A remark on "Some remarks on the problem of ergodicity of the Standard Map"

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Abstract We consider the standard family $F_g : (x,y) \mapsto (g\cos(2\pi x) - y, -x)$ of area-preserving maps defined on the two-torus $\mathbf{R}^2/\mathbf{Z}^2$, the parameter g ranging along the real axis $\mathbf{R}$. We prove that there exists a subset $\mathcal{E} \subset \mathbf{R}$ whose density tends to zero along the real axis, such that for any g in the complement to $\mathcal{E}$ the map F_g is ergodic an has nonzero Lyapunov exponents almost everywhere in $\mathbf{R}^2/\mathbf{Z}^2$ with respect to the Haar measure.

This note is an immediate continuation of the paper [1]. We investigate the map $F_g : \mathbf{R}^2/\mathbf{Z}^2 \to \mathbf{R}^2/\mathbf{Z}^2$ given by equations $F_g(x,y) = (x_1, y_1)$ where

$$\begin{aligned} x_1 &= g\cos(2\pi x) - y \bmod 1\,, \\ y_1 &= x \bmod 1\,, \end{aligned} \tag{1}$$

Our goal is to prove the following

Theorem 1 *There exists a subset $\mathcal{E} \subset \mathbf{R}$ and a constant $C > 0$ such that*

(i) the following estimate is true:

$$leb(\mathcal{E} \cap [p/2, (p+1)/2]) \leq C\,|p|^{-1}\,, \tag{2}$$

for all $p \in \mathbf{Z}$ (here leb denotes the Lebesgue measure on $\mathbf{R}$);

(ii) if $g \in \mathbf{R} \setminus \mathcal{E}$, then the map F_g is ergodic, and its Lyapunov exponents are nonzero almost everywhere in in $\mathbf{R}^2/\mathbf{Z}^2$ with respect to the Haar measure.

Let us recall in brief the construction, suggested in [1]. Evidently, it is sufficient to consider only sufficiently large positive values of g. The phase space can be reduced by means of additional symmetry generated by the maps $S_1 : (x,y) \mapsto (1/2 - x, 1/2 - y) \bmod 1$, $S_2 : (x,y) \mapsto (-x, y + 1/2) \bmod 1$, $S_3 : (x,y) \mapsto (x + 1/2, -y) \bmod 1$. The quotient space is the projective plane which can be represented by means of of a fundamental domain which is the square $[-0.25, 0.25] \times [-0.25, 0.25]$ with edges glued by the maps

$$g_1 : (-0.25, y) \mapsto (0.25, -y) \quad \text{and} \quad g_2 : (x, -0.25) \mapsto (-x, 0.25)\,. \tag{3}$$

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The angle points $(\pm 0.25, \pm 0.25)$ represent a hyperbolic periodic orbit of period 2 whose stable and unstable manifolds are the basic elements of our construction. There exist initial arcs, W_0^s and W_0^u which connect the points $(-0.25, 0.25)$ and $(0.25, -0.25)$ with a point H on the diagonal $x = y$ of the square close to $(0.25, 0.25)$, the following equality being fulfilled: $R(W_0^s) = W_0^u$ where R is a reversor defined by the equations $(x, y) \mapsto (y, x)$. Denote $W_n^u = F^n(W_0^u), n = 1, 2 \ldots$, and $W_n^s = F^{-n}(W_0^s), n = 1, 2 \ldots$. The partition $\mathcal{P} = \{A_1, A_2, \ldots, A_N$ of the phase space created by the cells of the web $W_0^u \cup W_0^s \cup W_1^s$ gives rise to a symbolic dynamics: almost each point z of the phase space acquires an infinite code $(\ldots, n_{-2}(z), n_{-1}(z), n_0(z), n_1(z), n_2(z), \ldots)$ by the rule: $F_g^i(z) \in A_{n_i(z)}, i \in \mathbf{Z}$.

