-sided sequences $\omega = (\omega_0 \omega_1 \ldots)$, $\omega_i \in \{0,1\}$ and denote the left shift map on Ω by T. We shall take the state 1 as a reference set and consider the *first passage function* $\tau : \Omega \to I\!N$ given by:

$$\tau(\omega) = 1 + \inf\{i \geq 0 \,:\, \omega_i = 1\}, \tag{2.1}$$

(with $\inf \emptyset = \infty$). The levelsets

$$A_k := \{\omega \in \Omega : \tau(\omega) = k\}, \qquad k \geq 1, \tag{2.2}$$

are both open and closed and will be called *partition sets*, as they form a partition of $\Omega_0 := \Omega \setminus \{0^\infty\}$, where 0^∞ denotes the singleton $(000\ldots)$. Let $0^n 1$ denote the word $\underbrace{0\ldots01}_{n+1}$. The A_k's will be sometimes denoted as cylinder sets:

$$A_k \equiv [0^{k-1}1] := \{\omega \in \Omega \mid \omega_i = 0,\, 0 \leq i < k-1,\, \omega_{k-1} = 1\}. \tag{2.3}$$

Let moreover

$$\tau_0 \equiv \tau, \quad \text{and} \quad \tau_k = \inf\{i > \tau_{k-1} \,:\, \omega_i = 1\}, \quad k > 0, \tag{2.4}$$

be the *sequence of successive entrance times in* 1, and let

$$\sigma_0 \equiv \tau, \quad \text{and} \quad \sigma_k = \tau_k - \tau_{k-1}, \quad k > 0,$$

(2.5)

be the *sequence of times between passages*. Both sequences are infinite on the residual subset $\Omega \setminus \cup_{k=0}^\infty T^{-k} 0^\infty$. Now, given a function $U : \Omega \to C\!\!\!\!I$, we define two different notions of variation:

$$\mathrm{var}_n U = \sup\Big\{ |U(\omega) - U(\omega')| \,:\, \inf\{k \geq 0 \,:\, \omega_k \neq \omega'_k\} = n \Big\}, \tag{2.6}$$

and

$$\mathrm{Var}_n U = \sup\Big\{ |U(\omega) - U(\omega')| \, : \, \inf\{k \geq 0 \, : \, \sigma_k(\omega) \neq \sigma_k(\omega')\} = n \Big\}. \tag{2.7}$$

We then say that U is *uniformly continuous* if $\forall\, \epsilon > 0$ there exists $n \geq 1$ such that $\mathrm{var}_n U < \epsilon$. On the other hand U will be called *locally Hölder continuous* if there is a constant $C > 0$ such that $\forall n \geq 1$, $\mathrm{Var}_n U \leq C\,\theta^n$ (notice that nothing is required for $\mathrm{Var}_0\, U$). Let $V : \Omega \to \mathbb{C}$ be given and define its *induced version* $W : \Omega \to \mathbb{C}$ as

$$W(\omega) = \sum_{k=0}^{\tau(\omega)-1} V(T^k \omega). \tag{2.8}$$

Conversely, we have

$$V(\omega) = \begin{cases} W(\omega) & \text{if } \omega \in A_1, \\ W(\omega) - W(T\omega) & \text{if } \omega \in A_k, \ k > 1, \end{cases} \tag{2.9}$$

We shall be interested in examples of uniformly continuous maps $V : \Omega \to \mathbb{R}$ whose induced version $W : \Omega \to \mathbb{R}$ is locally Hölder continuous. Whenever the function V is viewed as a *potential* corresponding to the map $T : \Omega \to \Omega$, its induced version W is naturally interpreted as the potential for the *induced map* $T^\tau : \Omega \to \Omega$.
We shall now introduce a class of real valued maps V on the space Ω whose regularity properties are dictated by those of their induced version W.

PROPERTIES. *We will be considering potential functions $V : \Omega \to \mathbb{R}$ having the following properties:*

1. *$V(\omega)$ is continuous and satisifes $-C_1 \leq V(\omega) \leq 0$ (for some positive constant C_1), with $V(0^\infty) = 0$;*

2. *there is a constant $C_2 > 0$ s.t.*

$$\sum_n \exp\left(\sup_{\omega \in A_n} W(\omega) \right) < C_2;$$

3. *there is $0 < \theta < 1$ and a constant $C_3 > 0$ such that for any $n \geq 1$,*

$$\mathrm{Var}_n W \leq C_3\, \theta^n;$$

4. *the limit*

$$\wp(V) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega_0, \omega_1, \ldots, \omega_{n-1}} \exp\left(\sup_{\omega'} \sum_{i=0}^{n-1} V(\omega_i \ldots \omega_{n-1} \omega') \right)$$

exists and is equal to 0.

Remark 1 One readily obtains from (2.9) that

$$\mathrm{var}_n V = \sup_{\omega \in A_{n+1}} |V(\omega)|, \tag{2.10}$$

so that Property 1 entails that V is uniformly continuous and, in particular, $\mathrm{var}_n V$ eventually decreases monotonically. On the other hand, by Property 2, V is not of summable variation:

$$\sum_{k<n} \mathrm{var}_k V \geq \inf_{\omega \in A_n} |W(\omega)| \to \infty \quad \text{as} \quad n \to \infty.$$

Moreover, by Property 3, its induced version $W : \Omega \to I\!\!R$ is locally Hölder continuous, even though not bounded from below.

Remark 2 The function $\wp(V)$ defined in Property 4 is called the (topological) pressure of V. Its existence follows from Property 1 along with the sub-additivity of the sequence $\{\sup_{\omega'} \sum_{i=0}^{n-1} V(\omega_i \ldots \omega_{n-1}\omega')\}_{n \in I\!\!N}$. Notice that for any real number c we have $\wp(V + c) = \wp(V) + c$, and therefore $\wp(V - \wp(V)) = 0$. Hence, it always possible to reduce to the case of potentials with zero pressure.

Example 1. Let $1 = p_0 > p_1 > p_2 > \cdots$ be a sequence of real numbers s.t. $p_k > 0$, $\sum_{k \geq 1} p_k = 1$ and $q_k = \log(p_{k-1}/p_k) \searrow 0$ (e.g., take the p_k's forming a strictly log-concave sequence: $p_k^2 > p_{k+1} p_{k-1}$). Define a continuous function $V : \Omega \to I\!\!R$ as

$$V(\omega) = \begin{cases} -q_k & \text{if } \omega \in A_k, \\ 0 & \text{if } \omega = 0^\infty, \\ 0 & \text{otherwise.} \end{cases}$$

One easily checks that $\mathrm{var}_n V = q_{n+1}$, so that V is uniformly continuous on Ω. Notice that V is not of summable variation: $\sum_{k \leq n} q_k = -\log p_n \to \infty$ as $n \to \infty$. On the other hand $W(\omega) = -\sum_{k=1}^{\sigma_0} q_k = \log p_{\sigma_0}$ so that $\mathrm{Var}_n W = 0$, $\forall n \geq 1$. Notice that using the correspondence $A_k \leftrightarrow k$ we can relate this example to a Markov chain with state space $I\!\!N$ and transition probabilities

$$p_{ij} = \begin{cases} p_j, & \text{if} \quad i = 1, j \geq 1, \\ 1, & \text{if} \quad 1 \leq j = i - 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.11}$$

To see this, we set $x_i = k$ if $T^i \omega \in A_k$ and $x = (x_0 x_1 \cdots) \in X = I\!\!N^{I\!\!N_0}$, and note that there is a one-to-one correspondence between points $\omega \in \Omega_0$ and points $x \in X$ satisfying the compatibility condition: given x_i then either $x_{i-1} = x_i + 1$ or $x_{i-1} = 1$. Such a correspondence induces a natural action of the shift T on X. With slight abuse of notation, we can thus write the function V as a function of x, which turns out to depend only on the first two coordinates:

$$V(x) = \log \left(\frac{p_{x_0} \, p_{x_0 x_1}}{p_{x_1}} \right).$$

This can be viewed as a 'two-body potential' function for the Markov shift (X, T). In particular, there is no mutual interaction between the 'spins' σ_k

given by the times between passages in the state 1: the sequence $\{\sigma_k\}$ is isomorphic to a sequence of i.i.d.r.v. and $\tau_n = \sigma_0 + \cdots + \sigma_n$ is a stationary renewal process under the probability measure $P(\sigma_k = \ell) = p_\ell$.

3 An operator-valued power series

We define the operation of *inducing* as the bijection $\iota : \Omega_0 \to \Sigma$ where $\Sigma := \iota(\Omega_0)$ is the non-compact set of all sequences $\sigma = \iota(\omega)$ given by times between passages, i.e. $\iota(\omega) = \sigma_0(\omega)\sigma_1(\omega)\dots$. The map $S := \iota^* T^\tau$ acts as a left shift on Σ. In this Section we shall exclusively work with objects (functions, measures, operators) living on the symbol space Σ, temporarily forgetting its origin as an induced space from Ω_0 through the map ι. We denote by $\mathcal{F}_\theta(\Sigma)$ the Banach space of complex valued functions on Σ which are finite with respect to the norm $\|U\|_\theta = |U|_\infty + |U|_\theta$, where $|U|_\theta$ is the least Lipschitz constant of U wrt the metric

$$d_\theta(\sigma, \sigma') = \theta^{\inf\{j \geq 0 \,:\, \sigma_j \neq \sigma'_j\}}. \tag{3.12}$$

that is

$$|U|_\theta = \sup\left\{ \frac{|U(\sigma) - U(\sigma')|}{\theta^n} \,:\, n = \min\{j \geq 1 \,:\, \sigma_j \neq \sigma'_j\} \right\}. \tag{3.13}$$

We shall denote with the same symbol W the projection $\iota^* W : \Sigma \to I\!\!R$ of the induced potential defined in (2.8). Notice that even though W is not bounded from below, the function

$$\psi(\sigma) := \exp W(\sigma) \tag{3.14}$$

clearly is. In particular, by Property 2 we have that

$$\sum_{S\sigma' = \sigma} \psi(\sigma') = \sum_{k=1}^{\infty} \psi(k\sigma) < C_2, \quad \text{for all} \quad \sigma \in \Sigma. \tag{3.15}$$

Here $k\sigma$ denotes the sequence $(k\sigma_0\sigma_1\dots) \in \Sigma$. This and Property 3 entail that $\psi \in \mathcal{F}_\theta(\Sigma)$. Let $z \in \mathbb{C}$ and $\mathcal{M}_z : \mathcal{F}_\theta(\Sigma) \to \mathcal{F}_\theta(\Sigma)$ be the operator-valued power series defined as follows (see [PS] for related objects):

$$\mathcal{M}_z = \sum_{n=1}^{\infty} z^n \mathcal{M}^{(n)}, \qquad (\mathcal{M}^{(n)} f)(\sigma) = \psi(n\sigma)\, f(n\sigma). \tag{3.16}$$

Alternatively, one can think of $\mathcal{M}_z$ as the transfer operator associated to the 'grand canonical' complex potential

$$W_z(\sigma) := W(\sigma) + \sigma_0 \log z \tag{3.17}$$

where we take the determination of $\log z$ which is real for $z > 0$.

Lemma 3.1 *The power series of $\mathcal{M}_z$ when acting on $\mathcal{F}_\theta$ has radius of convergence bounded from below by* 1 *and, moreover, it converges absolutely at every point of the unit circle.*

Proof. The radius of convergence of $\mathcal{M}_z$ is $\lim_{n\to\infty} \|\mathcal{M}^{(n)}\|_\theta^{-1/n}$. We have

$$|\mathcal{M}^{(n)} f(\sigma)| \leq \psi(n\sigma) |f|_\infty$$

and also

$$|\mathcal{M}^{(n)}f(\sigma)-\mathcal{M}^{(n)}f(\sigma')| \le \psi(n\sigma)\ (\theta\,|f|_\theta + C\,\theta\,|f|_\infty\,).$$

Hence,

$$\|\mathcal{M}^{(n)}\|_\theta \le C \sup_\sigma \psi(n\sigma) \tag{3.18}$$

and the assertion follows from (3.15). $\diamondsuit$

For any fixed $z \in [0,1]$ we now set

$$\Lambda_n(z) := \sup_{\sigma\in\Sigma} \sum_{k_1=1}^{\infty} \ldots \sum_{k_n=1}^{\infty} z^{k_1+\ldots+k_n} \prod_{i=1}^{n} \psi(k_i \ldots k_n \sigma) \tag{3.19}$$

and

$$P(z) := \lim_{n\to\infty} \frac{1}{n} \log \Lambda_n(z). \tag{3.20}$$

One easily checks that under our hypotheses on the the potential V (and for each z as above) the sequence $(\Lambda_n(z))_{n\in I\!N}$ is sub-multiplicative, and therefore the limit (3.20) exists and satifies $-\infty \le P(z) < \infty$. Also notice that, using (2.8) and (3.14) with $\sigma = \iota(\omega)$, we can write

$$\prod_{i=1}^{n} \psi(k_i \ldots k_n\sigma) = \prod_{j=0}^{k_1+\cdots+k_n-1} \varphi(T^j \underbrace{0\ldots01}_{k_1} \cdots \underbrace{0\ldots01}_{k_n} \omega),$$

where $\varphi = \exp V$. Property 4 then implies that

$$P(1) = \lim_{n\to\infty} \frac{1}{n} \log \Lambda_n(1) = 0.$$

We shall however say more. In the next theorem we prove that for any fixed $z \in [0,1]$ the function $\exp P(z)$ is equal to the spectral radius $r(\mathcal{M}_z)$ of $\mathcal{M}_z : \mathcal{F}_\theta \to \mathcal{F}_\theta$. The monotonicity of $z \to \mathcal{M}_z$ for $0 \le z \le 1$ thus implies that $\exp P(z)$ is strictly increasing, ranging from 0 to 1 when z ranges from 0 to 1.

Remark 3 For V and W as in Example 1 we find $\Lambda_n(z) = \left(\sum_{k=1}^{\infty} z^k\, p_k\right)^n$ so that $\exp P(z)$ is but the generating function of the numbers p_k.

For z in some complex neighbourhood of $(0,1]$ the quantity $P(z)$ will be interpreted as the *pressure* associated to the potential W_z. Before stating the next result we let $\mathcal{D}$ and $\overline{\mathcal{D}}$ denote the open unit disk $\{z : |z| < 1\}$ and its closure $\{z : |z| \le 1\}$, respectively. Moreover, we recall that the spectrum of a bounded linear operator K can be decomposed into a discrete part, made up of isolated

eigenvalues of finite multiplicity, and its complement, the essential spectrum, denoted by ess(K). The essential spectral radius is then defined as $r_{\text{ess}}(K) = \sup\{\,|\lambda| : \lambda \in \text{ess(K)}\,\}$.

Theorem 3.1 *Let $z \in \overline{\mathcal{D}}$ and $\mathcal{M}_z$ be acting on $\mathcal{F}_\theta$.*

1. *The spectral radius $r(\mathcal{M}_z)$ is bounded above by* $\exp P(|z|)$.

2. *The essential spectral radius $r_{\mathrm{ess}}(\mathcal{M}_z)$ is bounded above by $\theta \exp P(|z|)$.*

3. *There is at most one eigenvalue of modulus $\exp P(|z|)$, which is simple, and exactly one at $\exp P(z)$ if z is real and positive. The rest of the spectrum is contained in a disc of radius strictly smaller than $\exp P(|z|)$. In addition, if $|z| \le 1$ but $z \neq 1$ then 1 is not an eigenvalue of $\mathcal{M}_z$.*

Proof. It is easy to check that for all $f \in \mathcal{F}_\theta$

$$|\mathcal{M}_z^n f|_\infty \le \Lambda_n(|z|)\,|f|_\infty \le \Lambda_n(|z|)\,\|f\|_\theta \tag{3.21}$$

and

$$|\mathcal{M}_z^n f|_\theta \le \Lambda_n(|z|)\,(C\,|f|_\infty + \theta^n |f|_\theta) \le \Lambda_n(|z|)\,(C+1)\|f\|_\theta. \tag{3.22}$$

Therefore $\|\mathcal{M}_z^n\|_\theta \le (C+2)\Lambda_n(|z|)$, where we have also denoted by $\|\ \|_\theta$ the operator norm. Thus, the spectral radius formula implies that

$$r(\mathcal{M}_z) \le \exp P(|z|).$$

To estimate the essential spectral radius we now extend an argument given in [Ke] (see also [Pol1]) to the present infinite-alphabet situation. For any $n \ge 1$, let $\Sigma^{(n)}$ be the set of all words η of length n, i.e. words of the form $\eta = (\sigma_0 \ldots \sigma_{n-1})$. Moreover, given $N > 0$, set $G_n^N = \{\eta \in \Sigma^{(n)} \,:\, \sigma_i < N,\ 0 \le i < n\}$ and $B_n^N = \Sigma^{(n)} \setminus G_n^N$. Now choose $N = N(n)$ in such a way that

$$\sup_{\sigma\in\Sigma} \sum_{\eta\in B_n^N} |z|^{S_n(\eta\sigma)}\, e^{S_n W(\eta\sigma)} \le C\,\Lambda_n(|z|)\theta^n \tag{3.23}$$

where $S_n(\sigma) = \sum_{j=0}^{n-1} \sigma_j$ and $S_n W = \sum_{j=0}^{n-1} W \circ S^j$. Having fixed n and N, let E_N^n be the finite rank operator acting on $f \in \mathcal{F}_\theta$ as follows:

$$(E_N^n f)(\sigma) = \sum_{\eta\in G_n^N} f_\eta\, \chi_\eta(\sigma)$$

where $\chi_\eta(\sigma)$ is the characteristic function of the cylinder set $D_\eta = \{\sigma \in \Sigma \,:\, \sigma_j = \eta_j,\ j = 0,\ldots,n-1\}$ and f_η is the value of f at some (arbitrarily) chosen point of D_η. Then, for any $0 < |z| \le 1$ and for any pair $n, N > 0$, the operator $K_{z,N}^n = \mathcal{M}_{W_z}^n E_N^n$ is compact. Put moreover

$$\tilde f(\sigma) = \begin{cases} (f - E_N^n f)(\sigma), & \text{if } [\sigma_0,\ldots,\sigma_{n-1}] \in G_n^N; \\ f(\sigma), & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\sup_{\eta\in G_n^N} |\tilde f(\eta\sigma)| \le C\,|f|_\theta\,\theta^n \quad \text{and} \quad \sup_{\eta\in G_n^N} |\tilde f(\eta\sigma) - \tilde f(\eta\sigma')| \le C\,|f|_\theta\,\theta^n$$

