On the PCT–Theorem in the Theory of Local Observables

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Dedicated to Sergio Doplicher and John Roberts on the occasion of their 60th birthday

Abstract

We review the PCT-theorem and problems connected with its demonstration. We add a new proof of the PCT-theorem in the theory of local observables which is similar to that one of Jost in Wightman quantum field theory. We also look at consequences in case the PCT-symmetry is given on the algebraic level. At the end we present some examples which answer general questions and throw some light on open problems.

1 Introduction

Before starting to discuss the different aspects of the PCT-theorem we collect some terminology and notations that we shall be using. This is necessary because the notions in the literature are not unique.

1.1 On quantum field theory

The term Lagrangian or Wightman field theory means quantum field theory with "point like" localized fields. More precisely, the fields are operator

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valued distributions and the field operators are supposed to transform covariantly under a continuous unitary representation of the connected part $\mathcal{P}_+^\uparrow$ of the Poincaré group, or at least of the subgroup of space-time translations. The representation of the translations is required to fulfill the spectrum condition and the representation space shall contain a vacuum vector (mostly assumed to be unique). It is also assumed that this vacuum vector is cyclic for the algebra generated by the field operators. For space-like separation of the arguments the field operators shall either commute or anti-commute. The Bose-fields have to commute with themselves and all other fields. Fermi-fields anti-commute with Fermi-fields. The Bose-fields represent particles with integer spin and the Fermi-fields such with half-integer spin. Therefore, the phrase "spin and statistics" means in this context the same as "spin and commutation relations". Para-fields, introduced by O.W. Greenberg [26], will not be considered. Para-statistics are usually only discussed in the frame of quantum field theory of local observables, where a reduction to ordinary Bose- or Fermi commutation relations has been achieved by Doplicher and Roberts in [22].

Quantum field theory of local observables (QFTLO) in the sense of Araki, Haag and Kastler [31] is concerned with C^*-algebras $\mathcal{A}(O)$, associated with bounded open regions $O \subset \mathbb{R}^d$. These algebras shall fulfill isotony, i.e., $O_1 \subset O_2$ implies $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$, and locality, i.e., if O_1 and O_2 are space-like separated, then the algebras $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ commute element-wise. Space-like separation is here defined in terms of the Minkowski metric on $\mathbb{R}^d$. For unbounded domains G the algebra $\mathcal{A}(G)$ is defined as the C^*-inductive limit of $\{\mathcal{A}(O) : \ O \subset G\}$. Usually it is assumed that the net $\{\mathcal{A}(O)\}$ is covariant under a representation of the proper, orthochronous Poincaré group $\mathcal{P}_+^\uparrow$, or at least some subgroup $\mathcal{G} \subset \mathcal{P}_+^\uparrow$, by automorphism of $\mathcal{A}(\mathbb{R}^d)$, i.e., to every $g \in \mathcal{G}$ there is assigned an automorphism α_g such that $\alpha_{g_1}\alpha_{g_2} = \alpha_{g_1 g_2}$, and for all O

$$\alpha_g \mathcal{A}(O) = \mathcal{A}(gO). \tag{1.1}$$

A QFTLO may often be defined in terms of Wightman fields by taking as local algebras $\mathcal{A}(O)$ the von Neumann algebras generated by the polar decompositions of the smeared field operators with test functions supported in O. The local commutativity of these algebras is not a consequence of the locality of the field alone, but several conditions which guarantee this are known. See [10] for a review.

1.2 Representations.

A representation π of $\mathcal{A}(\mathbb{R}^d)$ on a Hilbert space $\mathcal{H}_\pi$ will always mean a non-degenerate representation. We shall denote the von Neumann algebra $\pi(\mathcal{A}(O))''$ by $\mathcal{M}_\pi(O)$, or simply by $\mathcal{M}(O)$ if π is a vacuum representation (see below). We say that *weak additivity* holds in the representations considered, i.e., the von Neumann algebra generated by $\cup\{\mathcal{M}_\pi(O+x) : x \in \mathbb{R}^d\}$ is equal $\mathcal{M}_\pi(\mathbb{R}^d)$ for each bounded, open O, independent of its size. Usually it is required only for vacuum representations. When the QFTLO is covariant under $\mathcal{P}_+^\uparrow$ or some subgroup $\mathcal{G} \subset \mathcal{P}_+^\uparrow$ the automorphisms α_g are *implemented* in the representation π if there is a continuous unitary representation U of $\mathcal{G}$ on $\mathcal{H}_\pi$ such that

$$U(g)\pi(A)U^*(g) = \pi(\alpha_g A), \qquad A \in \mathcal{A}(\mathbb{R}^d),\ g \in \mathcal{G}. \tag{1.2}$$

By (1.1), it is clear that

$$U(g)\mathcal{M}_\pi(O)U^*(g) = \mathcal{M}_\pi(gO). \tag{1.3}$$

For a classification of the representations π one often takes for $\mathcal{G}$ the translation group $\mathbb{R}^d$ or the subgroup of time translations. If a topological group $\mathcal{G}$ acts on a C^*-algebra $\mathcal{A}$ by automorphisms α_g one speaks of a C^* *dynamical system*. If a representation π of $\mathcal{A}$ is given, then it is of interest to know whether or not the group action is unitarily implemented in the representation. The answer is known up to multiplicity problems implying that eventually one might have to to change the multiplicity of the representation. Representations which differ at most in their multiplicity are called *quasi-equivalent*. Such representations have the same *folium* of states, i.e., states given by a density matrix in the representation. These are the ultra-weakly continuous (normal) states on the von Neumann algebra $\pi(\mathcal{A})''$. For a discussion of the problem of implemented group action and its history see [7], Section II.7. The main result is:

Theorem 1.1. (Borchers). *Let $\{\mathcal{A}, \mathcal{G}, \alpha\}$ be a C^*-dynamical system, with $\mathcal{G}$ a topological group, and let π be a representation of $\mathcal{A}$. The following are equivalent:*

(i) There exists a representation $\widehat{\pi}$ of $\mathcal{A}$ that is quasi-equivalent to π and such that α_g is implemented in the representation $\widehat{\pi}$ by a continuous unitary representation of $\mathcal{G}$.

(ii) The folium $F(\pi)$ of π is invariant under the adjoint action α_g^* and the action is strongly continuous in g on $F(\pi)$.

A *thermal representation* is characterized by the group of time translations. Its representation $U(t)$ must have an invariant vector $\Omega_\beta \in \mathcal{H}_\pi$ which is cyclic and separating for the representation. Moreover, the representation $U(t)$ shall fulfill the β-KMS condition, which will be explained together with the Tomita-Takesaki theory.

We call a representation π a *particle representation* if the whole translation group $\mathbb{R}^d$ is implemented by a unitary group $U(a)$ that fulfills the spectrum condition, i.e., the spectrum of U is contained in the closed, forward light-cone $\overline{V}^+$. The name "particle representation" will be used in accordance with the terminology in [7]. This does not imply that the mass operator has a discrete part. In nature probably most massive particles are infra-particles implying that there are representations such that the mass operator has a purely continuous spectrum. A particle representation π is called a *vacuum representation*, if $U(a)$ has an invariant vector Ω and if this vector is cyclic for the representation. Moreover, we will assume that vacuum representations are factor representations. A vacuum representation will always be denoted by π_0.

If π is a representation with cyclic vector Ω then we say that the *Reeh-Schlieder property* holds for (π, Ω), if Ω is cyclic for every $\mathcal{M}_\pi(O)$ with O open and nonempty. If π is a particle representation enjoying the weak additivity property and if $\psi \in \mathcal{H}_\pi$ has compact energy support then (π, ψ) has the Reeh-Schlieder property. For the vacuum vector it was proved by Reeh and Schlieder [42]

1.3 Maximal local algebras.

Domains of special interest in our discussion are the *wedges* in $\mathbb{R}^d$. They are defined in the following manner by two non-zero light-like vectors ℓ_1, ℓ_2, belonging to the boundary of the forward light-cone V^+ :

$$W(\ell_1, \ell_2) = \{\lambda\ell_1 + \mu\ell_2 + \ell^\perp : \lambda > 0, \mu < 0, (\ell^\perp, \ell_i) = 0, i = 1, 2\}. \quad (1.4)$$

The plane spanned by ℓ_1 and ℓ_2 will be called the *characteristic two-plane* of the wedge $W(\ell_1, \ell_2)$. The translated wedge $W(\ell_1, \ell_2) + a$ is denoted by $W(\ell_1, \ell_2, a)$. The C^*-algebra $\mathcal{A}(W)$ is the algebra generated by all $\mathcal{A}(O)$ with $O \subset W$. A representation π of the theory of local observables fulfills *wedge duality*, if for every wedge W one has

$$\mathcal{M}_\pi(W)' = \mathcal{M}_\pi(W'), \quad (1.5)$$

where W' denotes the interior of the space-like complement of W, i.e. $W(\ell_1, \ell_2)' = W(\ell_2, \ell_1)$ and $\mathcal{M}_\pi(G)$ the von Neumann algebra generated by $\pi(\mathcal{A}(G))$. The index π will be dropped if π is a vacuum representation.

Another important family of domains are the double cones or other domains which are the open interior of intersections of wedges. Double cones, denoted by D, play a special role. At several occasions one assumes or derives some properties of the algebras $\mathcal{M}_\pi(W)$ associated with wedges and one wants to deduce corresponding properties of a double cone algebra. This is only possible if the latter algebra can be expressed in terms of wedge algebras. (For short we often use the term "double cone algebra" instead of "algebra associated with a double cone", and analogously for other domains.) Namely, the double cone algebra must have the form

$$\mathcal{M}_\pi(D) = \cap\{\mathcal{M}_\pi(W) : \; D \subset W\}. \tag{1.6}$$

If this is the case then we call $\mathcal{M}_\pi(D)$ a *maximal local algebra.* It is easily checked that the theory constructed from the maximal local algebras fulfills all requirements of the QFTLO. If $\mathcal{M}_\pi(D)$ is not the maximal local algebra then one can *define* $\mathcal{M}_{\pi,\max}(D)$ by the right hand side of Eq. (1.6). If wedge duality holds, these algebras are, indeed, the maximal extensions of $\mathcal{M}_\pi(D)$ compatible with locality. In general the QFTLO's $\{\mathcal{A}(O)\}$ and $\{\mathcal{M}_{\pi,\max}(O)\}$ will have different families of representations. We shall consider the maximal algebras only for a vacuum representation and denote them in this case simply by $\mathcal{M}_{\max}(O)$.

For reasons we will see later, several aspects of QFTLO can be handled by taking as input von Neumann algebras $\mathcal{M}(W)$ associated only with wedges. Of course the algebras associated with W and W' have to commute. In this case one can always define double cone algebras as on the right hand side of (1.6) and these algebras will satisfy locality. The only problem is that in general one does not know the their size, in particular $\mathcal{M}(W)$ need not be generated by the $\mathcal{M}(D)$'s with $D \subset W$.

