Bosonic Monocluster Expansion

A. Abdesselam

Département de Mathématiques

Université Paris XIII, Paris-Nord, Villetaneuse

Avenue J.B. Clément, F93430 Villetanneuse, France

J. Magnen and V. Rivasseau

Centre de physique théorique, CNRS UMR 7644

Ecole Polytechnique, F91128 Palaiseau Cedex, France

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Abstract

We compute connected Green's functions of a Bosonic field theory with cutoffs by means of a "minimal" expansion which in a single move, interpolating a generalized propagator, performs the usual tasks of the cluster and Mayer expansion. In this way it allows a direct construction of the infinite volume or thermodynamic limit and it brings constructive Bosonic expansions closer to constructive Fermionic expansions and to perturbation theory.

Key words : Constructive quantum field theory, Bosons, Cluster expansions, Thermodynamic limit.

I Introduction

A key problem in physics is to construct the thermodynamic limit of large systems. Only intensive or normalized quantities have a well defined limit. For a Bosonic field theory the standard way to construct this limit is to introduce first a finite volume cutoff, then to perform a cluster expansion, which writes the theory as a polymer gas but with hardcore constraints, then to perform a Mayer expansion which removes these constraints by comparing this gas to a perfect gas [9]. It is still slightly frustrating for two reasons.

Firstly for Fermionic theories there is no need of such a sequence of two expansions on top of each other: a single tree formula expresses directly the infinite volume limit of normalized functions as a convergent series [3]. It is therefore desirable to have such a single formula computing directly the infinite volume limit of connected Green's functions in the Bosonic case too.

Secondly mathematically both the cluster and the Mayer expansions can be written elegantly using forest formulas [1]; they have therefore some common nature, which led us to suspect for quite a while that there should exist a single expansion performing both tasks at the same time. In fact the first example of such a formula was given in [1], but it is still really a somewhat artificial mixing of the two expansions (using a two stages formula technically called a "jungle" formula), and it is not obtained by interpolating propagators only.

In this paper we propose a much more natural solution to this problem, which writes directly the infinite volume limit of normalized functions as a convergent series. The Mayer expansion can be understood as taking place in some extended space of copies. Therefore we propose, for any space $\mathbb{R}^d$, to define the Mayer space as $\mathbb{R}^d \times \mathbb{N}$. In this extended space we introduce expansions steps which interpolate solely the (generalized) propagator of the extended theory. The outcome of our expansion is not exactly but almost a tree formula in this extended Mayer space-time. It generates a single cluster (hence we name our expansion a "monocluster" expansion), and the profile of this cluster in the Mayer space is a solid-on-solid profile, with no overhangs. This means that our expansion makes truly a minimal use of the Mayer copies.

We hope to extend this analysis in the future to multiscale expansions such as the one of [2], written for the infrared ϕ^4_4 model. This would suppress the need for iteration of Mayer expansions to perform renormalization (probably the most cumbersome aspect of explicit multiscale expansions). In this way we hope to obtain a completely explicit non-perturbative solution of the renormalization group induction for Bosonic theories (apart from the inductive computation of the effective constants). It would bring these

Bosonic theories to the same level of understanding than Fermionic theories, for which such explicit solutions are known [5]. For a review of rigorous renormalization group methods for bosonic field theory models we refer the reader to [4, 6, 7, 10].

II The Model

Let $C(x,y)$ be the smooth translation-invariant kernel of a covariance operator on $\mathbb{R}^d$, i.e. such that $(f,g) \mapsto < f, Cg >_{L^2(\mathbb{R}^d)}$ is a positive continuous bilinear form on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. By the Bochner-Minlos theorem (see [8]), there is an associated Gaussian measure $d\mu_C$ on $\mathcal{S}'(\mathbb{R}^d)$ with covariance C. The smoothness of C insures that $d\mu_C$ is supported on smooth functions.

