February $14^{\rm th}$, 2000

SOME EXACT RESULTS ON THE ULTRAMETRIC OVERLAP DISTRIBUTION IN MEAN FIELD SPIN GLASS MODELS (I)

Francesco Baffioni [1] and Francesco Rosati [2]

[1] Dipartimento di Matematica
[2] Dipartimento di Fisica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica 1, I-00133 Roma, Italy.

e-mail : baffioni@mat.uniroma2.it
Francesco.Rosati@roma2.infn.it

Abstract

The mean field spin glass model is analyzed by a combination of mathematically rigororous methods and a powerful *Ansatz*. The method exploited is general, and can be applied to others disordered mean field models such as, *e.g.*, neural networks.

It is well known that the probability measure of overlaps among replicas carries the whole physical content of these models. A functional order parameter of Parisi type is introduced by rigorous methods, according to previous works by F. Guerra. By the *Ansatz* that the functional order parameter is the correct order parameter of the model, we explicitly find the full overlap distribution. The physical interpretation of the functional order parameter is obtained, and ultrametricity of overlaps is derived as a natural consequence of a branching diffusion process.

It is shown by explicit construction that ultrametricity of the 3-replicas overlap distribution together with the Ghirlanda–Guerra relations determines the distribution of overlaps among s replicas, for any s, in terms of the one-overlap distribution.

1 Introduction.

Mean field spin glass models are considered as a prototype of disordered, frustrated systems and, more generally, of a large class of complex systems that can be successfully analyzed using the ideas developed in the study of spin glasses [2, 15]. Among these, the Sherrington–Kirkpatrick model [1] has a primary importance. This model is by now well understood in its general features, as described by Parisi with an ingenious method and the ultrametric *Ansatz* [2]. This picture has been confirmed by extensive numerical simulations [14, 15] and some rigorous results [3, 4, 5, 8, 9, 13]. In particular, F. Guerra has given a rigorous motivation for the introduction of a functional order parameter of Parisi type, and has shown how in this framework a simple *Ansatz* allows to express the thermodynamic variables and some physical observables in terms of that order parameter [5, 7].

In the present paper, the *Ansatz* of Guerra is extended, and is developed a method to express all physical observables in terms of the functional order parameter, in a mathematically rigorous framework. The method is general, and can be applied to other mean field disordered models such as the multi-spin interaction spin glass and the neural networks.

It is well known that the whole physical content of mean field spin glass models is contained in the overlap random variables. Given s replicas there are $s(s-1)/2$ overlaps between them, where s ranges on the natural numbers. Therefore, the physics of the model is fully contained in a probability distribution on an infinite-dimensional space. Overlaps do not fluctuate in the hight temperature phase : the Sherrington–Kirkpatrick solution turns out to be correct and the overlap distribution is trivial. In the low temperature phase this cannot happen : overlaps do fluctuate [2, 3, 5, 8].

Fluctuations are constrained by the symmetry under permutations of replicas and by the gauge symmetry. Thermodynamical constraints are expressed by Ghirlanda–Guerra relations, in the slightly stronger case when suitable infinitesimal interactions are added to the Hamiltonian [8, 11] (this is also known as the stochastic stability property [12, 13]). By the *Ansatz* that the overlap distribution is ultrametric, Parisi gave a solution of the model, in terms of a functional order parameter [2]. Ultrametricity is a simple constraint on the support that considerably simplifies the overlap distribution : together with the previously stated constraints, it reduces the problem to the determination of the mono-dimensional, one-overlap distribution P_{12}. This is proven in the last section of this paper.

A functional order parameter of Parisi type can be introduced rigorously to give a functional representation of the marginal martingale function, and therefore of the free energy [5]. This representation is not unique : there is an infinite set of functional order parameters giving rise to the same free energy. By an *Ansatz* on this representation, some overlap correlation functions has been expressed through the functional order parameter [7]. In this paper, by an extension of the *Ansatz*, we explicitly find the full overlap distribution in terms of the functional order parameter, and we show how ultrametricity naturally emerges.

The method (and the paper) goes as follows. We introduce a generating functional of physical observables (*i.e.*, expectations of overlap functions), derived from the marginal martingale function (sect. 3). Through the solution of a non-linear antiparabolic equation, and exploiting the *Ansatz*, we represent it in terms of the functional order parameter x (sect. 4). Then, we solve the antiparabolic equation by asymptotic expansion and explicitly find the overlap probability distribution. The physical interpretation of the functional order parameter is obtained, and ultrametricity of overlaps is derived as a natural consequence of the branching diffusion process underlying the equation (sect. 5).

Finally, it is shown that complete ultrametricity of overlaps results from ultrametricity of the 3-replicas overlap distribution. Moreover, it is proved that ultrametricity and the Ghirlanda–Guerra identities are *complementary* in order to determine the full overlap distribution, in the sense that one can hold independently of the other, but together they determine explicitly the overlap measure in terms of the one-overlap distribution P_{12} (sect. 6).

2 Overlaps in the Sherrington–Kirkpatrick model.

The mean field model of a spin glass is defined on sites $i = 1, 2, \dots, N$. To each site is assigned the Ising spin variable $\sigma_i = \pm 1$, so that a configuration of the system is described by the application $\sigma : \; i \to \sigma_i \in Z_2 = \{-1, 1\}$. The spins on two different sites i and j are coupled through the random variables J_{ij}, all independent from each other and equally distributed. For the sake of simplicity we assume a Gaussian distribution, with

$$\mathrm{E}(J_{ij}) = 0, \quad \mathrm{E}(J_{ij}^2) = 1, \tag{1}$$

where E denotes averages on the J variables. The J_{ij}'s are called *quenched* variables, because they do not participate to thermalisation. The Hamiltonian of the Sherrington–Kirkpatrick model is

$$H_N(\sigma, J) = -\frac{1}{\sqrt{N}} \sum_{(i,j)} J_{ij} \sigma_i \sigma_j, \tag{2}$$

where the sum extends over all the $N(N-1)/2$ couples of sites. The normalization factor $1/\sqrt{N}$ is needed to have the correct behavior of the thermodynamic variables in the limit $N \to \infty$. Denoting with β the inverse temperature (in proper units), we introduce the partition function $Z_N(\beta, J)$ and the free energy density $f_N(\beta, J)$:

$$\begin{aligned} Z_N(\beta, J) &= \sum_{\sigma_1 \dots \sigma_N} e^{-\beta H_N(\sigma, J)}, & (3) \\ -\beta f_N(\beta, J) &= \frac{1}{N} \log Z_N(\beta, J). & (4) \end{aligned}$$

The associated Boltzmann state $\omega_{N,\beta,J}$ is defined by

$$\omega_{N,\beta,J}(A) = \frac{1}{Z_N(\beta, J)} \sum_{\sigma_1 \dots \sigma_N} A(\sigma) e^{-\beta H_N(\sigma, J)}, \tag{5}$$

for a generic function A of the spin variables. Another relevant quantity is the average of internal energy density $u_N(\beta)$

$$u_N(\beta) = \frac{1}{N}\mathrm{E}\,\omega_{N,\beta,J}(H_N(\sigma,J)) = \mathrm{E}\frac{\partial}{\partial\beta}(\beta f_N(\beta,J)). \tag{6}$$

In the thermodynamic limit the free energy density is self-averaging in quadratic mean [3]. For the internal energy density the same property has been proven for almost all values of β, but is believed to hold without restrictions [8] .