1.4 Charge sectors.

Two representations π_1, π_2 of $\mathcal{A}(\mathbb{R}^d)$ are quasi-equivalent if they have the same kernel and if the isomorphism $\pi_1(\mathcal{A}(\mathbb{R}^d)) \leftrightarrow \pi_2(\mathcal{A}(\mathbb{R}^d))$ extends to an isomorphism between the von Neumann algebras generated by the two representations. For a C^*-algebra as complicated as (non-trivial) QFTLO there exists at least a continuum of non-equivalent representations. Therefore, one is interested in principles which group the set of representations into sub-families. The principle mostly used is that of *local equivalence.* Two representations π_1, π_2 are called *locally equivalent*, if for every bounded open region the representations $\pi_1(\mathcal{A}(O))$ and $\pi_2(\mathcal{A}(O))$ are quasi-equivalent. If one of them is a vacuum representation then one calls the second representation *locally normal*. The requirement of local normality is often used in order to select from the set of thermal representations (defined by other means) a suitable sub-family which, from some point of view, can be regarded as physically acceptable.

While in the Lagrangian or in Wightman's field theory charged fields are put in by hand, it is the philosophy of the QFTLO that representations of the observable algebra describing a finite number of charged particles shall be constructed from the algebra of observables. Also the charged fields connecting the vacuum representation with the representations describing charges should be constructed with help of the different representations. This can be worked out if one uses the equivalence relation introduced by Borchers [3]. If π_0 is a vacuum representation, then a factor representation π_1 is called a *charged sector* if π_0 and π_1 have the same kernel and if for every O the isomorphism $\pi_0(\mathcal{A}(O')) \leftrightarrow \pi_1(\mathcal{A}(O'))$ extends to an isomorphism of the corresponding von Neumann algebras. Here O' denotes the space-like complement of O. With this concept Doplicher, Haag and Roberts [19, 20], [22] have worked out the details of the mentioned program. The algebra generated by the local observables and by the localized charged fields is called the *field algebra* and the corresponding net is usually denoted by $\{\mathcal{F}(O)\}$. Within this setting also the concept of *conjugate charge sectors* has a precise meaning: If an element F of the field algebra generates a charged sector by applying it to the vacuum, then the conjugate sector is generated in the same way by F^*.

1.5 Gauge transformations.

We shall use the term *gauge transformation* for any unitary operator U on the Hilbert space of a vacuum representation π_0 of a QFTLO that fulfills $U\Omega = \Omega$ and

$$U\pi_0(\mathcal{A}(O))U^* = \pi_0(\mathcal{A}(O)) \tag{1.7}$$

for every bounded open O. In Lagrangian field theory such operators typically implement transformations of the form $\psi(x) \to \mathrm{e}^{\mathrm{i}\varphi}\psi(x)$ and generate a compact group. Since the QFTLO is given by a set of axioms and not by a finite number of fields the gauge group in the sense defined is not compact in general, however. In the last section an example of such a case will be considered. Therefore, a possible requirement for selecting a reasonable family of QFTLO is the assumption that the gauge group is compact. There are other conditions implying this property. Buchholz and Wichmann [18] have introduced the concept of *nuclearity.* This is the requirement that the number of states, which can be created locally, does not increase too fast with energy. Doplicher and Longo [21] have introduced the *split property.* This is equivalent to the following: Let $O_1 \subset O_2$ be two bounded open regions such that the closure of O_1 is still contained in O_2. Let ω_1 be a vector state on $\pi_0(\mathcal{A}(O_1))$ and ω_2 be a vector state on $\pi_0(\mathcal{A}(O_2'))$. Then exists a vector state on π_0 of the QFTLO which coincides with ω_1 on $\pi_0(\mathcal{A}(O_1))$ and with ω_2 on $\pi_0(\mathcal{A}(O_2'))$. Between these concepts one has the following relations:

$$\text{nuclearity property} \longrightarrow \text{split property} \longrightarrow \text{compact gauge group} \tag{1.8}$$

For the first arrow see [18] and [13], for the second see [21].

1.6 Tomita-Takesaki theory.

The main tool for handling the problems connected with the PCT-theorem in the QFTLO is the modular theory introduced by Tomita. It is usually called Tomita-Takesaki theory because the first presentation of this theory, beyond a preprint, is due to Takesaki.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a von Neumann algebra, acting on this space, with commutant $\mathcal{M}'$. A vector Ω is cyclic and separating for $\mathcal{M}$, if $\mathcal{M}\Omega$ and $\mathcal{M}'\Omega$ are dense in $\mathcal{H}$. If these conditions are fulfilled, then a modular operator Δ and a modular conjugation J are associated with the pair $(\mathcal{M}, \Omega)$, such that

(i) Δ is self-adjoint, positive and invertible,

$$\Delta\Omega = \Omega, \qquad J\Omega = \Omega.$$

(ii) The operator J is a conjugation, i.e., J is anti-linear, $J^* = J$, $J^2 = 1$, and J commutes with $\Delta^{\mathrm{i}t}$. This implies the relation

$$\mathrm{Ad}\, J\Delta = \Delta^{-1}.$$

(iii) For every $A \in \mathcal{M}$ the vector $A\Omega$ belongs to the domain of $\Delta^{\frac{1}{2}}$, and

$$J\Delta^{\frac{1}{2}}A\Omega = A^*\Omega =: SA\Omega.$$

(iv) The unitary group $\Delta^{\mathrm{i}t}$ defines a group of automorphisms of $\mathcal{M}$,

$$\mathrm{Ad}\,\Delta^{\mathrm{i}t}\mathcal{M} = \mathcal{M}, \qquad \text{for all } t \in \mathbb{R}.$$

(5) J maps $\mathcal{M}$ onto its commutant

$$\mathrm{Ad}\, J\mathcal{M} = \mathcal{M}'.$$

These results apply in particular to QFTLO in a vacuum representation because of the Reeh-Schlieder theorem which implies that the vacuum vector is cyclic and separating for every algebra $\mathcal{M}(G)$, where G is any domain which has a space-like complement with interior points.

The matrix elements of the modular group have the following important analyticity properties. If $A, B \in \mathcal{M}$ and $\sigma^t(A) := \mathrm{Ad}\,\Delta^{\mathrm{i}t} A$ then the continuous function

$$F(t) = (\Omega, B\sigma^t(A)\Omega)$$

has bounded analytic continuation into the strip $S(-1,0) = \{z \in \mathbb{C} : -1 < \Im m\, z < 0\}$. At the lower boundary one finds

$$F(t - \mathrm{i}) = (\Omega, \sigma^t(A)B\Omega) \tag{1.9}$$

This relation is called the *KMS-condition.* In QFTLO the group σ^t coincides for thermal states with scaled time translations. If τ denotes the automorphism of the time translations and Ω defines a thermal state at inverse temperature β the equations become

$$\begin{aligned} F(t) &= (\Omega, B\tau_t(A)\Omega), \\ F(t - \mathrm{i}\beta) &= (\Omega, \tau_t(A)B\Omega). \end{aligned} \tag{1.10}$$

The relations Eq. (1.10) are called the *β-KMS-condition.*

2 Review of the PCT-theorem

2.1 Point-fields

In Lagrangian or Wightman field theory the PCT-operator Θ is an anti-unitary operator implementing the PCT-symmetry (if present). For a scalar field one has the relation

$$\Theta\Phi(x)\Theta = \Phi^*(-x). \tag{2.1}$$

The formula for higher spin looks the same with respect to the space-time variable x, but in the index space one has in general to introduce an additional transformation independent of the variable x. In a suitable spinorial basis the matrix of this transformation is diagonal and each component simply gets multiplied by a phase factor. In addition Θ shall fulfill the following commutation rule with the Poincaré transformations

$$\Theta U(\Lambda, a)\Theta = U(\Lambda, -a). \tag{2.2}$$

These requirements imply

$$\Theta = \Theta^{-1} = \Theta^*. \tag{2.3}$$

PCT-theorems are results showing the existence of a PCT-operator Θ. The product PT represents an element of the Lorentz group. All proofs so far assume that PT is the element -1 and has the determinant $+1$. This implies that the Minkowski space must have even dimensions. In the odd dimensional case one replaces P by the total reflection in the space perpendicular to the one-direction, denoted P_1. We will not discuss this case and restrict ourselves to even dimensional Minkowski spaces. As remarked in the last section, however, also in odd dimensions, where PT has determinant -1, free single component hermitian fields have full PCT symmetry, even without Lorentz covariance.

The Lagrange function for a field theory is usually invariant under total reflection, time-reversal, and charge conjugation. But it does not necessarily mean that these symmetries are implemented separately by unitary or antiunitary operators. G. Lüders [37] discovered that the product of these symmetries is always implemented by an operator Θ, the PCT-operator. The input for this result is the implemented Poincaré covariance, spectrum condition, and the existence of charge conjugate partners. Pauli [41] was quite excited about this result, because it clarified the relation about spin and the commutation relations of the fields. It also gave an understanding why for Fermi-fields one should use a positive energy representation and not one with a "Dirac sea". (For the discussion of spin and statistics see also G. Lüders and B. Zumino [38]).

In Wightman's field theory [46] the PCT-theorem was proved by R. Jost [33]. This proof is based on the same assumptions as for the Lagrangian field theory which are

1. The theory is Poincaré covariant and the translations fulfill the spectrum condition. Moreover, the representation space contains a cyclic vacuum vector.

2. The Poincaré transformations of the fields induce a finite dimensional representation of the Lorentz group in the index space.

3. Every field in the theory is accompanied by its charge conjugate partner (which may be the field itself).

Jost's proof is based on a result of Hall and Wightman [32] which says that the analytic continuation of the Wightman functions are invariant under the complexified connected component of the Lorentz group. In even dimension the complex Lorentz group contains the element -1. The requirement 2 does not forbid that a certain field is infinitely degenerate. But if we deal with fields transforming with an irreducible, infinite dimensional representation of the Lorentz group in the index space the properties leading to the spin-statistics and PCT-theorem may be destroyed. R. Streater [43] constructed examples where fields with integer spin anti-commute and those with half-integer spin commuted. Oksak and Todorov [40] gave other examples where Jost's proof of the PCT theorem did not work. Requirement 3 implies a symmetry under complex conjugation for the set of Wightman functions.

2.2 Algebraic PCT-symmetry in QFTLO

Before discussing the PCT-theorem in the QFTLO let us assume that we have such symmetry on the algebraic level, i.e., there exists an anti-linear automorphism θ of $\mathcal{A}(\mathbb{R}^d)$ with

$$\begin{aligned} \theta(AB) = \theta(A)\theta(B), \qquad \theta(\lambda A) = \overline{\lambda}\theta(A), \\ \theta(\mathcal{A}(O)) = \mathcal{A}(-O), \qquad A, B \in \mathcal{A}(\mathbb{R}^d). \end{aligned} \tag{2.4}$$

The automorphism θ should also transform the translations in the correct manner, i.e.,

$$\theta(\alpha_a(A)) = \alpha_{-a}(\theta(A)). \tag{2.5}$$

Since θ maps the algebra $\mathcal{A}(\mathbb{R}^d)$ onto itself, its transpose, θ^*, maps the dual space $\mathcal{A}(\mathbb{R}^d)^*$ onto itself. This implies in particular that θ is represented by an anti-unitary operator on the standard representation of the enveloping von Neumann algebra $\mathcal{A}(\mathbb{R}^d)^{**}$. For later discussions it is of interest to know sub-families of representations which are mapped by the PCT-symmetry onto itself.