We assume that C satisfies a condition of rapid decay:

$$\forall r \geq 1, \exists K_1(r) > 0, \forall x, y \in \mathbb{R}^d, |C(x,y)| \leq K_1(r)(1 + |x-y|)^{-r} \tag{1}$$

Let $P(x)$ be a real polynomial with even degree $2m$ and positive leading coefficient. There is then a constant $K_2 > 0$ such that, for all $x \in \mathbb{R}$, $|P(x)| \leq K_2(1 + x^{2m})$. We introduce a discretization

$$\mathcal{D} \stackrel{\text{def}}{=} \left\{ \prod_{i=1}^{d} [k_i, k_i + 1[\ | \ (k_1, \ldots, k_d) \in \mathbb{Z}^d \right\} \tag{2}$$

of $\mathbb{R}^d$ with boxes Δ of unit size. If $x \in \mathbb{R}^d$, we denote by $\Delta(x)$ the unique $\Delta \in \mathcal{D}$ containing x. We denote by Λ a hypercube of $\mathbb{R}^d$ that is a union of boxes in $\mathcal{D}$, and by $|\Lambda|$ the number of these boxes, which also happens to be equal to $vol(\Lambda)$.

For any $\lambda \geq 0$, we introduce a partition function with free boundary conditions:

$$Z(\Lambda) = \int d\mu_C(\phi) \exp\left(-\lambda \int_\Lambda P(\phi(x))dx \right) \tag{3}$$

as well as unnormalized Schwinger functions, for $x_1, \ldots, x_n$ in $\mathbb{R}^d$:

$$S_{\Lambda,u}(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \int d\mu_C(\phi)\phi(x_1)\cdots\phi(x_n) \exp\left(-\lambda \int_\Lambda P(\phi(x))dx \right) \tag{4}$$

These are well defined quantities, besides $Z(\Lambda) > 0$. Indeed, by Jensen's inequality and Wick's theorem (see [8]),

$$Z(\Lambda) \ \geq \ \exp\left(\left(\int d\mu_C(\phi)(-\lambda) \int_\Lambda P(\phi(x))dx \right) \right. \tag{5}$$

$$\geq \exp\left(-K_2\lambda\int_\Lambda dx\int d\mu_C(\phi)(1+\phi(x)^{2m})\right) \quad (6)$$

$$\geq \exp\left(-K_2\lambda|\Lambda|\left(1+\frac{(2m)!}{2^m m!}C(0,0)\right)\right) > 0 \ . \quad (7)$$

One can thus consider the finite-volume *normalized* Schwinger functions, or correlation functions,

$$S_\Lambda(x_1,\ldots,x_n) \stackrel{\text{def}}{=} \frac{S_{\Lambda,u}(x_1,\ldots,x_n)}{Z(\Lambda)} \quad (8)$$

and study their thermodynamic limit when $\Lambda \nearrow \mathbb{R}^d$.

The typical example we have in mind is the ϕ^4 theory in a single slice of momenta, that is with both ultraviolet and infrared cut-offs as defined e.g. by the choice:

$$C(x,y) \stackrel{\text{def}}{=} \int \frac{d^dp}{(2\pi)^d} e^{ip(x-y)} \frac{e^{-p^2}}{p^2+1} \quad (9)$$

and $P(x) = x^4$.

One of the classical results we rederive using our new expansion scheme is

Theorem 1 *There exists $\lambda_0 > 0$, such that, for any $\lambda \in [0,\lambda_0]$, any $n \geq 1$, and $x_1,\ldots,x_n \in \mathbb{R}^d$, $S(x_1,\ldots,x_n) = \lim_{\Lambda\nearrow\mathbb{R}^d} S_\Lambda(x_1,\ldots,x_n)$ exists.*

Of course, more results can be obtained with our method, like Borel summability of perturbation theory, or complete asymptotic expansion of the decay rate of $S(x_1,x_2)$ etc... But as explained in the introduction, our purpose here is rather to present, at work, a new expansion scheme in the cluster expansion business that produces a sum over a single polymer (i.e. set of cubes), and therefore completely avoids the so-called Mayer expansion.