One of the main features of the mean field spin glass model is the existence of observables that do not self-average in the thermodynamic limit. This is one of the fundamental intuitions contained in the Parisi *Ansatz* of replica symmetry breaking. Indeed Pastur and Shcherbina have proven that if a suitably chosen order parameter (coming from the response of the system to an external random field) is self-averaging in the thermodynamic limit, then the solution of the model has the Sherrington–Kirkpatrick form [3, 4]: this is unphysical at high β, because it gives negative entropy. Moreover, self-averaging of the Edward–Anderson order parameter implies that the overlap distribution is the trivial one corresponding to the replica symmetric *Ansatz* of S.–K. [8].

Let us consider s copies (replicas) of the system, whose configurations are given by the Ising spin variables $\sigma_i^{(1)},\ldots,\sigma_i^{(s)}$, and denote with $\omega_J^{(a)}, a = 1,2,\ldots,s$ the relative Boltzmann states, the dependence on β and N being understood. We introduce the product state Ω_J by

$$\Omega_J = \omega_J^{(1)}\omega_J^{(2)}\cdots\omega_J^{(s)}, \tag{7}$$

where all the states $\omega_J^{(a)}$ are subject to the same values of the quenched variables J, and the same temperature β.

The overlap between the two replicas a and b, Q_{ab}, is defined by

$$Q_{ab} = \frac{1}{N}\sum_i \sigma_i^{(a)}\sigma_i^{(b)}, \tag{8}$$

with the obvious bounds $-1 \leq Q_{ab} \leq 1$.

The importance of overlaps lies in the fact that all physical observables can be expressed in the form

$$\mathrm{E}\Omega_J[F(Q_{12},Q_{13},\ldots)], \tag{9}$$

for some function F. For F smooth, we can introduce the random variables $q_{12},q_{13},\ldots$, through the definition of their averages

$$\langle F(q_{12},q_{13},\ldots)\rangle = \mathrm{E}\Omega_J[F(Q_{12},Q_{13},\ldots)]. \tag{10}$$

Notice that the expectation $\langle\cdot\rangle$ includes both the thermal average and the average E over disorder. The overlap distribution carries the whole physical content of the model [8].

Let us recall some considerations about the overlap distribution. The average E over quenched variables introduces correlation between different groups of replicas. For example we have, in general,

$$\langle q_{12}^2 q_{34}^2 \rangle \neq \langle q_{12}^2 \rangle \langle q_{34}^2 \rangle. \tag{11}$$

The $\langle \cdot \rangle$ average is obviously invariant under permutations of replica indices (e.g. $\langle q_{12}^2 q_{13}^2 \rangle = \langle q_{23}^2 q_{13}^2 \rangle$, $\langle q_{12}^2 \rangle = \langle q_{34}^2 \rangle$). Moreover, it is invariant under the gauge transformations defined by

$$q_{ab} \longrightarrow \varepsilon_a q_{ab} \varepsilon_b, \tag{12}$$

where $\varepsilon_a = \pm 1$. This is an easy consequence of the fact that each of the $\omega_J^{(a)}$ is an even state on the respective $\sigma^{(a)}$. It follows, for instance, that polynomials in the overlaps which are not gauge invariant have zero mean. These symmetries furnish important restrictions on the the overlap distribution, but even more important constraints have been given by [8, 11], using simple arguments based on convexity properties and positivity of fluctuations. Consider s replicas, and the $s(s-1)/2$ overlaps between them. Let us denote by $\mathcal{A}_s$ the associated algebra of observables. Introduce the overlap $q_{a,s+1}$, between replica a and an additional replica $s+1$, and consider the conditional probability distribution $\tilde{P}_{(a,s+1)}(q_{a,s+1}|\mathcal{A}_s)$ of $q_{a,s+1}$ given the overlaps among the first s replicas. By adding to the Hamiltonian suitable infinitesimal external fields, and taking the thermodynamic limit with a careful procedure, Guerra and Ghirlanda have demonstrated that the following theorem holds for a very general class of probability measures, including short range models [11].

Theorem 2.1 *Given the overlaps among s replicas, the overlap between one of these, let say a, and an additional replica $s+1$ is either independent of the former overlaps, or it is identical to one of the overlaps q_{ab}, with b running from 1 to s, excluding a. Each of these cases have probability $1/s$:*

$$\tilde{P}_{(a,s+1)}(q_{a,s+1}|\mathcal{A}_s) = \frac{1}{s} P_{12}(q_{a,s+1}) + \frac{1}{s} \sum_{b \neq a} \delta(q_{a,s+1} - q_{ab}). \tag{13}$$

Results of this kind have been obtained by Parisi in the frame of replica method [12], and by Aizenmann and Contucci [13].