Theorem 2.1. *Let $\{\mathcal{A}(O)\}$ be a QFTLO with algebraic PCT-symmetry θ. Assume in addition that the translations α_a act strongly continuous, i.e., the function $a \to \alpha_a(A)$ is continuous in the norm topology of $\mathcal{A}(\mathbb{R}^d)$ and this for every $A \in \mathcal{A}(\mathbb{R}^d)$. Then the family of particle representations is invariant under θ. Moreover, the sub-family of vacuum representations is also invariant.*

The proof will be given in the next section. The requirement that the translations act strongly continuous is not necessary. For the proof of the general case one would have to go into details of positive energy representations as described in [7]. But since we will look only at representations which are locally normal, the general version of Thm. (2.1) is not needed for the following reason: Let $\{\mathcal{A}(O)\}$ be a QFTLO and π_0 a vacuum representation. Then $\{\mathcal{M}_{\pi_0}(O)\}$ defines a new QFTLO, and the locally normal representations of the two nets are clearly in one to one correspondence with each other. Now let π_1 be a locally normal representation of $\mathcal{M}_{\pi_0}$. The corresponding local C^* algebras, $\mathcal{A}_1(O)$, contain dense subalgebras $\mathcal{A}_{\rm c}(O)$ on which the action of α_a is strongly continuous in a. For every bounded O, $\mathcal{A}_{\rm c}(O)'' = \mathcal{A}_1(O)''$ is isomorphic as a von Neumann algebra to $\mathcal{M}_{\pi_0}(O)$ because π_1 is locally normal. Hence $\{\mathcal{A}(O)\}$ and $\mathcal{A}_{\rm c}(O)$, which satisfies the hypothesis of Theorem 2.1, have the same family of locally normal representations. For the family of these representations one has the following result:

Theorem 2.2. *Let $\{\mathcal{A}(O)\}$ be a QFTLO with algebraic PCT symmetry θ. Let π_0 be a vacuum representation and assume it is invariant under θ. Then θ maps the family of locally normal representations onto itself. Moreover, the family of charge sectors is a θ invariant subfamily of representations.*

Also the proof of this result will be postponed to the next section. As a last point we want to look at the algebraic PCT-symmetry in thermal representations.

Theorem 2.3. *Let $\{\mathcal{A}(O)\}$ be QFTLO with PCT-symmetry θ. Assume that the time translations α_t act strongly continuously. Then the set of β-KMS-states is θ^* invariant.*

The proof will be given in the next section.

2.3 Preparations for the PCT-theorem

In the QFTLO one usually looks for the PCT-symmetry only in the vacuum representation of the local observables or of the corresponding field algebra. The requirements for a PCT operator Θ are:

- Θ is antiunitary and for all bounded O

$$\Theta\pi_0(\mathcal{A}(O))\Theta = \pi_0(\mathcal{A}(-O)). \tag{2.6}$$

- The relation (2.2) between Θ and the representation of $\mathcal{P}_+^\uparrow$ holds.

- For field algebras one replaces $\pi_0(\mathcal{A}(O))$ by $\mathcal{F}(O)$ in (2.6). Moreover, it is required that Θ transforms a charge sector into its conjugate sector.

For the discussion of the PCT-theorem we will mostly look at vacuum representations. This is sufficient because of Thm. 2.2 (see also [28]). For the construction of the PCT-operator on the field algebra one must go first to the maximal local algebras and construct the charged fields as described by Doplicher, Haag and Roberts [19, 20], [22]. Starting from the PCT-theorem in the vacuum sector one can construct the PCT-operator for the whole field algebra. This has been done by Guido and Longo [29], see also [28].

One more remark is in order. Also for the vacuum representation of the observable algebra we will use the phrases PCT-symmetry and PCT-theorem and not simply PT-symmetry and so on. The reason is the following: The axioms of the QFTLO are so general that one can not exclude that the algebras $\mathcal{A}(O)$ contain also charged Bose-fields. If they contain such fields then the group of gauge transformations contains a continuous representation of the circle group. But we do not know any manageable condition excluding the existence of such gauge transformations. If such a charged Bose field is present then one has a proper PCT-symmetry also in the vacuum sector.

For a long time it was impossible to prove the PCT-theorem in QFTLO because of the lack of proper mathematics. This changed with the appearance of the Tomita-Takesaki theory. The basic result, where the modular group and conjugation can be computed for some algebra of interest in QFTLO, was given in [1].

Theorem 2.4. (Bisognano and Wichmann). *Assume a Wightman field theory of a scalar neutral field is such that the smeared field operators generate the local algebras. Then:*

1. *The modular group of the algebra associated with a wedge and the vacuum vector coincides with the unitary representation of the group of Lorentz boosts which maps the wedge onto itself.*

2. *The modular conjugation of the wedge W is given by the formula*

$$J_W = \Theta U(R_W(\pi)). \tag{2.7}$$

Here Θ denotes the PCT-operator of the Wightman field theory and $U(R_W(\pi))$ is the unitary representation of the rotation which leaves the characteristic two-plane of the wedge invariant. The angle of rotation is π.

3. *The theory fulfills wedge duality.*

In case of the right wedge, $W_{\mathrm{r}} = \{x : |x^0| < x^1\}$, the group of Lorentz boosts that leave the wedge invariant is

$$\Lambda_{W_{\mathrm{r}}}(t) = \begin{pmatrix} \cosh 2\pi t & -\sinh 2\pi t & 0 & 0 \\ -\sinh 2\pi t & \cosh 2\pi t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.8}$$

In particular, $\Lambda_{W_{\mathrm{r}}}(-\frac{\mathrm{i}}{2}) =$ $\mathrm{diag}\,(-1,-1,1,1,1)$. Moreover, $R_{W_{\mathrm{r}}}(\pi) = \mathrm{diag}\,(1,1,-1,-1)$, so $\Lambda_{W_{\mathrm{r}}}(-\frac{\mathrm{i}}{2})R_{W_{\mathrm{r}}}(\pi) = -1$.

In a second paper [2] Bisognano and Wichmann extended this result to charged Bose and Fermi-fields transforming covariantly with respect to a finite dimensional representation of the proper, orthochronous Lorentz group in the index space. The remarkable feature of this result is the fact that modular group and the modular conjugation of the wedge regions map local algebras onto local algebras. This property has inspired the following

Definition 2.5. Let $\mathcal{M}(O)$ be the local algebras of a vacuum representation of a QFTLO.

1. We say that the *Bisognano-Wichmann property* holds in this representation if for any wedge W the modular group $\Delta_W^{\mathrm{i}t}$ associated with $\mathcal{M}(W)$ and the vacuum vector acts like the corresponding one-parameter group of Lorentz boosts $\Lambda_W(t)$ that leave W invariant, i.e.

$$\Delta_W^{\mathrm{i}t}\mathcal{M}(O)\Delta_W^{-\mathrm{i}t} = \mathcal{M}(\Lambda_W(t)O). \tag{2.9}$$

We remark that this property has been called *modular covariance* in [29].

2. Suppose the Poincaré symmetry is implemented on the representation space by a unitary representation $U(\Lambda, a)$ of $\mathcal{P}_+^\uparrow$. The U is called the *minimal representation* if for every wedge one has

$$\Delta_W^{\mathrm{i}t} = U(\Lambda_W(t)). \tag{2.10}$$

Since the Lorentz boosts of different wedges generate $\mathcal{P}_+^\uparrow$, the minimal representation is unique, when it exists. For some time it was not known whether the result of Bisognano and Wichmann holds only for Wightman field theories or if it holds in more general cases. A positive answer for the two-dimensional Minkowski space was given in [5].

Theorem 2.6. (Borchers). *Let π_0 be a vacuum representation of a QFTLO on the two-dimensional Minkowski space. If the theory fulfills wedge duality, then:*

1. *The modular group $\Delta_W^{\mathrm{i}t}$ and the translations $U(a)$ generate a representation of the two-dimensional Poincaré group.*
2. *If the local algebras are the maximal algebras, then the representation defined by the modular group and the translations is the minimal representation and π_0 has the Bisognano-property.*
3. *The theory* (with maximal local algebras) *is PCT-covariant. The PCT-operator coincides with J_W.*

If wedge duality does not hold for the QFTLO, it is in two dimensions always possible to extend the local algebras in such a way that it is fulfilled, but the extension is not necessarily unique.

The basis of this result is the commutation relation between the modular group and the translations. For the right wedge W_{r} then in light cone coordinates $x^{\pm} = \frac{1}{2}(x^0 \pm x^1)$ the relations read

$$\Delta_{W_{\mathrm{r}}}^{\mathrm{i}t} U(x^{\pm}) \Delta_{W_{\mathrm{r}}}^{-\mathrm{i}t} = U(\mathrm{e}^{\mp 2\pi t} x^{\pm}). \tag{2.11}$$

The spectrum condition for the translations is an essential ingredient for proving this result, see also [24]. Eq. (2.11) holds also in higher dimension if $x^{\pm}$ are the light cone coordinates defined by the two light rays ℓ_1, ℓ_2 defining the wedge W. Also Thm. 2.6 can be transcribed to higher dimensions: Let W be a wedge satisfying duality, i.e., $\mathcal{M}(W)' = \mathcal{M}(W')$. The same holds then for all translates of W. If D is a double cones in the characteristic two plane of the wedge and $K(D)$ denotes the cylinder, obtained by translating D in all directions perpendicular to the characteristic two plane of the wedge, then $\Delta_W^{\mathrm{i}t} \mathcal{M}(K(D)) \Delta_W^{-\mathrm{i}t} = \mathcal{M}(\Lambda_W(t) K(D))$. But the theorem does not imply that double cones are mapped onto double cones. In fact, as known from the examples of Yngvason [47] (see also [25]) the action of the modular group of a wedge does not need to be local in the perpendicular direction.

This result, as well as that of Bisognano and Wichmann, is an indication that wedge duality is important for the proof of the PCT-theorem. In fact, it is explicitly or implicitly contained in the hypothesis of all known proofs, although it is neither a necessary nor sufficient condition for PCT symmetry as we will see in the examples in the last section. In [6] Borchers derived necessary and sufficient conditions for the validity of wedge duality in QFLTO's with implemented Poincaré symmetry. Also here the cylindrical sets $K(D)$ just mentioned play an important role, because the wedge duality is essentially a two-dimensional problem. Before formulating this result let us first discuss the situation. This will help to understand the necessary concepts.