Hence,

$$\begin{aligned} &|\,(\mathcal{M}_z^n - K_{z,N}^n) f(\sigma)\,| = |\,\mathcal{M}_z^n \tilde f(\sigma)\,| \le \\ &\le |f|_\infty \sup_{\sigma\in\Sigma} \sum_{\eta\in B_n^N} |z|^{S_n(\eta\sigma)}\, e^{S_n W(\eta\sigma)} + \Lambda_n(|z|) \sup_{\eta\in G_n^N} |\tilde f(\eta\sigma)| \end{aligned}$$

Therefore, using (3.23), we get

$$|\,(\mathcal{M}_z^n - K_{z,N}^n) f\,|_\infty \le C\,\|f\|_\theta\,\theta^n\,\Lambda_n(|z|).$$

We now estimate the variation. Let $\sigma, \sigma' \in \Sigma$ be two sequences having the first k symbols in common. Then we have

$$\begin{aligned}
&|\,\mathcal{M}_z^n \tilde{f}(\sigma) - \mathcal{M}_z^n \tilde{f}(\sigma')\,| \le \\
&\le \sum_{\eta \in G_n^N} |z|^{S_n(\eta\sigma)} \left(e^{S_n W(\eta\sigma)} \tilde{f}(\eta\sigma) - e^{S_n W(\eta\sigma')} \tilde{f}(\eta\sigma') \right) + \\
&+ \sum_{\eta \in B_n^N} |z|^{S_n(\eta\sigma)} \left(e^{S_n W(\eta\sigma)} f(\eta\sigma) - e^{S_n W(\eta\sigma')} f(\eta\sigma') \right) = \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Now a routine calculation shows that

$$\mathrm{I} \le C\,\Lambda_n(|z|)\,|f|_\theta\,\theta^{n+k}.$$

Moreover

$$\begin{aligned}
\mathrm{II} &\le \sum_{\eta \in B_n^N} |z|^{S_n(\eta\sigma)}\, e^{S_n W(\eta\sigma)}\, (f(\eta\sigma) - f(\eta\sigma')) + \\
&+ \sum_{\eta \in B_n^N} |z|^{S(\eta\sigma)}\, e^{S_n W(\eta\sigma)}\, f(\eta\sigma') \left(1 - e^{S_n W(\eta\sigma') - S_n W(\eta\sigma)}\right) \\
&\le C\,\Lambda_n(|z|)\,|f|_\theta\,\theta^{n+k} + C\,\theta^k\,|f|_\infty \sum_{\eta \in B_n^N} |z|^{S_n(\eta\sigma)}\, e^{S_n W(\eta\sigma)}
\end{aligned}$$

so that, using (3.23), we get

$$|\,(\mathcal{M}_z^n - K_{z,N}^n) f\,|_\theta \le C\,\|f\|_\theta\,\theta^n\,\Lambda_n(|z|).$$

Putting together the above and Nussbaum formula [Nu] we have thus proved that

$$r_{\mathrm{ess}}(\mathcal{M}_z) \le \theta\,\exp P(|z|).$$

We are now going to prove the third statement. Let us first notice that, if z is real and positive, we have

$$r(\mathcal{M}_z) = \lim_{n\to\infty} \left(\,\|\mathcal{M}_z^n\|_\theta\,\right)^{1/n} \ge \lim_{n\to\infty} \left(\,|\mathcal{M}_z^n 1|_\infty\,\right)^{1/n} = \exp P(z)$$

and therefore $r(\mathcal{M}_z) = \exp P(z)$. Next, let us see that for z real and positive $\exp P(z)$ is a maximal simple eigenvalue of $\mathcal{M}_z$ and the remainder of the spectrum is contained in a disk of radius strictly smaller than $\exp P(z)$. To this end, we proceed as in ([CI], Theorem 2.1) and construct a sequence of compact spaces Σ_N, $N \in \mathbb{N}$, whose elements are sequences $\sigma = (\sigma_0\sigma_1\ldots)$ with $\sigma_j \in \{1, \ldots, N\}$. Clearly, $\Sigma_N \subset \Sigma_{N+1} \subset \ldots \subset \Sigma$. For any $0 < z \le 1$ define a family of operators $\mathcal{M}_{z,N} : \mathcal{F}_\theta(\Sigma_N) \to \mathcal{F}_\theta(\Sigma_N)$ by

$$\mathcal{M}_{z,N} = \sum_{k=1}^{N} z^k\, \mathcal{M}^{(k)}.$$

Now, having fixed $N \in I\!\!N$ and $z \in (0,1]$, a Ruelle-Perron-Frobenius theorem holds for $\mathcal{M}_{z,N}$ (see, e.g., [PP], Theorem 2.2). Let $\lambda_{z,N}$ be the simple eigenvalue with largest modulus of $\mathcal{M}_{z,N}$ and $h_{z,N} \in \mathcal{F}_\theta(\Sigma_N)$, $\nu_{z,N} \in \mathcal{F}_\theta{}^*(\Sigma_N)$ be such that

$$\mathcal{M}_{z,N} h_{z,N} = \lambda_{z,N} h_{z,N} \quad \text{and} \quad \mathcal{M}^*_{z,N} \nu_{z,N} = \lambda_{z,N} \nu_{z,N}.$$

Putting

$$\Lambda_{n,N}(z) = \sup_{\sigma \in \Sigma_N} \sum_{k_1=1}^{N} \dots \sum_{k_n=1}^{N} z^{k_1+\dots+k_n} \exp \sum_{i=1}^{n} W(k_i \dots k_n \sigma)$$

and

$$P_N(z) = \lim_{n\to\infty} \frac{1}{n} \log \Lambda_{n,N}(z) \tag{3.24}$$

we find

$$\begin{aligned} \log \lambda_{z,N} &= \overline{\lim_{n\to\infty}} \frac{1}{n} \log \mathcal{M}^n_{z,N} h_{z,N} \\ &= \overline{\lim_{n\to\infty}} \frac{1}{n} \log \mathcal{M}^n_{z,N} 1 = P_N(z) \end{aligned}$$

because $D^{-1}_{z,N} \le h_{z,N} \le D_{z,N}$ for some positive constant $D_{z,N}$. Now, reasoning as in ([CI], Theorem 2.1) one then shows that, having fixed $0 < z \le 1$, the triple $(\lambda_{z,N}, h_{z,N}, \nu_{z,N})$ converges uniformly to the triple (λ_z, h_z, ν_z) which is uniquely determined by the conditions

$$\begin{aligned} &\lambda_z > 0, \quad h_z > 0, \quad \int h_z \, d\nu_z = 1 \\ &\mathcal{M}_z h_z = \lambda_z h_z, \quad \mathcal{M}^*_z \nu_z = \lambda_z \nu_z. \end{aligned} \tag{3.25}$$

Clearly we have $\lambda_z = \exp P(z)$ where $P(z)$ is defined in (3.20). Furthermore,

$$\lambda_z^{-n} \mathcal{M}^n_z f \to h_z \int f \, d\nu_z \quad \text{uniformly} \quad \forall\, f \in \mathcal{F}_\theta(\Sigma). \tag{3.26}$$

The fact that the rest of the spectrum is contained in a disk of radius strictly smaller than $\exp P(z)$ now follows from the argument given in ([PP], p.26), which relies on the basic inequalities (3.21) and (3.22).
Finally, in order to deal with complex values of z, let us write $z = |z|e^{i\phi}$ and $W_z = a+ib$ where $a(\sigma) = W(\sigma) + \sigma_0 \log|z|$ and $b(\sigma) = \sigma_0 \phi$. Proceeding as in

([PP], Chapter 4), one then shows that if $\mathcal{M}_z$ has an eigenvalue of modulus $\exp P(|z|)$, then $\mathcal{M}_z = \vartheta\, \mathcal{S} \mathcal{M}_a \mathcal{S}^{-1}$ where $\mathcal{M}_a = \mathcal{M}_{|z|}$, $\mathcal{S}$ is a multiplication operator and $\vartheta \in \mathbb{C}$, $|\vartheta| = 1$, so that the spectral properties of $\mathcal{M}_z$ follow from those of $\mathcal{M}_a$ and hence from the above discussion; if, instead, $\mathcal{M}_z$ has no eigenvalues of modulus $\exp P(|z|)$, then its spectral radius is strictly smaller than $\exp P(|z|)$ and $\exp\left(-nP(|z|)\right) \mathcal{M}^n_z \to 0$ in the $\|\ \|_\theta$-operator topology. To complete the proof of the Theorem, suppose there exist $z = e^{i\phi}$ with $\phi(\text{mod} 2\pi) \neq 0$ and $h_z \in \mathcal{F}_\theta(\Sigma, \mathbb{C})$ such that $\mathcal{M}_z h_z = h_z$ (the case $|z| < 1$ follows by just observing that $\exp P(z)$ is strictly increasing in z when $0 \le z \le 1$ and $r(\mathcal{M}_z) \le \exp P(|z|)$). Using again an argument given in ([PP], Chapter 4) one then shows that $h_z \circ S^n(\sigma) = e^{i\,\phi\,\sum_{i=0}^{n-1}\sigma_i} h_z(\sigma)$. However, $S : \Sigma \to \Sigma$ is mixing so that we have found a contradiction. $\diamondsuit$

Some interesting consequences for the ergodic theory of the shift map S on Σ (together with the weight function ψ), can be obtained by putting $z = 1$ in the above theorem. First, we shall say that a measure ν is *conformal* if it satisfies

$$\int_\Sigma \mathcal{M} f\, d\nu = \int_\Sigma f\, d\nu, \qquad \forall f \in \mathcal{F}_\theta. \tag{3.27}$$

Now, from the above theorem it follows that the bounded linear operator $\mathcal{M} : \mathcal{F}_\theta \to \mathcal{F}_\theta$ given by $\mathcal{M} \equiv \mathcal{M}_1$, that is the usual transfer operator associated to the shift map S on Σ and the weight ψ, has spectral radius $r(\mathcal{M}) = 1$. Let $\nu \equiv \nu_1$, $h \equiv h_1$ $\lambda \equiv \lambda_1$ with ν_1, h_1, λ_1 defined in eq (3.25). Then the element ν of the dual space $\mathcal{F}_\theta^*(\Sigma)$ is a conformal measure. Furthermore, we recall that a probability measure ρ on Σ is called a Gibbs measure (in the sense of Bowen, see [Bo]) if there exists $\Psi \in \mathcal{C}(\Sigma)$ such that

$$A \le \frac{\rho([\sigma_0 \cdots \sigma_{n-1}])}{e^{-n\,C + \sum_{k=0}^{n-1} \Psi(S^k\sigma)}} \le B$$

for $n > 0$ and fixed constants $A, B > 0$ and $C \in I\!R$. Here $[\sigma_0 \cdots \sigma_{n-1}]$ denotes the cylinder set $\{\sigma' \in \Sigma \,:\, \sigma'_i = \sigma_i,\, 0 \le i < n\}$.

Corollary 3.1 *The measure $\rho := h \cdot \nu$ is a S-invariant Gibbs measure for $C = 0$ and $\Psi = W$. Moreover it is uniformly mixing, i.e. there is a constant $0 < \vartheta < 1$ such that for any pair of cylinders $E = \{\sigma \in \Sigma : \sigma_i = e_i, 0 \le i \le s\}$ and $F = \{\sigma \in \Sigma : \sigma_i = f_i, 0 \le i \le r\}$ we can find a constant $M = M(E, F, \vartheta)$ such that*

$$\left| \frac{\rho(S^{-n}F \cap E)}{\rho(E)} - \rho(F) \right| \le M\, \vartheta^n.$$

Proof. We have

$$\begin{aligned} \rho([\sigma_0 \cdots \sigma_{n-1}]) &= \int_\Sigma \chi_{[\sigma_0\cdots\sigma_{n-1}]}(\sigma) \cdot h(\sigma) d\nu(\sigma) \\ &= \int_\Sigma \mathcal{M}^n(\chi_{[\sigma_0\cdots\sigma_{n-1}]} \cdot h)(\sigma)\, d\nu(\sigma) \\ &= \int_\Sigma h(\sigma_0 \cdots \sigma_{n-1}\sigma)\psi(\sigma_0 \cdots \sigma_{n-1}\sigma)\, d\nu(\sigma) \end{aligned}$$

Now, since W is locally Hölder continuous, $\sum_{n\ge1} \mathrm{Var}_n W \le C_3\theta/(1-\theta) < \infty$. Therefore, for any pair $\sigma, \sigma' \in \Sigma$, we have

$$e^{\frac{-C_3\theta}{1-\theta}}\, \mathcal{M}^n 1(\sigma') \le \mathcal{M}^n 1(\sigma) \le e^{\frac{C_3\theta}{1-\theta}} \mathcal{M}^n 1(\sigma')\,. \tag{3.28}$$

Taking limits as $n \to \infty$ and recalling that $\mathcal{M}^n 1 \to h$ we get $\|\log h\|_\infty < \infty$ so that we can find a constant $D > 0$ s.t. $D^{-1} < h < D$. The assertion, with $B = A^{-1} = D\, e^{C_3\theta/(1-\theta)}$, now follows from the above identity. The S-invariance follows from the fact that h is a fixed function for $\mathcal{M}$ and the uniform mixing property from the existence of a spectral gap for

$\mathcal{M} : \mathcal{F}_\theta \to \mathcal{F}_\theta$ (see [Ru4]). $\diamondsuit$

We end this Section by studying the operator-valued function $z \to (1-\mathcal{M}_z)^{-1}$. We first recall that $z \to \mathcal{M}_z$ is holomorphic in $\mathcal{D}$ and continuous on $\overline{\mathcal{D}}$. Therefore, if $(1-\mathcal{M}_z)^{-1}$ exists (as a bounded operator acting on $\mathcal{F}_\theta$) for all z in some open subset $D \subseteq \mathcal{D}$ then $z \to (1-\mathcal{M}_z)^{-1}$ is holomorphic in D. We know from Theorem 3.1 that the spectral radius of $\mathcal{M}_z : \mathcal{F}_\theta \to \mathcal{F}_\theta$ is bounded above by $\exp P(|z|)$ which is < 1 for $0 \le |z| < 1$. In addition, from the last statement of Theorem 3.1 it easily follows that for $|z| \le 1$ and $z \neq 1$ there is no eigenvalue of modulus 1 (see also Proposition 4.3 below). This shows that the function $z \to (1-\mathcal{M}_z)^{-1}$ is holomorphic in $\mathcal{D}$ and extends continuously to $\overline{\mathcal{D}} \setminus \{1\}$.

Now, standard analytic perturbation theory ([Ka], Section 7.1) implies that the functions λ_z, h_z, ν_z exend to holomorphic functions in a neighbourhood J of $[0,1)$ so that

$$\lambda_z \neq 0, \quad \mathcal{M}_z h_z = \lambda_z h_z, \quad \mathcal{M}_z^* \nu_z = \lambda_z \nu_z, \quad \nu_z(h_z) = 1 \tag{3.29}$$

for $z \in J \cup \{1\}$. Given $f \in \mathcal{F}_\theta$ and $z \in J \cup \{1\}$ we decompose

$$\mathcal{M}_z\, f = \lambda_z\, \nu_z(f)\, h_z + \mathcal{N}_z\, f. \tag{3.30}$$

where the subspace generated by h_z is one-dimensional, whereas $\mathcal{N}_z$ maps $\mathcal{F}_\theta$ onto the subspace $\{f \in \mathcal{F}_\theta \,:\, \nu_z(f) = 0\}$.

Moreover its iterates can be written as $\mathcal{N}_z^n = \mathcal{M}_z^n \mathcal{Q}_z$ where $\mathcal{Q}_z$ is the spectral projection valued function $1 - \mathcal{P}_z$ with $\mathcal{P}_z := h_z \cdot \nu_z$. Now, for each $z \in J \cup \{1\}$ one has the decomposition $\mathcal{F}_\theta = \mathcal{F}_\theta^{(1,z)} \oplus \mathcal{F}_\theta^{(2,z)}$ with $\mathcal{F}_\theta^{(1,z)} = \mathcal{P}_z \mathcal{F}_\theta$ and $\mathcal{F}_\theta^{(2,z)} = \mathcal{Q}_z \mathcal{F}_\theta$. On the other hand, since h_z and ν_z are holomorphic there exists a bounded operator-valued function $\mathcal{U}_z : \mathcal{F}_\theta \to \mathcal{F}_\theta$ with the property that the inverse $\mathcal{U}_z^{-1}$ exists as a bounded operator on $\mathcal{F}_\theta$ and both $\mathcal{U}_z$ and $\mathcal{U}_z^{-1}$ are holomorphic in J and satisfy (see [Ka], Subsection 7.1.3)

$$h_z = \mathcal{U}_z\, h, \qquad \nu_z = \nu\, \mathcal{U}_z^{-1}. \tag{3.31}$$

This entails that the pair $\mathcal{F}_\theta^{(1,1)}$ and $\mathcal{F}_\theta^{(2,1)}$ decomposes the operator $\hat{\mathcal{M}}_z := \mathcal{U}_z^{-1} \mathcal{M}_z \mathcal{U}_z$ for all z in $J \cup \{1\}$, and the eigenvalue problem for the part of $\mathcal{M}_z$ in $\mathcal{F}_\theta^{(1,z)}$ is equivalent to that for the part of $\hat{\mathcal{M}}_z$ in $\mathcal{F}_\theta^{(1,1)}$ (the eigenvalue being λ_z in both cases). In particular, it is easily seen that $\mathcal{U}_z 1 = \mathcal{U}_z^{-1} 1 = 1$.