3 A generator of overlap distributions.

Let ω be a generic even state on the Ising spins $\sigma_1, \ldots, \sigma_N$, possibly depending on the quenched variables J_{ij} and let $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$ be an even, convex function, such that $|f_1(y)| \leq c|y|$ asymptotically with $|y| \to \infty$ for some positive c. Let the generating functional $\psi_N(\omega, f_1)$ be given by

$$\psi_N(\omega, f_1) = \mathrm{E} \log \omega(\exp f_1(h_N(\sigma, J))), \tag{14}$$

where $h_N(\sigma, J) = N^{-1/2} \sum_i J_i \sigma_i$ is the cavity field, and the J_i's are fresh noise with the same properties of J_{ij}. The functional $\psi_N(\omega, f_1)$ contains all informations on the distribution of the replicated cavity fields $h^{(a)} \equiv h_N(\sigma^{(a)}, J)$. That, in turn, is related to the overlap distribution through the well known formula

$$\mathrm{E}\Omega_J\left(\exp\left(i\sum_a k_a h^{(a)}\right)\right) = \left\langle \exp\left(-\sum_{a,b} k_a k_b q_{ab}/2\right)\right\rangle . \tag{15}$$

We expand the logarithm in power series and we introduce replicas:

$$\begin{aligned} \psi_N(\omega,\, f_1) &= \mathrm{E}\left[\ln\left(1-\omega\left(1-\exp\left(f_1(h)\right)\right)\right)\right] \\ &= -\sum_{s=1}^{\infty}\frac{1}{s}\mathrm{E}\left[\omega\left(1-\exp\left(f_1(h)\right)\right)\right]^s \\ &= -\sum_{s=1}^{\infty}\frac{1}{s}\mathrm{E}\Omega_J\left[\prod_{a=1}^{s}\left(1-\mathrm{e}^{f_1(h^{(a)})}\right)\right] \end{aligned} \tag{16}$$

where h denotes the cavity field and $h^{(a)}$ its replicas.
Let us introduce the generalized Fourier transform ϕ, which is a well defined even generalized function :

$$1-\exp\left(f_1(y)\right) = \int_{-\infty}^{\infty}\mathrm{d}k\,\phi(k)e^{iky} \tag{17}$$

By the convenient replacement $\varphi(k) \equiv \phi(k)\exp(-k^2/2)$, we finally have

$$\psi_N(\omega,\, f_1) = -\sum_{s=1}^{\infty}\frac{1}{s}\int \mathrm{d}^s k\prod_{a=1}^{s}\varphi(k_a)\left\langle \exp\left(-\sum_{(a,b)} k_a k_b \mathrm{q}_{ab}\right)\right\rangle \tag{18}$$

where the sum in the exponential is over the couples (a, b), for $1 \le a < b \le s$. The dependence of the r.h.s. on f_1 is through the function φ. Notice that terms $s = 2, 3$ contain the characteristic functions of the distributions of overlaps among 2 and 3 replicas, respectively.

It important to notice that the thermodynamic functions can be represented through the functional $\psi_N(\omega,\, f_1)$. Consider the case $f_1(y) = \log\cosh\beta y$, and the corresponding function $\psi_N^\star(\beta) \equiv \psi_N(\omega,\, \log\cosh\beta\,\cdot)$, where ω is the Boltzman state of SK model. Then the following holds [5].

Proposition 3.1 *Assume the existance of the limit* $\lim_{N\to\infty}\psi_N^\star(\beta) = \psi^\star(\beta)$*, uniformly on a compact region* $0 \le \beta \le \tilde{\beta}$*, with* $\psi^\star$ *continuous in* β*, as a consequence. Let us define*

$$\alpha(\beta) = \log 2 + \int_0^1 \psi^\star(\beta\sqrt{1-q})\; dq, \tag{19}$$

so that the β *derivative* $\alpha'(\beta)$ *exists and the following holds*

$$\alpha(\beta) + \beta\alpha'(\beta)/2 = \log 2 + \psi^\star(\beta). \tag{20}$$

Then, we have, for $0 \le \beta \le \tilde{\beta}$,

$$\lim_{N\to\infty}\frac{1}{N}\mathrm{E}\left(\log Z_N(\beta,\, J)\right) = \alpha(\beta), \quad \lim_{N\to\infty}\frac{1}{N}\partial_\beta\mathrm{E}\left(\log Z_N(\beta,\, J)\right) = \alpha'(\beta). \tag{21}$$

4 The functional order parameter.

In the frame of the cavity method, a functional order parameter of Parisi type was introduced by Guerra as a functional representation of the marginal martingale function [5]. Then, he showed that by a simple *Ansatz* some overlap correlation functions can be expressed in terms of the functional order parameter [7].

In this section we give an extension of the representation Theorem, thus obtaining a functional representation of the physical observables. Exploiting the *Ansatz*, the generating functional $\psi_N(\omega,\, f_1)$ is expressed in terms of the functional order parameter. Therefore, the explicit form of the overlap distribution can be extracted.

Let us introduce the convex set $\mathcal{X}$ of functional order parameters of the type

$$x:\ [0,1] \ni q \longrightarrow x(q) \in [0,1], \tag{22}$$

with the $L^1(dq)$ distance norm. We induce on $\mathcal{X}$ a partial ordering, by defining $x \le \bar{x}$ if $x(q) \le \bar{x}(q)$ for all $0 \le q \le 1$, and introduce the extremal order parameters $x_0(q) \equiv 0$ and $x_1(q) \equiv 1$, such that for any x we have $x_0(q) \le x(q) \le x_1(q)$.

For each x in $\mathcal{X}$, and for suitable f_1 (see the previuos section), let us define the function with values $f(q, y;\, x, f_1), 0 \le q \le 1,\ y \in \mathbb{R}$, as the solution of the nonlinear antiparabolic equation

$$\partial_q f + \frac{1}{2}(f'' + x(q) f'^2) = 0, \tag{23}$$

with final condition

$$f(1, y;\, x, f_1) = f_1(y) \tag{24}$$

In (23), $f' = \partial_y f$ and $f'' = \partial_y^2 f$.

With these definitions, the following representation theorem holds [5].

Theorem 4.1 *There exists a nonempty hyper-surface $\Sigma_N(\omega, f_1)$ in $\mathcal{X}$ such that, for any $x \in \Sigma_N(\omega, f_1)$ and f solution of (23,24), we have the following representation*

$$\psi_N(\omega,\, f_1) = f(0, 0;\, x, f_1). \tag{25}$$

Any family of functional order parameters, x_ϵ, depending continuously in the L^1 norm on the variable ϵ, $0 \le \epsilon \le 1$, with $x_0 \equiv 0$, and $x_1 \equiv 1$, and nondecreasing in ϵ, must necessarily cross $\Sigma_N(\omega, f_1)$ for some value of the variable ϵ (we say that $\Sigma_N(\omega, f_1)$ has the monotone intersection property). A similar representation holds also in the infinite volume limit.

Of particular interest are those states ω such that the representation (25) holds with some x, depending on ω, but independent on f_1, with some possible error vanishing in the limit $N \to \infty$. We call such states x-representable. Some examples of x-representable states are shown in [7].

