EXPONENTIAL STABILITY IN A NONLINEAR STRING EQUATION

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Abstract. We study the nonlinear wave equation

$$u_{tt} - c^2 u_{xx} = \psi(u) \qquad u(0,t) = 0 = u(\pi, t) \tag{0.1}$$

with an analytic nonlinearity of the type $\psi(u) = \pm u^3 + \sum_{k\geq 4} \alpha_k u^k$. On each small–energy surface we consider a solution of the linearized system with initial datum having the profile of an elliptic sinus: we show that solutions starting close to the corresponding phase space trajectory remain close to it for times growing exponentially with the inverse of the energy. To obtain the result we have to compute the resonant normal form of 0.1, and we think this could be interesting in itself.

1. Introduction and statement

In this paper we study small amplitude solutions of the equation 0.1 concentrating on the periodic behaviour.

Consider the solution of the *linear* string equation $u_{tt} - c^2 u_{xx} = 0$ with the initial datum:

$$u_\epsilon(x, 0) = \epsilon V_{\mathrm{m}} \mathrm{sn}(\omega x|\mathsf{m}) \quad \dot{u}(x,0) = 0 \ , \tag{1.1}$$

where sn is the elliptic sine [1], m, ω and V_{m} are constants whose values will be fixed later in the paper (see 4.1,4.4), and ϵ is a small parameter; its phase space trajectory is a closed curve that will be denoted by Γ_ϵ. The main result of the present paper is the forthcoming theorem 1.3, it ensures that solutions starting close to Γ_ϵ remain close to it for times exponentially long with ϵ^{-1}. In the corresponding statement we will make use of the distance $d(.,.)$ induced in the phase space $L^2(0,\pi) \times H_0^1(0,\pi) \ni (\dot{u}, u)$ by the norm

$$\|(\dot{u}, u)\|^2 := \frac{1}{2}\int_0^\pi (\dot{u}(x)^2 + c^2 u_x(x)^2) dx \ . \tag{1.2}$$

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Theorem 1.3 *Consider the nonlinear wave equation 0.1 with ψ analytic in a neighbourhood of the origin, assume $\psi(0) = \psi'(0) = \psi''(0) = 0$ and $\psi'''(0) \neq 0$. Then there exist strictly positive constants $\epsilon_*, C_1, ..., C_4$ such that the following holds true: fix $\epsilon < \epsilon_*$ and consider an initial datum $z_0 = (\dot{u}_0, u_0) \in L^2(0,\pi) \times H_0^1(0,\pi)$ close to Γ_ϵ, precisely such that*

$$d(z_0, \Gamma_\epsilon) \leq C_1 \epsilon^2 \ ,$$

then the corresponding solution $z(t)$ remains close to it for exponentially long times, precisely one has

$$d(z(t), \Gamma_\epsilon) \leq C_2 \epsilon^2$$

for times $|t| \leq C_3 \epsilon^2 \exp\left(\frac{C_4}{\epsilon}\right)$.

This result is obtained by applying the general theory developed in [4], and recalled in the next section, which requires the verification of a non degeneracy condition. From the technical point of view the verification of the non degeneracy condition is quite complicate, indeed as pointed out by Moser [10] in a similar context, non degeneracy is "a property which can be rarely verified"; this is a remarkable case where it is possible.

We think that besides its interest for the dynamics of the string equation, our result could be interesting as a first natural application of the theory of [4].

We recall the work [9] where existence of some periodic solutions for the equation

$$u_{tt} - u_{xx} + u^3 = 0$$

was obtained, but as far as we know no stability result are known for these solutions.

Finally we recall that the results usually obtained by KAM theory (see [8, 6, 5, 12]) or Nekhoroshev theory [5, 3] do not apply to 0.1 since it is a perturbation of a completely resonant system, and even the result of [2] does not apply since it requires the existence of an integrable first order normal form.

2. General setting

First we rescale u by introducing a variable u' defined by $\epsilon u' = u$. We recall that the equation for the rescaled variable is hamiltonian with hamiltonian

$$\mathcal{H}(p', u') = h_\omega(p', u') + \epsilon^2 f(u') + \epsilon^3 f_1(u') \ , \tag{2.1}$$

where

$$h_\omega(p', u') = \frac{1}{2}\int_0^\pi \left(p'^2(x) + c^2 u_x'^2(x)\right) dx \ ,$$

the main part of the perturbation given by

$$f(u') = \pm\frac{1}{4}\int_0^\pi u'(x)^4 dx \ , \tag{2.2}$$

and higher order corrections given by

$$f_1(u') = \sum_{k\geq 4} \frac{\alpha_k \epsilon^{k-4}}{k+1}\int_0^\pi u'^{k+1}(x) dx \ .$$

To fix ideas we will always consider the case where the sign in front of the integral in 2.2 is plus. In the minus case nothing changes.

