VARIATIONAL METHODS FOR QUASI-PERIODIC SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

RAFAEL DE LA LLAVE
Mathematics Department, The University of Texas at Austin
Austin, Texas 78712-1082, USA
llave@math.utexas.edu

We show how to use variational methods to prove two different results: Existence of periodic solutions with irrational periods of some hyperbolic equations in one dimension and existence of spatially quasi-periodic solutions of some elliptic equations.

1 Introduction

In this note, we will present two results. The first is a remark about a question of ([1]) on whether one can use variational methods to produce solutions with irrational period of

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial x^2} + V(u) = 0$$
$$u(0, t) = u(1, t)$$

We will show that for some particular numbers there is an extremely simple answer.

The second result is to show how one can use variational methods to produce quasiperiodic solutions to some difference equations as well as partial differential equations and to equations involving some pseudo-differential operators. The method of proof is related to some of the results of Aubry-Mather theory in dynamical systems and we recover the corresponding results in dynamical systems as particular results. Of course, the Aubry-Mather theory for dynamical systems is much more developed and it includes not only results about existence of minimizing periodic orbits, but also construction of connecting orbits, etc. Many of these results do not have an analogue yet for PDE's. The connection of Aubry-Mather theory with PDE's was pointed out in ([2]). Results for difference equations were obtained in ([3]).

Here we will present two proofs for partial difference equations that are based respectively in ([4]) and ([5]). These strategies can be adapted to the more complicated case of partial differential equations. As it will appear in the proof, the arguments rely just on translation invariance, comparison principles, periodicity, and a variational structure (for differential and pseudo-differential equations we will also need some compactness properties, which amount to regularity results) and, hence they work in great generality. For the purposes of these notes we will just present the simplest cases. We point out that an important element in our proof is the consideration of the heat flow as a tool. The relevance of the heat flow for Aubry-Mather theory seems to have originated in ([6]) and was used in ([7]).

2 Periodic solutions of irrational periods for wave equations

2.1 Statement of results

We recall that a number $T \in \mathbb{R}$ is called of constant type if $|Tk - l| \geq C|k|^{-1}$ $\forall\ k, l \in \mathbb{Z}$. It is well known (Liouville's theorem) that all irrational numbers which solve quadratic equations with rational coefficients are of constant type. We sketch a proof of the delightful classical argument.

In effect, if ξ is an irrational number and $P(\xi) \equiv a\xi^2 + b\xi + c = 0$ with $a, b, c \in \mathbb{Z}$, we have $P'(\xi) \neq 0$ because, otherwise $P(x) = a(x-\xi)^2 = ax^2 - 2\xi ax + a\xi^2$, and, then $\xi = -b/2a$ would be rational.

If $n, m \in \mathbb{Z}$ and $|m/n - \xi|$ is sufficiently small, by the mean value theorem

$$|P(n/m)| = |P(n/m) - P(\xi)| \leq 2|P'(\xi)|\ |n/m - \xi|$$

On the other hand

$$|P(n/m)| = \frac{1}{m^2}|an^2 + bnm + cm^2| \geq \frac{1}{m^2}$$

because the numerator is an integer and different from zero by hypothesis. Putting together the two inequalities above, we have:

$$|n/m - \xi| \geq \frac{1}{2|P'(\xi)|}\ \frac{1}{m^2}.$$

∎

Quadratic irrationals are not the only constant type numbers. It is easy to show that a number is constant type if and only if it has a bounded continued fraction expansion. In particular, the set of constant type numbers is uncountable (but it has measure zero).

As usual, given a function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfying $f(x,t) = f(x+1,t) = f(x, t+T)$. We will write

$$f(x,t) = \sum_{k,l\in\mathbb{Z}} \widehat{f}_{k,l} e^{2\pi i[kx + l(t/T)]}$$

and denote

$$H^s = \left\{ f \middle| \|f\|_s^2 = \sum_{k,l\in\mathbb{Z}} |\widehat{f}_{k,l}|^2 (1+k^2+l^2)^s < \infty \right\} .$$

We will also denote by $\square$ the operator $\frac{\partial}{\partial t^2} - \frac{\partial}{\partial x^2}$.