III The expansion

We first introduce a denumerable set of copies of the field ϕ. We let $\mathcal{L} \stackrel{\text{def}}{=} \mathcal{D}\times\mathbb{N}$ which we identify with a discretization of the "Mayer space" $\mathbb{R}^d\times\mathbb{N}$. For $\mathcal{M}$ a positive matrix with entries indexed by elements b of $\mathcal{L}$, we define the covariance operator on $\mathbb{R}^d\times\mathbb{N}$:

$$\mathcal{C}[\mathcal{M}](x,k;x'k') = C(x,x')\mathcal{M}(b(x,k),b(x',k')) \quad (1)$$

where $b(x,k) = (\Delta(x),k)$ denotes, with a slight abuse of terminology, the box of $\mathcal{L}$ containing the pair (x,k). In particular we consider $\mathcal{M}_\emptyset$ defined by

$$\mathcal{M}_\emptyset((\Delta,k);(\Delta',k')) = \begin{cases} 1 & \text{if } k=k'=0 \\ \delta_{\Delta,\Delta'} & \text{if } k=k'\geq 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

i.e. in block form

$$\mathcal{M}_\emptyset = \begin{array}{c} \begin{array}{cc} \mathcal{L}_0 & \mathcal{L}_{\geq 1} \end{array} \\ \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \end{array} \tag{3}$$

where $\mathcal{L}_0 \stackrel{\text{def}}{=} \mathcal{D}\times\{0\}$, $\mathcal{L}_{\geq 1} \stackrel{\text{def}}{=} \mathcal{D}\times\mathbb{N}^*$, $\mathbf{1}$ is the matrix with entries 1 everywhere and Id is the identity matrix. Clearly, $\mathcal{C}_\emptyset = \mathcal{C}[\mathcal{M}_\emptyset]$ is a positive covariance operator; and we can define $d\mu_{\mathcal{C}_\emptyset}(\Phi)$ the measure of a Gaussian random field $\Phi(x,k)$ on $\mathbb{R}^d\times\mathbb{N}$, with covariance $\mathcal{C}_\emptyset$. We introduce also the notations $\mathcal{D}_\Lambda \stackrel{\text{def}}{=} \{\Delta\in\mathcal{D}|\Delta\subset\Lambda\}$, and for any integer $N\geq 0$, $\mathcal{L}_{\Lambda,N} \stackrel{\text{def}}{=} \mathcal{D}\times\{0,1,\ldots,N\}\subset\mathcal{L}$.

Now consider

$$H_{\Lambda,N}(x_1,\ldots,x_n) \stackrel{\text{def}}{=} \int d\mu_{\mathcal{C}_\emptyset}(\Phi)\prod_{i=1}^n \Phi(x_i,0)\exp\left(-\lambda\sum_{(\Delta,k)\in\mathcal{L}_{\Lambda,N}}\int_\Delta P(\Phi(x,k))dx\right) . \tag{4}$$

We obviously have, due to the definition of $\mathcal{C}_\emptyset$, the factorization

$$H_{\Lambda,N}(x_1,\ldots,x_n) = S_{\Lambda,N}(x_1,\ldots,x_n)\cdot Z_0^{N|\Lambda|} \tag{5}$$

where

$$Z_0 \stackrel{\text{def}}{=} \int d\mu_{\mathbb{1}_\Delta C\mathbb{1}_\Delta}\exp\left(-\lambda\int_\Delta P(\Phi(x,k))dx\right) \tag{6}$$

the normalization of an isolated cube, does not depend on Δ, since the kernel C is translation-invariant. Here, $\mathbb{1}_\Delta$ denotes the sharp characteristic function of Δ. Note that Z_0 differs from $Z(\Delta)$ by a choice of boundary condition. We now proceed to write an expansion for $H_{\Lambda,N}(x_1,\ldots,x_n)$, after introducing some combinatorial definitions.

First we define the notion of a polymer. We let $\Gamma_0 \stackrel{\text{def}}{=} \{\Delta\in\mathcal{D}|\exists i, x_i\in\Delta\}\times\{0\}\subset\mathcal{L}_0$. We also define $\Gamma_{-1} \stackrel{\text{def}}{=} \emptyset$. We then say that a finite set $\Gamma\subset\mathcal{L}$ is polymer if, whenever $(\Delta,k)\in\Gamma$, we also have $(\Delta,k')\in\Gamma$ for any k', $0\leq k'\leq k$. We also introduce the altitude function h_Γ of a polymer, on $\mathcal{D}$ as:

$$h_\Gamma(\Delta) \stackrel{\text{def}}{=} \begin{cases} -1 & \text{if } \{k|(\Delta,k)\in\Gamma\}=\emptyset \\ \max\{k|(\Delta,k)\in\Gamma\} & \text{otherwise.} \end{cases} \tag{7}$$