Now, since $\mathcal{N}_z$ is holomorphic in J, its spectral radius is a lower semicontinuous function of z. Therefore, since for $z \in (0,1]$ the spectral radius of $\mathcal{N}_z$ is strictly smaller than λ_z, there is a neighbourhood H of $z = 1$ and $\epsilon > 0$ such that the spectral radius of $\mathcal{N}_z$ is smaller than $1 - 2\epsilon$ for all $z \in H \cap J$. Spectral radius formula then implies that $\|\mathcal{N}_z^n\|_\theta \le (1-\epsilon)^n$ for n large enough and thus $z \to (1-\mathcal{N}_z)^{-1}$ is a holomorphic operator-valued function of $z \in H \cap J$. The above discussion along with Theorem 3.1 and (3.30) yield the following result.

Proposition 3.1 *The function $z \to (1-\mathcal{M}_z)^{-1}$ is holomorphic in $\mathcal{D}$ and extends continuously to $\overline{\mathcal{D}} \setminus \{1\}$. In particular, for each $f \in \mathcal{F}_\theta$ and z in some neighbourhood of $z = 1$ (with $z \neq 1$), we have*

$$(1-\mathcal{M}_z)^{-1} f = Q(z)\, h_z\, \nu_z(f) + (1-\mathcal{N}_z)^{-1} f$$

where $Q(z) = (1-\lambda_z)^{-1}$ and $(1-\mathcal{N}_z)^{-1} f$ is continuous at $z = 1$.

4 Transfer operators, invariant measures, ergodic degree and more

We now go back to the original space Ω. We shall denote by $\mathcal{F}_\theta(\Omega_0)$ the lift of $\mathcal{F}_\theta(\Sigma)$ with the map ι (or simply $\mathcal{F}_\theta$ whenever the underlying space is clear). For notational simplicity'sake, we shall use the same symbols ψ, h_z, h, ν_z, ν, ρ to denote the lift of the corresponding objects dealt with in the previous Section, as well as $\mathcal{M}_z : \mathcal{F}_\theta(\Omega_0) \to \mathcal{F}_\theta(\Omega_0)$.
Now, the transfer operator $\mathcal{L} : \mathcal{C}(\Omega) \to \mathcal{C}(\Omega)$ associated to the shift map T on Ω and the weight function $\varphi(\omega) := \exp V(\omega)$, is defined by

$$\begin{aligned} (\mathcal{L}g)(\omega) &= \sum_{T(\omega')=\omega} \varphi(\omega')\, g(\omega') \\ &= \varphi(0\omega)\, g(0\omega) + \varphi(1\omega)\, g(1\,\omega) =: (\mathcal{L}_0 + \mathcal{L}_1)\, g(\omega). \end{aligned} \tag{4.32}$$

¿From Property 4 it follows that $r(\mathcal{L}) = 1$. Moreover we can write,

$$\begin{aligned} (\mathcal{M}_z f)(\omega) &= \sum_{k=1}^{\infty} z^k\, \psi(0^{k-1}1\ \omega) \cdot f(0^{k-1}1\ \omega) \\ &= \sum_{k=1}^{\infty} z^k \prod_{i=0}^{k-1} \varphi(0^{k-i-1}1\ \omega) \cdot f(0^{k-1}1\ \omega) \\ &= \sum_{k=1}^{\infty} z^k\, \mathcal{L}^k (f \cdot 1_{A_k})(\omega) \\ &= \sum_{k=1}^{\infty} z^k \left(\mathcal{L}_1 \mathcal{L}_0^{k-1} f\right)(\omega) \\ &= z\, \mathcal{L}_1 (1 - z\mathcal{L}_0)^{-1} f(\omega) \end{aligned} \tag{4.33}$$

where (2.8) and (4.32) have been used in the second and third equalities. ¿From (4.33) we obtain the following algebraic relation between $\mathcal{M}_z$ and $\mathcal{L}$ (see also [HI] where a similar relation has been exploited to study statistical properties of rational maps):

Proposition 4.2 *For $z \in \overline{\mathcal{D}}$ and for any f such that $(1 - \mathcal{L}_0)f \in \mathcal{F}_\theta$ we have*

$$(\,1 - \mathcal{M}_z\,)\,(\,1 - z\,\mathcal{L}_0\,)\,f = (1 - z\,\mathcal{L})\,f.$$

¿From this identity and Proposition 3.1 we get

Corollary 4.2 *The function $z \to (1 - z\mathcal{L})^{-1}$ is holomorphic in $\mathcal{D}$ and extends continuously to $\overline{\mathcal{D}} \setminus \{1\}$. For all f s.t. $(1 - \mathcal{L}_0)f \in \mathcal{F}_\theta$ and z in some neighbourhood of $z = 1$ (with $z \neq 1$) we have,*

$$(1 - z\,\mathcal{L})^{-1} f = Q(z)\, e_z\, \nu_z(f) + (1 - z\mathcal{L}_0)^{-1} (1 - \mathcal{N}_z)^{-1} f, \tag{4.34}$$

where $e_z = (1 - z\,\mathcal{L}_0)^{-1} h_z$.

Other consequences of Proposition 4.2 are the following:

Proposition 4.3 *Let $z \in \overline{\mathcal{D}} \setminus \{0\}$. Then 1 is an eigenvalue of $\mathcal{M}_z$ if and only if $1/z$ is an eigenvalue of $\mathcal{L}$ and they have the same geometric multiplicity. Furthermore, the corresponding eigenfunctions f_z of $\mathcal{L}$ and g_z of $\mathcal{M}_z$ are related by $g_z = (1 - z\mathcal{L}_0)f_z$ or else $f_z = \sum_{k=0}^{\infty} z^k \mathcal{L}_0^k g_z$.*

Proof. Assume that $\mathcal{M}_z g_z = g_z$. From Proposition 4.2 it follows that $(1 - z\mathcal{L}) \sum_{k=0}^{\infty} z^k \mathcal{L}_0^k g_z = 0$. Conversely, assume that $z\mathcal{L} f_z = f_z$, then $(1 - \mathcal{M}_z)(1 - z\mathcal{L}_0) f_z = 0$. $\diamondsuit$

Proposition 4.4 *The σ-finite measure $\mu = e \cdot \nu$ with*

$$e = \sum_{k=0}^{\infty} \mathcal{L}_0^k h \quad \textit{or else} \quad h = (1 - \mathcal{L}_0)e = \mathcal{L}_1 e$$

(here $\mathcal{M}h = h$ and thus $\mathcal{L}e = e$) is T-invariant. To any Borel subset E of Ω_0 it assigns the weight

$$\mu(E) = \sum_{k \geq 0} \rho \left(T^{-k} E \cap D_k \right)$$

where $D_k = \{\tau > k\} = \cup_{l>k} A_l$. In particular $\mu(A_k) = \rho(D_{k-1})$.

Conversely, the measure ρ is obtained by pushing backward μ with the map $T_1 : \Omega \to \Omega$ given by $T_1 \omega = 1\omega$, i.e.

$$\rho(E) = (\mu \circ T_1)(E). \tag{4.35}$$

In particular we have

$$\rho(A_n) = \mu(B_n), \tag{4.36}$$

with

$$B_n = T_1(A_n) = \{\omega \in A_1 \,:\, \min\{k \geq 1 \,:\, T^k \omega \in A_1\} = n\}. \tag{4.37}$$

Moreover, we have the following chain of formal identities:

$$\mu(\Omega) = \sum_k \mu(A_k) = \sum_k \rho(D_{k-1}) = \sum_k \sum_{l \geq k} \rho(A_l) = \sum_k k \cdot \rho(A_k) = \rho(\tau) =: M_1 \tag{4.38}$$

Here M_1 denotes the mean first passage time in the state 1 for the dynamical system (Ω, T, μ) (which might be infinite). Thus (4.38) is a version of Kac's formula. More generally, let M_γ, $\gamma \geq 0$, be the family of moments defined by

$$M_\gamma := \sum_k k^\gamma \cdot \rho(A_k). \tag{4.39}$$

Definition 4.1 *The triple (Ω, T, μ) is said to have* ergodic degree d *if $M_{d+1} = \infty$ but $M_{d+1-\epsilon} < \infty$, $\forall \epsilon > 0$.*

Remark 4 One could also define the ergodic degree using moments wrt the σ-finite measure μ by saying that (Ω, T, μ) has ergodic degree d if $\sum_k k^d \cdot \mu(A_k) = \infty$ but $\sum_k k^{d-\epsilon} \cdot \mu(A_k) < \infty$, $\forall \epsilon > 0$. The above definition may appear somewhat strict. In particular it may happen that setting $\tilde{d} = \inf\{\gamma : M_{\gamma+1} = \infty\}$ one has $\sum_k k^{\tilde{d}} \cdot \mu(A_k) < \infty$. In this case the ergodic degree does not esist. Take for instance $\mu(A_k) =$

$k^{-1}(\log k)^{-2}$. Then we have $M_{\gamma+1} = \infty$ for all $\gamma > 0$ (so that $\tilde{d} = 0$) but $\mu(\Omega) = M_1 < \infty$. On the other hand, the above definition has the advantage to make a neat distinction between finite measure case $\mu(\Omega) < \infty$, where $d > 0$, and the infinite one, where $d \le 0$. In particular, notice that $M_0 = 1$, so that the ergodic degree satisfies $d > -1$, provided it exists. For the Markov chain of Example 1, this notion is related to some already used in the literature (see, e.g., [Is2] and references therein). In particular, if $-1 < d \le 0$ one has a null-recurrent chain, whereas for $d > 0$ it is positive recurrent. The Markov measure μ is given by $\mu([x_0 \cdots x_n]) = \pi_{x_0} p_{x_0 x_1} \cdots p_{x_{n-1} x_n}$ with $\pi_k = \sum_{\ell \ge k} p_\ell$ and is infinite if $d \le 0$.

Remark 5 By Property 3 we have

$$e^{-C_3\theta} \le \frac{\psi(0^{n-1}1\omega)}{\psi(0^{n-1}1\omega')} \le e^{C_3\theta}. \tag{4.40}$$

Moreover, using normalization and conformality of the measure ν we have

$$\begin{aligned} \nu(A_n) &= \int_\Omega 1_{A_n}(\omega) d\nu(\omega) = \int_\Omega \mathcal{M} 1_{A_n}(\omega) d\nu(\omega) \\ &= \int_\Omega \psi(0^{n-1}1\omega) d\nu(\omega) = \psi(0^{n-1}1\omega^*) \end{aligned} \tag{4.41}$$

for some $\omega^* \in \Omega$. Since $h \asymp 1$ we therefore have[1]

$$\rho(A_n) \asymp \psi(0^{n-1}1\omega) \tag{4.42}$$

uniformly in $\omega \in \Omega$. Suppose now that $\psi(0^{n-1}1\omega) \asymp n^{-\alpha}$, for some $\alpha > 1$. Then one finds $d = \alpha - 2$. The case $\psi(0^{n-1}1\omega) \asymp n^{-\alpha} L(n)$ where $L(n)$ a function slowly varying at infinity, (i.e. $L(cn) \sim L(n)$ for every positive c) is more delicate. If for instance $L(n) = \log n$ then again $d = \alpha - 2$. If instead $L(n) = (\log n)^2$ then, as already noted, d does not exist because $\sum n^{d+1-\alpha}(\log n)^{-2} < \infty$ for $d = \alpha - 2$ although $\alpha - 2 = \inf\{d : M_{d+1} = \infty\}$.

Remark 6 For every finite d the fixed function e for the operator $\mathcal{L}$ extends to a unique extended-real-valued function on Ω, still denoted by e, such that e is finite except at 0^∞ where it takes the value $+\infty$. Moreover it is easy to check that for each k there is constant $D = D(k)$ so that $\mathrm{Var}_n\,(e \cdot 1_{A_k}) \le D\,\theta^n$. On the other hand the function $h = (1 - \mathcal{L}_0)e$ is uniformly bounded on Ω and $\mathrm{Var}_n h \le C\,\theta^n$. Finally, we may extend the measures ρ and μ to the whole space Ω by putting $\rho(\{0^\infty\}) = \mu(\{0^\infty\}) = 0$.

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[1] *Notational warning*: Here and in the sequel, for two sequences a_n and b_n we shall write:

- $a_n \approx b_n$ if the ratio a_n/b_n grows slower than any power of n, or decays slower than any inverse power of n, as $n \to \infty$;
- $a_n \asymp b_n$, if $C^{-1} \le a_n/b_n \le C$ for all n and fixed $C \ge 1$;
- $a_n \sim b_n$ if the quotient a_n/b_n tends to unity as $n \to \infty$.

Remark 7 As shown in the previous Section (see Corollary 3.1), the probability measure ρ can be viewed a Gibbs measure on Ω_0 which is invariant under the action of the induced shift T^τ. Now, if $d > 0$ the corresponding property for the T-invariant probability measure $\hat{\mu} := \mu/M_1$ would be

$$\frac{\hat{\mu}([\omega_0 \cdots \omega_{n-1}])}{\exp \sum_{k=0}^{n-1} V(T^k\omega)} \asymp 1. \tag{4.43}$$

On the other hand, if d is positive but finite, this is not the case. Indeed, consider the cylinder set A_n (see (2.3)). According to Proposition 4.4 and (4.42) we have

$$\mu([0^{n-1}1]) \asymp \sum_{l \geq n} \psi(0^{l-1}1\omega) \tag{4.44}$$

uniformly in $\omega \in \Omega$. Now, if (Ω, T, μ) has finite ergodic degree then for all $\omega \in \Omega$,

$$\psi(0^{n-1}1\omega) = o\left(\sum_{l \geq n} \psi(0^{l-1}1\omega)\right), \tag{4.45}$$

and therefore

$$\frac{\hat{\mu}([0^{n-1}1])}{\exp \sum_{k=0}^{n-1} V(0^{n-k-1}1\omega)} \asymp \Delta(n) \tag{4.46}$$

with $\Delta(n) \approx n$. This can be interpreted as a *weak Gibbs property* [MRTVV], [Yu]. It appears to be related to the fact that the potential V has (exactly) two equilibrium states (in the sense of Walters). Let us see this point in more detail. According to [Wal2], and using Property 4 of the potential function V, a T-invariant probability measure m on Ω is called an *equilibrium state* for V if it satisfies

$$h_m(T) + m(V) = 0, \tag{4.47}$$

where $h_m(T)$ denotes the measure-theoretic entropy of T wrt m. Now, on the one hand, by Property 1 it is plain that the point measure δ_{0^∞} concentrated at $\{0^\infty\}$ satisfies $h_{\delta_{0^\infty}}(T) + \delta_{0^\infty}(V) = 0$. On the other hand, if $M_1 < \infty$, it follows from the results of Section 3 and [Wal2]

that the measure ρ is the *unique* equilibrium state for W, namely $h_\rho(T^\tau) + \rho(W) = 0$ and any other T-invariant (and thus also T^τ-invariant) measure $\tilde{\rho}$ (with $\tilde{\rho}(\{0^\infty\}) = 0$) satisfies $h_{\tilde{\rho}}(T^\tau) + \tilde{\rho}(W) < 0$. Moreover we have

$$\begin{aligned}
\mu(V) &= \int_\Omega V(\omega)\, e(\omega) \nu(d\omega) = \int_\Omega V(\omega) \sum_{n=0}^{\infty} \mathcal{L}_0^n h(\omega)\, \nu(d\omega) \\
&= \sum_{n=0}^{\infty} \int_{D_n} V(T^n\omega)\, h(\omega)\, \nu(d\omega) = \sum_{n=1}^{\infty} \int_{A_n} \left(\sum_{k=0}^{n-1} V(T^k\omega)\right) h(\omega)\, \nu(d\omega) \\
&= \int_\Omega W(\omega)\, h(\omega)\, \nu(d\omega) = \rho(W).
\end{aligned} \tag{4.48}$$

In addition, taking natural extension and using Abramov's formula [Ab] we get $h_\rho(T^\tau) = M_1 \cdot h_{\hat{\mu}}(T)$ so that, finally,

$$h_{\hat{\mu}}(T) + \hat{\mu}(V) = \frac{h_\rho(T^\tau) + \rho(W)}{M_1} = 0. \tag{4.49}$$

Whence, any T-invariant measure m which satisfies (4.47) is a convex combination of δ_{0^∞} and $\hat{\mu}$ (see also [Ho] and [FL] for related examples).