Let W be the right wedge and $\mathcal{M}(W)$ the corresponding von Neumann algebra in a vacuum representation. This algebra is invariant under the representation of the boosts, $U(\Lambda_W(t))$, and since this group and the modular group have the vacuum vector as invariant vector, $U(\Lambda_W(t))$ and $\Delta_W^{\mathrm{i}t}$ commute. Therefore, the two groups differ by a one parameter group $V_W(t)$. Moreover, if wedge duality holds for W, Thm. 2.6 implies that $V_W(t)$ is a gauge which maps every $\mathcal{M}(K(D))$ onto itself. If $A \in \mathcal{M}(K(D))$ with $D \subset W$ then $\Delta_W^{\mathrm{i}t} A\Omega$ has an analytic continuation into the strip $S(-\frac{1}{2}, 0) = \{z \in \mathbb{C} : -\frac{1}{2} < \Im m\, z < 0\}$. At the lower boundary the Tomita-Takesaki theory implies $\Delta_W^{1/2} A\Omega = J_W A^* J_W \Omega$. If wedge duality does not hold then the operator $J_W A^* J_W$ belongs to $\widetilde{\mathcal{M}}(K(-D))$ where $\widetilde{\mathcal{M}}(K(-D))$ is defined as follows: In the two-dimensional Minkowski space a double cone is defined by the intersection of two wedges. Therefore, we can write $K(D) = W_a \cap W'_b$ with $b - a \in W$. So one has $\mathcal{M}(K(D)) = \mathcal{M}(W_a) \cap \mathcal{M}(W'_b)$. With this notation one defines $\widetilde{\mathcal{M}}(K(D)) = \mathcal{M}(W_a) \cap \mathcal{M}'(W_b)$ which is an algebra containing $\mathcal{M}(K(D))$ as proper subset in the non-duality case. In this situation $V_W(t)$ maps $\widetilde{\mathcal{M}}(K(D))$ onto itself. If $A \in \mathcal{M}(K(D))$ with $K(D) \subset W$ it can happen that $U(\Lambda_W(t))A\Omega$ also has a bounded analytic continuation into the strip $S(-\frac{1}{2}, 0)$. If this is the case it can be shown that there exists an element $\widehat{A} \eta \mathcal{M}(K(-D))$ with

$$U(\Lambda_W(-\tfrac{\mathrm{i}}{2}))A\Omega = \widehat{A}\Omega. \tag{2.12}$$

The symbol η means that the operator might be unbounded but is affiliated with $\mathcal{M}(K(-D))$. The group $V_W(t)$ is a continuous Abelian gauge-group. Therefore, if wedge duality holds the set of elements in $\mathcal{M}(K(D))$ such that $V_W(t) A V_W(-t)$ are entire analytic in t is *-strongly dense in $\mathcal{M}(K(D))$. Hence, in this situation the set fulfilling Eq. (2.12) is *-strongly dense in $\mathcal{M}(K(D))$. The converse is the content of the following result.

Theorem 2.7. (Borchers). *Assume we are dealing with a vacuum representation of a QFTLO where the action of $\mathcal{P}_+^\uparrow$ is unitarily implemented. Assume the local algebras are maximal. The theory fulfills duality for the wedges W, W', i.e., $\mathcal{M}(W)' = \mathcal{M}(W')$, if and only if for all D with $K(D) \subset W$ the set of elements $A \in \mathcal{M}(K(D))$ for which $U(\Lambda_W(t))A\Omega$ has a bounded analytic continuation into the strip $S(-\frac{1}{2}, 0)$ is *-strongly dense in $\mathcal{M}(K(D))$.*

By the results of Bisognano and Wichmann the group of the Lorentz boosts and the modular group coincide for the algebra of a wedge if the QFTLO is generated by Wightman fields transforming covariantly with respect to a finite dimensional representation of the Lorentz group. In the last theorem we saw that this is a general feature of theories in the two-dimensional situation, irrespective whether they are generated by Wighman fields or not. In higher dimensions the modular groups need not in general act like Lorentz boosts, but the assumption that they do was the starting point of the investigation of Brunetti, Guido and Longo [12]. They obtained the following result:

Theorem 2.8. (Brunetti, Guido and Longo). *Let $\{\mathcal{M}(O)\}$ be the local von Neumann algebras of some representation of a QFTLO and suppose there exists a vector Ω that is cyclic and separating for all $\mathcal{M}(O)$ (O open and bounded). If the Bisognano-Wichmann property* (2.9) *holds with respect to the modular groups defined by the wedge algebras and Ω then the local algebras transform covariantly under a representation of the covering group $\tilde{\mathcal{P}}_+^\uparrow$ of $\mathcal{P}_+^\uparrow$. This representation fulfills the spectrum condition. Moreover, wedge duality holds.*

The vector Ω is invariant under the representation of $\tilde{\mathcal{P}}_+^\uparrow$ and hence a vacuum vector. Hence one expects that if one deals only with one sector (the vacuum sector) the covering group $\tilde{\mathcal{P}}_+^\uparrow$ can be replaced by $\mathcal{P}_+^\uparrow$. The solution of this problem is due to Guido and Longo [29]. They handled the problem by looking at the field algebra $\{\mathcal{F}(O)\}$ and treating at the same time the spin and statistics problem.

Theorem 2.9. (Guido and Longo). *Assume we are dealing with a representation of a field algebra $\{\mathcal{F}(O)\}$. Let there exist a vector Ω in the representation space which is cyclic an separating for every $\mathcal{F}(O)$. If this representation fulfills the Bisognano-Wichmann property then:*

1. *The theory is covariant under a representation of the covering $\tilde{\mathcal{P}}_+^\uparrow$ of $\mathcal{P}_+^\uparrow$. This representation fulfills the spectrum condition.*
2. *In the vacuum sector the representation of the Poincaré group is single-valued and hence it is the minimal representation. (See Def. 2.5.)*
3. *The standard spin and statistics relations hold.*
4. *The exists a PCT-operator, which transforms the local field algebras correctly, has the correct commutation relations with the Poincaré transformations and maps every sector onto its conjugate sector.*

The question whether one deals in the vacuum sector with a representation of the Poincaré group or its covering inspired Borchers [8] to derive the minimal group representation directly from the Bisognano-Wichmann property. In this constructive proof it is assumed that the local algebras $\mathcal{M}(O)$ are the maximal algebras. From the previous discussions it is important to stress the following results:

- *The Bisognano-Wichmann property is equivalent to the existence of the minimal representation. These conditions imply wedge duality and the PCT-theorem.*

Because of the importance of the modular groups of the wedges one would therefore like to know whether other localization properties of these groups are possible, besides those described by Bisognano and Wichmann. This problem has been investigated by Kuckert [34].

Theorem 2.10. (Kuckert). *Consider a QFTLO which is covariant under translations in a vacuum representation. Let W be a wedge, $D \subset W$ a double cone and $\mathcal{M}(D)$ maximal.*

1. *If we assume*

$$\Delta_W^{\mathrm{it}} \mathcal{M}(D) \Delta_W^{-\mathrm{it}} = \mathcal{M}(G_{W,D,t})$$

with some domain $G_{W,D,t}$ then necessarily $G_{W,D,t} = \Lambda_W(t)D$.

2. *If we have*

$$J_W \mathcal{M}(D) J_W = \mathcal{M}(\tilde{G}_{W,D}))$$

with another domain $\tilde{G}_{W,D}$, then one finds $\tilde{G}_{W,D} = j_W D$. Here j_W denotes the reflection in the characteristic two-plane of the wedge which leaves the perpendicular directions unchanged.

It is important to note that these conclusions do not hold in general if the equality of the algebras is replaced by an inclusion. Kuckert assumed that the conditions of the theorem hold for all double cones. But this is not necessary because of the translation covariance and Eq. (2.11). This was first observed by Guido [27]. Recently Kuckert [35] has generalized his theorem by looking at individual operators instead of algebras. Making the additional assumption of wedge duality and introducing a slightly different notation of localization he derived the Bisognano-Wichmann property from the assumption that for every localized operator A the expression $\Delta_W^{\mathrm{it}} A \Delta_W^{-\mathrm{it}}$ is localized and this localization is continuous in t. For details see his paper.

2.4 Implemented Poincaré covariance and the PCT-theorem

Assume one deals with a vacuum representation of a QFTLO which is covariant under the Poincaré symmetry. In this situation it can happen that the symmetry is implemented by more than one continuous unitary representation of the Poincaré group. If there are two different representations $U_1(g), U_2(g), g \in \mathcal{P}_+^\uparrow$ then these differ by a cocycle $U_2(g) = V(g)U_1(g)$ with values in the gauge group. $V(g)$ fulfills the cocycle relation

$$V(g_1 g_2) = V(g_1)U_1(g_1)V(g_2)U_1(g_1^{-1}). \tag{2.13}$$

Moreover, the unitary family $V(g)$ does not depend on the translations. If $U_1(g)$ is the minimal representation then $U_1(g)$ and $V(g)$ commute for arbitrary g_1, g_2. This implies that $V(g)$ itself is a continuous unitary representation of the Lorentz group [9]. In good situations one has only one representation. Such a situation has been described in [11].

Theorem 2.11. (Brunetti, Guido and Longo). *Assume we are dealing with a vacuum representation of an implemented Poincaré covariant QFTLO. If this representation fulfills the (distal) split property, then the representation of the Poincaré group is unique.*

The word "distal" means that the split property explained earlier need only be assumed for localization regions sufficiently far from each other.

The last conclusion relies on a result in [21] and it does not imply the existence of the minimal representation. Therefore, a large part of the investigations on the PCT-theorem consists in finding conditions for the existence of the minimal representation. One such condition can be found in [9]. Recall from the discussion of the wedge duality that there must be sufficiently many elements $A \in \mathcal{M}(K(D))$ with $K(D) \subset W$ such that $U(\Lambda_W(t))A\Omega$ has a bounded analytic continuation into the strip $S(-\frac{1}{2}, 0)$. For such elements one has

$$U(\Lambda_W(-\tfrac{\mathrm{i}}{2}))A\Omega = \widehat{A}\Omega$$

with $\widehat{A}\eta\mathcal{M}(K(-D))$. We say A fulfills the *reality condition* if $U(\Lambda_W(t))A^*\Omega$ can also be bounded analytically continued and if

$$\widehat{A^*} = \widehat{A}^* \tag{2.14}$$

holds.

Theorem 2.12. (Borchers). *For a representation of a Poincaré covariant theory of local observables in the vacuum sector the modular group associated with the algebra of any wedge coincides with the corresponding Lorentz boosts iff the theory fulfills wedge duality and the set of elements $A \in \mathcal{M}(K(D))$ which fulfill the reality condition Eq.* (2.14) *is *-strong dense in $\mathcal{M}(K(D))$. This implies in particular the existence of a PCT operator.*

In a recent paper Guido and Longo [30] gave conditions for the Bisognano-Wichmann property which have some similarity with Thm. 2.12. Some of the results are more general (statement (iii) and (iv) of the following theorem) because the spectrum condition is not required.