A polymer Γ is uniquely determined by its altitude function h_Γ. We also introduce the roof $W(\Gamma) \subset \mathcal{L}$ of a polymer Γ as:

$$W(\Gamma) \stackrel{\text{def}}{=} \{(\Delta, h_\Gamma(\Delta)+1) | \Delta \in \mathcal{D}\} \tag{8}$$

and its sky $S(\Gamma) \stackrel{\text{def}}{=} \mathcal{L}\backslash(\Gamma \cup W(\Gamma))$. The sets Γ, $W(\Gamma)$ and $S(\Gamma)$ then form a partition of $\mathcal{L}$.

Let $\mathsf{g} = (l_1, \dots, l_p)$ be an ordered sequence of unordered pairs of the form $l = \{b, b'\}$ with b, b' distinct elements of $\mathcal{L}$. $p = 0$ corresponding to $\mathsf{g} = \emptyset$ is allowed too. We define, for $1 \le i \le p$, $\Gamma_{i,\mathsf{g}} \stackrel{\text{def}}{=} \Gamma_0 \cup l_1 \cup \dots \cup l_i$. We also set, by convention, $\Gamma_{0,\mathsf{g}} \stackrel{\text{def}}{=} \Gamma_0$ and $\Gamma_{-1,\mathsf{g}} \stackrel{\text{def}}{=} \Gamma_{-1} = \emptyset$. We say that g is a cluster-graph if, for any i, $1 \le i \le p$, the unordered pair l_i is of the form $\{b, b'\}$ for some b and b' that satisfy one of the following two conditions:

(i) $b \in \Gamma_{i-1,\mathsf{g}}$ and $b' \in W(\Gamma_{i-1,\mathsf{g}})$

(ii) b, $b' \in W(\Gamma_{i-1,\mathsf{g}})$ and $b \notin \mathcal{L}_0$.

It is easy to check that $\Gamma_{i,\mathsf{g}}$ defined previously is indeed a polymer, for any i, $1 \le i \le p$. A pair l_i, which is called a link of the graph g, is said of type *cluster-roof* or ΓW if (i) occurs, and of type *roof-roof* or WW if (ii) occurs (see Fig.1).

If $b \in \mathcal{L}$, we define the conception index of b with respect to g:

$$\mu_\mathsf{g}(b) \stackrel{\text{def}}{=} \inf\left(\{i | -1 \le i \le p, b \in W(\Gamma_{i,\mathsf{g}})\} \cup \{p+1\}\right) \tag{9}$$

and the creation index of b:

$$\nu_\mathsf{g}(b) \stackrel{\text{def}}{=} \inf\left(\{i | -1 \le i \le p, b \in \Gamma_{i,\mathsf{g}}\} \cup \{p+1\}\right) \tag{10}$$

Note that we always have $\mu_\mathsf{g}(b) < \nu_\mathsf{g}(b)$ if $b \in (\Gamma_{p,\mathsf{g}} \cup W(\Gamma_{p,\mathsf{g}}))$. Indeed, by definition of a cluster-graph $\Gamma_{i,\mathsf{g}} \backslash \Gamma_{i-1,\mathsf{g}} = l_i \backslash \Gamma_{i-1,\mathsf{g}} \subset W(\Gamma_{i-1,\mathsf{g}})$. In fact, $W(\Gamma_i)$ can be viewed as a solid-on-solid interface that elevates in $\mathcal{L}$ as the cluster $\Gamma_{i,\mathsf{g}}$ grows with increasing i. A cube b has to belong to a $W(\Gamma_{i,\mathsf{g}})$ before it belongs to a $\Gamma_{i,\mathsf{g}}$. If b, b' are two elements of $\mathcal{L}$ we let:

$$s\mu_\mathsf{g}(b, b') \stackrel{\text{def}}{=} \max(\mu_\mathsf{g}(b), \mu_\mathsf{g}(b')) \tag{11}$$

$$s\nu_\mathsf{g}(b, b') \stackrel{\text{def}}{=} \max(\nu_\mathsf{g}(b), \nu_\mathsf{g}(b')) \tag{12}$$

and

$$i\nu_\mathsf{g}(b, b') \stackrel{\text{def}}{=} \min(\nu_\mathsf{g}(b), \nu_\mathsf{g}(b')) \ . \tag{13}$$