The idea of [4] is to use averaging theory to transform 2.1 into a system of the form

$$\mathcal{H}(p', u') = h_\omega(p', u') + \epsilon^2\langle f\rangle(p', u') + \epsilon^3 Z_\epsilon(p', u') + \exp\left(-\frac{C}{\epsilon}\right)\mathcal{R}_\epsilon(p', u') \ , \tag{2.3}$$

where $\langle f\rangle$ is the average of f with respect the unperturbed flow (for a precise definition see equation 3.1), Z is a part of the Hamiltonian which commutes with h_ω, and $\mathcal{R}$ is an exponentially small remainder.

Consider then the *simplified system* obtained from 2.3 by neglecting $\mathcal{R}$, namely

$$\mathcal{H}_s(p', u') = h_\omega(p', u') + \epsilon^2\langle f\rangle(p', u') + \epsilon^3 Z_\epsilon(p', u') \ . \tag{2.4}$$

Then h_ω is an integral of motion independent of the energy for 2.4. The critical points of $\mathcal{H}_s$ constrained to the surface of constant $h_\omega = 1$ are invariant sets. It easy to realize that if they are non degenerate in the transversal directions they are trajectories of periodic solutions of 2.4. Moreover, if such non degenerate critical points are also extrema, the corresponding periodic orbits are stable.

Taking into account the remainder $\mathcal{R}$ one obtains that the above extrema are no more invariant sets, but, due to the very slow change in time of both $\mathcal{H}_s$ and h_ω for the complete dynamics, it turns out that solutions starting close to extrema remain close to them for exponentially long times.

Finally, remark that usually one can hope to compute $\langle f \rangle$ but not the exact expression of $\mathcal{H}_s$, so we will look for non degenerate critical points of $\mathcal{H}_s|_{h_\omega^{-1}(1)}$ as perturbations of non degenerate critical points of $\langle f \rangle|_{h_\omega^{-1}(1)}$.

Actually, applying the theory of [4] (see theorem 6.1), in the improved formulation by [11] one obtains the following theorem (stated in non rescaled variables), which gives a more precise description than theorem 1.3.

Theorem 2.5 *Consider the nonlinear wave equation 0.1 with ψ analytic in a neighbourhood of the origin, assume $\psi(0) = \psi'(0) = \psi''(0) = 0$ and $\psi'''(0) \neq 0$. There exist strictly positive constants $\epsilon_*, C_4, ..., C_9$ and a family of curves $\{\gamma_\epsilon\}_{0 \le \epsilon < \epsilon_*}$ with the following properties:*

1) For any initial datum $(\dot{u}_0, u_0) \equiv z_0 \in \mathcal{P}$ satisfying $d_0 = d(z_0, \gamma_\epsilon) \le C_5\epsilon$ the corresponding solution $z(t)$ satisfies

$$d\left(z(t), \gamma_\epsilon\right) \le \sqrt{C_6 d_0^2 + C_7\epsilon^2 |t| \exp\left(-\frac{C_4}{\epsilon}\right)} \ ,$$

$\forall t$ such that $|t| \le C_8 \exp\left(\frac{C_4}{\epsilon}\right)$;

2) γ_ϵ depends smoothly on ϵ and is close to Γ_ϵ, precisely one has

$$\sup_{z \in \Gamma_\epsilon} d\left(\gamma_\epsilon, z\right) \le C_9 \epsilon^2 \ .$$

3) If $(\dot{u}, u) \in \gamma_\epsilon$ then both $\dot{u}$ and u are C^∞ functions.

Theorem 1.3 is an obvious corollary of this result. The proof of theorem 2.5 requires the computation of $\langle f \rangle$, of the extrema of $\langle f \rangle|_{h_\omega^{-1}(1)}$ and the verification that they are non degenerate. This occupy the rest of the paper.

Remark 2.6 *¿From the estimate of the point 2) of the theorem it follows that the ratio between the distance $d(\Gamma_\epsilon, \gamma_\epsilon)$ and the size of the energy surface tends to zero as $\epsilon \to 0$.*

3. Normal form

In this section we give the explicit form of the average $\langle f \rangle$ of the perturbation and give the precise statement of the normal form theorem for the case of eq. 0.1.