An attractive conjecture is that the Boltzmann state of mean field spin glass models is x-representable. Indeed, this must be the case if x is the correct order parameter. We will refer to this as the *tomographic Ansatz* : in the $\mathcal{X}$ space the hyper-surfaces $\{\Sigma_\infty(\omega, f_1), f_1 \in F_1\}$ have a common point x, which gives the physical content of the theory. By this *Ansatz*, we can express the full probability distribution of overlaps in terms of the functional order parameter. Let us state the following theorem, one of the main results of this paper, leaving the proof to the next sections.

Theorem 4.2 *Let ω be an even state on the Ising spin variables σ_i, depending on the quenched variables J, and suppose it is x-representable, with $x(0) = 0$ and $x(1) = 1$. Then the following holds.*

a) The probability distributions of overlaps among $s = 2, 3$ replicas are given in terms of the functional order parameter x by the following expressions :

$$P_{12}(q) \equiv P(q) = \frac{d}{dq}x(q), \tag{26}$$

$$\begin{aligned} P_{12,23,13}(q_{12}, q_{23}, q_{13}) &= \frac{1}{2}x(q_{12})P(q_{12})\delta(q_{12} - q_{23})\delta(q_{12} - q_{13}) + \\ &+ \frac{1}{2}\left(P(q_{12})P(q_{23})\theta(q_{12} - q_{23})\delta(q_{13} - q_{23}) + \text{cyclic perm.}\right) \end{aligned} \tag{27}$$

b) Assume in addition the hypothesis of Theorem 2.1. Then the overlap distribution is uniquely determined in terms of the functional order parameter x, and the s-replicas marginals (i.e., the distribution of overlaps among s replicas) can be given explicitly for any s (see section 6).

We have used Dirac's δ function and the step function θ. Extension to regions of negative q's is made by gauge symmetry, as shown in the next section. Equation (26) gives the physical meaning of the functional order parameter; equation (27) corresponds to ultrametricity of the overlap distribution, as is proven in the following. For other values of $x(0)$ and $x(1)$, slightly different results can be obtained.

As is shown extensively in the next section, ultrametricity arises naturally as a consequence of the branching diffusion process underlying equation (23, 24).

All results are in full agreement with those found in the frame of replica symmetry breaking method with Parisi *Ansatz* [2].

5 Asymptotic solution of the antiparabolic equation.

The results (26,27) of Theorem 4.2 are obtained by equation (25), and the tomographic *Ansatz*. Both members of equation (25) are expressed as asymptotic series, which are then compared term by term. The first one is given by equation (18), the second is obtained in this section.

Let us transform equation (23, 24)

$$\begin{cases} \partial_q \ f_q + \dfrac{1}{2}\left(f_q'' + x_q \ f_q'^2\right) = 0 \\ f(1,y;x,f_1) = f_1(y) \end{cases}$$

into an equivalent form. When $x(0) = 0$ and $x(1) = 1$, satisfied by physical order parameters, it is convenient to make the substitution

$$g_q\,(y) = \left[1 - \exp\left(x_q \ f_q(y)\right)\right]/x_q \tag{28}$$

the x and f_1 dependence of f being understood. The resulting equation is

$$\begin{cases} \partial_q \ g_q + \dfrac{1}{2} \ g_q'' = \rho_q \left[x_q \ g_q + (1 - x_q \ g_q)\ln(1 - x_q \ g_q)\right] / \ x_q^2 \\ g(1,y;x,f_1) \equiv g_1(y) = 1 - \exp\left(f_1(y)\right) \end{cases} \tag{29}$$

where $x_q \equiv x(q)$ and $\rho_q \equiv \mathrm{d}x_q/\mathrm{d}q$. Notice that the final condition for g is equal to the function used in the expansion of $\psi_N(\omega, f_1)$ (eq. 16), and that $g(0,y) \ = \ f(0,y)$. Let us re-write equation (29) in integral form:

$$g_q = N_{1-q} * g_1 + \int_q^1 \mathrm{d}q' \ \frac{\rho_{q'}}{x_{q'}^2} \ N_{q'-q} * \left[x_{q'} \ g_{q'} + (1 - x_{q'} \ g_{q'})\ln(1 - x_{q'} \ g_{q'})\right] \tag{30}$$

as one can straightforwardly see by simple inspection. Here $N_q \equiv N(q,y) = \exp\left(-y^2/2q\right)/\sqrt{2\pi q}$ is the usual heat kernel and the symbol $*$ is the convolution operation on y variable [1].

Equation (30) can be handled by asymptotic expansion of the r.h.s. term under square brackets :

$$g_q = N_{1-q} * g_1 + \sum_{i=2}^{\infty} \frac{1}{i(i-1)} \int_q^1 \mathrm{d}q' \ \rho_{q'} \ x_{q'}^{(i-2)} \ N_{q'-q} * \left[(g_{q'})^i\right] \tag{31}$$

We write the above equation in the "moment space" : let η_q be the Fourier transform of $N_q * g_q$ in the y variable and φ that of $N_1 * \ g_1$. Thus we have, after simple algebraic manipulation

$$\eta_z = \varphi_z + \mathrm{F}_z\,[\eta] \tag{32}$$

where z is a collective variable for (q,k), $\varphi_z \equiv \varphi(k)$ is the same function appearing in eq. (18), $\mathrm{F}_z\,[\eta]$ is a function of z and a functional of η

$$\mathrm{F}_z\,[\eta] \equiv \sum_{i=2}^{\infty} \frac{1}{i!} \ \hat{\mathrm{O}}_z^{(i)} \ [\eta,\ldots,\eta] \tag{33}$$

[1] $(f * g)(y) \equiv \int \mathrm{d}y' f(y-y') \ g(y')$

and the $\hat{\mathrm{O}}_z^{(i)}$ are well defined multi-linear integral operators

$$\begin{aligned}
&\hat{\mathrm{O}}_z^{(i)}\,[\varphi_1,\ldots,\varphi_i] \\
&\quad \equiv (i-2)! \quad \int \mathrm{d}^i k\ \delta(k_1+\ldots+k_i-k)\ \prod_{a=1}^{i}\ \varphi_a(k_a) \\
&\qquad\qquad \int_q^1 \mathrm{d}q'\rho_{q'}\ {x_{q'}}^{i-2}\ \exp(-q'\sum_{(a,b)} k_a k_b)
\end{aligned}$$

Every term in the asymptotic expansion is well defined. Notice that the representation Theorem 4.1 can be rephrased as

$$\psi_N(\omega, f_1) = -g(0,0) = -\int \mathrm{d}k\ \eta(0,k) \tag{34}$$

and this is the form that we will use in the sequel.