5 The pressure function $P(z)$

We now characterize the behaviour of the pressure function $P(z)$ of its derivatives when $z \uparrow 1$ (and thus of the leading eigenvalue $\lambda_z = e^{P(z)}$) in terms of suitable expectations wrt the equilibrium state ρ. We first introduce some further quantities. For $\gamma \geq 0$ and $|z| < 1$, set

$$M_\gamma(z) := \sum_{k=1}^{\infty} z^k \, k^\gamma \, \rho(A_k) \tag{5.50}$$

so that $M_\gamma(1) \equiv M_\gamma$ (provided it exists). Notice that $M_0(z) = \nu(\mathcal{M}_z h)$. More generally define, for $m \geq 0$,

$$\mathcal{M}_z^{(m)} f(\omega) := \sum_{k=1}^{\infty} z^k \, k^m \, \left(\mathcal{L}_1 \mathcal{L}_0^{k-1} f\right)(\omega) = \mathcal{M}_z(f \cdot \tau^m)(\omega), \tag{5.51}$$

so that $\mathcal{M}_z^{(0)} \equiv \mathcal{M}_z$. Reasoning as in the proof of Lemma 3.1 one sees that if (Ω, T, μ) has ergodic degree $d > 0$ then, for all integers m with $0 \leq m < d+1$, the power series of $\mathcal{M}_z^{(m)}$ when acting on $\mathcal{F}_\theta$ has radius of convergence bounded from below by 1 and, moreover, it converges absolutely at every point of the unit circle. In addition, from (5.50) and (5.51) we have, for $m > 0$,

$$M_m(z) = (zD)^m M_0(z) = \nu\left(\mathcal{M}_z^{(m)} h\right), \tag{5.52}$$

where zD denotes the differential operator $z(d/dz)$. It is not difficult to realize that, whenever it is defined for $0 \leq z \leq 1$, the operator $\mathcal{M}_z^{(m)}$ has leading eigenvalue $\lambda_z^{(m)} = (zD)^m \lambda_z$.

Theorem 5.2 *The function $z \to \lambda_z \equiv e^{P(z)}$ is analytic in a complex open neighbourhood of $[0,1)$. Moreover, we have the following properties:*

1. *If $-1 < d \leq 0$ then $(zD)\lambda_z \sim M_1(z)$ as $z \uparrow 1$;*

2. *if $d > 0$ then $P(z) \in C^1((0,1])$ and $(zD)P(z)\Big|_{z=1} = M_1 = \rho(\sigma_0)$;*

3. *if $d > 1$, then $P(z) \in C^2((0,1])$ and*

$$(zD)^2 P(z)\Big|_{z=1} = \sigma^2 := \sum_{n \geq 0} [\rho(\sigma_0 \sigma_n) - \rho(\sigma_0)\rho(\sigma_0)];$$

4. more generally, if $m-1<d\le m$ for some $m\in\mathbb{Z}_+$ then $P(z)\in C^m((0,1])$ and, for $1\le \ell\le m$,

$$(zD)^\ell P(z)\Big|_{z=1} = \sum_{n_1\ge 0}\cdots\sum_{n_{\ell-1}\ge 0} U_\ell(\sigma_0,\sigma_{n_1},\ldots,\sigma_{n_{\ell-1}}) \,<\, \infty,$$

where

$$U_\ell(\sigma_{k_1},\ldots,\sigma_{k_\ell}) := \frac{\partial^\ell}{\partial t_1\ldots\partial t_\ell}\log\rho\left(\exp(\sum_{i=1}^\ell t_i\sigma_{k_i})\right)\Big|_{t_1=\cdots=t_\ell=0}.$$

In addition, $(zD)^{m+1}\lambda_z\Big|_{z=1}\sim M_{m+1}(z)$ as $z\uparrow 1$.

Proof. In this proof we shall use the notation of Section 3 and consider $\mathcal{M}_z$ as acting on $\mathcal{F}_\theta(\Sigma)$. We first notice that, as a consequence of Lemma 3.1, $z\to\mathcal{M}_z$ is an analytic family in the sense of Kato for z in the open unit disk. Theorem 3.1 and Kato-Rellich Theorem (see, e.g., Thm. XII.8 in [RS]) then imply that $\exp P(z)$ extends analytically in a complex open neighbourhood of $[0,1)$. In addition, since $\Lambda_n(z)$ is monotonically increasing for $0\le z\le 1$, so is $\exp P(z)$, and therefore $\exp P(z)<\exp P(1)=1$ for $z\in[0,1)$. This proves the first statement. Now, from the proof of Theorem 3.1 we know that for z real and positive $\lambda_{z,N}=\exp P_N(z)$ is the (simple) eigenvalue with largest modulus of $\mathcal{M}_{z,N}:\mathcal{F}_\theta(\Sigma_N)\to\mathcal{F}_\theta(\Sigma_N)$, where $P_N(z)$ is defined in (3.24). We now define a function $K_N(z)$ by setting

$$\exp P_N(z) = \exp P_N(1)\left(\sum_{k=1}^N z^k\,\rho_N(A_k)+K_N(z)\right) \tag{5.53}$$

where $\rho_N=h_{1,N}\cdot\nu_{1,N}$ is the equilibrium state on Σ_N for the function W and $\rho_N(A_k)=\rho_N\{\sigma\in\Sigma_N:\sigma_0=k\}$. Clearly we have $K_N(0)=K_N(1)=0$. Furthermore, from the proof of Theorem 3.1 it follows that for all $0\le z\le 1$ the function $\exp P_N(z)$ converges to $\exp P(z)$ as $N\to\infty$. We also know from ([CI], Theorem 2.1) that $\exp P_N(1)\to 1$ and $\rho_N\to\rho$ uniformly as $N\to\infty$, where $\rho=h\cdot\nu$ is the equilibrium state on Σ for the function W. Therefore, for $0\le z\le 1$ and $N\to\infty$ we have that $K_N(z)$ tends pointwise to a function $K(z)$ so that

$$\exp P(z) = \sum_{k=1}^\infty z^k\rho(A_k)+K(z), \tag{5.54}$$

The function $K(z)$ is analytic in a complex open neighbourhood of $[0,1)$ and satisfies $K(0)=K(1)=0$. We now substitute $z=e^t$ into (5.53) and expand the sum $\sum_{k=1}^N z^k\,\rho_N(A_k)$ in powers of t:

$$\exp P_N(e^t) = \exp P_N(1)\left(\sum_{m=0}^\infty \frac{\rho_N(\sigma_0^m)}{m!}t^m + K_N(e^t)\right) \tag{5.55}$$

where $\rho_N(\sigma_0^m)=\sum_{k=1}^N k^m\,\rho_N(A_k)$. Clearly $(zD)^\ell = d^\ell/dt^\ell$ for $z=e^t$. Now observe that $P_N(e^t)$ is the pressure of the function $W+t\,\sigma_0$ restricted to Σ_N. It is a standard result in the theory of equilibrium states (see [Ru4], Chapter 5) that, under the conditions assumed here, $P_N(e^t)$ is analytic in some neighbourhood of $t = 0$ and its derivatives at $t = 0$ are related to suitable moments of ρ_N (see [Ru4], p.99-100). In particular, we have

$$\frac{dP_N(e^t)}{dt}\Big|_{t=0} = \rho_N(\sigma_0), \tag{5.56}$$

and

$$\frac{d^2P_N(e^t)}{dt^2}\Big|_{t=0} = \sum_{n\geq 0}[\rho_N(\sigma_0\sigma_n) - \rho_N(\sigma_0)\rho_N(\sigma_0)]. \tag{5.57}$$

Now, differentiating twice (5.55) at $t = 0$, and using (5.56) and (5.57), we get

$$\frac{dK_N(e^t)}{dt}\Big|_{t=0} = 0 \quad \text{and} \quad \frac{d^2K_N(e^t)}{dt^2}\Big|_{t=0} = \sum_{n>0}[\rho_N(\sigma_0\sigma_n) -$$

$\rho_N(\sigma_0)\rho_N(\sigma_0)]. Notice that, for any fixed \mathrm{N}, the expression in square brackets decreases exponentially with \mathrm{k}. More specifically, set \mathrm{M}_N \equiv$ $\mathcal{M}_{1,N}$, $\lambda_N \equiv \lambda_{1,N}$, $h_N \equiv h_{1,N}$, $\nu_N \equiv \nu_{1,N}$ where $\lambda_{1,N}, h_{1,N}, \nu_{1,N}$ are as above, and define

$$X_0^N = \frac{\mathcal{M}_N\,(\,h_N\,\sigma_0\,)}{\lambda_N\,\rho_N(\sigma_0)}. \tag{5.58}$$

Reasoning as in the proof of Lemma 3.1, one can find a positive constant C, independent of N, such that, for N large enough, we have $\|X_0^N\|_\theta < C$. On the other hand, from the proof of Theorem 3.1 it follows that there exist two constants $M > 0$ and $0 < \vartheta < 1$, independent of N, such that, for N large enough and for all $v \in \mathcal{F}_\theta(\Sigma_N)$, we have

$$\|\lambda_N^{-k}\mathcal{M}_N^k v - h_N \cdot \nu_N(v)\|_\theta \leq \; M\,\vartheta^k\,\|v\|_\theta.$$

Whence, taking $v = X_0^N$ and using $\nu_N(X_0^N) = 1$, we get

$$\|\lambda_N^{-k}\mathcal{M}_N^k X_0^N - h_N\|_\theta \leq \; C\,M\,\vartheta^k$$

and therefore, for $k > 0$,

$$|\rho_N(\sigma_0\sigma_k) - \rho_N(\sigma_0)\rho_N(\sigma_0)| \leq C\,M\,\vartheta^{k-1}\,\rho_N(\sigma_0)\rho_N(\sigma_0).$$

This gives

$$\frac{d^2K_N(e^t)}{dt^2}\Big|_{t=0} \leq \frac{C\,M}{1-\vartheta}\,\rho_N(\sigma_0)\rho_N(\sigma_0). \tag{5.59}$$

¿From this discussion we obtain that the following properties hold uniformly in N: for all $d > -1$, we have

$$\lim_{t\to 0_-}\frac{dK_N(e^t)}{dt} = 0 \tag{5.60}$$

and moreover, if $d > 0$, then

$$\frac{dK_N(e^t)}{dt}\Big|_{t=0} = 0 \quad \text{and} \quad \frac{d^2K_N(e^t)}{dt^2}\Big|_{t=0} < \infty. \tag{5.61}$$

We can actually say more. First observe that if $d \le 0$ then $\rho_N(\sigma_0) \to \infty$ as $N \to \infty$. In addition, if $m-1 < d \le m$, with $m > 0$, then the expectations $\rho_N(\sigma_0^\ell)$ are bounded uniformly in N for $1 \le \ell \le m$, but $\rho_N(\sigma_0^{m+1}) \to \infty$ as $N \to \infty$. On the other hand, if $d > m-1$ the derivatives $d^\ell K_N(e^t)/dt^\ell|_{t=0}$ are bounded uniformly in N for $1 \le \ell \le m+1$, as we are now going to show (notice however that the case $m=1$ has already been discussed).
Using induction, one can compute the ℓ-th derivative (with $\ell > 1$) of the pressure $P_N(e^t)$ at $t=0$ as:

$$\frac{d^\ell P_N(e^t)}{dt^\ell}\Big|_{t=0} = \sum_{k_1 \ge 0} \dots \sum_{k_{\ell-1} \ge 0} U_{\ell,N}(\sigma_0, \sigma_{k_1}, \dots, \sigma_{k_{\ell-1}}), \tag{5.62}$$

where $U_{\ell,N}$ is given by

$$U_{\ell,N}(\sigma_{k_1}, \dots, \sigma_{k_\ell}) = \frac{\partial^\ell}{\partial t_1 \dots \partial t_\ell} \log \rho_N \left(\exp(\sum_{i=1}^{\ell} t_i \sigma_{k_i}) \right) \Big|_{t_1 = \dots = t_\ell = 0}.$$

The function $U_{\ell,N}$ can be considered as a particular version of what in statistical mechanics is called the ℓ-th Ursell function (see, e.g., [Si], Section II.12). Let us write

$$\frac{d^{m+1} P_N(e^t)}{dt^{m+1}}\Big|_{t=0} = U_{m+1,N}(\sigma_0, \dots, \sigma_0) + V_{m+1,N}. \tag{5.63}$$

¿From the characterization of the $U_{\ell,N}$'s given in ([Si], Corollary II.12.7) we have that, for any $\ell < m+1$, $U_{\ell,N}(\sigma_0, \sigma_{k_1}, \dots, \sigma_{k_{\ell-1}})$ can be written as a linear combination of products

$$\rho_N(\sigma_0^{r_1}) \dots \rho_N(\sigma_0^{r_k}) \, (\, \rho_N(X_1 X_2) - \rho_N(X_1)\rho_N(X_2) \,)$$

for suitable $r_1, \dots, r_k$ and with X_1 a product of functions from among $\sigma_0, \dots, \sigma_{k_{\ell-1}}$ and X_2 a product of functions from among $\sigma_{k_\ell}, \dots, \sigma_{k_m}$. Thus, reasoning as above, one obtains that, for N large enough, $|V_{m+1,N}|$ is bounded from above by a linear combination, whose coefficients depend of $m+1$ but not of N, of products $\rho_N(\sigma_0^{r_1}) \dots \rho_N(\sigma_0^{r_n})$, with $r_1 + \dots + r_n = m+1$ and $m+1 \ne r_i \ge 0$ for all $i \in \{1, \dots, n\}$. Notice that the leading term (with respect to the limit $N \to \infty$) is given by $\rho_N(\sigma_0^m)\, \rho_N(\sigma_0)$ and corresponds to the choice $k_1 = k_2 = \dots = k_m \ne 0$ in (5.62). Moreover, differentiating (5.55) m times at $t=0$ and using (5.62), along with the combinatorial features of the $U_{\ell,N}$'s (see [Si], Section II.12), one sees, inductively in ℓ, that $d^{m+1} K_N(e^t)/dt^{m+1}$ at $t=0$ is equal to $V_{m+1,N}$ plus a linear combination of the $V_{k,N}$'s with $k = 2, \dots, m$. This finishes the proof. $\diamondsuit$

6 A Markov approximation

We now consider the *renewal sequence* $a_0, a_1, \dots$ associated with the sequence $p_1, p_2, \dots$, with $p_n \equiv \rho(A_n)$, which is generated by the recurrence relation:

$$a_0 = 1 \quad \text{and} \quad a_n = p_n + a_1\, p_{n-1} \cdots + a_{n-1}\, p_1 \quad \text{for} \quad n \ge 1. \tag{6.64}$$

Its generating function $A(z)$ is given by

$$A(z) = \sum_{n=0}^{\infty} a_n z^n = \left(1 - \sum_{n=1}^{\infty} p_n z^n\right)^{-1} = \left((1-z)\sum_{n=0}^{\infty} d_n z^n\right)^{-1}, \tag{6.65}$$

where we have written $d_n = \sum_{k>n} p_k = \rho(D_n)$ and the D_n's are defined in Proposition 4.4. If $d > 0$, another generating function of interest is that of the numbers $b_n := M_1\, a_n - 1$, that is

$$B(z) = \sum_{n=0}^{\infty} b_n z^n = M_1\, A(z) - \frac{1}{1-z} = \frac{D^{(1)}(z)}{D(z)}, \tag{6.66}$$

where we have put

$$D^{(1)}(z) = \sum_{n\geq 0} d_n^{(1)} z^n, \qquad d_n^{(1)} := \sum_{l>n} d_l = \mu(D_{n+1}) = \mu(\tau > n+1), \tag{6.67}$$

and in the last identity we have used Proposition 4.4. We shall see in a moment that $a_n \to 1/M_1$, so that b_n measures the rate at which this limit is reached.