Theorem 2.1 *Let T be a number of constant type. Let $V : \mathbb{R} \to \mathbb{R}$ satisfy*

i) $0 < \alpha \le V' \le \beta$, *where α is any number bigger than zero and β depends on T in a way that will be made explicit*

ii) $|V''(x)| \le K$

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy

iii) $f(x+1,t) = f(x,t+T) = f(x,t)$

iv) $f \in L^2$

Then, there exist a $u \in L^2$ such that:

a) $\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) u + V(u) = f$

b) $u(x+1,t) = u(x,t+T) = u(x,t)$

As we will see later, the smallness hypothesis on β are optimal.

Of course, u satisfies a) only in the weak sense at this stage (as it is typical of variational results). For rational T, ([1,8]) contain a beautiful argument that shows that weak solutions are strong solutions. This argument does not seem to be available in our case. Nevertheless, we can obtain smooth solutions for constant type periods imposing smoothness in V and f and smallness in f.

Theorem 2.2 *Let T be an irrational constant type number as before. Assume that $f \in H^r$ and $V \in C^{r+2}$ and that its C^{r+2} norm is sufficiently small. Then, there is one $u \in H^r$ which satisfies*

$$\square u + V(u) = f$$

The solution is unique in a ball in H^r around the origin.

Note that, in particular, when $V(0) = 0$, $f = 0$, the only solution with period T is $u \equiv 0$ contained in a sufficiently small ball around the origin. This should be compared with the results of ([9]) which construct solutions of $\square u + V(u)$ for many frequencies.

2.2 Proof of Theorems 2.1, 2.2

The main observation is that, under the hypothesis that T is a constant type number $\Box$ has closed range in H^s. In effect, the range is just the closed subspace of functions f such that $\widehat{f}_{0,0} = 0$.

Note that $\Box$ is diagonal on trigonometric functions and that if f is such that $\widehat{f}_{0,0} = 0$ if we denote by

$$\Box^{-1} f = \sum_{k,l} \widehat{f}_{k,l} \frac{1}{(2\pi i)^2(-k^2 + l^2/T^2)} E_{k,l}(x,t)$$

where $E_{k,l}(x,t) = \exp[2\pi i(kx + l(t/T)]$, we have that on functions f with zero average

$$\Box\Box^{-1} f = \Box^{-1}\Box f = f.$$

(Even if the first can only be defined for functions with zero average, we can define the second for any f and have $\Box^{-1}\Box f = f - \hat{f}_{0,0}$.)

The crucial remark is that when T is constant type we have

$$|k^2 - (l/T)^2| \geq C > 0 \quad \forall (k,l) \in \mathbb{Z}^2 - (0,0)$$

.

Since

$$k^2 - \frac{l^2}{T^2} = T^{-2}(Tk - l)(Tk + l)$$

we see that when k, l are of the same sign, we can bound $|Tk - l| \geq C|k|^{-1}$ and $|Tk + l| > T|k|$, therefore the product can be bounded from below by CT. When k, l are of opposite sign, $|Tk - l| \geq T|k|$ and $|Tk + l| > C|k|^{-1}$. ∎

This shows clearly that $\Box^{-1}$ can be defined as a bounded operator on the space of functions with zero average. Since it is clear that the average of a function in the range is zero, Ran$\Box$ is precisely the set of functions in H^r with zero average. Clearly, it is a closed subspace. Note also that Ran$(\Box)^{\perp} = \{$constant functions$\}$.

The rest of the proof of Theorem 2.1 is very similar to parts of the corresponding result in ([1,8]), which we reproduce for the sake of completeness. Unfortunately, the bootstrap of regularity in these references, does not go through in our case, indeed, we suspect that the conclusions of the bootstrap may not be true in our case.

Note that $\Box u + V(u) = f$ implies that $V(u) - f \in$ Ran$(\Box)$ and that, since $0 < \alpha \leq V' \leq \beta$ we can find V^{-1} and that $\beta^{-1} \leq (V^{-1})' \leq \alpha^{-1}$. If we write $V(u) - f = w$, $u = V^{-1}(w - f)$ the original problem becomes

$$V^{-1}(w-f) - \frac{1}{T}\int V^{-1}(w-f) + \square^{-1} w = 0$$

where $w \in \mathrm{Ran}(\square)$, that is

$$V^{-1}(w-f) + \square^{-1} w \in \mathrm{Ran}(\square)^{\perp}$$

This problem has a variational structure.