The i–th functional derivative of $\mathrm{F}_z[\eta]$ w.r.t. η calculated in zero gives the integral kernel of the $\hat{\mathrm{O}}_z^{(i)}$ operator. In particular

$$\mathrm{F}_z\,[\eta]|_{\eta=0} = 0 \quad ; \quad \left.\frac{\delta \mathrm{F}_z\,[\eta]}{\delta \eta_w}\right|_{\eta=0} = 0 \tag{35}$$

Replacing $\eta = \mathrm{L}[\varphi]$ in (32) we have

$$\mathrm{L}_z\,[\varphi] \equiv \varphi_z + \mathrm{F}_z\,[\mathrm{L}[\varphi]]\,, \tag{36}$$

which defines iteratively the inverse functional $L_z\,[\,.\,]$:

$$\mathrm{L}_z\,[\varphi] = \sum_{s=1}^{\infty} \frac{1}{s!} \int \mathrm{d}^i k\ \prod_{a=1}^{s} \varphi(k_a)\quad L_z^{(s)}(k_1,\ldots,k_s) \tag{37}$$

It is easy to check that

$$\mathrm{L}_z\,[\varphi]|_{\varphi=0} = 0 \quad ; \quad \left.\frac{\delta \mathrm{L}_z\,[\varphi]}{\delta \varphi_w}\right|_{\varphi=0} = \delta_{z-w} \tag{38}$$

where δ_{z-w} is the usual Dirac's function, and that, by derivating (36) w.r.t. φ,

$$\frac{\delta \mathrm{L}_z\,[\varphi]}{\delta \varphi_w} \equiv \delta_{z-w} + \int \mathrm{d}w' \frac{\delta \mathrm{F}_z}{\delta \eta_{w'}}\,[\mathrm{L}[\varphi]]\ \ \frac{\delta \mathrm{L}_{w'}\,[\varphi]}{\delta \varphi_w} \tag{39}$$

Subsequent functional derivatives w.r.t. φ, calculated in $\varphi \equiv 0$, and the properties (35) and (38) allow us to obtain straightforwardly all the integral kernels $L_z^{(s)}(k_1,\ldots,k_s)$ for any s, in terms of $\hat{\mathrm{O}}_z^{(i)}$ operators. We thus obtain

$$\int \mathrm{d}k\,\eta(0,k) = \sum_{s=1}^{\infty} \frac{1}{s} \int \mathrm{d}^s k \prod_{a=1}^{s} \varphi(k_a) \frac{1}{(s-1)!} \mathrm{L}_o^{(s)}(k_1,\ldots,k_s) \tag{40}$$

where $\mathrm{L}_o^{(s)}(k_1,\dots,k_s)$ are the integral kernels, calculated for $q=0$ and integrated on the overall delta dependence in the k variable. They are of the form:

$$\frac{1}{(s-1)!}\mathrm{L}_o^{(s)}(k_1,\dots,k_s) = \int \prod_{(a,b)} \mathrm{d}\mathrm{y}_{ab}\ \rho_s^{(+)}(\{\mathrm{y}_{ab}\}) \exp\left(-\textstyle\sum_{(a,b)} k_a k_b \mathrm{y}_{ab}\right) \tag{41}$$

where $\rho_s^{(+)}$ has support on a subset of $[0,1]^{s(s-1)/2}$. In appendix we report the explicit expressions of $\rho_s^{(+)}$, for $s=2,3,4$ and the recipe to construct it for a generic s.

By eq. (34) we can compare the asymptotic expansions in eq. (18) and eq. (40). As the function $\varphi(k)$ is even, only the even part in the k's of integral kernels from both sides can be equated. Let us define on $[-1,1]^{s(s-1)/2}$ the function ρ_s, extending by 'gauge symmetry' the function $\rho_s^{(+)}$

$$\begin{aligned}\rho_s(\{\mathrm{y}_{ab}\}) &= 2^{-s}\sum_{\{\varepsilon\}}\rho_s^{(+)}(\{\varepsilon_a \mathrm{y}_{ab}\varepsilon_b\}) \\ &= 2^{-(s-1)}\sum_{\{\varepsilon\}:\varepsilon_1=1}\rho_s^{(+)}(\{\varepsilon_a \mathrm{y}_{ab}\varepsilon_b\})\end{aligned} \tag{42}$$

where the sums run over all the $\varepsilon_a = \pm 1,\ a = 2,\dots,s$ and $\rho_s^{(+)} = 0$ if its argument is outside $[0,1]^{s(s-1)/2}$; we have used the invariance $\varepsilon_a \longrightarrow -\varepsilon_a$ to fix $\varepsilon_1 = 1$ in the second line. We finally have

$$\left\langle \exp\left(-\textstyle\sum_{(a,b)} k_a k_b \mathrm{q}_{ab}\right)\right\rangle = \int \prod_{(a,b)} \mathrm{d}\mathrm{y}_{ab}\ \rho_s(\{\mathrm{y}_{ab}\}) \exp\left(-\textstyle\sum_{(a,b)} k_a k_b \mathrm{y}_{ab}\right) \tag{43}$$

which proves part a of Theorem 4.2.

For $s > 3$ the l.h.s. of eq. (43) does not correspond to the characteristic function of the overlap distribution, as the number of the k's parameters is not sufficient, but to its restriction on a hyper-surface of dimension s. In the next section, assuming the hypothesis of Theorem 2.1, we show that the results obtained so far allow us to construct the full overlap distribution function. The resulting $s=4$ overlap distribution coincides with ρ_s. This is a strong indication that ρ_s is the correct distribution also in the case of no additional interactions, as required by Theorem 2.1.

For a generic s the distribution ρ_s has the ultrametric form

$$\rho_s(\{\mathrm{y}_{ab}\}) = \sum_{i:A_i^s\subset A^s} p_i\ \rho_s^{(i)}(\{\mathrm{y}_{ab}\}|A_i^s) \tag{44}$$

Here A_i^s are disjoint sets, made by portions of hyper-planes in $[-1,1]^{s(s-1)/2}$, with dimension $|A_i^s| \le s-1$; A^s is the union set; p_i are positive numbers, which sum to one, and $\rho_s^{(i)}(\cdots|A_i^s)$ are probability densities, whose supports are the sets A_i^s. The r.h.s. can thus be interpreted as a composite probability formula: p_i is the probability that the *ultrametric event* A_i^s happens and $\rho_s^{(i)}(\cdots|A_i^s)$ is the *overlap* probability, conditioned to A_i^s. The events A_i^s are disjoint and each $\rho_s^{(i)}$ effectively depends only on at most $s-1$ variables.