In [1] we constructed a decreasing sequence $\mathrm{FLE}(n), n = 1, 2, \ldots$, of subsets of the phase space with the following properties:

1. $\mathrm{FLE}(n)$ is a union of some cells of $W_k^s \cup W_k^u$;

2. the complement to $\bigcup_{r=-n+1}^{r=n-1} F_g^r(\mathrm{FLE}(n))$ possesses a cone structure: there are fields of invariant stable and unstable cones (see [3]).

A crucial role in proving Theorem 1 plays the estimate

$$leb(\mathrm{FLE}(n)) \leq C_1 \mu^{n+1} \tag{4}$$

where $C_1 > 0$ and $0 < \mu < 1$ do not depend on n. Of course (4) is not true for all values of the parameter g. An obstacle to (4) is the existence of so-called "islands of stability" created by KAM circles. Indeed, a periodic chain of islands of positive measure possesses at least one component inside $\mathrm{FLE}(n)$. In [1] we discovered the following mechanism of creating islands. The cells of $W_k^s \cup W_k^u$ which belong to $\mathrm{FLE}(n)$ form a kind of one-dimensional structure placed in a vicinity of the coordinate axes $x = 0$ and $y = 0$. The picture is symmetric with respect to the reversor R (reflection with respect to the diagonal), so we can restrict ourselves with considering only the horizontal branch of $\mathrm{FLE}(n)$. Its cells can be aggregated in "units", $c_k, k = (k_1, k_2, \ldots, k_n)$ consisting of all points $z \in \mathrm{FLE}(n)$ such that $n_{i-1}(z) = k_i, i = 1, 2, \ldots, n$. We found in [1] that each c_k contains a "central spot", $w_k^* \subset c_k$, whose sizes are much less than those of c_k and which possess the following property: if c_k contains an island such that its n-th iterate contains in the vertical part of $\mathrm{FLE}(n)$ then the mentioned island is contained in w_k^*. The sizes of w_k^* admit the estimates:

$$\text{the width of } w_k^* \sim g^{-1-2n}, \quad \text{the height of } w_k^* \sim g^{-1-n}, \tag{5}$$

It is not difficult to prove that $F^n(w_k^*$ is contained in a cell c' of the vertical part of $\mathrm{FLE}(n)$ such that $R(c') \subset c_{\overline{k}}$ where $\overline{k}$ is the "inverse" sequence to k: $\overline{k} = (\sigma(k_n), \sigma(k_{n-1}, \ldots \sigma(k_1)$ where $\sigma : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$ is the involution defined by the equalities: $RF_f(A_i) = A_{\sigma(i)}, i = 1, 2, \ldots, N$.

A necessary condition for the existence of an island of the considered type reads

$$w_{\overline{k}}^{\prime\prime *} \cap w_l^* \neq \emptyset \tag{6}$$

where $w_{\overline{k}}^{\prime\prime *} = F_g^{n+1}(w_k^*)$, a small horseshoe, and w_l^* is another central spot with the code l. In [1] we tried to destroy *all* collisions of the form (6) by throwing out intervals on the axis of the parameter g for which the collisions take place. Thus we obtained the estimate

$$leb(\mathcal{E}_n) \leq const \cdot g^{-1} \tag{7}$$

where $\mathcal{E}_n$) is the union of all intervals in $[p/2,(p+1)/2]$ containing the values of g for which (6) occurs in FLE(n). This does not suit to our aim because the sum of the right hand sides of (7) for all n diverges and cannot be small To overcome this difficulty we formulated in [1] a conjecture which concerns possible codes of cells containing islands. Unfortunately this conjecture hardly can be true.

In the present note we make another step: we eliminate the *double* collisions of the form

$$w''^{*}_{\overline{k}} \cap w^*_l \neq \emptyset \qquad \text{and} \qquad RF_g^{-1}(w^*_k) \cap w^*_m \neq \emptyset\,. \tag{8}$$

If $n \geq 2$, any periodic chain of islands of considered type has a fragment contained in the spots entering (8) and their iterates. Hence the condition (8) also is necessary for existing islands. Denote by $\mathcal{E}_n^{(2)}$ the set of all g in the interval $[p/2,(p+1)/2]$ such that (8) fails for all triples m,l,m of multi-codes of the length n.