Theorem 2.13. (Guido and Longo). *Let a represented theory of local observables be covariant under a representation of the Poincaré group $\mathcal{P}_+^\uparrow$ with an invariant vector Ω. Let W be a wedge and $S \subset W$ be a spacelike cone. Assume Ω is cyclic for $\mathcal{M}(S)$ and $\mathcal{M}(W)$. By the covariance it is also separating for both algebras. Let $\mathcal{A}$ be a weakly dense *-subalgebra of $\mathcal{M}(S)$. The following are equivalent:*

(i) The Bisognano-Wichmann relation $\Delta_W^{\frac{1}{2}} = U(\Lambda_W(-\frac{\mathrm{i}}{2}))$ holds.

(ii) $U(\Lambda_W(-\frac{\mathrm{i}}{2}))\mathcal{A}_1\Omega$ is bounded and the translations fulfill the spectrum condition.

(iii) $U(\Lambda_W(-\frac{\mathrm{i}}{2}))\mathcal{M}(W)_1\Omega$ is bounded.

(iv) $\|U(\Lambda_W(-\frac{\mathrm{i}}{2}))\mathcal{M}(W)_1\Omega\| \leq 1$.

(The index 1 at the algebras denotes their unit ball.) Moreover, if the boundary of S intersects the edge W in a half-line, then the spectrum condition in (ii) follows already from the boundedness of $U(\Lambda_W(-\frac{\mathrm{i}}{2}))\mathcal{A}_1\Omega$.

We shall now discuss another aspect of the analyticity requirement that will lead to a more direct proof of the PCT theorem in Theorem 2.15. Assume we are dealing with a vacuum representation of a Poincaré covariant theory which fulfills wedge duality and the Bisognano-Wichmann property. If D is a double cone in a wedge W and away from the boundary, then D belongs to many wedges. Let $\Gamma(D)$ be the set of Lorentz transformations such that $D \subset gW$. If $A \in \mathcal{M}(D)$ then $\Delta_{gW}^{\mathrm{i}t} A\Omega$ has an analytic continuation into $S(-\frac{1}{2}, 0)$ for every $g \in \Gamma(D)$. Recall that for the wedge duality it was necessary that there are enough elements in $\mathcal{M}(K(D))$ such that $U(\Lambda_W(t)$ can be bounded analytically continued into $S(-\frac{1}{2}, 0)$. Now we define:

Definition 2.14. Let D be a double cone with center at the origin. An element $A \in \mathcal{M}(D)$ is called *fully analytic* if for any x and every wedge W such that $D + x \subset W$ the element $T(x)AT(-x)$ has the above analyticity property, i.e., the expression $U(\Lambda_W(t))T(x)A\Omega$ has a bounded analytic continuation into $S(-\frac{1}{2}, 0)$. The set of fully analytic elements of $\mathcal{M}(D)$ will be denoted by $\mathcal{M}^{\mathrm{fa}}(D)$.

$T(x)$ denotes the representation of the translation. With this concept we will show

Theorem 2.15. *Let a theory of local observables be covariant under a representation of the Poincaré group which fulfills the spectrum condition. Assume the fully analytic elements have the following properties:*

(i) The set of fully analytic elements is covariant under the adjoint operation of the Lorentz group, i.e., $A \in \mathcal{M}^{\mathrm{fa}}(D)$ *implies* $U(\Lambda)AU(\Lambda)^{-1} \in \mathcal{M}^{\mathrm{fa}}(\Lambda D)$

(ii) Let $\mathcal{M}^{\mathrm{fa}}(W) = \cup\{T(x)\mathcal{M}^{\mathrm{fa}}(D)T(-x) : D + x \subset W\}$ *then* $\mathcal{M}^{\mathrm{fa}}(W)\Omega$ *is a core for* $U(\Lambda_W(-\frac{\mathrm{i}}{2}))$ *as well as for* $\Delta_W^{1/2}$*.*

If these conditions are fulfilled then a PCT-operator exists.

We remark that by Nelson's theorem $\mathcal{M}^{\mathrm{fa}}(W)\Omega$ is a core for $U(\Lambda_W(-\frac{\mathrm{i}}{2}))$ if this domain is dense in the Hilbert space and invariant under the Lorentz boosts. It is a core for $\Delta_W^{1/2}$ if $\mathcal{M}^{\mathrm{fa}}(W)$ is *-strongly dense in $\mathcal{M}(W)$.

The proof of this result will also be postponed to the next section. One other result concerning the fully analytic elements is the following:

Theorem 2.16. *Let a theory of local observables be covariant under a representation of the Poincaré group fulfilling the spectrum condition. Then the assumptions of the last theorem hold if and only if the given representation of the Poincaré group is the minimal one. This implies that every localized element is fully analytic.*

Also the proof of this result will be given in the next section.

One obtains a direct application of the result of Bisognano and Wichmann [2] if one deals with a QFTLO with isolated masses giving rise to complete asymptotic fields. In this situation J. Mund [39] has shown that the modular groups of any wedge coincide for the interacting and the asymptotic fields. This result holds also if the charged fields are localized in space like cones as described by Buchholz and Fredenhagen [17]. The proof is possible since one knows the the commutation relations between the modular transformations and the translations (2.11).

Theorem 2.17. (Mund). *Assume we are dealing with a representation of a field algebra $\{\mathcal{F}(O)\}$ which is covariant under a representation of the connected Poincaré group. Let there exist a vacuum vector Ω in the representation space which is cyclic an separating for every $\mathcal{F}(O)$ and let the theory fulfil (twisted) wedge duality. Assume moreover, that the spectrum of the translations has isolated masses in the sectors describing elementary charges, and that the corresponding asymptotic fields have finite multiplicity and are complete. Then the theory has the Bisognano-Wichmann property.*

2.5 Supplements to the PCT-theorem

In the previous investigations we looked at properties of the modular groups of the wedge algebras. Therefore, some results can be obtained by looking at wedges alone. Buchholz and Summers initiated a program where they only used the modular conjugations of all wedges [14]. The idea is based on the well known fact that the Poincaré group $\mathcal{P}_+$ is generated by reflections. Therefore, the modular conjugations J_W should generate a representation of $\mathcal{P}_+$. The original Ansatz has been generalized considerably in recent papers [15] and [16]. In order to discuss their results we must introduce the concepts used. (We restrict ourselves to the case of the four-dimensional Minkowski space.)

By $\mathcal{W}$ we denote the set of all wedges (including the translated ones). For $W \in \mathcal{W}$ there shall exist a bijection $\tau_W : \mathcal{W} \to \mathcal{W}$ with

(i) $\tau_W^2 = id.$

(ii) $W_1 \subset W_2 \to \tau_W(W_1) \subset \tau_W(W_2)$.

(iii) $\overline{W}_1 \cap \overline{W}_2 = \emptyset \to \tau_W(W_1) \cap \tau_W(W_2) = \emptyset$.

If these conditions are fulfilled then one concludes [15]:

1. $\tau_W(W_1') = (\tau_W(W_1))'$.
2. To τ_W there exists an element g of the Poincaré group with $\tau_W(W_1) = \alpha_g(W_1)$.
3. If the group generated by the τ_W acts transitively on $\mathcal{W}$ and if for one W there holds $\tau_W W = W'$ then the group generated by the τ_W is $\mathcal{P}_+$.

Next we look at the theory defined by the wedges. Let $\{\mathcal{M}(W).\Omega\}$ be a representation such that Ω is cyclic for every $\mathcal{M}(W)$. Assume to every W exists an anti-unitary operator J_W such that

$$J_W \mathcal{M}(W_1) J_W = \mathcal{M}(\tau_W(W_1))$$

holds. Moreover it is required that

(iv) $W \to \mathcal{M}(W)$ is an order preserving bijection.

(v) $W_1 \cap W_2 \neq \emptyset \to \Omega$ is cyclic for $\mathcal{M}(W_1) \cap \mathcal{M}(W_2)$. If Ω is cyclic for $\mathcal{M}(W_1) \cap \mathcal{M}(W_2)$ then holds $\overline{W}_1 \cap \overline{W}_2 \neq \emptyset$.

Using this the authors [15] obtained:

Theorem 2.18. (Buchholz, Dreyer, Florig and Summers). *Assume we are dealing with a family of wedge algebras on a Hilbert space $\mathcal{H}$ and mappings τ_W, such that the conditions (i)–(v) above are fulfilled. Let a vector $\Omega \in \mathcal{H}$ be cyclic for every $\mathcal{M}(W)$. Assume to every W there exists an antilinear involution J_W such that*

$$J_W \mathcal{M}(W_1) J_W = \mathcal{M}(\tau_W W_1).$$

Furthermore, assume $J_W \Omega = \Omega$ for every W and that $\{\tau_W\}$ has the transitivity property mentioned under 3 above. Then the operators J_W generate a projective representation of the group $\mathcal{P}_+$. If we denote by $J(g)$ the representant of $g \in \mathcal{P}_+$ then

$$J(g)\mathcal{M}(W)J^*(g) = \mathcal{M}(gW).$$

Moreover, the theory fulfills wedge duality, and since $\mathcal{P}_+$ contains the element -1 the theory is PCT-covariant with $\Theta = J(-1)$.

Since by assumption (ii) the $\mathcal{M}(W)$ generate the algebras of the double cones one obtains Poincaré covariance of the QFTLO. In [16] it was shown that the representation of the translation group is continuous. But the representation of the translations does not need to fulfill the spectrum condition. The spectrum condition holds only if the group generated by the J_W contains also the modular groups of the wedges.

Another aspect of the wedge algebras and the modular theory is the construction of a QFTLO out of a finite set of wedge algebras fulfilling some requirements among each other. This construction leads to the family of all wedge algebras. This set is covariant under a representation of the Poincaré group $\mathcal{P}_+^\uparrow$. Since this representation is generated by the modular groups of the wedges it is the minimal representation. These results are due to Wiesbrock [44, 45] and to Kähler and Wiesbrock [36]. We do not want to go into the details of this program because this would need the introduction of too many new concepts.