6 Ultrametric distributions.

Consider the set Φ of random variables $q_{a,b} \in [-1,\, 1]$:

$$\Phi = \{q_{a,b}\,,\ \ (a,b) \in \varphi \subset C\}, \tag{45}$$

where C is the set of couples (a,b) of natural numbers $a, b \in \mathbb{N},\ a < b$. To the set φ are then associated a probability space and the probability distribution $P_\varphi(\Phi)$ on it. The distribution functions P_φ satisfy the consistency conditions

$$\int P_{\varphi,\varphi'}(\Phi,\, \Phi') \prod_{\alpha\in\varphi'} dq_\alpha = P_\varphi(\Phi), \tag{46}$$

for all disjoint sets $\varphi, \varphi' \subset C$. In the following we will often write $\{A, B\} \equiv A \cup B$ and $ab \equiv (a,b)$ when not ambiguous. Let us introduce the operator $\Pi_{l,m}$ that, acting on φ, permutates the indices l and m. E.g.: $\Pi_{12}\{(1,2),\,(2,3)\} = \{(1,2),\,(1,3)\}$. Let $\varphi' = \Pi_{l,m}\varphi$, and denote by Φ' the associated set of q's. According to the symmetries of the $\langle - \rangle$ average, we ask the probability measure P to be gauge invariant, and symmetric under permutations of indices, in the following sense:

$$P_{\varphi'}(\Phi') = P_\varphi(\Phi) = P_\varphi(\Pi_{l,m}\Phi') \equiv (\Pi_{l,m}\, P_\varphi)\ (\Phi'). \tag{47}$$

This defines the operator $\Pi_{l,m}$ on the space of distributions.

Let us consider a very particular class of distributions for the overlaps between three replicas, *i.e.*, the ultrametric distributions :

$$\begin{aligned} P_{12,23,13}(q_{12},\, q_{23},\, q_{13}) &= B(q_{12}, q_{23})\, \theta(q_{12} - q_{23})\, \delta(q_{13} - q_{23}) \\ &+ B(q_{23}, q_{12})\, \theta(q_{23} - q_{12})\, \delta(q_{13} - q_{12}) \\ &+ B(q_{13}, q_{23})\, \theta(q_{13} - q_{23})\, \delta(q_{12} - q_{23}), \end{aligned} \tag{48}$$

where B is a distribution. This simply states that among the three overlaps, two are equal and the third is greater or equal. From eq. (48), a simple application of symmetries and of the consistency conditions (46), leads to

Proposition 6.1 *If the distribution $P_{12,23,13}$ has the form (48), then for any tern of replicas, (a,b,c), the operator $\mathcal{F}_{a,b,c}$ is defined such that*

$$P_{ab,ac,\varphi} = \mathcal{F}_{a,b,c}(P_{ab,\varphi},\, P_{ac,\varphi}), \tag{49}$$

where $\varphi \subset C,\ (b,c) \in \varphi$ and $(a,b),\, (a,c) \notin \varphi$. The operator $\mathcal{F}_{a,b,c}$ is defined through its values

$$\begin{aligned} &\mathcal{F}_{a,b,c}(P_{ab,\varphi},\, P_{ac,\varphi})\, (q_{ab},\, q_{ac};\, \Phi) = \\ =\ & P_{ab,\varphi}(q_{ab}; \Phi)\ [\theta(q_{ab} - q_{bc})\delta(q_{ac} - q_{bc}) + \theta(q_{bc} - q_{ab})\delta(q_{ac} - q_{ab})] + \\ +\ & P_{ac,\varphi}(q_{ac};\, \Phi)\, \theta(q_{ac} - q_{bc})\delta(q_{ab} - q_{bc}) \\ -\ & \delta(q_{ab} - q_{bc})\delta(q_{ab} - q_{ac}) \int P_{ac,\varphi}(q_{ac};\, \Phi)\theta(q_{ac} - q_{bc})\, dq_{ac}. \end{aligned} \tag{50}$$

Note that when the set φ is symmetric under permutation of indices b, c, we can introduce the operator $\widetilde{\mathcal{F}}_{a,b,c}$,

$$P_{ac,\varphi'} = \widetilde{\mathcal{F}}_{a,b,c}(P_{\varphi'}) \equiv \mathcal{F}_{a,b,c}(P_{\varphi'},\, \Pi_{b,c} P_{\varphi'}), \tag{51}$$

where $\varphi' \equiv \{(a,b),\, \varphi\}$.

This property of the overlap distribution corresponds to ultrametricity. In fact, eq. (50) simply states that for *any* triangle of overlaps in a given set $\tilde{\varphi}$, two overlaps are equal and the third is greater or equal. The proof of the theorem and of the subsequent lemma as well, does not depend on the nature of the $q_{\cdot,\cdot}$ variables, but only on symmetries and general properties of probability spaces.

Lemma 6.2 *In the hypothesis of theorem 6.1 we can express the probability distribution of the overlaps between $s+1$ replicas in terms of the distribution of the overlaps between s replicas and $q_{1,s+1}$ (for $s \geq 3$).*

The proof goes as follows. Given $s \geq 3$ and $l \leq s$, we define the set $\varphi_{s,l}$ by

$$\varphi_{s,l} \equiv \{(a,b),\ 1 \leq a < b \leq s\} \cup \{(c, s+1),\ 1 \leq c \leq l\}, \tag{52}$$

such that the following simple relations hold:

$$\{(l+1, s+1),\, \varphi_{s,l}\} = \varphi_{s,l+1}, \tag{53}$$

$$\varphi_{s,s} = \varphi_{s+1,0}. \tag{54}$$

Applying formula (51), with $l+1 \leq s$, we have

$$P_{\varphi_{s,l+1}} = P_{(l+1,s+1),\varphi_{s,l}} = \widetilde{\mathcal{F}}_{s+1,1,l+1}(P_{\varphi_{s,l}}). \tag{55}$$

By iteration we have the thesis

$$P_{\varphi_{s+1,0}} = \widetilde{\mathcal{F}}_{s+1,1,s} \ldots \widetilde{\mathcal{F}}_{s+1,1,2}\,(P_{\varphi_{s,1}}). \tag{56}$$

Moreover, by definition of conditional probability we have

$$P_{\varphi_{s,1}} = \tilde{P}_{(1,s+1)}\, P_{\varphi_{s,0}}, \tag{57}$$

where $\tilde{P}_{(1,s+1)}$ is given by eq.(13). Therefore we have proven the following

Theorem 6.3 *If the 3-replicas overlap distribution $P_{12,23,13}$ is ultrametric (i.e., of the form (48)), and in the limits of validity of theorem 2.1, the overlap distribution is uniquely determined in terms of P_{12}. The explicit form of the distributions of overlaps among s replicas, for any s (i.e., the s-replicas marginals $P_{\varphi_{s,0}}$), can be calculated by repeated applications of eqs. (56,57).*

Since Theorem 4.2.a proves the hypothesis of Theorem 6.3 in the case of mean field spin glass models, this completes the proof of its part b.