Now given a decreasing vector $\mathbf{h}$ of $p+1$ parameters $1 > h_1 > \dots > h_p > h_{p+1} > 0$ with the additional convention $h_0 \stackrel{\text{def}}{=} 1$ and $h_{-1} \stackrel{\text{def}}{=} +\infty$ so that

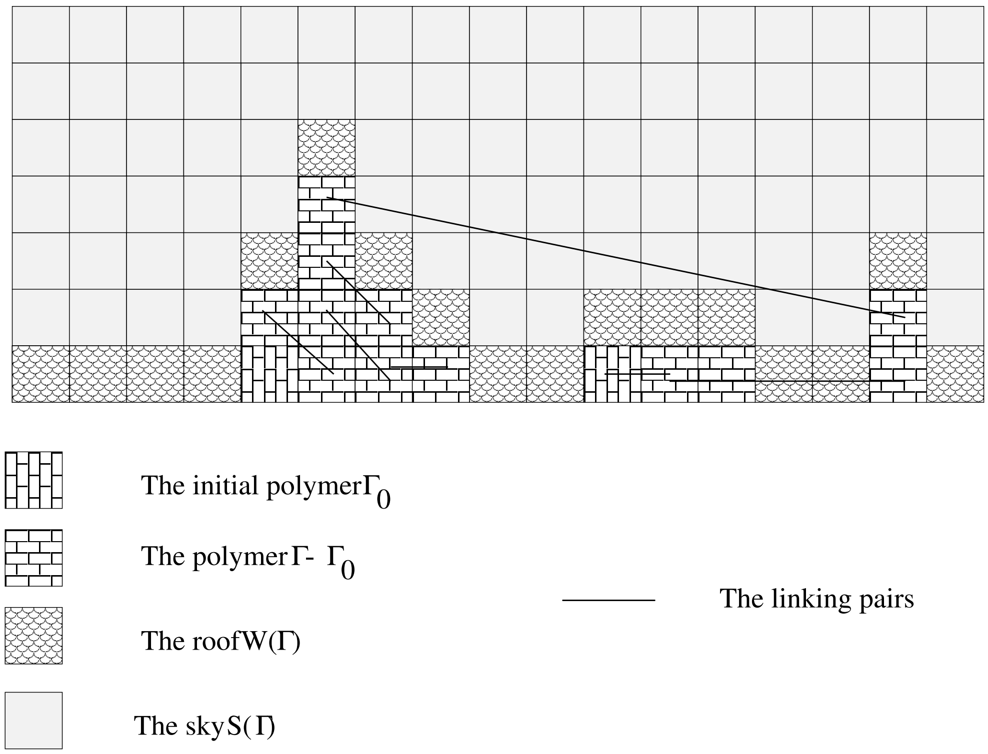

Figure 1: A cluster graph

$\frac{1}{h_{-1}} = 0$, we define the following matrix $\mathcal{M}_{\mathbf{g},\mathbf{h}}$ on $\mathcal{L}$. For b, b' in $\mathcal{L}$ we let

$$\mathcal{M}_{\mathbf{g},\mathbf{h}} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } b = b' \\ 0 & \text{if } b \neq b' \text{ and } s\mu_{\mathbf{g}}(b,b') \geq i\nu_{\mathbf{g}}(b,b') \\ h_{s\nu_{\mathbf{g}}(b,b')} \left(\frac{1}{h_{i\nu_{\mathbf{g}}(b,b')}} - \frac{1}{h_{s\mu_{\mathbf{g}}(b,b')}} \right) & \text{if } b \neq b' \text{ and } s\mu_{\mathbf{g}}(b,b') < i\nu_{\mathbf{g}}(b,b'). \end{cases} \tag{14}$$