First we recall the precise definition of $\langle f \rangle$, which is the average of f with respect to the linear flow; denoting by $\Phi^t(p, u)$ the flow of the linear string equation with initial data $\dot{u}(x, 0) = p(x)$, $u(x, 0) = u(x)$, we define

$$\langle f \rangle (p, u) := \frac{1}{T} \int_{-T/2}^{T/2} f(\Phi^t(p, u)) dt \ . \tag{3.1}$$

In order to be able to do explicit calculation, one needs good coordinates. To introduce them we proceed as follows. First we extend both u' and $p' \equiv \dot{u}'$ to $[-\pi, \pi]$ by skew symmetry, and define $P(x)$ as the unique function with zero average such that

$$p'(x) = cP_x(x) \ ;$$

remark that $P(x)$ is an even function. Define $\bar{\mathcal{P}} \subset H^1(\mathbb{T}^1)$ as the space of periodic functions of period 2π with zero average. Then the map

$$\mathcal{P} \ni (p, u') \mapsto V := u' + P \in \bar{\mathcal{P}}$$

is an isomorphism. Remark that, under this isomorphism the linear flow Φ^t takes the form

$$(\Phi^t V)(x) = V(x+ct) \ .$$

Using V to parameterize phase points, the hamiltonian f of the main part of the perturbation takes the form

$$f(V) = \frac{1}{8}\int_{-\pi}^{\pi}\left(\frac{V(x+y)-V(-x+y)}{2}\right)^4 dxdy \ .$$

In the following we will use the notation

$$[g] = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)dx$$

for the average of a function.

Proposition 3.2 *There exist open domains $U_2 \subset U_1$, containing the sphere of radius 1 in the phase space, constants C_4 and an analytical canonical transformation $T : U_2 \to U_1$ such that $H \circ T$ takes the form*

$$h_\omega(V) + \epsilon^2\langle f\rangle(V) + \epsilon^3 Z_\epsilon(V) + \exp\left(-\frac{C_4}{\epsilon}\right)\mathcal{R}_\epsilon(V) \ ,$$

with

$$\langle f\rangle(V) = \frac{\pi}{32}\left(\left[V^4\right] + 3\left[V^2\right]^2\right) \ , \tag{3.3}$$

and with Z_ϵ, $\mathcal{R}_\epsilon$, and their vector fields, bounded on U_2, uniformly with respect to ϵ.

Remark 3.4 *We recall the paper* [13, 7] *where the computation of the normal form of a finite dimensional approximation of 0.1 was done. In that paper the authors also studied some features of the dynamics of the so obtained system.*

Proof. This is an application of Lemma 8.1 of ref. [4]. The only point to be proved is the explicit formula 3.3. This is obtained by a calculation that we are going to summarize. Using the changes of variables $y = ct$ one has

$$\begin{aligned}\langle f\rangle(V) =& \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{4}\left(\int_0^{\pi} u'(x,t)^4 dx\right)dt = \\ =& \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{8}\int_{-\pi}^{\pi}\left(\frac{V(x+y)-V(-x+y)}{2}\right)^4 dxdy \ .\end{aligned}$$

Making the change of variables $x+y=\xi$, $-x+y=\eta$ and using the periodicities to double the interval of integration the above quantity takes the form

$$\begin{aligned}=&\frac{1}{256\pi}\frac{1}{2}\int_{-2\pi}^{2\pi}\int_{-2\pi}^{2\pi}\left[V^4(\xi)+V^4(\eta)+6V^2(\xi)V^2(\eta)-4V^3(\xi)V(\eta)+\right.\\ &\left.-4V(\xi)V^3(\eta)\right]\frac{d\xi d\eta}{2} \ ,\end{aligned}$$

which, taking into account that V has zero average, is easily seen to coincide with 3.3.

□

Remark 3.5 *With these variables one can easily check that every nonlinearity $\psi(u) = u^{2n}$ has zero average, so the theory of* [4] *does not apply directly to the case of even power.*

4. Critical Points

Using the explicit expression of $\langle f\rangle$, we can look for its critical points constrained to the constant harmonic energy surface $\mathcal{S}_1$, defined by $\mathcal{S}_E = \{V \in \bar{\mathcal{P}} \mid h_\omega(V) = E\}$ with $E = 1$.

Actually we study the critical points of $\langle f\rangle$ constrained to the surface $\mathcal{S}_E$ with an arbitrary E. Such critical points are expressed in terms of elliptic functions (for definitions and properties of elliptic functions and elliptic integrals see [1]). In what follows we will denote by $\mathsf{K} = \mathsf{K}(\mathsf{m})$ the complete elliptic integral of the first kind and by $\mathsf{E} = \mathsf{E}(\mathsf{m})$ the complete elliptic integral of second kind; moreover we fix the value of the parameter m as the only solution of the equation

$$\frac{\mathsf{E}}{\mathsf{K}} = \frac{7+\mathsf{m}}{6} \ . \tag{4.1}$$

It is not difficult to verify that $\mathsf{m} \approx -0.2554$.