3 Proofs

In this section we present the proofs of the new results, Thms. 2.1–2.3 and Thms. 2.15 and 2.16. For the proofs of Thms. 2.1–2.3 one should notice that the antilinearity of θ implies the relation $(\theta^*\omega)(A) = \omega(\theta(A^*))$. This is best seen by looking at a situation where θ is implemented. In this case one has

$$\omega_\psi(\theta(A)) = (\psi, \Theta A \Theta \psi) = (\Theta\psi, A^* \Theta \psi) = (\theta^*\omega_\psi)(A^*).$$

Proof. **of Theorem 2.1** For $f \in \mathcal{L}^1(\mathbb{R}^d)$ and $A \in \mathcal{A}(\mathbb{R}^d)$ set $A(f) = \int d^dx f(x)\alpha_x(A)$. This is defined as a strong integral since α_x acts strongly continuously. For $a \in V^+$ let $\mathcal{L}_a$ be the left-ideal defined by the set

$$\{A(f) : A \in \mathcal{A}(\mathbb{R}^d), f(x) \in \mathcal{L}^1(\mathbb{R}^d), \text{ and support } \tilde{f}(p) \subset \mathbb{R}^d \setminus (-a + \overline{V}^+)\}.$$

If ω is a state on $\mathcal{A}(\mathbb{R}^d)$ with $\omega(\mathcal{L}_a) = 0$ then π_ω defines a particle representation. If $E(p)$ denotes the spectral family of the corresponding representation of the translation group one has $\omega(E(D_{0,a}) = 1$ where $D_{0,a} = V^+ \cap (-V^+ a)$. Conversely we know that the set of all states, fulfilling $\omega(\mathcal{L}_a) = 0$ for some $a \in V^+$, is norm-dense in the set of all normal states (folium) of the universal particle representation.

Since θ and the translations fulfill (2.5), we have

$$\begin{aligned}\theta(A(f)) &= \theta(\int dx\{\int dp\, \mathrm{e}^{\mathrm{i}px}\alpha_x(A)\}) \\ &= \int dx\{\int dp\, \mathrm{e}^{-\mathrm{i}px}\alpha_{-x}(\theta(A))\} = (\theta(A))(f).\end{aligned}$$

Hence we get $\theta(\mathcal{L}_a) = \mathcal{L}_a$. This implies θ^* maps a norm-dense set of normal states of the universal particle representation onto itself. This implies the invariance of the set of particle representations.

The vacuum states are characterized by the annihilation of $\mathcal{L}_0$. This ideal is θ invariant, implying the invariance of the vacuum representations.

For details on the universal particle representation see [7]. The left-ideals $\mathcal{L}_a$ were introduced in [4]. □

Proof. **of Theorem 2.2** Let ω be a locally normal state on $\mathcal{A}(\mathbb{R}^d)$ with respect to a vacuum representation π_0. Since this representation is θ-invariant there exists an anti-unitary operator Θ on $\mathcal{H}_0$ with

$$\Theta\pi_0(A)\Theta = \pi_0(\theta A).$$

Since π_0 is a vacuum representation it has the Reeh-Schlieder property. Consequently every normal state on $\pi_0(\mathcal{A}(O))$ is a vector state. In particular local normality implies that for every O there exists a vector $\psi(O)$ with $\omega(A) = (\psi(O), \pi_0(A)\psi(O))$ for all $A \in \mathcal{A}(O)$. Let now O be symmetric with respect to the origin. Then $A \in \mathcal{A}(O)$ implies $\theta(A) \in \mathcal{A}(O)$ and then obtain

$$(\theta^*\omega)(A) = \omega(\theta(A^*)) = (\psi(O), \pi_0(\theta(A^*))\psi(O)) =$$
$$(\psi(O), \Theta\pi_0(A^*)\Theta\psi(O)) = (\Theta\psi(O), \pi_0(A)\Theta\psi(O)).$$

Hence $(\theta^*\omega)$ is a normal functional on $\pi_0(\mathcal{A}(O))$. This implies the first statement.

For the second statement notice that in a particle representation the translations are unitarily implemented and we know the commutation relations (2.5) of θ with the translations. Since every double cone can be translated in such a way that its center is at the origin we find that $(\theta^*\omega)$ is normal on $\pi_0(\mathcal{A}(O' + x))$ if and only if it is normal on $\pi_0(\mathcal{A}(O'))$. Hence it is sufficient to look at the complement of symmetric double cones. Since the algebras if these sets are invariant under θ the method of the locally normal case can be applied. Consequently the set of particle representations is θ-invariant. □

Proof. **of Theorem 2.3** Let ω be a β-KMS-state, then for $A, B \in \mathcal{A}(\mathbb{R}^d)$ the expression $F(t) = \omega(B\alpha_t(A))$ can be analytically continued into $S(-\beta, 0)$. At the lower boundary this function has the value $F(t - i\beta) = \omega(\alpha_t(A)B)$. Since the algebra is translation invariant and since the translations act strongly continuously, there is a norm-dense sub-algebra $\mathcal{A}^{\mathrm{an}}(\mathbb{R}^d)$ such that for $A \in \mathcal{A}^{\mathrm{an}}(\mathbb{R}^d)$ the expression $\alpha_t(A)$ is entire analytic in t. From the relation $\alpha_t \circ \theta = \theta \circ \alpha_{-t}$ we see that $\mathcal{A}^{\mathrm{an}}(\mathbb{R}^d)$ is invariant under θ. Moreover, the antilinearity of θ implies for complex z the equation

$$\alpha_z(\theta(A)) = \theta(\alpha_{-\overline{z}}(A)).$$

Inserting for $A, B \in \mathcal{A}^{\mathrm{an}}(\mathbb{R}^d)$ the operators $\theta(A)$ and $\theta(B)$ into the expression for $F(t)$ one obtains for complex z

$$F(z) = \omega(\theta(B)\alpha_z(\theta(A))) = \omega(\theta(B)\alpha_{-\overline{z}}(A))) = (\theta^*\omega)(\alpha_{-z}(A^*)B^*).$$

Since θ commutes with the time translations it follows that also $\theta^*\omega$ is translation invariant. Hence we obtain

$$(\theta^*\omega)(\alpha_{-z}(A^*)B^*) = (\theta^*\omega)(A^*\alpha_z(B^*)).$$

From

$$F(t - \mathrm{i}\beta) = \omega(\alpha_t(\theta(A))\theta(B)) = \omega(\theta(\alpha_{-t}(A)B)) = \\ (\theta^*\omega)(B^*\alpha_{-t}(A^*)) = (\theta^*\omega)(\alpha_t(B^*)A^*),$$

we see that $(\theta\omega)$ fulfills the β-KMS-condition for elements $A, B \in \mathcal{A}^{\mathrm{an}}(\mathbb{R}^d)$. Since $\mathcal{A}^{\mathrm{an}}(\mathbb{R}^d)$ is norm-dense in $\mathcal{A}(\mathbb{R}^d)$, the β-KMS-conditions holds for arbitrary elements. This implies that $\theta^*\omega$ is again a β-KMS-state. Hence the set of β-KMS-representations is θ invariant. □

Proof of Theorem 2.15 If the representation of the Poincaré group is the minimal one, then Eq. (2.7) holds, i.e.,

$$\Theta = J_W U(R_W(\pi)),$$

provided the origin is contained in the edge of the wedge. $R_W(\alpha)$ denotes the rotation in the two-plane perpendicular to the characteristic two-plane of the wedge, and J_W the modular conjugation of the algebra of the wedge. Therefore, one has to solve two problems:

We know from Thm. 2.7 that the wedge duality is equivalent to analyticity properties of sufficiently many $A \in \mathcal{M}(D),\ D \subset W$ and this condition is in particular fulfilled by our assumption (ii). For these elements

$$U(\Lambda_W(-\tfrac{\mathrm{i}}{2}))A\Omega = \widehat{A}\Omega,$$

holds with $\widehat{A}\eta\mathcal{M}(K(P_W D))$, where K is a cylindrical set introduced in the remarks following Thm. 2.6 and P_W the reflection in the characteristic two-plane of the wedge. In order to show the PCT-theorem one must prove that the map

$$A\Omega \longrightarrow \widehat{A}\Omega, \quad A \in \mathcal{M}(D),\ D \subset W$$

sends double cone algebras into double cone algebras and not only into cylindrical set algebras.

The problem of the double cones can be looked at as follows: The element $R_W(\pi)\Lambda_W(-\frac{\mathrm{i}}{2})$ considered as an element of the complex Lorentz group is the unique element -1. Looking at representations, then $U(\Lambda_W(t)$ has an analytic continuation when applied to vectors $A\Omega$ with sufficient analyticity properties. Starting from different wedges it must be shown that one does not end by -1 on different sheets of an analytic manifold.

The second problem is the following: From the discussion following Thm. 2.6 we know that one can write

$$U(\Lambda_W(t)) = \Delta_W^{\mathrm{i}t} V_W(t),$$

where $V_W(t)$ maps every translated cylinder set algebra $\mathcal{M}(K(D)+x)$, $D \in W$ onto itself. We would like to know conditions implying that $V_W(t)$ is a gauge transformation, which means that it maps the algebra of every double cone onto itself.

We start with the first problem. Let D be a double cone with center at the origin. Let $D+x \subset W$ and define

$$\Gamma(D+x) = \{g \in \mathcal{P}_+^\uparrow : \overline{D}+x \subset gW\}. \tag{3.1}$$

Since W is open, $\Gamma(D+x)$ is open and contains the identity of the group. The first observation is:

Lemma 3.1. *There exist $g_1, g_2, ..., g_6 \in \Gamma(D+x)$ and $T_1, ...T_6 > 0$, such that*

$$D+x \subset \Lambda_{g_6 W}(t_6) \cdots \Lambda_{g_1 W}(t_1) W$$

for $|t_i| < T_i$, $i = 1, ..., 6$. The elements $g_1, ..., g_6$ can be chosen in such a way that the generators of the groups $\Lambda_{g_i W}(t_i)$ are linearly independent.

Proof. Let a neighborhood of the identity $\Gamma_1(D+x)$ be a subset of $\Gamma(D+x)$ such that $g_1, ..., g_6 \in \Gamma_1(D+x)$ implies $g_1 ... g_6 \in \Gamma(D+x)$. Since $\Gamma_1(D+x)$ is a neighborhood of the identity there exists $g_1 = 1, g_2, ..., g_6 \in \Gamma_1(D+x)$, such that the generators of $\Lambda_{g_i W}(t)$ are linearly independent. Choosing T_i such that $\Lambda_{g_i W}(t_i) \in \Gamma_1(D+x)$ for $|t_i| < T_i$ then the statements of the lemma are fulfilled. $\square$

With help of the last lemma we can construct an analytic function on parts of the complex Lorentz group $\hat{\mathcal{P}}_+$. (Its elements will be denoted by $\hat{g}$.)

Proposition 3.2. *Let $A \in \mathcal{M}^{\mathrm{fa}}(D)$ and let $g_1, \ldots, g_6$ and $D+x$ be as in the last lemma. Then the function*

$$U(\Lambda_{g_6 W}(t_6)) ... U(\Lambda_{g_1 W}(t_1)) A\Omega, \quad A \in \mathcal{M}(D+x), \quad D+x \subset W \tag{3.2}$$

has an analytic continuation into all t–variables. The function $U(\Lambda_{g_j W}(-\frac{\mathrm{i}}{2}))A\Omega$ is the boundary value of an analytic function and the product

$$U(R_{g_j W}(\pi)) U(\Lambda_{g_j W}(-\tfrac{\mathrm{i}}{2})) A\Omega \tag{3.3}$$

is independent of g_j.