Proposition 6.5 *Suppose the ergodic degree d exists and is finite. Then the power series expansions of (6.65) and (6.66) define holomorphic functions $A(z)$ and $B(z)$ in $\mathcal{D}$ which converge uniformly at every point of the unit circle with the exception of $z = 1$, where they have a non-polar singular point. Moreover, one has the following asymptotic behaviour of their coefficients: for every $d > -1$ we have*

$$a_n \to \frac{1}{M_1} \quad \textit{as} \quad n \to \infty,$$

where $1/M_1 \equiv 0$ if $M_1 = \infty$. Furthermore, if $-1 < d \leq 0$ we have

$$a_n \approx n^d,$$

whereas, if $d > 0$,

$$b_n \equiv M_1\, a_n - 1 \sim \left[M_1^{-1}\mu(\tau > n)\right] \approx n^{-d}.$$

Proof. We first show that the function $A(z)$ is analytic in $|z| < 1$ and converges at every point of the unit circle besides $z = 1$. Indeed, we have $|\sum_{n=1}^{\infty} p_n z^n| < 1$ for $|z| < 1$ because $p_n \geq 0$. Moreover, $|\sum_{n=1}^{\infty} p_n z^n| < 1$ also for $|z| = 1$, $z \neq 1$. This follows from the aperiodicity condition g.c.d.$\{n,\, p_n > 0\} = 1$ which, in turn, is a consequence eq. (2.8), Corollary 3.1 and Property 1, that give

$$p_n \geq A \exp\left(\inf_{\omega \in A_n} S_n V(\omega)\right) \geq A\, e^{-C_1 n}.$$

Now set

$$D(z) = \sum_{n=0}^{\infty} d_n z^n, \tag{6.68}$$

and notice that $D(z)$ converges absolutely in $|z| \leq 1$ and has no zeros on $|z| = 1$. In addition $\sum_{n=0}^{\infty} d_n = M_1$. It then follows (see, e.g., [Pos], p.88) that the function

$$\frac{1}{D(z)} = (1-z)A(z) \tag{6.69}$$

has a power series expansion which converges absolutely in the closed unit disk and, moreover, its value at $z = 1$ is M_1^{-1}. Now observe that if $d \leq 0$ (so that $M_1^{-1} = 0$) we have

$A(z) \to \infty$ but $(1-z)A(z) \to 0$ as $z \uparrow 1$. If $d > 0$, generalising (6.67), we can define recursively a family of formal 'tail sequences' $d_n^{(m)}$, $m \geq 0$, as $d_n^{(0)} = d_n$ and $d_n^{(m)} = \sum_{l>n} d_l^{(m-1)}$ for $m > 0$, and formal power series $D^{(m)}(z) = \sum d_n^{(m)} z^n$. It is easy to realize that if (Ω, T, μ) has ergodic degree $d > 0$ then $\{d_n^{(m)}\}$ is defined for $m \leq [d]+1$ (here $[d]$ denotes the integral part of d), $D^{([d])}(z)$ is absolutely convergent at $z = 1$, with $D^{([d])}(1) = d_0^{([d]+1)}$, but $D^{([d]+1)}(z)$ diverges at $z = 1$. Define

$$H^{([d])}(z) = A(z)\,(d_0^{([d]+1)} - D^{([d])}(z)) = \frac{D^{([d]+1)}(z)}{D^{(0)}(z)}. \tag{6.70}$$

In particular $H^{(0)}(z) = B(z)$. We then have $(1-z)H^{([d])}(z) \to 0$ but $H^{([d])}(z) \to \infty$ as $z \uparrow 1$. This shows that for each of these functions, but in particular for $A(z)$ and $B(z)$, the point $z = 1$ is a non-polar singular point.
To show the asymptotic behaviour of the coefficient we set

$$\frac{1}{D(z)} =: \sum_{n=0}^{\infty} \gamma_n\, z^n, \qquad \sum_{n=0}^{\infty} |\gamma_n| < \infty. \tag{6.71}$$

Now, if $d \leq 0$ then from (6.69) we immediately obtain $a_n = o(1)$. On the other hand the assumption $d > 0$ implies that $d_n = o(n^{-1})$. We may then use Lemma 3.II in [Ro] to obtain $\gamma_n = o(n^{-1})$ as well. By an Abelian theorem (see, e.g., [Chu], p.55) we then have

$$a_n = \sum_{k=0}^{n} \gamma_n \to M_1^{-1}, \qquad n \to \infty. \tag{6.72}$$

But we can say more. We first point out that

$$d_0 = 1, \quad d_n > 0, \quad \frac{d_n}{d_{n-1}} = \left(1 + \frac{p_n}{d_n}\right)^{-1} < 1 \quad (n \geq 1).$$

Therefore, if d is finite the sequence p_n/d_n eventually decreases with n so that d_n/d_{n-1} is eventually increasing and tends to 1 as $n \to \infty$. Therefore we can find an integer $n_0 \geq 0$ and coefficients $\tilde{d}_n$ so that $\tilde{d}_n \equiv d_n$, $\forall n \geq n_0$, and $\tilde{d}_n/\tilde{d}_{n-1}$ is increasing for all $n \geq 1$. Let $\tilde{\gamma}_n$ be such that $\tilde{D}(z) := \sum_n z^n \tilde{d}_n = (\sum_{n=0}^{\infty} z^n\, \tilde{\gamma}_n)^{-1}$. We can then apply ([Har], Theorem 22) to get $\tilde{\gamma}_0 = \gamma_0 = 1$ and $\tilde{\gamma}_n \leq 0$ for $n \geq 1$. In addition we have $-\sum_{n\geq 1} \tilde{\gamma}_n \leq 1$ with equality iff $M_1 = \infty$. Setting $\epsilon(z) = \sum_{n \leq n_0} (d_n - \tilde{d}_n) z^n$ and $\tilde{\Gamma}(z) = \sum_{n=0}^{\infty} z^n \tilde{\gamma}_n$ we have

$$A(z)\left(1 + \epsilon(z) \cdot \tilde{\Gamma}(z)\right) = \frac{\tilde{\Gamma}(z)}{1-z}, \tag{6.73}$$

If $M_1 = \infty$ then $\tilde{\Gamma}(1) = 0$ and therefore the coefficients of $A(z) \cdot \epsilon(z) \cdot \tilde{\Gamma}(z)$ are $o(a_n)$. This implies that a_n is eventually monotone decreasing and $a_n \sim 1 + \tilde{\gamma}_1 + \cdots + \tilde{\gamma}_n$. Now, if (Ω, T, μ) has ergodic degree d then $\sum n^d\, d_n = \infty$ but $\sum n^{d-\epsilon}\, d_n < \infty$, $\forall \epsilon > 0$. Therefore $d_n \sim C\, L(n)\, n^{-1-d}$ with $L(n)$ some slowly varying function. Formula (6.65), eventual monotonicity of the a_n's and a repeated application of a Tauberian theorem for power series (see, e.g., [Fe], Chap XIII.5, Theorem 5) then gives $a_n \approx n^d$ (for $d = 0$ this means that a_n vanishes slower than any inverse power of n).
Finally, if $d > 0$ (but $d < \infty$) it follows from (6.66), (6.67) and [Ro] (see also [Is2], Lemma 2) that

$$\frac{M_1^2\, a_n - M_1}{\sum_{\ell > n} d_\ell} \to 1 \quad \text{as} \quad n \to \infty. \tag{6.74}$$

By (6.67) this gives the asymptotic behaviour of b_n. To finish the proof we observe that if (Ω, T, μ) has ergodic degree $d > 0$ then $n^d \cdot \sum_{\ell > n} d_\ell$ varies slower than any power of n. $\diamondsuit$

Now, if we view the partition set A_n as the n-th 'state' for the dynamical system (Ω, T, μ), the quantity $p_n \equiv \rho(A_n)$ can be interpreted as the ρ-probability that a first passage in the state 1 occurs after n iterates. We may then consider the quantity

$$u_n := \mu(A_1 \cap T^{-n} A_1), \tag{6.75}$$

that is the μ-probability to observe a *return* in the state 1 after n iterates (recall that $\mu(A_1) = 1$). Using (4.35) backward we have

$$u_n = \mu(T_1(T^{-(n-1)} A_1)) = \rho\left(T^{-(n-1)} A_1\right) \equiv \rho\left(T^{n-1}(x) \in A_1\right), \tag{6.76}$$

the last quantity being the ρ-probability to observe a *passage* in the state 1 after $n-1$ iterates (for the first time or not). Another interpretation of u_n is the following. Let $N_n(\omega) := 1_{A_1}(\omega) + \cdots + 1_{A_1}(T^{n-1}(\omega))$ be the number of passages in the state 1 up to the n-th iterate of the map T. Let also $s_n(\omega) = \sigma_0(\omega) + \sigma_1(\omega) + \cdots + \sigma_{n-1}(\omega)$ be the total number of iterates of T needed to observe n passages in the state 1. Then notice that $(N_n = k) = (s_k \le n < s_{k+1}) = (s_k \le n) - (s_{k+1} \le n)$. Thus $\rho(N_n = k) = \rho(s_k \le n) - \rho(s_{k+1} \le n)$, which is the same as $\rho(s_k \le n) = \sum_{r=k}^n \rho(N_n = r)$. Moreover $\rho(s_k = n) = \rho(s_k \le n) - \rho(s_k \le n-1)$ for $k < n$ and $\rho(s_n = n) = \rho(s_n \le n)$. Therefore we have the following chain of identities

$$\begin{aligned} u_n &= \sum_{k=1}^n \rho(s_k = n) = \sum_{k=1}^n \rho(s_k \le n) - \sum_{k=1}^{n-1} \rho(s_k \le n-1) \\ &= \sum_{k=1}^n \sum_{r=k}^n \rho(N_n = r) - \sum_{k=1}^{n-1} \sum_{r=k}^{n-1} \rho(N_{n-1} = r) = \rho(N_n) - \rho(N_{n-1}) \end{aligned} \tag{6.77}$$

where $\rho(N_n)$ denotes the mean of the random variable N_n (set $N_0 = 0$). Thus, u_n may be regarded as the expected number of passages in the state 1 per iteration of the map T (after $n-1$ iterations), namely as a 'renewal density' for (Ω, T, μ).
A further decomposition using a first passage argument is the following:

$$u_n = \sum_{r=1}^{n} p_r \; \rho \left(T^{n-1}\omega \in A_1 \,|\, \omega \in A_r \right). \tag{6.78}$$

Now suppose that the sequence of random variables $\sigma_0(\omega), \sigma_1(\omega), \sigma_2(\omega), \ldots$ were mutually independent. In this case the iteration process $\{T^n\omega\}_{n\geq 0}$ would 'start afresh' at each passage in the state 1 and

$$\rho \left(T^{n-1}\omega \in A_1 \,|\, \omega \in A_r \right) = \rho \left(T^{n-r-1}\omega \in A_1 \right) = u_{n-r}. \tag{6.79}$$

Direct comparision of (6.64), (6.78) and (6.79) shows that under the above supposition we would have $u_n \equiv a_n$.
Furthermore, it has been observed (see, e.g., [Ki]) that any renewal sequence, that is any sequence generated as in (6.64) with $p_1, p_2 \ldots$ satisfying $p_n \geq 0$ and $\sum p_n \leq 1$, can arise as the diagonal transition probabilites corresponding to a given state in some Markov chain. In our case, it is easy to check that a Markov chain which does the job is that discussed in Example 1 with the identification $p_n = \rho(A_n)$. According to the above discussion, we shall say that the Markov chain of Example 1 is the *Markov approximation of* (Ω, T, μ) *with respect to the reference set* A_1. Its statistical properties are partially described by Proposition 6.5 (see also [Is1], [Is2]; and [Che] for a related approximation scheme).

¿From Theorem 5.2 and Proposition 6.5 we obtain a first result about the proximity of (Ω, T, μ) and its Markov approximation.

Corollary 6.3 *For all finite $d > -1$, the function $Q(z) := (1 - \exp P(z))^{-1}$ has a non-polar singular point at $z = 1$ with $Q(z) \sim A(z)$ as $z \uparrow 1$, where $A(z)$ is the generating function defined in (6.65). Moreover we have $Q(z) - A(z) = \mathcal{O}(1)$ and $(zD)\,[Q(z) - A(z)] = \mathcal{O}\left(M_2(z)/M_1(z)\right) = \mathcal{O}\left((1-z)^{-1}\right)$ as $z \uparrow 1$.*
For $d > 0$, the function $U(z) := M_1\, Q(z) - (1-z)^{-1}$ has the following properties:

1. *if $0 < d \leq 1$ then $U(z) \sim B(z)$ as $z \uparrow 1$, where $B(z)$ is the generating function defined in (6.66);*

2. *if $1 < d \leq 2$ we have $U(z) \to \dfrac{\sigma^2 + M_1^2 - M_1}{2M_1}$ and $(zD)U(z) \sim (zD)B(z)$ as $z \uparrow 1$;*

3. *more generally, if $m - 1 < d \leq m$, for $m > 0$, then $(zD)^\ell U(z)$ is uniformly bounded for $0 \leq \ell < m-1$ but $(zD)^{m-1}U(z) \sim (zD)^{m-1}B(z)$ as $z \uparrow 1$. In addition we have $(zD)^{m-1}\,[U(z) - B(z)] = o\,(1)$ and $(zD)^m\,[U(z) - B(z)] = \mathcal{O}(M_{m+1}(z)) = o\left((1-z)^{-1}\right)$ as $z \uparrow 1$.*

Proof. ¿From (5.54) we can write

$$Q(z) = A(z) \cdot (1 + J(z)) \quad \text{with} \quad J(z) := \left(\frac{A(z) \cdot K(z)}{1 - A(z) \cdot K(z)} \right). \tag{6.80}$$

Now, from the proof of Theorem 5.2 we have that when $z \uparrow 1$ and for all $d > -1$

$$\begin{aligned}(zD)K(z) &= (zD)\lambda_z - M_1(z) = \lambda_z (zD)P(z) - M_1(z) \\ &= \lambda_z \nu_z (\mathcal{M}_z^{(1)} h_z) - M_1(z) \sim (\lambda_z - 1)\, M_1(z),\end{aligned} \tag{6.81}$$

so that $\lim_{z\uparrow 1} K(z)/(1-z) = 0$ for all $d > -1$ and $\lim_{z\uparrow 1} K(z)/(1-z)^2 < \infty$ for $d > 0$. Hence, for all $d > -1$, we have $\lim_{z\uparrow 1} J(z) = 0$. This gives the asymptotic behaviour of $Q(z)$ and the nature of the singularity at $z = 1$ follows from Proposition 6.5.
More generally, again from Theorem 5.2 we have that when $z \uparrow 1$ and $d > m - 1$,

$$\frac{(zD)^{m+1}K(z)}{1-\lambda_z} = \frac{(zD)^{m+1}\lambda_z - M_{m+1}(z)}{1-\lambda_z} = \mathcal{O}(M_{m+1}(z)). \tag{6.82}$$

This gives $(zD)^m J(z) = o\,(1)$ and $(zD)^{m+1}J(z) = \mathcal{O}(M_{m+1}(z))$ as $z \uparrow 1$, so that the behaviour of $U(z)$ follows from the identities

$$U(z) = Q(z)\left(M_1 - \frac{1 - e^{P(z)}}{1 - z}\right) = B(z) + M_1 A(z) J(z)\,. \tag{6.83}$$

More specifically, differentiating the term $A(z)J(z)$ we get

$$(zD)\,[A(z)J(z)] = A(z)\cdot\left\{\left((zD)(e^{P(z)})\right) A(z)J(z) + (zD)J(z)\right\} \tag{6.84}$$

which, by the above, is $\mathcal{O}\left(M_2(z)/M_1(z)\right)$ as $z \uparrow 1$. This implies that if $-1 < d \le 0$ the term $A(z)J(z)$ is continuous at $z = 1$ and $(zD)\,[A(z)J(z)] = \mathcal{O}\left((1-z)^{-1}\right)$ when $z \uparrow 1$. For $d > 0$ we have $A(z)J(z) = o\,(1)$ and $(zD)\,[A(z)J(z)] = \mathcal{O}\left(M_2(z)\right) = o\left((1-z)^{-1}\right)$ when $z \uparrow 1$. In particular, the last expression is integrable in a neighbourhood of $z = 1$. More generally, setting $H_\ell(z) := A(z)^{-1}(zD)^\ell\,[A(z)J(z)]$, so that $H_{\ell+1}(z) = (zD)H_\ell(z) + H_\ell(z)\,(zD)\,[\log A(z)]$, one readily sees inductively that if $d > m-1$ for some $m > 0$ then the term $A(z)\cdot H_m(z)$ is $\mathcal{O}((zD)^{m+1}J(z)) = \mathcal{O}\left(M_{m+1}(z)\right)$ as $z \uparrow 1$, and thus integrable as above. $\diamondsuit$

7 Renewal vs scaling and mixing properties

We are now going to make use of the results of the previous sections to study the generating function of the numbers u_n's (see (6.75)), that is the function

$$S(z) := \sum_{n=0}^{\infty} z^n\, u_n. \tag{7.85}$$

We have the following

Proposition 7.6 *$S(z)$ is holomorphic in $\mathcal{D}$ and extends continuously to $\overline{\mathcal{D}}\backslash\{1\}$. Moreover, for z in some neighbourhood of $z = 1$, with $z \neq 1$, it can be written as*

$$S(z) = \nu(\mathcal{P}_z h)\cdot Q(z) + \nu\left((1-\mathcal{N}_z)^{-1} h\right),$$

where $\mathcal{P}_z = h_z \cdot \nu_z$ is the spectral projection of $\mathcal{M}_z$ corresponding to the eigenvalue λ_z.