The explicit construction (56, 57) clearly shows that ultrametricity and the Ghirlanda–Guerra relations can be considered as *complementary* in order to determine the full overlap distribution, in the sense that one can hold independently of the other, but together they determine explicitly the overlap measure in terms of the one-overlap distribution.

Results of this kind were obtained by Parisi in the case $s = 3$ [12].

7 Conclusions

It has been shown how mean field disordered models can be successfully analyzed in a mathematically rigorous framework, with a simple *Ansatz* which is completely different from the Replica Simmetry Breaking *Ansatz* . In the S.–K. spin glasses case, the main features of the accepted physical solution – the Parisi solution – have been obtained. The method exploited, due to F. Guerra, is based on the cavity method and general theorems, and can therefore be applied to other disordered mean field models such as the multi-spin interaction spin glasses or neural networks.

The functional order parameter $x(q)$ has been introduced in the S.–K. model. By the *Ansatz* that x is indeed the correct order parameter, all physical observables have been expressed in terms of it. The physical interpretation of the functional order parameter (*i.e.* $dx(q)/dq = P(q)$) results, and ultrametricity of overlaps is derived as a natural consequence of a branching diffusion process.

It has been shown by explicit construction that ultrametricity of the 3-replicas overlap distribution together with the Ghirlanda–Guerra relations determines the distribution of overlaps among s replicas, for any s, in terms of P_{12}.

Acknowledgments

The authors wish to express their warmest thanks to F. Guerra, for fruitful suggestions and stimulating discussions.

8 Appendix

We report the explicit expressions of $\rho_s^{(+)}(\{\mathrm{y}_{ab}\})$ for $s = 2, 3, 4$. For two replicas we have

$$\rho_2^{(+)}(\mathrm{y}_{12}) = \int_0^1 dq\ \rho(q)\ \delta(\mathrm{y}_{12} - q) \tag{58}$$

for three replicas

$$\begin{aligned}\rho_3^{(+)}(\{\mathrm{y}_{ab}\}) &= \frac{1}{2}\int_0^1 dq\ \rho(q)\, x(q) \prod_{(a,b)\subset G_3} \delta(\mathrm{y}_{ab} - q)\ + \\ &+ \frac{1}{2}\sum_\pi{}^{(3)} \int_0^1 dq \int_q^1 dq'\ \rho(q)\,\rho(q')\,\delta(\mathrm{y}_{\pi_1\pi_2} - q') \prod_{(a,b)\subset G_3\backslash(\pi_1,\pi_2)} \delta(\mathrm{y}_{ab} - q)\end{aligned} \tag{59}$$

and for four replicas

$$\begin{aligned}&\rho_4^{(+)}(\{\mathrm{y}_{ab}\}) = \\ &= \frac{1}{3}\int_0^1 dq\ \rho(q) x^2(q) \prod_{(a,b)\subset G_4} \delta(\mathrm{y}_{ab} - q)\ + \\ &+ \frac{1}{6}\sum_\pi{}^{(6)} \int_0^1 dq \int_q^1 dq'\ \rho(q) x(q) \rho(q')\ \delta(\mathrm{y}_{\pi_1\pi_2} - q') \prod_{(a,b)\subset G_4\backslash(\pi_1,\pi_2)} \delta(\mathrm{y}_{ab} - q)\ + \\ &+ \frac{1}{6}\sum_\pi{}^{(4)} \int_0^1 dq \int_q^1 dq'\ \rho(q)\rho(q') x(q') \prod_{(a,b)\subset G_3(\pi_1,\pi_2,\pi_3)} \delta(\mathrm{y}_{ab} - q')\ \times \\ &\times \prod_{(a,b)\subset G_4\backslash G_3(\pi_1,\pi_2,\pi_3)} \delta(\mathrm{y}_{ab} - q)\ + \\ &+ \frac{1}{6}\sum_\pi{}^{(3)} \int_0^1 dq \int_q^1 dq' \int_q^1 dq''\ \rho(q)\rho(q')\rho(q'')\ \ \delta(\mathrm{y}_{\pi_1\pi_2} - q')\delta(\mathrm{y}_{\pi_3\pi_4} - q'')\ \times \\ &\times \prod_{(a,b)\subset G_4\backslash\{(\pi_1,\pi_2),(\pi_3,\pi_4)\}} \delta(\mathrm{y}_{ab} - q)\ + \\ &+ \frac{1}{6}\sum_\pi{}^{(12)} \int_0^1 dq \int_q^1 dq' \int_{q'}^1 dq''\ \rho(q)\rho(q')\rho(q'')\ \ \delta(\mathrm{y}_{\pi_1\pi_2} - q'')\ \times \\ &\times \prod_{(a,b)\subset G_3(\pi_1,\pi_2,\pi_3)\backslash(\pi_1,\pi_2)} \delta(\mathrm{y}_{ab} - q') \prod_{(a,b)\subset G_4\backslash G_3(\pi_1,\pi_2,\pi_3)} \delta(\mathrm{y}_{ab} - q)\end{aligned} \tag{60}$$

Here $G_r(i_1, \dots, i_r)$ is the complete graph with vertices $(i_1, \dots, i_r) \subseteq \{1, \cdots, s\}$ [2]; $\sum_\pi{}^{(n)}$ indicates the sum on all different n permutations π on G_r vertices' indexes, which render permutation invariant the associated measure. The numbers p_i, the probabilities of different ultrametric events,

[2] clearly $G_s \equiv G_s(1, \dots, s) = \varphi_{s,0}$

are obtained by normalizing the corresponding measures; counting together the permutations of variables they are, for three replicas, $(1/4, 3/4)$ and for four replicas $(1/9, 1/6, 2/9, 1/6, 1/3)$.