Proof. In the variable t_i the above function can be analytically extended in the into the strip $S(-\frac{1}{2},0)$. provided we keep $t_6,...,t_{i+1},t_{i-1},...,t_1$ real and in their proper domain. An analytic continuation in all t variables is obtained with help of the Malgrange–Zerner theorem (see e.g. [23]) The domain into which the function can be continued has still to be determined. It is clear from the construction that the real function is the boundary value of the analytic continuation.

Next we want to determine the domain of holomorphy of this function. This calculation will be done by mapping the strip $S(-\frac{1}{2},0)$ bi–holomorphically onto itself in such a way that the interval $|x|<T$ is mapped onto $\mathbb{R}$ and the rest of the boundary onto $-\frac{\mathrm{i}}{2}+\mathbb{R}$. This is achieved by the transformation

$$\zeta = \frac{1}{2\pi}\log\frac{1-\mathrm{e}^{-2\pi T}}{\mathrm{e}^{2\pi T}-1}\frac{\mathrm{e}^{2\pi T}-\mathrm{e}^{-2\pi z}}{\mathrm{e}^{-2\pi z}-\mathrm{e}^{-2\pi T}}. \tag{3.4}$$

With these new variables the domain of holomorphy becomes

$$0 > \sum_{i=1}^{6}\Im m\,\zeta_i > -\frac{1}{2}. \tag{3.5}$$

If the elements $g_1,...,g_6$ are properly chosen then an interior point of the ζ variables corresponds to an interior point in the $\hat{g}$ variables.

In the ζ–variable the domain (3.5) is convex and hence simply connected. Since the transformation (3.4) is bi–holomorphic, it follows that also the image in the t–variables is simply connected. Hence there are no monodromy problems in these variables.

Note that the symbol $U(\Lambda)$ denotes a representation of the Lorentz group and therefore, the first expression of Eq. (3.2) can be written as

$$U(\Lambda_{g_6W}(t_6)\cdots\Lambda_{g_1W}(t_1))A\Omega. \tag{3.6}$$

The arguments inside of U are defined for all elements of the complex Lorentz group. U applied to this product is defined if it belongs to the domain Eq. (3.5) (transformed into the variable z), eventually multiplied from the left with an element of the real Lorentz group.

In particular we can look at the product $U(R_{g_jW}(\pi))U(\Lambda_{g_jW}(-\frac{\mathrm{i}}{2}))U(g)A\Omega$ for g sufficiently close to 1 in (3.1) and obtain

$$U(R_{g_jW}(\pi)\Lambda_{g_jW}(-\tfrac{\mathrm{i}}{2})g)A\Omega = U(gR_{g_jW}(\pi)\Lambda_{g_jW}(-\tfrac{\mathrm{i}}{2}))A\Omega = U(g)U(R_{g_jW}(\pi))U(\Lambda_{g_jW}(-\tfrac{\mathrm{i}}{2}))A\Omega.$$

This implies $U(g)U(R_{g_jW}(\pi))U(\Lambda_{g_jW}(-\frac{\mathrm{i}}{2}))U(g^{-1}) = U(R_{gg_jW}(\pi))U(\Lambda_{gg_jW}(-\frac{\mathrm{i}}{2}))$. Choosing $\Gamma_1(D+x)$ in such a way that it contains with g_j also its inverse, then the statement of the proposition is obtained by choosing $g = g_i g_j^{-1}$. □

Collecting the result of the discussion we obtain with $A_x = T(x)AT(-x)$

Proposition 3.3. *Let D be a double cone centered at the origin and x such that the closure of $D+x$ does not contain the origin. Then for $A \in \mathcal{M}^{\mathrm{fa}}(D)$ and g such that $D+x \subset gW$ the vector function*

$$U(\Lambda_{gW}(-\tfrac{\mathrm{i}}{2}))U(R_{gW}(\pi))A_x\Omega$$

is independent of g.

Proof. From the above discussion we know that the statement is true for g in a sufficiently small neighborhood of the identity in $\mathcal{P}_+^\uparrow$. But this implies by varying the wedges and the neighborhoods that it is true for all $g \in \Gamma(D+x)$. □

Using Prop. 3.3 we find

Theorem 3.4. *Let $A \in \mathcal{M}^{\mathrm{fa}}(D)$ with D centered at the origin, and let $D+x \subset W$. Then one has*

$$U(\Lambda_W(-\tfrac{\mathrm{i}}{2}))A_x\Omega = \widehat{A}_{P_Wx}\Omega$$

with $\widehat{A}_{P_Wx} \in \mathcal{M}(D + P_Wx)$.

Proof. From Eq. (3.4) we know $U(R_{gW}(\pi))U(\Lambda_{gW}(-\frac{\mathrm{i}}{2}))A_x\Omega$ is independent of g as long as g belongs to the set $\Gamma(D+x)$ (see Eq. (3.2)). Hence we obtain

$$U(R_W(\pi))U(\Lambda_W(-\tfrac{\mathrm{i}}{2}))A_x\Omega = U(R_W(\pi))\widehat{A}_x\Omega$$

with

$$U(R_W(\pi)\widehat{A}_xU(R_W(\pi) \subset \bigcap_{g\in\Gamma(D+x)} \mathcal{M}(-K_{gW}(D+x)).$$

Next observe that a translation in the characteristic two-plane of W commutes with $U(R_W(\pi))$ and is mapped onto its negative by $U(\Lambda_W(-\frac{\mathrm{i}}{2}))$. This implies

$$U(R_W(\pi))\widehat{A}_xU(R_W(\pi)) \subset \bigcap_{D+x\subset W}\ \bigcap_{g\in\Gamma(D+x)} \mathcal{M}(-K_{gW}(D+x) = \mathcal{M}(-D-x).$$

Using the fact that D is symmetric and that $R_W(\pi)(-x) = P_Wx$ holds we get the result of the theorem. □

Now we are prepared to show Thm. 2.15.

Proof. First a remark: If $\mathcal{A} \in \mathcal{M}(W)$ and $U(\Lambda_W(t)A\Omega$ has an analytic continuation into the strip $S(-\frac{1}{2}, 0)$ then the same holds for $V_W(t)A\Omega$ because $\Delta_W^{\mathrm{i}t} A\Omega$ has an analytic continuation and the relation

$$U(\Lambda_W(t)) = V_W(t)\Delta_W^{\mathrm{i}t} \tag{3.7}$$

holds. (For a detailed proof see [7].) Since all groups in (3.7) leave the wedge W invariant it follows that they commute. If $A \in \mathcal{M}(D)^{\mathrm{fa}}$ then $U(R_{gW}(\pi))U(\Lambda_{gW}(-\frac{\mathrm{i}}{2}))$ acts locally on $A_x\Omega$ i.e., it maps $A_x\Omega$ into $\mathcal{M}(D-x)$ and is independent of gW, as long as $g \in \Gamma(D+x)$. By Eq. (3.7) we can replace $U(\Lambda_{gW}(-\frac{\mathrm{i}}{2})$ by $R_{gW}(-\frac{\mathrm{i}}{2})\Delta_{gW}^{\frac{1}{2}}$. This implies that $U(R_{gW}(\pi))V_{gW}(-\frac{\mathrm{i}}{2})\Delta_{gW}^{\frac{1}{2}}$ acts locally on $A_x\Omega$ and is independent of g, as long as $g \in \Gamma(D+x)$. Replacing A_x by its adjoint A_x^* and observing that the Tomita conjugation S_{gW} acts locally on $A_x\Omega$ and is independent of g in the same range as before, we find, that $U(R_{gW}(\pi))V_{gW}(-\frac{\mathrm{i}}{2})J_{gW}$ has the good properties, i.e., it maps $A_x\Omega$ into $\mathcal{M}(D-x)\Omega$ and is independent of g for $g \in \Gamma(D+x))$. We define

$$\widehat{\Theta}_W = U(R_W(\pi))V_W(-\tfrac{\mathrm{i}}{2})J_W = U(R_W(\pi))J_W V_W(\tfrac{\mathrm{i}}{2}).$$

Notice that J_W commutes with $V_W(t)$. Hence we obtain $J_W V_W(-\frac{\mathrm{i}}{2})J_W = V_W(\frac{\mathrm{i}}{2})$
$= V_W(-\frac{\mathrm{i}}{2})^{-1}$. This implies $\widehat{\Theta}_W^2 = \mathbf{1}$. By assumption the fully analytic elements applied to the vacuum are a core for $\Delta_W^{-1/2}, U(\Lambda_W(-\frac{\mathrm{i}}{2})$ and hence also for $V_W(-\frac{\mathrm{i}}{2})$ and $\widehat{\Theta}_W$. (Remember that $U(\Lambda_W(t))V_W(t) = \Delta_W^{\mathrm{i}t}$.) Denoting the closure by the same symbol, $\widehat{\Theta}_W$ has a polar decomposition

$$\widehat{\Theta}_W = \Theta_W T_W.$$

By the uniqueness of the decomposition and the positivity of $V_W(\frac{\mathrm{i}}{2})$ it follows that the relation $T_W = V_W(\frac{\mathrm{i}}{2})$ holds. Since $\widehat{\Theta}_W$ maps $\mathcal{M}^{\mathrm{fa}}(D+x)$ onto $\mathcal{M}^{\mathrm{fa}}(D-x)$ the same must hold for Θ_W. Since by analytic continuation these expression are independent of W we get that $\Theta_W = \Theta$ is the PCT-operator and $V_W(\frac{\mathrm{i}}{2})$ is independent of W and maps $\mathcal{M}^{\mathrm{fa}}(D+x)$ onto itself. □

Proof. **of theorem 2.16** During the proof of the last theorem we had constructed the operator $\widehat{\Theta} = U(R_W(\pi))J_W V_W(\frac{\mathrm{i}}{2})$. By the uniqueness of the polar decomposition we see that the positive operator $V_W(\frac{\mathrm{i}}{2})$ and the PCT-operator $\Theta = U(R_W(\pi))J_W$ act locally and are independent of W. This implies that J_W acts locally and maps $\mathcal{M}(D+x)$ onto $\mathcal{M}(D+P_W x)$. Writing $V_W(t) = \mathrm{e}^{\mathrm{i}X_W t}$ then since $\mathrm{e}^{-X_W/2}$ is independent of W we conclude that also $V_W(t) = \mathrm{e}^{\mathrm{i}X_W t} = V(t)$ is independent of W. This implies

$$[V(t), U(\Lambda)] = 0$$

for all Λ. We know that $V(t)$ maps $\mathcal{M}(K_W(D+x))$ onto itself for every W. Choosing $x = 0$ and varying W we find that $V(t)$ maps $\mathcal{M}(D)$ onto itself. Since $V(t)$ commutes with the translations it is a local gauge. Hence also $\Delta_W^{\mathrm{i}t}$ acts locally for every W. Therefore $V(t)$ must define a representation of the Lorentz group. Since this group representation is Abelian it must be trivial. This implies that $U(g)$ is the minimal representation. $\square$

4 Examples

Free quantum fields, i.e., fields that fulfill the Klein Gordon equation, provide simple examples illustrating several of the points discussed in the previous sections.