We will later prove that $\mathcal{M}_{\mathbf{g},\mathbf{h}}$ is a positive matrix. Before that, we introduce the following operation on covariance matrices on $\mathcal{L}$. If Γ is a polymer, and $\mathcal{M}$ is a matrix on $\mathcal{L}$, we define the new matrix $T_\Gamma[\mathcal{M}]$ by

$$T_\Gamma[\mathcal{M}](b,b') \stackrel{\text{def}}{=} \begin{cases} \mathcal{M}(b,b') & \text{if } b,b' \in \Gamma \\ 1 & \text{if } b,b' \in W(\Gamma) \\ \delta_{b,b'} & \text{if } b,b' \in S(\Gamma) \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

or in block form

$$T_\Gamma(\mathcal{M}) = \begin{array}{c} \begin{array}{ccc} \Gamma & W(\Gamma) & S(\Gamma) \end{array} \\ \begin{pmatrix} \mathcal{M}|_\Gamma & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix} \end{array} . \tag{16}$$

Obviously $T_\Gamma[\mathcal{M}]$ is positive if $\mathcal{M}$ is.

Lemma 1 *If $\mathsf{g} = (\mathsf{g}', l_p)$ is a cluster-graph of length $p \geq 1$, and $\mathbf{h} = (\mathbf{h}', h_{p+1})$ is a decreasing vector of parameters, we have*

$$\mathcal{M}_{\mathbf{g},\mathbf{h}} = \frac{h_{p+1}}{h_p} \mathcal{M}_{\mathbf{g}',\mathbf{h}'} + \left(1 - \frac{h_{p+1}}{h_p}\right) T_{\Gamma_{p,\mathbf{g}}}[\mathcal{M}_{\mathbf{g}',\mathbf{h}'}] \tag{17}$$

Proof : We check the equality for every pair of boxes b, b' in $\mathcal{L}$. The case $b = b'$ holds trivially.

• If $b \neq b'$ are both in $\Gamma_{p,\mathbf{g}}$, then the choice of upper cut-off on the infimum in (9) and (10) readily implies that $\mu_{\mathbf{g}}(b) = \mu_{\mathbf{g}'}(b) \leq p$ and $\nu_{\mathbf{g}}(b) = \nu_{\mathbf{g}'}(b) \leq p$. Therefore,

$$\mathcal{M}_{\mathbf{g},\mathbf{h}}(b,b') = \mathcal{M}_{\mathbf{g}',\mathbf{h}'}(b,b') = T_{\Gamma_{p,\mathbf{g}}}[\mathcal{M}_{\mathbf{g}',\mathbf{h}'}](b,b') \tag{18}$$

so that (17) holds.

• If $b \neq b'$ are both in $W(\Gamma_{p,\mathbf{g}})$, then $\mu_{\mathbf{g}}(b) = \mu_{\mathbf{g}'}(b) \leq p$ whereas $\nu_{\mathbf{g}}(b) = p+1$, $\nu_{\mathbf{g}'}(b) = p$ and likewise for b'. Therefore

$$\mathcal{M}_{\mathbf{g},\mathbf{h}}(b,b') = h_{p+1} \left(\frac{1}{h_{p+1}} - \frac{1}{h_{s\mu_{\mathbf{g}'}(b,b')}} \right) \tag{19}$$

$$\mathcal{M}_{\mathbf{g}',\mathbf{h}'}(b,b') = h_p \left(\frac{1}{h_p} - \frac{1}{h_{s\mu_{\mathbf{g}'}(b,b')}} \right) \tag{20}$$

whereas $T_{\Gamma_{p,\mathbf{g}}}[\mathcal{M}_{\mathbf{g}',\mathbf{h}'}](b,b') = 1$, and thus

$$\begin{aligned} &\frac{h_{p+1}}{h_p}\mathcal{M}_{\mathbf{g}',\mathbf{h}'}(b,b') + \left(1 - \frac{h_{p+1}}{h_p}\right) T_{\Gamma_{p,\mathbf{g}}}[\mathcal{M}_{\mathbf{g}',\mathbf{h}'}](b,b') \\ &\quad = \frac{h_{p+1}}{h_p} h_p \left(\frac{1}{h_p} - \frac{1}{h_{s\mu_{\mathbf{g}'}(b,b')}} \right) + \left(1 - \frac{h_{p+1}}{h_p}\right) \qquad (21) \\ &\quad = h_{p+1} \left(\frac{1}{h_{p+1}} - \frac{1}{h_{s\mu_{\mathbf{g}'}(b,b')}} \right) \qquad (22) \end{aligned}$$

so that (17) holds.