Proposition 4.2 *Consider the function*

$$V(x) = V_{\mathrm{m}} \ \mathrm{sn}(\omega \ x|\mathsf{m}) \tag{4.3}$$

with

$$\omega := \frac{2\mathsf{K}}{\pi} \ , \quad V_{\mathrm{m}}^2 := \frac{9\pi^2}{2} \frac{\mathsf{m}}{(\mathsf{m}^2+14\mathsf{m}+1)\mathsf{K}^3} \ \frac{E}{c^2} \ ; \tag{4.4}$$

then the critical points of $\langle f\rangle|_{\mathcal{S}_E}$ *are*

$$V_n(x) := \frac{1}{n} V(nx) \quad n = 1, 2, \dots \ .$$

Proof. Defining the L^2–gradient ∇f of a function f by

$$\langle \nabla f(V); h\rangle_{L^2} = df(V)h$$

and using the method of Lagrange multipliers one obtains that the equation for the critical points of $\langle f\rangle|_{\mathcal{S}_E}$ is given by

$$-\lambda c^2 V_{xx}(x) + \frac{1}{8}V^3(x) + \frac{3}{16\pi}\left(\int_{-\pi}^{\pi} V^2(x)dx\right) V(x) = \kappa \ . \tag{4.5}$$

with the following conditions:

$$h_\omega(V) = E \ ,$$

$$V(-\pi) = V(\pi) \ , \quad V_x(-\pi) = V_x(\pi) \ , \tag{4.6}$$

$$\int_{-\pi}^{\pi} V(x)dx = 0 \ . \tag{4.7}$$

The above problem can be written in the following form

$$mV_{xx} = \kappa - \mu V - \frac{1}{8}V^3 \ , \tag{4.8}$$

$$E = \frac{c^2}{4}\int_{-\pi}^{\pi} V_x^2(x)dx \ , \tag{4.9}$$

$$\mu = \frac{3}{16\pi}\int_{-\pi}^{\pi} V^2(x)dx \ , \tag{4.10}$$

together with the boundary conditions 4.6 and the condition of zero average 4.7. The unknown being (V, μ, E, m, κ).

Remark 4.11 *If* (V, μ, E, m, κ) *is a solution of the above system, then, for any* $\rho > 0$, *also* $(\rho V, \rho^2\mu, \rho^2 E, \rho^2 m, \rho^3\kappa)$ *is a solution of the same system.*

So, we will remove the condition 4.9 and look for a solution of the system 4.8, 4.10, 4.6, 4.7, with the unknowns (V, μ, m, κ).

Remark that it is possible to find by quadratures the solution of 4.8: define $\Phi_\kappa(s) := -\kappa s + \frac{1}{2}\mu s^2 + \frac{1}{32}s^4$, then $V(x)$ is obtained by

$$x = \int_0^V \frac{ds}{\sqrt{\frac{2}{m}\left(E' - \Phi_\kappa(s)\right)}} \ , \tag{4.12}$$

where E' is a parameter representing the mechanical energy of the nonlinear oscillator 4.8.

We state now a lemma which will be useful in the following. Its proof is deferred to the appendix.

Lemma 4.13 *The equation 4.8 with the condition 4.7 has solutions only if $\kappa = 0$. In such case condition 4.7 is implied by 4.6.*

So we are left with the system 4.8, 4.6, 4.10, with $\kappa = 0$, for the unknowns (V, μ, m).

From

$$x = \int_0^V \frac{dV}{\sqrt{\frac{2}{m}\left(E' - \Phi(V)\right)}} \ ,$$

with $\Phi := \Phi_0$, one gets

$$x = \frac{4\sqrt{m}}{\sqrt{16\mu + V_{\mathrm{m}}^2}} F\left(\varphi \,\middle|\, -\frac{V_{\mathrm{m}}^2}{16\mu + V_{\mathrm{m}}^2}\right) \qquad \sin\varphi = \frac{V}{V_{\mathrm{m}}} \ ,$$

where F is the incomplete elliptic integral of first kind, and $E' = \Phi(V_{\mathrm{m}})$. Using the definition of elliptic sine, which is the inverse of such elliptic integral, we get the solution of 4.8 (with $\kappa = 0$) in the form

$$V(x) = V_{\mathrm{m}} \,\mathrm{sn}(\omega\, x|\mathsf{m}) \ , \tag{4.14}$$

with

$$\begin{aligned} \omega &= \frac{\sqrt{16\mu + V_{\mathrm{m}}^2}}{4m} \ , \\ \mathsf{m} &= -\frac{V_{\mathrm{m}}^2}{16\mu + V_{\mathrm{m}}^2} \ . \end{aligned} \tag{4.15}$$

Since the natural period of the elliptic sine is $4\mathsf{K}$, (4.6) gives:

$$\omega\frac{2\pi}{n} = 4\mathsf{K} \quad \Rightarrow \quad \omega_n = \frac{2\mathsf{K}}{\pi} n \ . \tag{4.16}$$

Now consider condition 4.10: using properties of elliptic functions we have

$$\mu = \frac{3}{8}\frac{\mathsf{K} - \mathsf{E}}{\mathsf{mK}} V_{\mathrm{m}}^2 \ ; \tag{4.17}$$

inserting in the second of 4.15, we obtain 4.1 which, as stated before, is the equation fixing the value of the parameter m.