If $\mathcal{V}$ is a primitive of V^{-1}, then (2.2) is the equation for the critical points of

$$\mathcal{S}(w) = \int \mathcal{V}(w-f) + (1/2)(w, \square^{-1} w)$$

considered as a functional on the space $Y = \mathrm{Ran}(\square)$. Since Y is closed, it is a Banach space.

Now we turn to computing the second derivative of the functional. We leave to the reader the easy verification that the functional admits second Gateaux derivatives and that they are what one would expect through a formal calculation. The only non-trivial part is to check that $w \to \int \mathcal{V}(w-f)$ is differentiable, which follows easily from the Moser estimates for Sobolev spaces (See ([10]) 13 §3.)

We have that:

$$D^2\mathcal{S}(w)\gamma = \int \mathcal{V}''(w-f)\gamma^2 + (\gamma, \square^{-1}\gamma)$$

Since the second derivative of $\mathcal{V}$ is the first derivative of V^{-1} we see that

$$\int \mathcal{V}''(w-f)\gamma^2 \geq \beta^{-1} T \|\gamma\|_{H_0}^2$$

Since $(\gamma, \square^{-1}\gamma) \leq T^{-1} C \|\gamma\|_{H_0}^2$ if

$$\beta^{-1} T > T^{-1} C 4\pi^2$$

, the second derivative of the functional $\mathcal{S}$ is strictly positive definite and, therefore the functional is strictly convex.

The condition above is the condition on β that we alluded to in the statement of the theorem. Note that it involves T and its number theoretic properties.

If this condition holds $\mathcal{S}$ is a convex functional, moreover, we have

$$\mathcal{S}(w) \geq a\|w\|_{H_0}^2 - b\|w\|_{H_0} - c \tag{1}$$

for some $a, b, c \in \mathbb{R}^+$.

The functional is bounded from below and we can find a sequence w_n such that $\mathcal{S}(w_n) \to \inf_{w \in \mathrm{Ran}(\Box)} \mathcal{S}(w)$. The bounds (1) show that $\|w_n\|_{H_0}$ is bounded, hence we can extract a weakly convergent subsequence by Banach-Alaoglu theorem.

As it is well known, convex differentiable functionals in Hilbert spaces are lower semi-continuous in the weak topology. Hence, the infimum is reached by the limit point and this limit point satisfies the variational equations.

This finishes the proof of Theorem 2.1. ∎

Note that the conditions in β we obtained are optimal for theorems of the form indicated. If we consider $V(u) = \lambda u$, and λ is an eigenvalue of $\Box$, we do not expect any solutions. The eigenvalues of $\Box$ get as close as desired to the value we claimed.

The proof of Theorem 2.2 is quite elementary. If we write $u = u_1 + \delta$, where $u \in \mathrm{Ran}(\Box)$, $\delta \in \mathbb{R} = \mathrm{Ran}(\Box)^{\perp}$, the equation $\Box u + V(u) = f$ is equivalent to

$$\begin{aligned} u_1 &= -\Box^{-1}[V(u_1+\delta) - f) \\ \delta &= \tfrac{1}{T}\textstyle\int[V(u_1+\delta) - f] \end{aligned}$$

We only have to check that the right hand side of the equation is a contradiction if we consider it as an operator acting on (u_1, δ) with a contradiction constant that can be controlled by $\|V\|_{C^{r+2}}$. This follows from the Moser estimates standard in Sobolev spaces. See e.g. ([10]) 13 §3) ∎

3 Aubry-Mather theory for configurations on lattices

The classical Aubry-Mather theory is concerned with variational problems for functions defined on the integers. A prototype example is the variational principle

$$L(u) = \sum_{i \in \mathbb{Z}} (1/2)(u_i - u_{i+1})^2 - S(u_i)$$

with $S(u+1) = S(u)$.

The Euler equations become

$$-u_{i+1} - u_{i-1} + 2u_i - S'(u_i) = 0 \tag{2}$$

which is the well known "standard map" of dynamical systems.

Notice that, besides being a variational principle for the standard map, we could consider $L(u)$ as the energy of a chain of penduli coupled by springs. The u_i would be the angle between the i^{th} pendulum and the rest position,

$S(u_i)$ the potential energy due to the position of the pendulum. Another alternative physical interpretation (which appeared in solid state physics) is a chain of atoms in a one dimensional periodic potential coupled by a harmonic interaction. With the latter physical interpretation, this is called the Frenkel-Kontorova model, which is a qualitative model of deposition on the surface of (one-dimensional) crystals.