• If $b \in \Gamma_{p,\mathbf{g}}$ and $b' \in W(\Gamma_{p,\mathbf{g}})$, then $\mu_{\mathbf{g}}(b) = \mu_{\mathbf{g}'}(b) \le p$, $\nu_{\mathbf{g}}(b) = \nu_{\mathbf{g}'}(b) \le p$, $\mu_{\mathbf{g}}(b') = \mu_{\mathbf{g}'}(b') \le p$, $\nu_{\mathbf{g}}(b') = p+1$ and $\nu_{\mathbf{g}'}(b') = p$. Therefore $s\mu_{\mathbf{g}}(b,b') = s\mu_{\mathbf{g}'}(b,b')$ and $i\nu_{\mathbf{g}}(b,b') = i\nu_{\mathbf{g}'}(b,b')$. Besides $T_{\Gamma_{p,\mathbf{g}}}[\mathcal{M}_{\mathbf{g}',\mathbf{h}'}](b,b') = 0$. So if $s\mu_{\mathbf{g}}(b,b') \ge i\nu_{\mathbf{g}}(b,b')$ both sides of (17) vanish; else we have

$$\mathcal{M}_{\mathbf{g},\mathbf{h}}(b,b') = h_{p+1} \left(\frac{1}{h_{i\nu_{\mathbf{g}'}(b,b')}} - \frac{1}{h_{s\mu_{\mathbf{g}'}(b,b')}} \right) \tag{23}$$

and

$$\mathcal{M}_{\mathbf{g}',\mathbf{h}'}(b,b') = h_p \left(\frac{1}{h_{i\nu_{\mathbf{g}'}(b,b')}} - \frac{1}{h_{s\mu_{\mathbf{g}'}(b,b')}} \right) \tag{24}$$

which implies (17).

• Finally if $b \in S(\Gamma_{p,\mathbf{g}}) \subset S(\Gamma_{p-1,\mathbf{g}'})$ and $b' \ne b$ is anywhere in $\mathcal{L}$, we have $T_{\Gamma_{p,\mathbf{g}}}[\mathcal{M}_{\mathbf{g}',\mathbf{h}'}](b,b') = 0$, $\mu_{\mathbf{g}}(b) = \nu_{\mathbf{g}}(b) = p+1$ and $\mu_{\mathbf{g}'}(b) = \nu_{\mathbf{g}'}(b) = p$. Thus $s\mu_{\mathbf{g}}(b,b') \ge i\nu_{\mathbf{g}}(b,b')$ and $s\mu_{\mathbf{g}'}(b,b') \ge i\nu_{\mathbf{g}'}(b,b')$ so that both sides of (17) vanish again.

This completes the check in every case. ∎

Lemma 2 *For any cluster-graph* $\mathbf{g}$ *of length* $p \ge 0$ *and associated decreasing parameter vector* $\mathbf{h}$ *of length* $p+1$*, the matrix* $\mathcal{M}_{\mathbf{g},\mathbf{h}}$ *is positive.*

Proof : Convex combinations and the operation $\mathcal{M} \mapsto T_\Gamma[\mathcal{M}]$ preserve positivity; so, by induction thanks to the previous lemma, we only need to check the $p=0$ situation. But then $\mathbf{g} = \emptyset$, $\mathbf{h} = (h_1)$, and for $b \in \mathcal{L}$ we have

$$\mu_\emptyset(b) = \begin{cases} -1 & \text{if } b \in \mathcal{L}_0 \\ 0 & \text{if } b \in (W(\Gamma_0) \backslash \mathcal{L}_0) \subset \mathcal{D} \times \{1\} \\ 1 & \text{if } b \in S(\Gamma_0) \end{cases} \tag{25}$$

and

$$\nu_\emptyset(b) = \begin{cases} 0 & \text{if } b \in \Gamma_0 \\ 1 & \text{if } b \in W(\Gamma_0) \cup S(\Gamma_0) \ . \end{cases} \tag{26}$$