To complete the picture we give the value of the other parameters involved. From 4.9 we obtain a relation between E and m

$$m_n = \frac{9\pi^4}{128} \frac{-1}{(\mathsf{m}^2 + 14\mathsf{m} + 1)\mathsf{K}^5} \frac{1}{n^4}\frac{E}{c^2} \ . \tag{4.18}$$

¿From this last equation and using the 4.15 we have

$$\left(V_{\mathrm{m}}^2\right)_n = \frac{9\pi^2}{2} \frac{\mathsf{m}}{(\mathsf{m}^2 + 14\mathsf{m} + 1)\mathsf{K}^3} \frac{1}{n^2}\frac{E}{c^2} \ , \tag{4.19}$$

and than also

$$\mu_n = \frac{9\pi^2}{32}\frac{-(1+\mathsf{m})}{(\mathsf{m}^2+14\mathsf{m}+1)\mathsf{K}^3}\ \frac{1}{n^2}\frac{E}{c^2}\ . \tag{4.20}$$

□

5. Extrema

In this section we will show that, among all the critical point found, there's only one extremum, while all the other points are saddle points.

Proposition 5.1 *$\langle f\rangle$ constrained to $\mathcal{S}_E$ has only one extremum, V_1 which is a maximum; the other critical points $V_n,\ n\neq 1$ are saddle points.*

Proof. We will show that all the critical points but the first are saddle points. Then we will ensure the presence of a maximum by variational methods.

First of all, we calculate the second differential of $\langle f\rangle|_{\mathcal{S}_E}$ at the critical points: it is given by the following formula

$$d^2\left(\langle f\rangle|_{\mathcal{S}_E}\right)(V_n)(h,h) = \lambda d^2 h_\omega(V_n)(h,h) + d^2\langle f\rangle(V_n)(h,h)\ ,$$

where λ is the lagrange multiplier at the critical point and h belongs to the tangent space to $\mathcal{S}_E$.

Thus we have

$$d^2\left(\langle f\rangle|_{\mathcal{S}_E}\right)(V_n)(h,h) = c^2\lambda\int_{-\pi}^{\pi} h_x^2(x)dx + \frac{3}{8}\int_{-\pi}^{\pi} V_n^2(x)h^2(x)dx +$$

$$+\frac{3}{16\pi}\int_{-\pi}^{\pi} V_n^2(x)dx\int_{-\pi}^{\pi} h^2(x)dx + \frac{3}{8\pi}\left(\int_{-\pi}^{\pi} V_n(x)h(x)dx\right)^2,$$

which can be expressed as an inner product $\langle h, A_n h\rangle_{L^2}$, with

$$\begin{aligned} A_n h =& B_n h + \frac{3}{4}\left[V_n h\right]V_n, \\ B_n h =& m_n h_{xx} + \frac{3}{8}\left(V_n^2 + \left[V_n^2\right]\right)h. \end{aligned} \tag{5.2}$$

We must evaluate this second differential along directions lying on the tangent space to the energy surface.

In the forthcoming computation we use the remark that if i is an even integer, and j is an odd integer then V_i is L^2 and H^1 orthogonal to V_j. So, having fixed n, consider a value of i with a different parity. One has

$$\begin{aligned} \langle V_i, A_n V_i\rangle_{L^2} &= \langle V_i, m_n V_{i\,xx} + \frac{3}{8}\left(V_n^2 + \frac{1}{2\pi}\int V_n^2\right)V_i + \frac{3}{8\pi}V_n\int V_nV_i\rangle_{L^2} \\ &= \left(\mu_n - \frac{m_n}{m_i}\mu_i\right)\int V_i^2 + \frac{3}{8}\int V_n^2V_i^2 - \frac{1}{8}\frac{m_n}{m_i}\int V_i^4 + \frac{3}{8\pi}\left(\int V_nV_i\right)^2 \\ &= \pi c_4^2\frac{E^2}{c^4}\frac{1}{n^4}\left[3(1+\mathsf{m})^2\frac{n^2-i^2}{i^2} + 4(1+5\mathsf{m}+\mathsf{m}^2)\right] + \frac{3}{8}\int V_n^2V_i^2\ , \end{aligned}$$

with

$$c_4 := \frac{3\pi^2}{8}\frac{1}{(\mathsf{m}^2+14\mathsf{m}+1)\mathsf{K}^3}\ .$$

When i is sufficiently large the above expression is negative, indeed $1+5\mathsf{m}+\mathsf{m}^2$ is easily seen to be negative (using the value of m), and all the positive terms vanish as $i\to\infty$.