The goal in this lecture is to investigate the existence of quasi-periodic solutions of similar equations when the independent variables are in $\mathbb{Z}^d$ not just in $\mathbb{Z}$. This could be interpreted physically as arrays of penduli coupled to their nearest neighbors. That is, we want to consider the variational principle for maps $u : \mathbb{Z}^d \to \mathbb{R}$

$$L(u) = \sum_{|i-j|=1} (1/2)(u_i - u_j)^2 - \sum_i S(u_i)$$

and its variational equation

$$(-\Delta u)_i - S'(u_i) = 0 \tag{3}$$

where Δ is the discrete Laplacian $(\Delta u)_i = \sum_{|j-i|=1} u_j - 2du_i$ and, again $S(u+1) = S(u)$ which we will assume is a C^2 function corresponding to the position and $(1/2)(u_i - u_{i+1})^2$ the elastic energy of the spring between two consecutive penduli.

One of the first results of Aubry-Mather theory is that, for every $\omega \in \mathbb{R}$, (2) has a solution u_i such that $\sup_i |u_i - \omega i| < \infty$. This result is somewhat surprising since for the apparently very similar equation $u_{i+1} + u_{i-1} - 2u_i = a$ with $a \neq 0$, the solutions grow parabolically.

We refer to ([11]) for an up-to-date review of Aubry-Mather theory that, nowadays, includes not only the existence of solutions as indicated above, but much deeper geometric characterizations of the quasi-periodic invariant sets, their stable manifolds and many other geometric and dynamical properties of these sets.

We will present two proofs of the following result. The proofs are based on ([4,5]). There, the reader will find more details as well as generalizations. Another proof can be found in ([12]).

Theorem 3.1 *With the notations above, for every $\omega \in \mathbb{R}^d$ there exists a solution of (3) such that*

$$\sup_i |u_i - \langle \omega, i \rangle| < \infty$$

The proofs we will present make use of the heat flow, whose importance was noted in ([6,7]).

Our goal here is to present the argument in its simplest form so that it is clear that it is a consequence of only:

- Variational structure
- The variational structure is somewhat local
- Periodicity in space of the variational problem
- Periodicity in the u's of the variational problem
- Twist conditions (comparison theorems for the gradient flow)
- Some mild regularity assumptions that guarantee the existence of a flow for all times as well as differentiability of functionals

As detailed in ([4,5]) there are several other contexts where we have the properties above and, hence we have an analogue of Theorem 3.1.

As for the locality of the variational principle, we point out that it is very mild. It will be apparent in the details of the proof. The locality requirements in the proof in ([4]) are slightly stronger than those in ([5]).

We again note that the variational structure of the problem with periodicity conditions is crucial since the equation $-\Delta u + a = 0$ leads to quadratic growth at ∞. From the point of view of variational theory these are somewhat non-standard problems since the spaces of trial functions are modeled in the spaces $\ell^\infty(\mathbb{Z}^d)$ and, the functionals are only formal — the sums are not meant to converge. If we try to regularize the problem by well defined ones, we see that we have to get some control at ∞ to guarantee the linear growth. This is conveniently achieved by introducing the customarily called Birkhoff configurations.

Definition 3.1 *We say that u is a Birkhoff configuration if for every $k \in \mathbb{Z}^d$, $l \in \mathbb{Z}$ we have either*

$$u_{i+k} + l \geq u_i \ \forall\ i \in \mathbb{Z}^d \qquad or \qquad u_{i+k} + l \leq u_i \ \forall\ i \in \mathbb{Z}^d$$

That is, if we translate the graph horizontally and vertically by integers, the graph does not cross itself.

Given $\omega \in \mathbb{R}^d$ we denote

$$\mathcal{B}_\omega = \{u \mid u \text{ is Birkhoff } u_i - \langle \omega, i \rangle \in \ell^\infty\} \ .$$

We note that $\mathcal{B}_\omega$ is not empty since $\langle \omega, i \rangle$ belongs to it. We also note that if $u \in \mathcal{B}_\omega$ then $u_k - u_0$ is restricted to lie in an interval of length 1. In effect, if $u_k - u_0 + l \geq 0$ then because of the Birkhoff property $u_{nk+k} - u_{nk} +$

$l \geq 0$ and adding we obtain $u_{(n+1)k} - u_0 + nl \geq 0$ (similarly for $\leq$). Since $u_{(n+1)k} - \langle \omega, (n+1)k \rangle$ is to remain bounded independently of n we see that the inequality we have to pick in Definition 3.1 is the same as that when we compare $\langle \omega, k \rangle$ with l.