Proof. We start with the identity

$$S(z)=\sum_{n=0}^{\infty} z^n\,\nu(1_{A_1}\cdot\mathcal{L}^n(1_{A_1}\cdot e)\,)=\nu\left(1_{A_1}\cdot(1-z\mathcal{L})^{-1}(1_{A_1}\cdot e)\,\right), \tag{7.86}$$

so that Corollary 4.2 yields the first part of the theorem along with the expression, valid in some neighbourhood of $z=1$,

$$S(z)=\nu(1_{A_1}\cdot e_z)\cdot\nu_z(1_{A_1}\cdot e)\cdot Q(z)+R(z) \tag{7.87}$$

with $R(z)=\nu\Big(1_{A_1}\cdot(1-z\mathcal{L}_0)^{-1}(1-\mathcal{N}_z)^{-1}(1_{A_1}\cdot e)\,\Big)$. Now notice that

$$\begin{aligned}\nu(1_{A_1}\cdot e_z) &= \nu(\mathcal{M}(1_{A_1}\cdot e_z))=\nu(\mathcal{L}_1 e_z)=\nu(\mathcal{L}_1(1-z\mathcal{L}_0)^{-1}h_z)\\ &= \nu\left(\frac{\mathcal{M}_z h_z}{z}\right)=\frac{\lambda_z}{z}\,\nu(h_z),\end{aligned} \tag{7.88}$$

and also

$$\nu_z(1_{A_1}\cdot e)=\nu_z\left(\frac{\mathcal{M}_z(1_{A_1}\cdot e)}{\lambda_z}\right)=\frac{z}{\lambda_z}\,\nu_z(\mathcal{L}_1 e)=\frac{z}{\lambda_z}\,\nu_z(h). \tag{7.89}$$

Therefore $\nu(1_{A_1}\cdot e_z)\cdot\nu_z(1_{A_1}\cdot e)=\nu(h_z)\cdot\nu_z(h)$. Using (3.31) we then have

$$\nu(h_z)\cdot\nu_z(h)=\nu(\mathcal{U}_z h)\cdot\nu(\mathcal{U}_z^{-1}h)=\nu\left(\mathcal{U}_z\, h\,\nu\,\mathcal{U}_z^{-1}\,h\right)=\nu(\mathcal{P}_z h). \tag{7.90}$$

Finally, since $\mathcal{M}_z$ and $\mathcal{N}_z$ are commuting we have

$$\begin{aligned}R(z) &= \nu\Big(1_{A_1}\cdot(1-z\mathcal{L}_0)^{-1}(1-\mathcal{N}_z)^{-1}(1_{A_1}\cdot e)\,\Big)\\ &= \nu\Big(\mathcal{L}_1\,(1-z\mathcal{L}_0)^{-1}(1-\mathcal{N}_z)^{-1}(1_{A_1}\cdot e)\,\Big)\\ &= \nu\Big(z^{-1}\mathcal{M}_z(1-\mathcal{N}_z)^{-1}(1_{A_1}\cdot e)\,\Big)\\ &= \nu\Big((1-\mathcal{N}_z)^{-1}z^{-1}\mathcal{M}_z(1_{A_1}\cdot e)\,\Big)\\ &= \nu\Big((1-\mathcal{N}_z)^{-1}h\Big).\end{aligned}$$

$\diamondsuit$

The above and formula (6.80) allow us to write

$$S(z)=A(z)+C(z), \tag{7.91}$$

where $A(z)$ is defined in (6.65) and $C(z)$ is a function which is holomorphic in $\mathcal{D}$ and in a neighbourhood of $z=1$ can be written as

$$C(z)=A(z)\cdot J(z)+Q(z)\cdot\Delta(z)+\nu\Big((1-\mathcal{N}_z)^{-1}h\Big) \tag{7.92}$$

where we have set $\Delta(z) := \nu(\mathcal{P}_z h) - 1$. We now investigate the behaviour of $C(z)$ on the unit circle. To this end, we can evaluate the function $\Delta(z)$ when $z \uparrow 1$ as follows. First, using (3.18), (4.42) along with $h \asymp 1$, and Corollary 6.3 we get

$$\|\mathcal{M}_z - \mathcal{M}_1\|_\theta \le \sum_{n\ge1}(1-z^n)\|\mathcal{M}^{(n)}\|_\theta \le C\,(1-\lambda_z). \tag{7.93}$$

On the other hand we have that $|\Delta(z)| \le \|\mathcal{P}_z - \mathcal{P}_1\|_\theta$. By the spectral properties of $\mathcal{M}_z$ the last quantity is of the same order as $\|\mathcal{M}_z - \mathcal{M}_1\|_\theta$ and thus, by the above, of order not larger than $1-\lambda_z = Q(z)^{-1}$. Furthermore, since $\mathcal{N}_1 h = 0$ the same holds true for the quantity $\nu\Big((1-\mathcal{N}_z)^{-1}h\Big)$. Now notice that by (5.52) we have, for $z \uparrow 1$,

$$\Delta(z) = \nu(\mathcal{P}_z h) - 1 \sim \nu(\mathcal{M}_z h - \mathcal{M}_1 h) \sim \nu(\mathcal{M}_z^{(1)} h)(z-1) = M_1(z)(z-1) \tag{7.94}$$

and therefore $Q(z)\cdot\Delta(z)$ is continuous at $z=1$ with

$$\lim_{z\uparrow1} Q(z)\cdot\Delta(z) = -1. \tag{7.95}$$

In particular, if $d>0$ then $\Delta(z)$ is differentiable at $z=1$ and $(zD)\Delta(z)|_{z=1} = M_1$. Differentiation of $Q(z)\cdot\Delta(z)$ leads to the expression

$$(zD)\,[Q(z)\cdot\Delta(z)\,] = Q(z)\cdot\Big\{\Big((zD)(e^{P(z)})\Big)\,Q(z)\Delta(z) + (zD)\Delta(z)\Big\}, \tag{7.96}$$

so that proceeding as in the proof of Corollary 6.3 we obtain that for all $d > -1$ the function $(zD)[Q(z)\cdot\Delta(z)]$ is $\mathcal{O}\,(M_2(z)/M_1(z))$ as $z\uparrow 1$. A similar reasoning shows that for all $d>-1$ the function $(zD)\nu\Big((1-\mathcal{N}_z)^{-1}h\Big)$ is $\mathcal{O}\,(M_1(z))$ as $z\uparrow1$.

A first consequence of the above discussion is that the function $C(z)$ is holomorphic in $\mathcal{D}$ and extends continuously to $\overline{\mathcal{D}}$. In particular, the function $C(e^{i\phi})$ is integrable on $[-\pi,\pi]$. By the theorem that Fourier coefficients tend to zero (see [Zig], p.45) we then have that the coefficient of z^n in the power series expansion of $C(z)$ is $o\,(1)$.

Let us consider first the infinite measure case, i.e. $-1<d\le0$. ¿From the previous discussion and Proposition 6.5 we have that the coefficient of z^n in the power series expansion of $S(z)$ tends to zero as $n\to\infty$. But we can say more. Multiplying (7.91) by the function $D(z)$ dealt with in Proposition 6.5 we obtain $D(z)\cdot S(z) = (1-z)^{-1} + D(z)\cdot C(z)$. This function is clearly continuous on $\overline{\mathcal{D}}\setminus\{1\}$. Moreover, from Corollary 6.3 and the above it follows that $D(z)\cdot C(z) = \mathcal{O}\,(M_2(z)(1-z))$ in a neighbourhood of $z=1$. This easily implies that the function $D(z)\cdot S(z)-(1-z)^{-1}$ is integrable on the unit circle $|z|=1$, so that the coefficients of its power series expansion tend to zero. We then have $\sum_{k=0}^n d_k\,u_{n-k} = 1 + o\,(1)$ and thus, by (6.65), $\sum_{k=0}^n d_k\,(u_{n-k}-a_{n-k}) = o\,(1)$. By a simple lemma (see [Chu], Chap. I.5, Lemma A) and Proposition 6.5 we then have that for $-1<d\le0$,

$$u_n \sim a_n \approx n^d, \qquad n\to\infty, \tag{7.97}$$

where, again, for $d=0$ this relation means that u_n tends to zero slower than any inverse power of n.

We now examine the finite measure case $d > 0$. Here, putting together (7.91), Proposition 6.5 and the vanishing of the coefficients of the power series expansion of $C(z)$ established so far we get

$$u_n \to \frac{1}{M_1} \quad \text{as} \quad n \to \infty. \tag{7.98}$$

In view of (6.77), (7.98) can be regarded as a *renewal property* as well as a *mixing property* for the dynamical system (Ω, T, μ). Indeed, in this case we can rewrite (7.98) as

$$\hat{\mu}(A_1 \cap T^{-n} A_1) \to (\hat{\mu}(A_1))^2 \quad \text{as} \quad n \to \infty, \tag{7.99}$$

where $\hat{\mu}$ is the probability measure μ/M_1.
In order to investigate the speed of convergence in the limit (7.99) we shall consider the numbers

$$v_n = M_1 u_n - 1 = \frac{\hat{\mu}(A_1 \cap T^{-n} A_1) - (\hat{\mu}(A_1))^2}{(\hat{\mu}(A_1))^2} \tag{7.100}$$

along with their generating function $S_1(z) = \sum_{n=0}^{\infty} z^n \, v_n$. By Proposition 7.6, $S_1(z)$ is holomorphic in $\mathcal{D}$, extends continuously to $\overline{\mathcal{D}} \setminus \{1\}$ and can be written as

$$S_1(z) = B(z) + M_1 \, C(z), \tag{7.101}$$

where $B(z)$ is defined in (6.66) and its behaviour is described in Proposition 6.5 while $C(z)$, defined in (7.91), is continuous on the unit circle *and* its derivative is integrable on the same domain. Therefore the coefficient of z^n in the power series expansion of $C(z)$ is $o\,(n^{-1})$. Comparing to Proposition 6.5 we then see that for all d satisfying $0 < d \leq 1$ one has $v_n \sim b_n$. More generally, using (5.51), (7.86) and (7.91) we see that if $d > m - 1$ for some $m > 0$ then $(zD)^m S(z)$ still extends continuously to $\overline{\mathcal{D}} \setminus \{1\}$ and clearly the same can be said for $(zD)^m B(z)$. One can now proceed inductively, as in the proof of Corollary 6.3, and realize that if $d > m - 1$ for some $m > 0$ then the function $(zD)^m C(z)$ is integrable on the unit circle so that the coefficient of z^n in its power series expansion is $o(1)$ and thus that in the power series expansion of $C(z)$ is $o(n^{-m})$. Again comparing to Proposition 6.5 one then sees that for all d satisfying $m - 1 < d \leq m$ one has $v_n \sim b_n$.
We have thus proved that the behaviour of both u_n and v_n for (Ω, T, μ) is the same as that of the corresponding quantities a_n and b_n for the Markov approximation:

Theorem 7.3 *For $-1 < d \leq 0$ we have*

$$u_n = \mu(A_1 \cap T^{-n} A_1) \sim a_n \approx n^d$$

wheras for $d > 0$ we have

$$v_n = M_1 u_n - 1 = \frac{\hat{\mu}(A_1 \cap T^{-n} A_1) - (\hat{\mu}(A_1))^2}{(\hat{\mu}(A_1))^2} \sim b_n \sim \hat{\mu}(\tau > n) \approx n^{-d}.$$

If we now consider the partition set A_ℓ for some $\ell > 1$, we have that $\mu(A_\ell \cap T^{-n} A_\ell) = 0$ for $0 < n < \ell$ and a computation similar to that in the proof of Proposition 7.6 leads to the following identity (valid for z the vicinity of 1, $z \neq 1$):

$$\sum_{n\geq\ell} z^n\, \mu(A_\ell \cap T^{-n}A_\ell) = \nu\left(\mathcal{M}_{z,\ell}\,\mathcal{P}_z\, h_\ell\right)\cdot Q(z) + \nu(\mathcal{M}_{z,\ell}\,(1-\mathcal{N}_z)^{-1}h_\ell) \tag{7.102}$$

where

$$\mathcal{M}_{z,\ell}\, u(\omega) := \sum_{k=\ell}^{\infty} z^k \left(\mathcal{L}_1\mathcal{L}_0^{k-1}u\right)(\omega) \quad \text{and} \quad h_\ell := \mathcal{M}_{1,\ell}\, h. \tag{7.103}$$

Notice that the terms $\nu\left(\mathcal{M}_{z,\ell}\,\mathcal{P}_z\, h_\ell\right)$ and $\nu(\mathcal{M}_{z,\ell}\,(1-\mathcal{N}_z)^{-1}h_\ell)$ are $\mathcal{O}(z^\ell)$ when $z\downarrow 0$ and satisfy

$$\lim_{z\uparrow 1}\nu\left(\mathcal{M}_{z,\ell}\,\mathcal{P}_z\, h_\ell\right) = (\nu(h_\ell))^2 = \left(\sum_{k\geq\ell}\rho(A_k)\right)^2 =$$

$(\mu(A_\ell))^2$, (7.104)and, since $\mathcal{N} h_\ell = (1-\mu(A_\ell))\, h$,

$$\lim_{z\uparrow 1}\nu(\mathcal{M}_{z,\ell}\,(1-\mathcal{N}_z)^{-1}h_\ell) = \mu(A_\ell)(1-\mu(A_\ell)), \tag{7.105}$$

respectively.
On the other hand, a short calculation yields the decompositions

$$\sum_{n\geq\ell} z^n\,\frac{\mu(A_\ell\cap T^{-n}A_\ell)}{(\mu(A_\ell))^2} = z^\ell\cdot A(z) + \frac{1}{(\hat\mu(A_\ell))^2}\,C_\ell(z), \tag{7.106}$$

and, for $d>0$,

$$\sum_{n\geq\ell} z^n \left(\frac{\hat\mu(A_\ell\cap T^{-n}A_\ell)-(\hat\mu(A_\ell))^2}{(\hat\mu(A_\ell))^2}\right) = z^\ell\cdot B(z) + \frac{1}{M_1(\hat\mu(A_\ell))^2}\,C_\ell(z), \tag{7.107}$$

where $C_\ell(z)$ is holomorphic in $\mathcal{D}$, extends continuously to $\overline{\mathcal{D}}\setminus\{1\}$, and for z in a neighbourhood of $z=1$ gets the expression

$$\begin{aligned} C_\ell(z) \;&=\; z^\ell\cdot(\hat\mu(A_\ell))^2\,A(z)J(z) + Q(z)\cdot\left[\nu\left(\mathcal{M}_{z,\ell}\,\mathcal{P}_z\,h_\ell\right) - z^\ell\,(\mu(A_\ell))^2\right] \\ &\quad+\; \nu(\mathcal{M}_{z,\ell}\,(1-\mathcal{N}_z)^{-1}h_\ell)\,. \end{aligned} \tag{7.108}$$

Therefore, the same reasoning leading to Theorem 7.3 can be applied here to give

$$\frac{\mu(A_\ell\cap T^{-n}A_\ell)}{(\mu(A_\ell))^2} \sim u_{n-\ell}\sim u_n, \tag{7.109}$$

and, for $d>0$,

$$\frac{\hat\mu(A_\ell\cap T^{-n}A_\ell)-(\hat\mu(A_\ell))^2}{(\hat\mu(A_\ell))^2} \;\sim\; v_{n-\ell}\sim v_n, \tag{7.110}$$

where the last asymptotic equivalences in both expressions hold for each fixed $\ell\in\mathbb{Z}_+$. It is now an easy matter to realize that for any given $\ell\geq 1$ and any Borel set $E\subseteq A_\ell$ with $\mu(E)>0$ one has

$$\sum_{n\geq\ell} z^n\,\frac{\mu(E\cap T^{-n}E)}{(\mu(E))^2} = z^\ell\cdot A(z) + \frac{1}{(\hat\mu(E))^2}\,C_E(z), \tag{7.111}$$

and, for $d > 0$,

$$\sum_{n\geq\ell} z^n \left(\frac{\hat\mu(E\cap T^{-n}E) - (\hat\mu(E))^2}{(\hat\mu(E))^2}\right) = z^\ell \cdot B(z) + \frac{1}{M_1\,(\hat\mu(E))^2}\,C_E(z), \tag{7.112}$$

where the expression of $C_E(z)$ in a neighbourhood of $z = 1$ is as in (7.108) with $\mu(A_\ell)$ replaced by $\mu(E)$ and the quantities $\mathcal{M}_{z,\ell}$, h_ℓ replaced by $\mathcal{M}_{z,E}$, h_E given by

$$\mathcal{M}_{z,E}\, u(\omega) := \sum_{k=\ell}^{\infty} z^k \, \left(\mathcal{L}_1\mathcal{L}_0^{k-1}(u\cdot 1_{T^{-(k-\ell)}E\cap A_k})\right)(\omega) \quad \text{and} \quad h_E := \mathcal{M}_{1,E}\, h. \tag{7.113}$$

Moreover, from these relations we see that the limits corresponding to (7.104) and (7.105) hold with $\mu(A_\ell)$ replaced by $\mu(E)$. The uniform distortion property (4.42) now allows us to repeat exactly the same reasoning as above to obtain, for any fixed $\ell \in \mathbb{Z}_+$ and $E \subseteq A_\ell$ with $\mu(E) > 0$,

$$\frac{\mu(E\cap T^{-n}E)}{(\mu(E))^2} \sim u_{n-\ell} \sim u_n, \tag{7.114}$$

and, for $d > 0$,

$$\frac{\hat\mu(E\cap T^{-n}E) - (\hat\mu(E))^2}{(\hat\mu(E))^2} \;\sim\; v_{n-\ell} \sim v_n. \tag{7.115}$$

In an entirely analogous way one shows that (7.114) and (7.115) hold true for $E \subseteq \cup_{\ell\in J} A_\ell$ where $J \subset \mathbb{Z}_+$ is any given finite set.
Let $\mathcal{B}(\Omega)$ be the Borel σ-algebra on Ω. Given $E \subset \mathcal{B}(\Omega)$, for $-1 < d \leq 0$ we define the *scaling rate* $\sigma_n(E)$ of E as

$$\sigma_n(E) := \frac{\mu(E\cap T^{-n}E)}{(\mu(E))^2}, \tag{7.116}$$

and, for $d > 0$, the *mixing rate* $\mu_n(E)$ of E as

$$\mu_n(E) := \frac{\hat\mu(E\cap T^{-n}E) - (\hat\mu(E))^2}{(\hat\mu(E))^2}. \tag{7.117}$$

This quantities are not uniform in $E \subset \mathcal{B}(\Omega)$. As already noted, the last asymptotic equivalences in (7.114) and (7.115) follow for each fixed $\ell \in \mathbb{Z}_+$, but not uniformly in ℓ. To recover uniformity, we consider the set $D_N = \{\omega \in \Omega : \tau(\omega) > N\} = \cup_{\ell>N} A_\ell$ and define

$$B_+ := \cup_N \{E \in \mathcal{B}(\Omega) :\, \mu(E) > 0,\, E \subseteq \Omega\setminus D_N\}. \tag{7.118}$$

An easy consequence of the above discussion is the following

Lemma 7.2 *Let $E, F \in B_+$. If $-1 < d \leq 0$ then $\sigma_n(E) \sim \sigma_n(F)$. If instead $d > 0$ then $\mu_n(E) \sim \mu_n(F)$.*

Therefore, one may give the following definition,

Definition 7.2 *For $-1 < d \leq 0$ the scaling rate $\sigma_n(T)$ of (Ω, T, μ) is the rate of asymptotic decay of the sequences $\{\sigma_n(E)\}$, with $E \in B_+$. Similarly, for $d > 0$, the mixing rate $\mu_n(T)$ of (Ω, T, μ) is the rate of asymptotic decay of the sequences $\{\mu_n(E)\}$, with $E \in B_+$.*

We summarize the previous findings in the following

Theorem 7.4 *For $-1 < d \leq 0$ we have $\sigma_n(T) \approx n^d$, wheras for $d > 0$ we have $\mu_n(T) = \hat{\mu}(\tau > n) \approx n^{-d}$.*

Remark 8 Let $E \subset \mathcal{B}$, with $0 < m(E) < 1$, and $E^c = \Omega \setminus E$. It is an easy observation that (see [Is1])

$$\hat{\mu}(E \cap T^{-n}E) - (\hat{\mu}(E))^2 = \hat{\mu}(E^c \cap T^{-n}E^c) - (\hat{\mu}(E^c))^2.$$

Now assume that $E \subset B_+$ so that $E^c \not\subset B_+$. The above identity and Theorem 7.4 then imply that

$$\mu_n(E^c) \sim \left(\frac{\hat{\mu}(E)}{\hat{\mu}(E^c)}\right)^2 \hat{\mu}(\tau > n), \quad E \subset B_+, \quad 0 < m(E) < 1.$$

This can be used, for instance, to evaluate the mixing rate of the set $D_N = \cup_{\ell > N} A_\ell = \{\tau > N\}$, for any $N \in \mathbb{Z}_+$.