Concerning the non existence of maxima it is easy to see that for any $n\geq 2$ there are directions h making the value of the second differential positive. For even n take $i=1$, and for odd $n\neq 1$ take $i=2$ using $3(1+\mathsf{m})^2+4(1+5\mathsf{m}+\mathsf{m}^2)>0$ one obtains that the square bracket is positive.

Up to now we showed that, among all the critical points, only V_1 can be an extremum, in particular a maximum; by variational techniques we now prove that a maximum actually exists, and therefore it must coincide with V_1.

We exploit the fact that $\langle f\rangle$ is positive and homogeneous, and can be extended to a continuous functional on L^∞.

Introduce the 0 degree homogeneous functional

$$F(V):=E^{\frac{3}{2}}\frac{\langle f\rangle(V)}{\left(\int V_x^2\right)^{\frac{3}{2}}}\ ,\qquad\qquad V\in\bar{\mathcal{P}}\setminus\{0\}.$$

We have

$$\sup_{V\in\bar{\mathcal{P}}\setminus\{0\}}F=\sup\ F|_{\mathcal{S}_E}=\sup\ \langle f\rangle|_{\mathcal{S}_E}\ .$$

¿From Sobolev embedding theorem $H^1(\Omega)\subset C(\overline{\Omega})$ it follows $\sup F\equiv\sup\ f|_{\mathcal{S}_E}<\infty$, while from positivity of $\langle f\rangle$ we have $\sup F>0$.

Let $\{V_n\}\subset\bar{\mathcal{P}}$ be a maximizing sequence, and $v_n\in\mathcal{S}_E$ be given by:

$$v_n:=\sqrt{E}\frac{V_n}{\sqrt{\int V_{nx}^2}}\ ,$$

which is also maximizing, since F is homogeneous. $\{v_n\}$ is bounded in H^1, and thus there exists a subsequence (that we call again $\{v_n\}$) weakly converging to a point v: $v_n\rightharpoonup v$.

By compact embedding we have $v_n\xrightarrow{L^\infty}v$.

Now, by construction, $\langle f\rangle(v_n)=F(v_n)\to\sup F$; by continuity of $\langle f\rangle$ in L^∞, $\langle f\rangle(v_n)\to\langle f\rangle(v)$, and so $\langle f\rangle(v)=\sup F>0$, which gives $v\neq 0$. It remains to prove that v satisfies the constraint.

¿From weak lower semicontinuity of the H^1 norm it follows

$$\left(\int v_x^2\right)^{\frac{3}{2}}\leq\liminf\left(\int v_{n\,x}^2\right)^{\frac{3}{2}}=E^{\frac{3}{2}}\ .$$

Since $v\neq 0$ we also have

$$\frac{E^{\frac{3}{2}}}{\left(\int v_x^2\right)^{\frac{3}{2}}}\geq\limsup\frac{E^{\frac{3}{2}}}{\left(\int v_{x\,n}^2\right)^{\frac{3}{2}}}=1\ ,$$

and remembering the definition of F

$$F(v)=E^{\frac{3}{2}}\frac{f(v)}{\left(\int v_x^2\right)^{\frac{3}{2}}}\geq f(v)=\sup F$$

which implies $F(v)=\sup F$, and $\sqrt{\int v_x^2}=E$.

$\square$

6. Non degeneration

Finally, to apply the theory of [4] we have to verify that the second differential of $\langle f\rangle|_{\mathcal{S}_E}$ is non degenerate, in the directions transversal to the orbit of the linearized system, at $V_1 \equiv V$. We know that $V_1(x+s)$ is again a critical point at the same level, so V_{1x} is a null direction of the second differential of the functional (remark that $A_1 V_{1x} = 0$ because V and V_x are L^2 orthogonal and $B_1 V_{1x} = 0$ being the derivative of the equation defining V). We have to prove that this is the only null direction. So we have to find all solutions $h \in T_{V_1} S_E$ of the equation

$$0 = d^2\left(\langle f\rangle|_{\mathcal{S}_E}\right)(V)(h,h) \equiv \langle h, Ah\rangle \ , \tag{6.1}$$

with $A := A_1$, and A_1 defined by 5.2 with $n = 1$. It is easy to see that 6.1 is equivalent to

$$0 = d^2\left(\langle f\rangle|_{\mathcal{S}_E}\right)(V)(\tilde{h},h) \equiv \left\langle \tilde{h}, Ah\right\rangle \ , \quad \forall \tilde{h} \in T_{V_1}\mathcal{S}_E \ , \tag{6.2}$$

which in turn is equivalent to the following equation for $h \in T_{V_1}\mathcal{S}_E$ and $\alpha \in \mathbb{R}$

$$Ah = -\alpha V_{xx} \ . \tag{6.3}$$

We study equation 6.3 in the space $\bar{\mathcal{P}}$ adding the constrain that the solutions must belong to $T_{V_1}\mathcal{S}_E$, i.e. that its H^1 scalar product with V must vanish or, more explicitly that

$$[h_x V_x] = 0 \ . \tag{6.4}$$

We will prove that the only solution of this problem is $h \propto V_x$.