For $k \in \mathbb{Z}^d$, $l \in \mathbb{R}$ we introduce $(\mathcal{C}^k u)_i = u_{i+k}$; $\mathcal{R}^l u_i = u_i + l$. Note that $\mathcal{C}^k \mathcal{C}^j = \mathcal{C}^{k+j}$, $\mathcal{R}^l \mathcal{R}^n = \mathcal{R}^{l+n}$ and $\mathcal{C}^k \mathcal{R}^l = \mathcal{R}^l \mathcal{C}^k$.

A more compact notation can be obtained as follows: We say that $u \geq v \Leftrightarrow u_i \geq v_i$ for all i (similarly for $\leq$) [note that this is not a total order, there are many non-comparable elements]

Using this notation, we can say that a configuration u is Birkhoff if and only if, for every $k \in \mathbb{Z}^d$, $l \in \mathbb{Z}$

$$\mathcal{C}^k \mathcal{R}^l u \geq u \qquad \text{or} \qquad \mathcal{C}^k \mathcal{R}^l u \leq u \ .$$

This makes it clear that

$$\mathcal{C}^k \mathcal{B}_\omega = \mathcal{R}^l \mathcal{B}_\omega = \mathcal{B}_\omega$$

We also note that $\mathcal{B}_\omega$ is closed under pointwise limits.

It will be important to note that the fact that a configuration is Birkhoff implies a priori bounds for the components. If $u \in B_\omega$, then $u_{k+i} + l - u_i$ has to have the same sign as $\langle \omega, k \rangle + l$. (Otherwise, applying the Birkhoff property, we would get a contradiction with $u_i - \langle \omega, k \rangle \in \ell^\infty$)

Therefore, evaluating for $i = 0$ we obtain that when $u \in \mathcal{B}_\omega$, $u_k - u_0$ has to lie on an interval of length two (which we can write explicitly as a function of k).

Hence, if we consider the subset of $u \in \mathcal{B}_\omega$ with $u_0 \in [a, b]$, applying Tychonov theorem, we conclude it is compact under the pointwise limit topology and, since it is closed, we obtain it is compact.

Relatedly, if we consider two configurations as equivalent if they differ by an integer — this is natural since this does not affect the variational principle — we can choose a representation with $u_0 \in \mathbb{T}^1 \equiv \mathbb{R}/\mathbb{Z}$. Hence,

$$\mathcal{B}_\omega / \mathcal{R} \ \text{ is compact.}$$

The next ingredient in the proofs is the heat flow. This is the solution of the equation

$$\frac{\partial u}{\partial t} = \Delta u - S'(u)$$

Note that, formally, this is the gradient flow for the (strictly speaking, meaningless) variational principle.

There is no problem in showing that the heat flow is defined for all time since the R.H.S. of the equation is Lipschitz on $\ell^\infty(\mathbb{Z}^d)$. We denote by $\Phi_t(u)$ the solution at time t of initial condition u.

An important result for the proof is the comparison principle

Lemma 3.1 *If $u \geq v$ then $\Phi_t(u) \geq \Phi_t(v)$.*

The proof of this lemma is not difficult. The crucial observation is that the twist condition implies that the off-diagonal terms in the variational equations of the flow are non-negative.

A simple calculation shows that the elements of the matrix of the variational equation are: $-\frac{\partial^2 L(u)}{\partial u_i \partial u_j}$. The non-negativity of this quantity is the twist hypothesis.

Now we observe that comparison is true for linear equations with diagonal coefficients (they become exponentials) and for those who have positive entries — use the Taylor expansion of the exponential. Using Trotter product formula, it is true for sums of diagonal and positive entries, hence for the variational equations of heat flow. Once the result is true for the variational equations it is true for the full equations by interpolation.

We refer to ([4]) for more details if needed. ∎

We also note that $\mathcal{R}^l \mathcal{C}^k \Phi_\tau = \Phi_\tau \mathcal{C}^k \mathcal{R}^l$. Hence, the heat flow of a Birkhoff configuration is Birkhoff. Note also that if $u \in \langle \omega, i \rangle + \ell^\infty$ then $\Delta(u) + S(u) \in \ell^\infty$, therefore

$$\Phi_\tau \mathcal{B}_\omega \subset \mathcal{B}_\omega$$

and, moreover, the heat flow projects to $\mathcal{B}_\omega / \mathcal{R}$.