We consider first the case of a single, hermitian Bose field Φ with mass $m > 0$ on $\mathbb{R}^d$, $d \geq 2$, transforming covariantly with respect to the translation group, but not necessarily w.r.t. the Lorentz group. The Fock space representation of Φ is determined by the two point function,

$$\mathcal{W}_2(x-y) = \langle \Omega, \Phi(x)\Phi(y)\Omega \rangle. \tag{4.1}$$

Locality, spectrum condition and positivity imply (by the Jost-Lehmann Dyson representation [7]) that its Fourier transform can be written

$$\tilde{\mathcal{W}}_2(p) = M(p)\theta(p^0)\delta(p\cdot p - m^2) \tag{4.2}$$

where $M(p)$ is a polynomial in p satisfying

$$M(p) = M(-p) \quad \text{and} \quad M(p) \geq 0 \tag{4.3}$$

for p on the positive mass shell $H_m^+ = \{p :\ p^0 > 0,\ p\cdot p = m^2\}$. Conversely, every such M defines a free field satisfying all Wightman axioms except possibly Lorentz covariance. The field Φ gives rise to a QFTLO where the local von Neumann algebras $\mathcal{M}(O)$ are generated by the Weyl operators $\exp(\mathrm{i}\Phi(f)$ with real test functions f supported in O.

According to the results of [25] wedge duality is always violated unless M is constant on the mass shell, i.e., unless Φ is the usual Poincaré covariant, scalar free field. In particular, neither the condition of geometric modular action nor the Bisognano Wichmann property hold for fields with non-constant M. On the other hand, it is trivial that PCT symmetry *always* holds in these examples, irrespective of M and d. In fact, since $\Phi(x) = \Phi(x)^*$ and $\mathcal{W}_2$ depends only on $x - y$, we have

$$\langle \Omega, \Phi(x)\Phi(y)\Omega \rangle = \langle \Omega, \Phi(-y)\Phi(-x)\Omega \rangle \tag{4.4}$$

which is exactly PCT symmetry. Space and time inversion are only symmetries separately, however, if M is even in p^0. These examples thus shows that

- Wedge duality is not a necessary condition for PCT symmetry.
- Full PCT symmetry is not in conflict with odd dimensionality of space-time.

More generally we may consider free fields Φ_α that have an arbitrary number of components and are not necessary hermitian. As before we assume translation covariance and spectrum condition, but not necessarily Lorentz covariance. It is convenient to take the set of indices α, that label the components of the field including the adjoint operators, as a basis of a complex vector space $\mathcal{K}$. The field operators $\Phi(f, \sigma)$ thus depend linearly on $\sigma \in \mathcal{K}$ besides the test function f, and we can write the adjoints as $\Phi(f, \sigma)^* = \Phi(\bar{f}, \sigma^*)$, where $\bar{f}$ is the complex conjugate test function and $\sigma \to \sigma^*$ is an antilinear involution on $\mathcal{K}$. (In the case of hermitian fields σ^* is simply complex conjugation of the components of σ with respect to the basis.) The Fourier transform of the two point function

$$\mathcal{W}_2(x - y; \sigma, \rho) = \langle \Omega, \Phi(x, \sigma)\Phi(y, \rho)\Omega \rangle \tag{4.5}$$

can now be written as

$$\tilde{\mathcal{W}}_2(p; \sigma, \rho) = M_{\sigma,\rho}(p)\theta(p^0)\delta(p \cdot p - m^2) \tag{4.6}$$

where $M_{\sigma,\rho}(p)$ is a polynomial in p that depends bilinearly on σ and ρ. For Bose fields locality is equilvalent to

$$M_{\sigma,\rho}(p) = M_{\rho,\sigma}(-p) \tag{4.7}$$

and positivity to

$$M_{\sigma^*,\sigma}(p) \geq 0, \qquad \text{for } p \in H_m^+, \tag{4.8}$$

which implies in particular the hermiticity condition

$$M_{\sigma^*,\rho}(p) = \overline{M_{\rho^*,\sigma}(p)}, \qquad \text{for } p \in H_m^+. \tag{4.9}$$

PCT symmetry of the field, on the other hand, requires that

$$M_{\sigma,\rho}(p) = \overline{M_{\rho,\sigma}(p)} = M_{\sigma^*,\rho^*}(p). \tag{4.10}$$

It is easy to give examples that satisfy (4.7) and (4.8) but not (4.10). For instance, on can take an n component field, $n \geq 2$, with $M(p)$ given by a positive definite $n \times n$ matrix whose diagonal elements are even polynomials in p and whose off diagonal elements are odd polynomials that are purely imaginary for $p \in H_m^+$. It has to be remarked, however, that not all such examples violate PCT symmetry *for the algebra of observables* generated by the field. In fact, two different multicomponent fields $\Phi_\alpha^{(1)}$ and $\Phi_\beta^{(2)}$ may generate the same local algebras of observables, and one of them can fulfill (4.10) while the other does not. To discuss this in a little more detail let us equip $\mathcal{K}$ with a scalar product $\langle\cdot,\cdot\rangle$ and write

$$M_{\sigma^*,\rho}(p) = \langle\sigma, M(p)\rho\rangle \tag{4.11}$$

where $M(p)$ is a hermitian linear operator on $\mathcal{K}$. If $M^{(1)}(p)$ and $M^{(2)}(p)$ correspond to the two different fields, then the generated local algebras are clearly equal if

$$M^{(1)}(p) = L(p)^* M^{(2)}(p) L(p) \tag{4.12}$$

where $L(p)$ is an invertible linear operator on $\mathcal{K}$ for all $p \in H_m^+$ whose matrix elements, together with those of $L(p)^{-1}$, are polynomials in p. In fact, $L(\mathrm{i}\partial)$ is then an invertible matrix of differential operators (at least when operating in fields satisfying the Klein-Gordon equation) and the two fields are related by

$$\Phi_\alpha^{(1)}(x) = \textstyle\sum_\beta L_{\alpha\beta}(\mathrm{i}\partial)\Phi_\beta^{(2)}(x), \quad \Phi_\beta^{(2)}(x) = \textstyle\sum_\alpha L_{\beta\alpha}^{-1}(\mathrm{i}\partial)\Phi_\alpha^{(1)}(x). \tag{4.13}$$

For a true counterexample to PCT symmetry of the QFTLO generated by a free field $\Phi_\alpha^{(1)}$ one must therefore pick $M^{(1)}(p)$ in such a way that it can *not* be written as (4.12) with a PCT symmetric $M^{(2)}(p)$. This, however can easily be shown e.g. for M defined by the matrix

$$M(p) = \begin{pmatrix} (p^0)^2 & imp^0 \\ -imp^0 & (p^0)^2 \end{pmatrix}. \tag{4.14}$$

It should even be possible to find within the class of Lorentz non-covariant, finite component free fields examples satisfying wedge duality, but still violating PCT symmetry. In fact, it is easy to check that wedge duality certainly holds, if $M(p)^{-1}$ exists for all for all $p \in H_m^+$ and has polynomial matrix elements. A sufficient, and by Lemma V.2 in [25] also necessary, condition for this is that $\det M(p)$ is constant on the mass shell. PCT symmetry, on the other hand, requires that $M(p)$ can be written as $L(p)^* M'(p) L(p)$ with $M'(p)$ satisfying (4.10) and $L(p)$ polynomial in p. Although an explicit example satisfying wedge duality without the latter property is not known to us there is hardly a doubt that such examples exist.

If one allows infinite dimensional index spaces $\mathcal{K}$ Oksak and Todorov [40] have given examples of free fields violating PCT symmetry, but with Lorentz covariance, i.e., where there is a representation $\Lambda \to V(\Lambda)$ of the Lorentz group on $\mathcal{K}$ such that

$$M_{\sigma,\rho}(\Lambda p) = M_{V(\Lambda^{-1})\sigma, V(\Lambda^{-1})\rho}(p). \tag{4.15}$$

All examples with infinite dimensional $\mathcal{K}$ violate the split property, but the status of wedge duality in this example, i.e., the validity of the analyticity condition of Theorem 2.6, is not known. The full analyticity condition of Theorem 2.15 is certainly violated.

Another instructive example is the case of an infinite number of copies of the neutral scalar free field. Here $\mathcal{K}$ is infinite dimensional, but $M_{\sigma,\rho}(p)$ is independent of p:

$$M_{\sigma^*,\rho}(p) = \langle \sigma, \rho \rangle \tag{4.16}$$

where $\langle \sigma, \rho \rangle$ is some scalar product on $\mathcal{K}$. This theory is clearly invariant under the Poincaré transformations that simply ignore the index space $\mathcal{K}$:

$$U_1(a,\Lambda)\Phi(f,\sigma)U_1^*(a,\Lambda) = \Phi(f_{\{a,\Lambda\}},\sigma) \tag{4.17}$$

with $f_{\{a,\Lambda\}}(x) = f(\Lambda^{-1}(x-a))$. This theory is also covariant under other representations of the Poincaré group. Let $V(\Lambda)$ be a continuous unitary representation of the Lorentz group acting on $\mathcal{K}$. Then we may define a representation U_2 by $U_2(a,\Lambda)\Omega = \Omega$ and

$$U_2(a,\Lambda)\Phi(f,\sigma)U_2^*(a,\Lambda) = \Phi(f_{\{a,\Lambda\}}, V(\Lambda^{-1})\sigma) \tag{4.18}$$

In this situation the cocycle linking U_1 and U_2 is itself a group representation and U_1 is the minimal representation. This follows from the fact that U_1 acts only on the test-functions which implies that the Wightman functions fulfill the Hall–Wightman analyticity [HW57] on the complex Lorentz group.

This is an example with wedge duality and the existence of two representations of the Poincaré group. One of them is the minimal one so that the Bisogano Wichmann property holds. For the second representation one has the partial analyticity property for $U_2(\Lambda_W(t))$ required for wedge duality, but not the full analyticity property required for Theorem 2.15. The analyticity for a fixed wedge W holds because $V(\Lambda_W(t))$ is a one-parametric group, which has sufficiently many analytic elements. On the other hand, since $V(\Lambda)$ is a non-trivial unitary representation of the Lorentz group it does not have an analytic continuation onto the complex Lorentz group and therefore there are no fully analytic elements.

Finally we mention an example given in [15], Section 5.3, where the Bisognano-Wichmann property is violated, but the conditions of Theorem 2.18 are fulfilled so that Poincaé covariance (without spectrum condition) and PCT symmetry hold. In this example the split property is fulfilled. One may ask whether Poincaré covariance, spectrum condition and nuclearity (which implies the split property) are sufficient to derive the PCT theorem. No counterexamples are known and Theorems 2.12 and 2.13 may be regarded as a step in this direction, but this question is otherwise open.

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