To find the solutions of 6.3 we introduce a new variable g defined by

$$h = V_x g \ .$$

Remark that g could be discontinuous, but this will not cause any problem. The equations we will write will be intended to hold at the points x such that g is continuous and differentiable ($x \neq \pm\pi/2$). In terms of g, 6.3 takes the form

$$m\partial_x\left(V_x^2 g_x\right) = \partial_x\left(-\frac{3}{8}[VV_x g]V^2 - \frac{\alpha}{2}V_x^2\right) \ ,$$

from which

$$mV_x^2 g_x = -\frac{3}{8}[VV_x g]V^2 - \frac{\alpha}{2}V_x^2 + \gamma \ , \tag{6.5}$$

with a real γ. In terms of g, eq. 6.4 takes the form $[V_x^2 g_x] = 0$. So, averaging 6.5, we get

$$-\frac{3}{8}[VV_x g] = \frac{\alpha}{2}\frac{[V_x^2]}{[V^2]} - \frac{\gamma}{[V^2]} \ , \tag{6.6}$$

from which, inserting in 6.5 we obtain

$$mg_x = \frac{\alpha}{2}\left(\frac{[V_x^2]}{[V^2]}\frac{V^2}{V_x^2} - 1\right) + \gamma\left(\frac{1}{V_x^2} - \frac{V^2}{[V^2]V_x^2}\right) \ .$$

Remembering that $V = V_{\mathrm{m}}\mathrm{sn}(\omega x|\mathsf{m})$ and using known expressions for the integrals of elliptic functions one obtains the general solution of the above equation. This is given by

$$mg(x) = \xi + \frac{\alpha}{2}\left[\frac{\mathsf{m}^2 + 14\mathsf{m} + 1}{\omega(3+\mathsf{m})(1-\mathsf{m})^2}\Big(A - B\Big) - x\right] + \gamma\left[\frac{(1+7\mathsf{m})A - \mathsf{m}(7+\mathsf{m})B}{V_{\mathrm{m}}^2\omega^3(1+\mathsf{m})(1-\mathsf{m})^2}\right] \tag{6.7}$$

with

$$
\begin{aligned}
A &\equiv -\mathsf{E}(\omega x) + (1-\mathsf{m})\omega x + \frac{\mathrm{sn}(\omega x|\mathsf{m})\mathrm{dn}(\omega x|\mathsf{m})}{\mathrm{cn}(\omega x|\mathsf{m})} \ , \\
B &\equiv \mathsf{E}(\omega x) - \mathsf{m}\frac{\mathrm{sn}(\omega x|\mathsf{m})\mathrm{cn}(\omega x|\mathsf{m})}{\mathrm{dn}(\omega x|\mathsf{m})} \ ,
\end{aligned}
$$

and ξ is a real parameter.

Imposing now the conditions of periodicity of g and that of orthogonality (see 6.4 and 6.6) we get a linear homogeneous system in the variables α and γ; the determinant of such system is given by

$$
\mathsf{K}^4 \frac{(1+\mathsf{m})(255 - 3444\mathsf{m} - 32661\mathsf{m}^2 - 27448\mathsf{m}^3 - 6147\mathsf{m}^4 - 180\mathsf{m}^5 - 7\mathsf{m}^6)}{216} \ ,
$$

and this quantity is different from zero (remember that m, and hence K, are real numbers fixed by 4.1).

The conclusion is than that α and γ must be zero, and the only null direction is $h = \xi V_x$.

Remark 6.8 *It is possible to prove the non degeneracy of all the other critical points V_n exactly in the same way, obtaining for the determinant precisely the same expression, which turns out to be independent from n.*

7. Appendix

Proof of lemma 4.13. The idea is to show that, when $\kappa \neq 0$, due to the broken symmetry of the potential Φ_κ, the average of the function V is no more zero.

We can write

$$
\begin{aligned}
[V] &= \frac{n}{\pi}\int_{V_{\min}}^{V_{\max}} \frac{V dV}{\sqrt{\frac{2}{m}\left(E' - \Phi_\kappa(V)\right)}} \\
&\propto \int_{s_m}^{s_M} \frac{s ds}{\sqrt{p(s, E', c, \mu)}} \ ,
\end{aligned} \tag{7.1}
$$

where we changed the variable of integration from x to V and where we denoted $p(s, E', \kappa, \mu) = E' - \Phi(s, \kappa, \mu)$, and again $p(s_m) = p(s_M) = 0$. Call this last integral $I(\kappa)$, and observe that $I(-\kappa) = -I(\kappa)$, so we restrict to the case $\kappa > 0$. We restrict also to the case $E' > 0$, because otherwise the inversion points are both positive and the proof is finished.