As remarked before, the heat flow is the gradient flow of the variational principle. But recall that since $L(u) = L(\mathcal{R}^l u)$ the variational principle is "defined" on the compact set $\mathcal{B}_\omega / \mathcal{R}$. Hence, the gradient should vanish at one point in the set.

Unfortunately, the previous argument does not work since the variational principle L is not "defined" in anything except a formal sense. So we need to regularize somehow and study the convergence of the regularizations.

The first solution that we present is that in ([4]). For a cube Λ we define:

$$L_\Lambda(u) = \sum_{\substack{i \in \Lambda \\ |i-j|=1}} (1/2)|u_i - u_j|^2 - \sum_{i \in \Lambda} S(u_i)$$

We note that $L_\Lambda(u)$ is a well defined function in $\mathcal{B}_\omega / \mathcal{R}$ continuous when $\mathcal{B}_\omega / \mathcal{R}$

is given the pointwise convergence topology. Moreover we have:

$$\frac{d}{dt}L_\Lambda(\Phi_c u) = -\sum_{i\in\Lambda}(-\Delta - S'(u))^2 + \sum_{\substack{i\in\partial\Lambda\\|i-j|=1}} J_i(u) \tag{4}$$

where the J_i are "gradients."

The intuition is that for large enough cubes the boundary terms should be ignorable. A moment's reflection shows that this should be the case, except when the bulk terms $\sum_{i\in\Delta}$ are very small, but in this case, we should have a critical point! More precisely,

Lemma 3.2 *Assume that there is no solution of (3) in $\mathcal{B}_\omega$ then, we can find $\varepsilon > 0$, N such that if $u \in \mathcal{B}_\omega$ $|\Lambda| > N$ then $\sum_{i\in\Lambda}(-\Delta u + S'(u))^2 \geq \varepsilon$.*

We show that the negation of the conclusion implies the negation of the assumptions. The negation of the conclusions is:

- $\exists\{u^{(n)}\}_{n=0}^\infty, \{\Lambda_n\}_{n=0}^\infty$ such that $u^{(n)} \in \mathcal{B}_\omega$, $|\Lambda_n| \to \infty$,
- $\Gamma_{\Lambda_n}(u^{(n)}) \equiv \sum_{i\in\Lambda_n}[-\Delta u^{(n)} + S'(u^{(n)})]^2 \to \infty$.

In such a case, we can find k_n such that $\widetilde{\Lambda}_n = \mathcal{C}^{k_n}\Lambda_n$ is centered at the origin. Since

$$\Gamma_{\mathcal{C}^{k_n}\Lambda_n}(\mathcal{C}^{k_n}u^{(n)}) = \Gamma_{\Lambda_n}(u^{(n)})$$

we see that the sequence $\mathcal{C}^{k_n}u^{(n)}$ — consisting of configurations in $\mathcal{B}_\omega$ — makes arbitrarily small. Hence $\Gamma_{\tilde{\Lambda}_n}(\tilde{u}^n) \to 0$ where $\widetilde{\Lambda}_n \subset \widetilde{\Lambda}_{n+1}$.

By passing to a subsequence (the Tychonov topology is metrizable in this case) we can assume that $\tilde{u}^{(n)} \to u^*$. Since $\Gamma_{\tilde{\Lambda}_{-(n+m)}}(u) \geq \Gamma_{\tilde{\Lambda}_n}(u)$ for all u, we have

$$\Gamma_{\Lambda_n}(\tilde{u}^{(n+m)}) \underset{m\to\infty}{\longrightarrow} 0$$

and therefore $\Gamma_{\Lambda_n}(u^*) = 0$ for all n. Hence u^* satisfies (3) ∎

Corollary 3.1 *In the assumptions of Lemma 3.2 we can find $C > 0$ such that*

$$\Gamma_\Lambda(u) \geq C(|\Lambda| - 1) \ \forall\ u \in \mathcal{B}_\omega$$

Given a large enough cube, divide it in cubes of size N and apply to each of the cubes Lemma 3.2. ∎

It is very easy to show that the terms J_i are uniformly bounded, hence if there was no critical point, applying Corollary 1 to estimate the right hand side of (4) we have:

$$\frac{d}{dt} L_\Lambda \Phi_\tau(u) \leq -C\left[(|\Lambda|-1)+|\Lambda|^{1-(1/d)}\right] \; \forall \; u \in \mathcal{B}_\omega$$

Hence, for big enough $|\Lambda|$, the right hand side would be negative. But this is impossible since L_Λ is a continuous function in the compact set $\mathcal{B}_\omega/\mathcal{R}$. We, therefore conclude that there is a critical point.