The recipe to construct $\rho_s^{(+)}(\{\mathrm{y}_{ab}\})$ is based on the costruction of abstact trees with a root and s "leaves", which carry the indices y_{ab} The $\rho_s^{(+)}$ is given by a sum on all such trees constructed by elementary branchings: each element in the sum is an integral on at most $s-1$ variables of the weight $w_T(\ .\ )$ associated with the tree T. For T given, w_T is the product of the combinatorial factor $[(s-1)!]^{-1}$ times the weights of the branchings forming the tree [3] and suitable θ and δ functions on the integral variables and the the output variables y_{ab}, according to the tree structure.

A simple way to deduce the number of structurally equivalent graphs, goes as follow: we use a scale transformation in (36) to obtain the generic term $\mathrm{L}_z^{(s)}[\varphi, \ldots, \varphi]$ in terms of $\{\mathrm{L}_{\cdot}^{(s')}[\varphi, \ldots, \varphi]\}$, for $1 \leq s' < s$ in the expansion of $\mathrm{L}_z[\varphi] \equiv \sum \mathrm{L}_z^{(s)}[\varphi, \ldots, \varphi]/s!$. Let $\varphi \longrightarrow \lambda\,\varphi$ be this scale transformation: it is $\mathrm{L}_z^{1}[\varphi] = \varphi_z$ and, for $s \geq 2$

$$\sum_{s=2}^{\infty} \frac{\lambda^s}{s!} \mathrm{L}_z^{(s)}[\varphi, \ldots, \varphi] = \sum_{i=2}^{\infty} \frac{\lambda^i}{i!} \hat{\mathrm{O}}_z^{(i)} \left[\sum_{s_1=2}^{\infty} \frac{\lambda^{s_1-1}}{s_1!} \mathrm{L}_{\cdot}^{(s_1)}[\varphi], \ldots, \sum_{s_i=2}^{\infty} \frac{\lambda^{s_i-1}}{s_i!} \mathrm{L}_{\cdot}^{(s_i)}[\varphi] \right] \tag{61}$$

By multilinearity of the operators, equating terms with equal powers of λ, we have

$$\mathrm{L}_z^{(s)}[\varphi, \ldots, \varphi] = \sum_{i=2}^{s} \sum_{\{m_j\}}{}' \left(\hat{\mathrm{O}}_z^{(i)} \left[(\mathrm{L}^{(1)}[\varphi])^{m_1}, \ldots, (\mathrm{L}^{(s-1)}[\varphi])^{m_{s-1}} \right] \right)_{(S)} \tag{62}$$

the sum $\sum'_{\{m_j\}}$ is on all $m_j \geq 0$ with the bounds $\sum_j m_j = i$ and $\sum_j j m_j = s$; $(\mathrm{L}_j[\varphi])^{m_j}$ is briefly for m_j repetitions of $\mathrm{L}_j[\varphi]$ operator as argument of $\hat{\mathrm{O}}_z^{(i)}$ and finally (S) is the symmetrical factor in the φ's given by

$$S = \frac{s!}{m_1! \cdots m_{s-1}!\ 2!^{m_2} \cdots (s-1)!^{m_{s-1}}} \tag{63}$$

which counts all structurally equivalent graphs.

[3] a branching formed by an input and, say, i outputs, has a weight $w_i(q) = (i-2)!\ \rho_q\ x_q{}^{i-2}$

References

[1] S. Kirkpatrick, D. Sherrington, *Infinite-ranged models of spin-glasses* Phys. Rev. B **17**, 4384, (1978)

[2] M. Mezard, G. Parisi, M.A. Virasoro, *Spin Glass Theory and Beyond* World Scientific, (1986)

[3] L. Pastur, M. Shcherbina, *The absence of self-averaging of the order parameter in the Sherrington–Kirkpatrick model*, J. Stat. Phys., **62**,1, (1992)

[4] M. Shcherbina, *On the replica symmetric solution for the Sherrington-Kirkpatrick model*, Helv. Phys. Acta, **70**,838-853, (1997)

[5] F. Guerra, *Fluctuation and Thermodynamic variables in Mean Field Glass* in "Stochastic Processes, Physics and Geometry, II", S. Albeverio *et al.*, eds., Singapore (1995); also avalaible through http://romagtc.roma1.infn.it

[6] F. Guerra, *Functional Order Parameters for the Quenched Free in Mean Field Spin Glass Models* in "Field Theory and Collective Phenomena", S. De Lillo *et al.*, eds., Singapore (1995)

[7] F. Guerra, *The Cavity Method in the Mean Field Spin Glass Model* in "Advances in Dynamical Systems and Quantum Physics" S. Albeverio *et al.*, eds., Singapore (1995)

[8] F. Guerra, *About the Overlap Distribution in Mean Field Spin Glass Models* Int. Jou. Mod. Phys. B **10**, 1675–1684 (1996)

[9] M. Talagrand, *The Sherrington Kirkpatrick model : a challenge for mathematicians*, Prob. and Rel. Fields, **110**, 109-176 (1998) and M. Talagrand, *Replica symmetry breaking and exponential inequalities for the Sherrington Kirkpatrick model*, Ann. Prob., to appear

[10] F. Guerra, *On the mean field spin glass model*, in preparation

[11] S. Ghirlanda, F. Guerra, *General properties of overlap probability distributions in disordered spin systems. Toward Parisi ultrametricity.*, J. Phys. A, **31**, 9149-9155 (1998), and S. Ghirlanda, *Sulle proprietà ultrametriche degli stati di sistemi disordinati*, Thesis at Università di Roma "La Sapienza" (in Italian), (December 1996)

[12] G. Parisi, *On the probabilistic formulation of the replica approach to spin glasses*, cond-mat/9801081, (1998)

[13] M. Aizenmann, P. Contucci, *On the stability of the quenched state in mean field spin glass models.*, J. Stat. Phys. **92**, 765-783 (1998)

[14] E. Marinari, G. Parisi and J. J. Ruiz-Lorenzo, *Numerical Simulations of Spin Glass Systems*, in "Spin Glasses and Random Fields" A. P. Young ed., World Scientific, Singapore (1998), and cond-mat/9701016; A. Billoire and E. Marinari, cond-mat/9910352.

[15] E. Marinari, G. Parisi, F. Ricci Tersenghi, J. Ruiz Lorenzo and F. Zuliani, *Replica Symmetry Breaking in Short Range Spin Glasses: A Review of the Theoretical Foundations and of the Numerical Evidence*, to be published on J. Stat. Physics, cond-mat/9906076.