Statement 1 *The following estimate is true uniformly for all n and p:*

$$leb(\mathcal{E}_n^{(2)}) \leq (C_3 p)^{-n} \tag{9}$$

where C_3 is a positive number.

Theorem 1 follows from Statement 1 if we set $\mathcal{E} = [-R,R] \cup \mathcal{E}' \cup (-\mathcal{E}')$ where R is a sufficiently large positive number and $\mathcal{E}' = \mathcal{E}_1 \cup \bigcup_{n=2}^{\infty} \mathcal{E}_n^{(2)}$. If g does not belong to $\mathcal{E}$ then F_g does not contain islands, and there is no obstacle for (4) to be true. The rigorous proof of (4) is cumbersome. The mentioned cone structure on the complement to FLE(n) and its forward and backward iterates up to the order $n-1$ gives rise to the positive Lyapunov exponents.

An idea of the proof of Statement 1. Denote by x_k the x-coordinate of the center of w^*_k. Let $g = p/2 + t$ where the variable t ranges through the interval $[0,1/2]$. In view of (1) and (5) the condition (8) is equivalent to the following Diophantine equations:

$$x_{\overline{k}} + x_l = t + O\left(p^{-1-2n}\right)\,, \tag{10}$$

$$x_k + x_m = t + O\left(p^{-1-2n}\right)\,. \tag{11}$$

We have to evaluate the measure of the set of the values of the parameter t for which the equations (10) and (11) admit a solution. The problem can be interpreted as a dynamical problem in the following manner: how much time the "diagonal" trajectory on $\mathbf{R}^2$ given by the equation $t \mapsto (t,t)$ stays in the $O\left(p^{-1-2n}\right)$-neighbourhood of the discrete set $\mathcal{D} = \{z_{(k,l,m)} = (x_{\overline{k}} + x_l, x_k + x_m) : (k,l,m) \in [0,N]^{3n}\} \subset \mathbf{R}^2$. If we fix the value of k the corresponding subset $\mathcal{D}_{||}$ of $\mathcal{D}$ becomes a two-dimensional discrete nonlinear "fractal lattice" whatever such words mean. Locally it can be approximated by a linear discrete fractal of the form

$$x = x^0 + \sum_{j=1}^{n}(l_i - l_i^0)\prod_{i=1}^{j}\lambda_i^{-1}\,, \qquad y = y^0 + \sum_{j=1}^{n}(m_i - m_i^0)\prod_{i=1}^{j}\mu_i^{-1}\,,$$

where λ_i and μ_i are kinds of multiplicators along the trajectories starting with the centers of w^*_l and w^*_m respectively (the formulae for the multiplicators can be extracted from the paper [2], but approximately $\lambda_i = -2\pi g \sin(2\pi x_i)$ where x_i is the first coordinate of the i-th point of the mentioned trajectory).

The linear problem can be resolved by successive application of the ergodic theorem to a particle moving in a rectangular lattice, the edges of which can be chosen incommensurate due to an appropriate choice of the center of the linearisation. The result should be the following: the time which the particle stays in the mentioned neighbourhood of the lattice asymptotically (if g tends to infinity) is equal to the relative area of the neighbourhood for the majority of the initial data (=values of k). Since the lattice is two-dimensional and contains $O\left(p^{2n}\right)$ points, it gives the following estimate: if we denote by $\mathcal{E}_{n,k}$ the mentioned above time then

$$leb(\mathcal{E}_{n,k}) = \left[O\left(p^{-1-2n}\right)\right]^2 \times O\left(p^{2n}\right)$$

Multiplying by the number of possible choices of the multi-vector k that is by the number of the order p^n , gives us the desired estimate (9)

The detailed proof is forthcoming.

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