A second proof of the result of Theorem 3.1 can be obtained by observing that if $\omega = (1/N)(n_1, \dots, n_d)$ with $N \in \mathbb{N}$, $n_1, \dots, n_d \in \mathbb{Z}$ we can define

$$L_N(u) = \sum_{\substack{|i-j|=1 \\ |i|_\infty \leq N}} (1/2)|u_i - u_j|^2 - \sum_{|i| \leq N} S(u_i) \tag{5}$$

In this case, the Birkhoff orbits are periodic and satisfy:

$$u_{i_1,\dots,i_j+N,\dots,i_d} = u_{i_1,\dots,i_j,\dots,i_d} + n_j$$

This implies that

$$\frac{d}{dt} L_N(u) = -\sum_{|i|_\infty \leq N} (-\Delta u + S'(u))_i^2$$

Hence, we can conclude that the heat flow leads to a solution of 3 when $\omega \in \mathbb{Q}^d$.

To prove the result for any ω we observe that if we find a sequence $\omega^{(n)} \to \omega$ with $u^{(n)} \in \mathcal{B}_{\omega^n}$ solving 3, (we can assume without loss of generality $|u_0^{(n)}| \leq 1$). By the fact that $u^{(n)}$ in in $\mathcal{B}_\omega$ for every $j \in \mathbb{Z}^d$ we know that $u_j^{(n)}$ lies in an interval of size $|j| \max_n(\omega^{(n)}, \omega)$ that is, it remains bounded.

Hence, by passing to a subsequence, we can obtain that the $u^{(n)} \to \tilde{u}$ pointwise. Since the $u^{(n)}$'s satisfy (3) so does $\tilde{u}$.

Remark 3.2 The proofs above do not need that the interaction is nearest neighbor. They go just through without changes assuming

1. $$L(u) = \sum_{B \subset \mathbb{Z}^d} H_B(u) \tag{6}$$

   where $H_B(u)$ is a C^2 function which depends only on $u|_B$ and the sum is formal.

2. The twist condition: $\frac{\partial^2 H_B(u)}{\partial u_i \partial u_j} \geq 0$ when $i \neq j$.

3. Translation invariance: $H_{\mathcal{R}_k B}(\mathcal{R}_k u) = H_B(u) \ \ \forall u, \forall B \subset \mathbb{Z}^d, k \in \mathbb{Z}^d$.

4. Periodicity: $H_B(\mathcal{C}_l u) = H_B(u) \ \ \forall u, \forall B \subset \mathbb{Z}^d, l \in \mathbb{Z}$.

5. Some conditions of decay for H_B when B is large.

Note that (6) is quite common in Statistical Mechanics. H_B describes the interaction between the bodies in B. The conditions of translation invariance appear very frequently in Statistical Mechanics since it is a reflection that the interactions do not depend on the translations. The twist condition appears in statistical mechanics as ferromagnetism.

As for the condition on decay, we will just say that it is easy to check that the conditions are satisfied when the interaction is finite range (i.e. $H_B \equiv 0$ when the diameter of B is large enough.)

Once we prove the result for $L_A(u) = \sum_{B \subset \mathbb{Z}^d, \mathrm{diam} B \leq A} H_B(u)$ we can consider taking the limit $A \to \infty$. In order to do that, roughly, we need that $\nabla L(u)\ A$ admits bounds independent of A in all Birkhoff configurations.

The conditions of decay alluded in the last point are enough to guarantee that. Note that in this argument, we do not need that the heat flow is differentiable uniformly in A since we only take the limit of the solutions of the equilibrium equations.

We refer to ([4,5]) for further details and for generalizations. ■

Remark 3.3 One can also generalize the results to other graphs that are not just lattices.

The generalization of $\mathcal{R}_k$ is that there is a group G acting effectively. The analogue of quasiperiodic solutions are going to be the characters of the group.

Note that the first proof requires that, when we increase the size, the boundary elements become negligible.