Let's write the polynomial p in the following way

$$
\begin{aligned}
&E' + \kappa s - \frac{\mu^2}{2}s^2 - \frac{1}{32}s^4 = \\
=&(s - s_m)(s - s_M)\left(\frac{E'}{s_m s_M} - \frac{s_m + s_M}{32}s - \frac{1}{32}s^2\right) \ ,
\end{aligned} \tag{7.2}
$$

and comparing the coefficient of s and s^2 deduce the following relations:

$$
\kappa = -(s_m + s_M)\left(\frac{E'}{s_m s_M} + \frac{s_m s_M}{32}\right) \ , \tag{7.3}
$$

$$
-\frac{\mu^2}{2} - \frac{(s_m + s_M)^2}{32} = \left(\frac{E'}{s_m s_M} - \frac{s_m s_M}{32}\right) \ . \tag{7.4}
$$

¿From 7.3 you see that when $\kappa > 0$, then $s_M > |s_m|$. Now look at the other two roots of p

$$s_{3,4} = -\frac{s_m + s_M}{2} \pm \frac{\sqrt{(s_m + s_M)^2 + \frac{128E'}{s_m s_M}}}{2}\ , \tag{7.5}$$

which are not real (use 7.4).

Now define

$$\begin{aligned} \Omega &:= \frac{s_M + s_m}{2} > 0 \qquad (\kappa > 0) \\ \omega &:= \frac{s_M - s_m}{2} > 0 \\ \Delta &:= (s_m + s_M)^2 + \frac{128E'}{s_m s_M} < 0 \end{aligned} \tag{7.6}$$

and write p in the following way

$$\begin{aligned} p(s, E', \kappa, \mu) &= -(s - s_M)(s - s_m)\left(s + \Omega + i\frac{\sqrt{-\Delta}}{2}\right)\left(s + \Omega - i\frac{\sqrt{-\Delta}}{2}\right) \\ p(\Omega \pm t, E', \kappa, \mu) &= (\omega^2 - t^2)\, Q^{\pm}(t) \\ Q^{\pm}(t) &:= (2\Omega \pm t)^2 - \frac{\Delta}{4} \end{aligned} \tag{7.7}$$

Our integral takes now the form

$$\begin{aligned} I(\kappa) &= \int_{s_m}^{s_M} \frac{s ds}{\sqrt{p(s, E', \kappa, \mu)}} \\ &= \frac{1}{2}\int_{-\omega}^{\omega}\left[\frac{\Omega + t}{\sqrt{p(\Omega + t, E', \kappa, \mu)}} + \frac{\Omega - t}{\sqrt{p(\Omega - t, E', \kappa, \mu)}}\right] dt \\ &= \int_0^{\omega} \frac{dt}{\sqrt{\omega^2 - t^2}\sqrt{Q^+Q^-}}\left[t\left(\sqrt{Q^-} - \sqrt{Q^+}\right) + \Omega\left(\sqrt{Q^-} + \sqrt{Q^+}\right)\right] \\ &= \int_0^{\omega} \frac{\Omega\, w(t)\, dt}{\sqrt{\omega^2 - t^2}\sqrt{Q^+Q^-}\left(\sqrt{Q^-} + \sqrt{Q^+}\right)} \end{aligned} \tag{7.8}$$

with

$$\begin{aligned} w(t) &:= -2t^2 - \frac{\Delta}{2} + 8\Omega^2 + 2\sqrt{Q^+Q^-} \\ &= -2t^2 - \frac{\Delta}{2} + 8\Omega^2 + 2\sqrt{\left(t^2 - \frac{\Delta}{4}\right)^2 + 2\,\Omega^2\, s(t)} \\ s(t) &:= -4t^2 - \Delta + 8\Omega^2 \end{aligned} \tag{7.9}$$

If we show that $w(t) > 0$, then the proof is finished. One has

$$s(t) \ge s(\omega) = 4\left[s_M s_m - \frac{32E'}{s_M s_m}\right] > 0\ , \tag{7.10}$$

using again 7.4. Thus

$$\begin{aligned} w(t) &> -2t^2 - \frac{\Delta}{2} + 8\Omega^2 + 2\sqrt{\left(t^2 - \frac{\Delta}{4}\right)^2} \\ &\ge -\Delta + 8\Omega^2 > 0 \end{aligned} \tag{7.11}$$

□

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