Remark 9 When $M_1 = \infty$ the scaling rate $\sigma_n(T)$ satisfies

$$\sum_{k=1}^{n} \sigma_k(T) \cdot \sum_{k=1}^{n} \mu(\tau = k) \sim n \quad \text{as} \quad n \to \infty. \tag{7.119}$$

This establishes a (asymptotic) relation between the scaling rate and the behaviour of the partial sums $\sum_{k=1}^{n} \mu(\tau = k)$ which, in turn, is related to quantities such as the *wandering rate* and the *return sequence* that naturally arise in the context of infinite ergodic theory and for which we refer to [Aa] (see also [Is1]).

Remark 10 The part for $d > 0$ of Theorem 7.4 was proved in [Is1] for the Markov chain of Example 1 (see also [FL]). We point out that it gives the (asymptotically) exact rate of mixing for (Ω, T, μ), not just a bound for it, and can be viewed as a statement about the decay of correlations for test functions as simple as indicators of sets in B_+. This makes the mixing rate (as defined above) determined by nothing else than the distribution of return times: $\hat{\mu}(\tau > n)$. On the other hand, when dealing with correlation functions of a broader class of observables, one expects a richer behaviour depending also of the smoothness properties of the functions involved along with their behaviour in the vicinity of the fixed point $\{0^\infty\}$. In particular one may obtain faster decays.

8 Weak Bernoulli and polynomial cluster properties

In this Section we shall consider only the ergodic case $d > 0$ and obtain some further properties of the probability measure $\hat{\mu}$. To this end we first observe that the reasoning of the previous Section can be easily extended to study the behaviour of the quantity $\mu(A_\ell \cap T^{-n} A_k)$, with $k \neq \ell$, yielding (notation as in the previous Section):

$$\sum_{n \geq \ell} z^n \, \mu(A_\ell \cap T^{-n} A_k) = z^{\ell-k} \left[\nu \left(\mathcal{M}_{z,k} \, \mathcal{P}_z \, h_\ell \right) \cdot Q(z) + \nu(\mathcal{M}_{z,k} \, (1 - \mathcal{N}_z)^{-1} h_\ell) \right] \quad (8.120)$$

and therefore

$$\sum_{n \geq \ell} z^n \left(\frac{\hat{\mu}(A_\ell \cap T^{-n} A_k) - \hat{\mu}(A_k)\hat{\mu}(A_\ell)}{\hat{\mu}(A_k)\hat{\mu}(A_\ell)} \right) = z^\ell \cdot B(z) + \frac{1}{M_1 \hat{\mu}(A_k)\hat{\mu}(A_\ell)} \, C_{k,\ell}(z), \quad (8.121)$$

where $C_{k,\ell}(z)$ is holomorphic in $\mathcal{D}$, extends continuously to $\overline{\mathcal{D}} \setminus \{1\}$, and for z in a neighbourhood of $z = 1$ writes

$$\begin{aligned} C_{k,\ell}(z) \;\; &= \;\; z^\ell \cdot \hat{\mu}(A_k)\hat{\mu}(A_\ell) \, A(z) J(z) + Q(z) \cdot \left[z^{\ell-k} \nu \left(\mathcal{M}_{z,k} \, \mathcal{P}_z \, h_\ell \right) - z^\ell \, \mu(A_k)\mu(A_\ell) \right] \\ &\quad + \;\; z^{\ell-k} \cdot \nu(\mathcal{M}_{z,k} \, (1 - \mathcal{N}_z)^{-1} h_\ell) \; . \end{aligned} \quad (8.122)$$

The same reasoning as above then yields

$$\frac{\hat{\mu}(A_\ell \cap T^{-n} A_k) - \hat{\mu}(A_k)\hat{\mu}(A_\ell)}{\hat{\mu}(A_k)\hat{\mu}(A_\ell)} \;\sim\; v_{n-\ell} \; . \quad (8.123)$$

Now recall that the partition sets A_ℓ are particular cylinder sets: $A_\ell = [0^{\ell-1}1]$. On the other hand, given an arbitrary ℓ-cylinder set $E = [\omega_0, \cdots, \omega_{\ell-1}]$ with $E \neq D_\ell \equiv [0, \cdots, 0] = \{\tau > \ell\}$, then either $E = A_\ell$ or $E \subset A_i$ for some $i < \ell$. Thus, a straightforward consequence of the previous discussion is that we can find a positive constant C such that for any pair of ℓ-cylinders E, F both $\neq D_\ell$ we have

$$|\hat{\mu}(E \cap T^{-n} F) - \hat{\mu}(E)\hat{\mu}(F)| \leq C \, \hat{\mu}(E)\hat{\mu}(F) \, v_{n-\ell} \, . \quad (8.124)$$

Moreover, noting that for all $F \subset B_+$ (see also remark 8)

$$\begin{aligned} |\hat{\mu}(D_\ell \cap T^{-n} F) - \hat{\mu}(D_\ell)\hat{\mu}(F)| \;\; &= \;\; |\hat{\mu}(D_\ell^c \cap T^{-n} F) - \hat{\mu}(D_\ell^c)\hat{\mu}(F)| \\ &\leq \;\; C \, \hat{\mu}(D_\ell^c)\hat{\mu}(F) \, v_{n-\ell}, \end{aligned} \quad (8.125)$$

(the same inequality holds with D_ℓ and F interchanged) and summing over all ℓ-cylinder sets E, F we get (recall that $\hat{\mu}(\Omega) = 1$)

$$\sum_{E,F} |\hat{\mu}(E \cap T^{-n} F) - \hat{\mu}(E)\hat{\mu}(F)| \leq C(1 + 2 \, \hat{\mu}(\{\tau \leq \ell\})) \cdot v_{n-\ell} \quad (8.126)$$

Therefore, since $v_n \approx n^{-d} \to 0$, we obtain (see [Bo] for definitions)

Theorem 8.5 *The partition $\{[0],[1]\}$ is weak-Bernoulli (and hence Bernoulli) for T and $\hat{\mu}$.*

We now proceed as in [Bo] to carry the polynomial convergence (8.124) over to functions of polynomial variation. Given $a > 0$ let $\mathcal{H}_a(\Omega)$ be the family of $f \in C(\Omega)$ with $\mathrm{var}_n f \le C n^{-a}$. The space $\mathcal{H}_a$ becomes a Banach space under the norm

$$\|f\|_a = |f|_\infty + \sup_{n>0} \{n^a \cdot \mathrm{var}_n f\}\,. \tag{8.127}$$

We have the following polynomial cluster property:

Theorem 8.6 *For $d > 0$ and fixed $a \ge d$ there is a constant $D > 0$ such that*

$$|\hat{\mu}(f \cdot g \circ T^n) - \hat{\mu}(f)\hat{\mu}(g)| \le D \|f\|_a \|g\|_a n^{-(d-\epsilon)}$$

for all $n > 0$, $\epsilon > 0$ and $f, g \in \mathcal{H}_a$.

Proof. Given a pair $f, g \in \mathcal{H}_a(\Omega)$ we let f_ℓ and g_ℓ be the conditional expectations of f and g wrt the σ-algebra generated by ℓ-cylinder sets, i.e.

$$f_\ell = \sum_E f_E \, 1_E \quad \text{with} \quad f_E = \frac{1}{\hat{\mu}(E)} \int_E f(\omega) \hat{\mu}(d\omega) \tag{8.128}$$

and similar expression for g. We have $\hat{\mu}(f_\ell) = \hat{\mu}(f)$ and $|f - f_\ell| \le \|f\|_a \cdot n^{-a}$. A routine estimate (see [Bo], p.39) then gives

$$|\hat{\mu}(f \cdot g \circ T^n) - \hat{\mu}(f)\hat{\mu}(g)| \le |\hat{\mu}(f_\ell \cdot g_\ell \circ T^n) - \hat{\mu}(f_\ell)\hat{\mu}(g_\ell)| + 2n^{-a}\|f\|_a\|g\|_a. \tag{8.129}$$

Moreover, by (8.126), we have

$$\begin{aligned}
|\hat{\mu}(f_\ell \cdot g_\ell \circ T^n) - \hat{\mu}(f_\ell)\hat{\mu}(g_\ell)| &\le \sum_{E,F} |f_E|\,|g_F|\,|\hat{\mu}(E \cap T^{-n}F) - \hat{\mu}(E)\hat{\mu}(F)| \\
&\le C\,|f|_\infty\,|g|_\infty\,(1 + 2\,\hat{\mu}(\{\tau \le \ell\})) \cdot v_{n-\ell} \\
&\le 3\,C\,\|f\|_a\|g\|_a\,v_{n-\ell}\,.
\end{aligned}$$

The assertion now follows putting together the last two inequalities with the choice $\ell = [n/2]$. $\diamondsuit$

9 Zeta functions

We consider the dynamical zeta functions $\zeta(\varphi, z)$ and $\zeta(\psi, w)$ associated to the pairs (T, φ) and (S, ψ), respectively (here, as in Section 3, S denotes the induced map T^τ when acting on the symbol space Σ) and defined by the following formal series (see [Ru1], [Ru2], [Ba2]):

$$\zeta(\varphi, z) = \exp \sum_{n=1}^\infty \frac{z^n}{n} Q_n \quad \text{and} \quad \zeta(\psi, w) = \exp \sum_{n=1}^\infty \frac{w^n}{n} Z_n,$$

where the 'partition functions' Q_n and Z_n are given by

$$Q_n = \sum_{\omega = T^n\omega} \prod_{k=0}^{n-1} \varphi(T^k\omega) \quad \text{and} \quad Z_n = \sum_{\sigma = S^n\sigma} \prod_{k=0}^{n-1} \psi(S^k\sigma).$$

Remark 11 Notice that the correspondence $\sigma_j(\omega)$ defined in (2.5) may associate to a periodic sequence an eventually periodic one. More precisely, let $\omega \in \Omega_0$ be a periodic sequence of period n for the shift T. We write it in the form $\omega = (\overline{\omega_0\omega_1\ldots\omega_{n-1}})$. Now, if $\omega_0 = 0$, it may happen that, for some $k \geq 1$,

$$\omega_0\omega_1\ldots\omega_{n-1} \;=\; \underbrace{00\ldots01}_{l_0\geq 2}\,\underbrace{1\ldots1}_{r_1\geq 0}\,\underbrace{00\ldots01}_{l_1\geq 2}\;\cdots\;\underbrace{1\ldots1}_{r_k\geq 0}\,\underbrace{00\ldots0}_{l_k\geq 1}$$

and one would find the eventually periodic sequence $\sigma(\omega) = (\sigma_0\overline{\sigma_1\sigma_2\ldots\sigma_m})$ where

$$\sigma_0 = l_0 \quad \text{and} \quad \sigma_1\sigma_2\ldots\sigma_m \;=\; \underbrace{1\ldots1}_{r_1\geq 0}\,l_1\;\cdots\;\underbrace{1\ldots1}_{r_k\geq 0}\,(l_k+l_0)$$

whose ultimate period is $m = k + r_1 + \ldots + r_k$ and which satisfies $n = \sigma_1 + \ldots + \sigma_m$. Thus, in the latter case, in order to obtain a correspondence between periodic sequences it is necessary to apply the aformentioned rule to some iterate of the original sequence.

Let us now examine how $\zeta(\varphi, z)$ and $\zeta(\psi, w)$ are related to one another. To this end we observe that if $\omega \in \Omega_0$ is a periodic sequence of period n for the shift T and $\sigma(\omega) \in \Sigma$ is the periodic sequence of period $m = \sum_{k=0}^{n-1} 1_{A_1}(T^k\omega)$ corresponding to ω or to some iterate of it (see the above remark), then we have $n = \sum_{j=0}^{m-1} \sigma_j$ and

$$\prod_{k=0}^{n-1} \varphi(T^k\omega) = \prod_{k=0}^{m-1} \psi(S^k\sigma).$$

Using this facts we write Q_n as follows:

$$Q_n = 1 + \sum_{m=1}^{n} \frac{n}{m} \sum_{\sigma=S^m\sigma} \prod_{k=0}^{m-1} \psi(S^k\sigma)$$

where the 1 comes from the fixed point 0^∞. The second sum ranges over the $\binom{n-1}{m-1}$ ways to write the integer n as a sum of m positive integers, counting all permutations. Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{z^n}{n} Q_n &= \log(\frac{1}{1-z}) + \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{1}{m} \sum_{\sigma=S^m\sigma} z^n \prod_{k=0}^{m-1} \psi(S^k\sigma) \\ &= \log(\frac{1}{1-z}) + \sum_{\ell=1}^{\infty} \frac{1}{\ell} \sum_{\sigma=S^\ell\sigma} z^{\sum_{j=0}^{\ell-1}\sigma_j} \prod_{k=0}^{\ell-1} \psi(S^k\sigma)\end{aligned}$$

Putting together these observations we have the following result, which can be viewed as the counterpart of Proposition 4.2:

Proposition 9.7 *Consider the 'grand partition function' $\Xi_\ell(z)$ defined by*

$$\Xi_\ell(z) := \sum_{\sigma=S^\ell\sigma} z^{\sum_{j=0}^{\ell-1}\sigma_j} \prod_{k=0}^{\ell-1} \psi(S^k\sigma)$$

and the two-variable zeta function given by

$$\zeta_2(w,z) = \exp \sum_{\ell=1}^{\infty} \frac{w^\ell}{\ell} \Xi_\ell(z).$$

$$Then we have$$

$\zeta_2(1,z) = (1-z)\,\zeta(\varphi,z) \quad and \quad \zeta_2(w,1) = \zeta(\psi,w) wherever the series expansions converge absolutely.$

We now study the grand partition function $\Xi_\ell(z)$. We first rewrite it in the form

$$\begin{aligned} \Xi_\ell(z) &= \sum_{\sigma=S^\ell\sigma} z^{\sum_{j=0}^{\ell-1}\sigma_j} \prod_{k=0}^{\ell-1} \psi(S^k\sigma) \\ &= \sum_{n=0}^{\infty} z^{\ell+n} \sum_{\substack{\sigma=S^\ell\sigma \\ \sigma_0+\cdots+\sigma_{\ell-1}=\ell+n}} \prod_{k=0}^{\ell-1} \psi(S^k\sigma) \end{aligned}$$

Notice that for $n=0$ the last sum yields only one term corresponding to the fixed point 1^∞ of S (and of T). More generally, the sum over periodic points yields $\binom{n+\ell-1}{\ell-1} = \binom{n+\ell-1}{n}$ terms, corresponding to the number of ways of distributing n identical objects into ℓ distinct boxes.

Lemma 9.3 *The sequence $\{\,\Xi_\ell(z)\,\}$ is uniformly bounded in $\overline{\mathcal{D}}$. In particular, for each $\ell>0$, the power series expansion of $\Xi_\ell(z)$ converges absolutely in $\overline{\mathcal{D}}$.*

Proof. Recalling that $\sum_n \mathrm{Var}_n W \le C_3\theta/(1-\theta) < \infty$ we have, for any real z such that $0 \le z \le 1$,

$$C^{-1}\,\Lambda_\ell(z) \le \Xi_\ell(z) \le C\,\Lambda_\ell(z)$$

where $\Lambda_\ell(z)$ is defined in (3.19) and $C = e^{\frac{C_3\theta}{1-\theta}}$. Moreover, from the previous Section it follows that $\mathcal{M}^\ell 1 \to h$ in the supremum norm. Therefore, since the power series expansion of $\Xi_\ell(z)$ has positive coefficients, we have for all $z \in \overline{\mathcal{D}}$ and all $\ell > 0$,

$$|\Xi_\ell(z)| \le \Xi_\ell(|z|) \le \Xi_\ell(1) \le C\,\Lambda_\ell(1) \le C\,(\|h\|_\infty + o(1)). \qquad \diamondsuit$$

¿From the above result we have that for any $z \in \overline{\mathcal{D}}$ the function $\zeta_2(w,z)$, viewed as function of the variable w, converges absolutely for $|w| < 1/\exp P(|z|)$. To obtain more information we shall establish in the following theorem a correspondence between the analytic properties of $\zeta_2(w,z)$ and the spectral properties of the operator-valued function $\mathcal{M}_z$ studied in Section 3.