The second proof requires an analogue of the rational vectors. This can be achieved for the "residually finite" groups. (The whole lattice can be exhausted by considering larger and larger cells which are the quotient of the group by a subgroup of finite index).

This allows to consider some graphs such as the homogeneous branching tree (called the Bethe lattice in Statistical Mechanics) ■

The following remark is based on unpublished research by the author.

Remark 3.4 The second proof presented above generalizes to partial differential equations.

The conditions of translation invariance, for the variational principle is quite straightforward to generalize.

Of course, to study the gradient flow one has to rely on PDE methods (parabolic regularity).

The analogue of the twist condition is that the heat flow satisfies a comparison principle.

To justify passages to the limit etc. we need something that shows that the equilibrium configurations are equicontinuous in small enough scales. (The equicontinuity in the large is still obtained from considering Birkhoff configurations as before.) For PDE's this can be obtained using elliptic regularity theory.

Some equations to which the the above conditions apply are

$$\Delta u + V'(x,u) = 0$$

where $V(x+e, u+l) = V(x,u)\ \forall x \in \mathbb{R}^d, u \in \mathbb{R}, e \in \mathbb{Z}^d, l \in \mathbb{Z}$

$$\sum_{i=1}^{k} L_i^2 + V'(x,u) = 0$$

where L_i are $\mathbb{Z}^d$ periodic vector fields satisfying Hörmander's hypoellipticity conditions and V is in the previous case.

$$(-\Delta)^{1/2} u + V'(x,u) \tag{7}$$

with V as above.

We note that the last equation is not a PDE, since $(-\Delta)^{1/2}$ is a non-local operator. Nevertheless all the argument can be carried though.

The comparison principle for the heat flow follows because $e^{t(-\Delta)^{1/2}}$ is convolution with the Poisson kernel, as it is well known in harmonic analysis, which is positive. Then, one can use Trotter product formula as indicated in the proof. (An alternative, more general, argument, suggested by C. Gutierrez is to use the *subordination* identity.)

Problems involving $(-\Delta)^{1/2}$ similar to (7) appear in several applications. Let us just note that they appear in the study of geostrophic flows and in Levy diffusions.

One can also combine this treatment of PDE's with the previous remarks and obtain quasiperiodic solutions in other homogeneous spaces besides the torus. We need to assume that the fundamental group satisfies the condition of being residually finite.

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References

1. H. Brezis. Periodic solutions of nonlinear vibrating strings and duality principles. *Bull. Amer. Math. Soc. (N.S.)*, 8(3):409–426, 1983.
2. J. Moser. Minimal solutions of variational problems on a torus. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 3(3):229–272, 1986.
3. M. L. Blank. Metric properties of minimal solutions of discrete periodical variational problems. *Nonlinearity*, 2(1):1–22, 1989.
4. H. Koch, R. de la Llave, and C. Radin. Aubry-Mather theory for functions on lattices. *Discrete Contin. Dynam. Systems*, 3(1):135–151, 1997.
5. A. Candel and R. de la Llave. On the Aubry-Mather theory in statistical mechanics. *Comm. Math. Phys.*, 192(3):649–669, 1998.
6. S. B. Angenent. Monotone recurrence relations, their Birkhoff orbits and topological entropy. *Ergodic Theory Dynamical Systems*, 10(1):15–41, 1990.
7. C. Golé. A new proof of the Aubry-Mather's theorem. *Math. Z.*, 210(3):441–448, 1992.
8. H. Brezis and L. Nirenberg. Forced vibrations for a nonlinear wave equation. *Comm. Pure Appl. Math.*, 31(1):1–30, 1978.
9. W. Craig and C. E. Wayne. Newton's method and periodic solutions of nonlinear wave equations. *Comm. Pure Appl. Math.*, 46(11):1409–1498, 1993.
10. M. E. Taylor. *Partial differential equations. III.* Springer-Verlag, New York, 1997. Nonlinear equations, Corrected reprint of the 1996 original.
11. J. N. Mather and Giovanni Forni. Action minimizing orbits in Hamiltonian systems. In *Transition to chaos in classical and quantum mechanics (Montecatini Terme, 1991)*, pages 92–186. Springer, Berlin, 1994.
12. M. L. Blank. Chaos and order in the multidimensional Frenkel-Kontorova model. *Teoret. Mat. Fiz.*, 85(3):349–367, 1990.