Theorem 9.7 *The two-variables zeta function defined in Proposition 9.7 has the following analytic properties:*

1. *for $z \in \overline{\mathcal{D}}$, $1/\zeta_2(w,z)$, considered as a function of the variable w, is holomorphic in the disk of radius $1/\theta \exp P(|z|)$. Its zeroes in this disk, counted with multiplicity, are the inverses of the eigenvalues of $\mathcal{M}_z : \mathcal{F}_\theta \to \mathcal{F}_\theta$ in the corresponding annulus. Moreover, for $0 < z \le 1$, the zero of smallest modulus is simple and located at $1/\exp P(z)$;*

2. *for $w \in \overline{\mathcal{D}}$, $1/\zeta_2(w,z)$, considered as a function of the variable z, is holomorphic in the disk of radius 1. Its zeroes in this disk are located at those values of z such that $\mathcal{M}_z : \mathcal{F}_\theta \to \mathcal{F}_\theta$ has $1/w$ as an eigenvalue.*

Proof. Fixing $n > 0$ we let $\sum_\eta$ be the sum over words η of length n and denote by $\sigma^{(\eta)}$ the periodic concatenation $(\overline{\sigma_0\sigma_1\ldots\sigma_{n-1}})$. Let moreover $1_\eta \in \mathcal{F}_\theta(\Sigma)$ be such that $1_\eta(\sigma) = 1$ if σ begins with the word η, $1_\eta(\sigma) = 0$ otherwise. Then we have the following relation for the grand partition function $\Xi_\ell(z)$:

$$\Xi_\ell(z) = \sum_{\substack{\sigma\in\Sigma \\ S^\ell\sigma=\sigma}} \exp\sum_{j=0}^{\ell-1} W_z(T^j\sigma) = \sum_\eta (\mathcal{M}_z^\ell 1_\eta)(\sigma^{(\eta)}). \tag{9.130}$$

The assertion now follows by putting together (9.130), Theorem 3.1 and a straightforward extension of ([Hay], Theorem 4) to the present situation $\diamondsuit$.

¿From Proposition 4.3 and Proposition 9.7 we then have the following,

Corollary 9.4

1. *$1/\zeta(\psi,w)$ is holomorphic in the disk of radius $1/\theta$. Its zeroes in this disk, counted with multiplicity, are the inverses of the eigenvalues of $\mathcal{M} : \mathcal{F}_\theta \to \mathcal{F}_\theta$ in the annulus $\{\,\lambda : \theta < |\lambda| \le 1\,\}$.*

2. *$1/\zeta(\varphi,z)$ is holomorphic in the disk of radius 1. In this disk, $1/\zeta(\varphi,z) = 0$ if and only if $1/z$ is an eigenvalue of $\mathcal{L} : \mathcal{F}_\theta \to \mathcal{F}_\theta$.*

The above result yields no zeroes of $1/\zeta(\varphi,z)$ but the point $z = 1$. In this case we know

that the eigenvalue 1 of $\mathcal{L}$ is not isolated (i.e. there is no 'gap'). Nevertheless, one may investigate the singular behaviour of $\zeta(\varphi,z)$ when $z \uparrow 1$. To this end, consider again eq. (9.130) and use (3.30) to rewrite it in the following way:

$$\Xi_\ell(z) \;\; = \;\; \lambda_z^\ell \cdot \sum_\eta h_z(\sigma^{(\eta)}) \cdot \nu_z(1_\eta) +$$

$\sum_\eta (\mathcal{N}_z^\ell 1_\eta)(\sigma^{(\eta)})$
$=\lambda_z^\ell + \lambda_z^\ell \cdot \nu_z \left(\sum_\eta h_z(\sigma^{(\eta)}) \cdot 1_\eta - h_z\right) + \sum_\eta (\mathcal{N}_z^\ell 1_\eta)(\sigma^{(\eta)})$
$=:\lambda_z^\ell + R_\ell^{(1)}(z) + R_\ell^{(2)}(z)$.Now, using the fact that $h_z \in \mathcal{F}_\theta$ for $z \in J \cup \{1\}$ and reasoning as in the proof of Proposition 3.1, one gets $|R_\ell^{(1)}(z)| \le C_1\,\gamma^\ell$ and $|R_\ell^{(2)}(z)| \le C_2\,\gamma^\ell$ for $z \in H$ and $\gamma = \max\{\theta, 1-\epsilon\}$ (the notation is as in Section 4). Therefore putting together (9.130), the above observations, Corollary 6.3, Proposition 9.7 and Corollary 9.4, we obtain the following (see [Ga], [Rug], [Is1] for related results):

Theorem 9.8 *For all finite $d > -1$ the zeta function $\zeta(\varphi,z)$ is holomophic in $\mathcal{D}$ and extends continuously to $\overline{\mathcal{D}} \setminus \{1\}$, whereas in $z = 1$ it has non-polar singularity. More specifically, it can be written as*

$$\zeta(\varphi,z) = \frac{A(z)\cdot L(z)}{(1-z)},$$

where $A(z)$ is as in (6.65) and $L(z)$ is continuous on the unit circle $|z| = 1$, with $L(1) \neq 0$.

Remark 12 In the situation considered in Example 1 a straightforward calculation gives

$$\Xi_\ell(z) = \left(\sum_{n=1}^{\infty} z^n p_n\right)^\ell \quad \text{and} \quad \zeta_2(w,z) = \left(1 - w\sum_{n=1}^{\infty} z^n p_n\right)^{-1}.$$

Using Proposition 9.7 we then get for $\zeta(\varphi, z)$ the expression given in Theorem 9.8 with $A(z) = (1 - \sum_{n=1}^{\infty} z^n p_n)^{-1}$ and $L(z) \equiv 1$.

Remark 13 A singularity of non-polar nature in the ζ-function should perhaps be related to the presence of some kind of 'phase transition' in the system (see remark 7 above and also [PS]). This is the case for instance in [Ga] where a direct connection with the model introduced by Fisher [FF] is emphasized.

10 Appendix: interval maps with indifferent fixed points

We shall consider a class of non-uniformly expanding interval maps $F : [0,1] \to [0,1]$ satisfying the following assumptions:

1. there is a number $q \in (0,1)$ such that the restrictions $F|_{(0,q)}$ and $F|_{(q,1)}$ extend to C^1-diffeomorphisms on $I_0 = [0,q]$ and $I_1 = [q,1]$, which are C^2 for $x > 0$, and such that $F(0) = 0$ and $F(I_0) = F(I_1) = [0,1]$;

2. there are numbers $0 < \beta < 1$ and $m \in \mathbb{Z}^+$ such that $(F^\ell)' \geq 1/\beta$ on I_1 for all $\ell \geq m$; whereas $F' > 1$ on $(0,q)$ and $F'(0) = 1$;

3. F has the following asymptotic behaviour when $x \to 0_+$:

$$F(x) = x + r\, x^{1+s}(1 + u(x))$$

   for some constant $r > 0$, exponent $1 + s > 1$ and where $u(x)$ satisfies $u(0) = 0$ and $u'(x) = \mathcal{O}(x^{t-1})$ as $x \to 0_+$ for some $t > 0$.

The partition of $[0,1]$ whose elements are the intervals I_0 and I_1 is a Markov partition for F. Let Ω be as in Section 2. The map $\pi : \Omega \to [0,1]$ defined by

$$\pi(\omega) = x \quad \text{according to} \quad F^j(x) \in I_{\omega_j}, \;\; j \geq 0 \tag{10.131}$$

is a coding map which is a homeomorphism on the residual set of points in $[0,1]$ which are not preimages of 1 with the map F. Moreover $F \circ \pi = \pi \circ T$. Let F_i be the inverse branch of F on I_i, $i = 0,1$. Given $\omega \in \Omega_0$ we set

$$V(\omega) = \log[(F_{\omega_0})'(\pi(\omega_1\omega_2\ldots))]. \tag{10.132}$$

It is then easy to realize that its induced version $W(\omega)$ can be written as

$$W(\omega) = \log[(G_{\sigma_0(\omega)})'(\pi(\sigma_1(\omega)\sigma_2(\omega)\ldots))], \tag{10.133}$$

where $\sigma_0(\omega)\sigma_1(\omega)\ldots = \iota(\omega)$ (see Section 3) and G_i, $i \geq 1$, are the (countably many) inverse branches of the induced map

$$x \to G(x) = F^{\tau(x)}(x), \tag{10.134}$$

with

$$\tau(x) = 1 + \min\{n \geq 0 \;:\; F^n(x) \in A_1\;\}. \tag{10.135}$$

For notational simplicity' sake we shall denote with the same symbol the first passage function (2.1) and its lift (10.135) with π, as well as the level sets

$$A_n = \{x \in [0,1] : \tau(x) = n\} = [F_0^n(1), F_0^{n-1}(1)]\,. \tag{10.136}$$

Notice that G (once suitably extended to the set $\{F_0^n(1)\}_{n\geq 0}$) maps A_n onto $[0,1]$, for all $n \geq 1$.
It turns out that the overall statistical behaviour of the map F depends on the way the lengths $|A_n|$ of the levelsets A_n vanish as $n \to \infty$.

Lemma 10.4 *Under the hypotheses (1)-(3) on the map F we have for $n \to \infty$*

$$|A_n| \sim r(rsn)^{-1-1/s}\,.$$

Proof. The last property of T gives for the inverse function:

$$F_0(x) = x - rx^{1+s}(1 + v(x))$$

where $v(x)$ is such that $v(0) = 0$ and $v'(x) = \mathcal{O}(x^{t-1})$ as $x \to 0_+$. We write this expression in a more manageable form, that is:

$$F_0(x) = \Big(x^{-s} + rs(1 + \tilde{v}(x))\Big)^{-1/s}$$

where $\tilde{v}(x)$ is another function such that $\tilde{v}(0) = 0$ and $v'(x) - \tilde{v}'(x) = \mathcal{O}(x^{u-1})$ where $u = \min\{s, t\}$. It is then easy to check that

$$F_0^n(x) = \left(x^{-s} + rsn\Big(1 + \frac{1}{n}\sum_{l=0}^{n-1}\tilde{v}(F_0^l(x))\Big)\right)^{-1/s}$$

whence

$$F_0^n(1) = (rsn)^{-1/s}\,(1 + o(1)))^{-1/s} \qquad (10.137)$$

and the assertion follows. $\diamondsuit$

We now list some properties of the induced map G which are relevant for our discussion.

Proposition 10.8

1. *$G_{|A_n}$ is a C^2-diffeomorphism of A_n onto $[0,1]$, for all $n \geq 1$;*

2. *$\exists m \in \mathbb{Z}^+$ so that*
$$\inf_{\substack{x\in\mathcal{A}_n \\ n\geq 1}} |(G^m)'(x)| = 1/\beta > 1;$$

3. $$\inf_{\substack{x,y,z\in\mathcal{A}_n \\ n\geq 1}} \left|\frac{G''(x)}{G'(y)G'(z)}\right| = K < \infty.$$

Proof. Statements 1) and 2) are immediate consequences of the definition. To show 3) we first observe that the chain rule yields

$$\frac{G''(x)}{(G')^2(x)} = \sum_{k=0}^{\tau(x)-1}\frac{F''(F^k(x))}{(F')^2(F^k(x))}\cdot\frac{1}{\prod_{j=k+1}^{\tau(x)-1}F'(F^j(x))}.$$

On the other hand, the properties of F imply that

$$\frac{|F''(x)|}{|(F')^2(x)|} \asymp x^{s-1}$$

Moreover, if ξ_n is any point in A_n, then $\xi_n^{s-1} \asymp n^{-1+\frac{1}{s}}$ and $\prod_{j=0}^{n-1} |F'(F^j(\xi_n))| \equiv |G'(\xi_n)| \asymp n^{1+\frac{1}{s}}$. Putting together these remarks we get

$$\left| \frac{G''(\xi_n)}{(G')^2(\xi_n)} \right| \leq C_1 \sum_{k=0}^{n-1} (n-k)^{-2} \leq C_2 \sum_{k=1}^{\infty} k^{-2} \leq C_3,$$

and the assertion follows by noting that $G'(\xi_n) \asymp G'(\eta_n)$ for any choice of $\xi_n, \eta_n \in A_n$ and any $n \geq 1$. $\diamondsuit$

The above properties yield a uniform bound for the buildup of non-linearity in the induction process.

Corollary 10.5 *Let $x, y \in [0,1]$ be such that $G^j(x)$ and $G^j(y)$ belong to the same partition set A_{k_j}, for $0 \leq j \leq n$ and some $n \geq 1$. Then there are two constants $C > 0$ and $\alpha < 1$ such that*

$$\left| \log \frac{G'(x)}{G'(y)} \right| \leq C\, \alpha^n \,.$$

Proof. Taking $x, y \in A_{k_0}$, let $\eta \in A_{k_0}$ be such that $|G'(\eta)| = |A_{k_0}|^{-1}$. Then, using Proposition 10.8(3), we have

$$\begin{aligned} \left| \log \frac{G'(x)}{G'(y)} \right| &= \left| \frac{G''(\xi)}{G'(\xi)} \right| \cdot |x-y| \quad \text{for some} \quad \xi \in [x,y] \subseteq A_{k_0} \\ &= \left| \frac{G''(\xi)}{G'(\xi) G'(\eta)} \right| \cdot \frac{|x-y|}{|A_{k_0}|} \leq K \, \frac{|x-y|}{|A_{k_0}|} \cdot \end{aligned}$$

Now, using Proposition 10.8(2), we can find a constant $C_4 > 0$ such that, under the above hypotheses, $|x-y| \leq C_4 \, |A_{k_0}| \, \alpha^n$ with $\alpha = \beta^{\frac{1}{m}} < 1$. $\diamondsuit$

These results and (10.133) imply that the potential V defined in (10.132) satisfies the properties listed in Section 2 for every $\theta \geq \alpha$. We can then apply the whole subsequent theory. In particular, there is an unique absolutely continuous probability measure $\rho(dx) = h(x)\, dx$ which is invariant for the dynamical system $([0,1], G)$ and whose density h is Lipschitz continuous and satisfies $h \asymp 1$ (see also [Wal2]). In turn, this and Lemma 10.4 imply (see, e.g., [CI2], Lemma 2.4) that as $n \to \infty$,

$$\rho(A_n) \sim \; h(0)\, |A_n| \sim h(0)\, r(rsn)^{-1-1/s} \,. \tag{10.138}$$

Moreover, the σ-finite absolutely continuous measure $\mu(dx) = e(x)\, dx$ with

$$e = \sum_{n=0}^{\infty} h \circ F_0^n \cdot (F_0^n)' \tag{10.139}$$

is invariant for $([0,1], F)$. It can be shown (see [Th]) that $e(x) \asymp x^{-s}$. Clearly, both e and h are the lifts with the map π of the corresponding quantities considered e.g. in Section 4. ¿From (10.138) and Definition 4.1 it follows that $([0,1], F, \mu)$ has ergodic degree

$$d = \frac{1}{s} - 1. \tag{10.140}$$

However, the asymptotic equivalence expressed by (10.138) is somewhat stronger than the mere knowledge of the ergodic degree. Indeed, (10.138) implies that all symbols $\approx$ in Proposition 6.5 can be turned into true asymptotic equivalences $\sim$. In particular, application of a Tauberian theorem for power series to the functions $A(z)$ and $B(z)$ is now more informative and gives

$$a_n \sim \begin{cases} c_1\, n^{-1+1/s} & \text{if } s > 1, \\ c_2\, (\log n)^{-1} & \text{if } s = 1, \end{cases} \tag{10.141}$$

and

$$b_n \sim c_3\, n^{1-1/s} \;\text{ if }\; 0 < s < 1, \tag{10.142}$$

for some positive constants c_1, c_2, c_3. According to Theorem 9.8 the behaviour of a_n given above can be used to establish (again by a Tauberian theorem for power series) the precise asymptotic behaviour when $z \uparrow 1$ of the dynamical zeta function for the map F. However we shall leave this easy task to the interested reader and end this Section by discussing some consequences of the construction outlined in the main part of the paper on scaling and mixing properties of $([0,1], F, \mu)$. To this end we set

$$B_+ := \cup_\epsilon \{ E \in \mathcal{B}([0,1]) : \, m(E) > 0, \; E \subseteq [0,1] \setminus (0, \epsilon) \, \} \, . \tag{10.143}$$

The following sharpening of Theorem 7.4 is then a straightforward consequence of application of the arguments of Section 7 to this situation.

Theorem 10.9 *For all $E \subset B_+$ we have*

- $\mu(E \cap T^{-n}E)/(\mu(E))^2 \sim c_1\, n^{-1+1/s}$ *if $s > 1$;*
- $\mu(E \cap T^{-n}E)/(\mu(E))^2 \sim c_2\, (\log n)^{-1}$ *if $s = 1$;*
- $[\hat\mu(E \cap T^{-n}E) - (\hat\mu(E))^2]/(\hat\mu(E))^2 \sim c_3\, n^{1-1/s}$ *if $0 < s < 1$;*

where, for $s < 1$, we have set $\hat\mu = \mu/\mu([0,1])$.

We finally concentrate on the finite measure case $s < 1$. Given a function $f : [0,1] \to I\!R$ which is Hölder continuous with exponent $\gamma > 0$, that is: $|f(x) - f(y)| \le K|x-y|^\gamma$, let us denote by $\tilde f$ its projection $f \circ \pi : \Omega \to I\!R$. It is easy to realize that, for n large,

$$\mathrm{var}_n \tilde f = \sup_{x \in A_{n+1}} |f(x) - f(0)| \le 2K(rsn)^{-\gamma/s} \tag{10.144}$$

where the last inequality comes from the Hölder property along with (10.137). Therefore, in order to have $\tilde f \in \mathcal{H}_a$ with $a \ge d$ it is sufficient that $\gamma \ge 1 - s$. Thus, using Theorem 8.6 and taking into account that due to Theorem 10.9 the bounds for the weak-Bernoulli property are truly polynomial (no slowly varying functions being involved) we have the following (see [Yo] and references therein for related results):

Theorem 10.10 *Let $0 < s < 1$ and $f, g : [0,1] \to I\!R$ be Hölder continuous with exponent $\gamma \ge 1 - s$. Then*

$$|\hat\mu(f \cdot g \circ F^n) - \hat\mu(f)\hat\mu(g)| \; = \; \mathcal{O}(n^{1-1/s}) \, .$$

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