Now a straight-forward calculation using (14) show that, in block form, we have

$$\mathcal{M}_{\emptyset,(h_1)} = \begin{array}{c} \begin{array}{cccc} \Gamma_0 & W(\Gamma_0)\cap\mathcal{L}_0 & W(\Gamma_0)\backslash\mathcal{L}_0 & S(\Gamma_0) \end{array} \\ \begin{pmatrix} \mathbf{1} & h_1\mathbf{1} & 0 & 0 \\ h_1\mathbf{1} & \mathbf{1} & (1-h_1)\mathbf{1} & 0 \\ 0 & (1-h_1)\mathbf{1} & (1-h_1)\mathbf{1}+h_1\mathrm{Id} & 0 \\ 0 & 0 & 0 & \mathrm{Id} \end{pmatrix} \end{array} \tag{27}$$

i.e.

$$\mathcal{M}_{\emptyset,(h_1)} = h_1 \begin{pmatrix} \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 & \mathrm{Id} \end{pmatrix} + (1-h_1) \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathrm{Id} \end{pmatrix} \tag{28}$$

or

$$\mathcal{M}_{\emptyset,(h_1)} = h_1\mathcal{M}_\emptyset + (1-h_1)T_{\Gamma_0}[\mathcal{M}_\emptyset] \tag{29}$$

which is clearly positive. ■

Remark that we have showed, *en passant*, that (17) really starts at $p=0$, $\mathcal{M}_\emptyset$ being the matrix corresponding to a cluster-graph of “length -1”. We need some more notation to proceed. Here $\mathsf{g} = (l_1, \ldots, l_p)$, $p \geq 0$, is a cluster-graph, $\mathbf{h} = (h_1, \ldots, h_{p+1})$ is a decreasing vector of parameters. For any $b \in \mathcal{L}$, and any α, $0 \leq \alpha \leq p+1$, we let

$$\mu_{\mathsf{g},\alpha}(b) \stackrel{\text{def}}{=} \inf\left(\{i | -1 \leq i \leq \alpha-1, b \in W(\Gamma_{i,\mathsf{g}})\} \cup \{\alpha\}\right) \tag{30}$$

and

$$\nu_{\mathsf{g},\alpha}(b) \stackrel{\text{def}}{=} \inf\left(\{i | -1 \leq i \leq \alpha-1, b \in \Gamma_{i,\mathsf{g}}\} \cup \{\alpha\}\right) \ . \tag{31}$$

This is the same as the previously defined $\mu_\mathsf{g}(b)$ and $\nu_\mathsf{g}(b)$, using the truncation $(l_1, \ldots, l_{\alpha-1})$ of g instead of the full graph g. We also denote for b, b' in $\mathcal{L}$,

$$s\mu_{\mathsf{g},\alpha}(b,b') \stackrel{\text{def}}{=} \max(\mu_{\mathsf{g},\alpha}(b), \mu_{\mathsf{g},\alpha}(b')) \tag{32}$$

$$s\nu_{\mathsf{g},\alpha}(b,b') \stackrel{\text{def}}{=} \max(\nu_{\mathsf{g},\alpha}(b), \nu_{\mathsf{g},\alpha}(b')) \tag{33}$$

and

$$i\nu_{\mathsf{g},\alpha}(b,b') \stackrel{\text{def}}{=} \min(\nu_{\mathsf{g},\alpha}(b), \nu_{\mathsf{g},\alpha}(b')) \ . \tag{34}$$

We next define for any q, $1 \leq q \leq p$, the expression $\omega(\mathsf{g}, \mathbf{h}, q)$ as follows.

Let $l_q = \{b, b'\}$ for some $b \neq b'$ in $\mathcal{L}$.

• If $b \in \Gamma_{q-1,\mathsf{g}}$ and $b' \in W(\Gamma_{q-1,\mathsf{g}})$, we let

$$\omega(\mathsf{g}, \mathbf{h}, q) \stackrel{\mathrm{def}}{=} \begin{cases} 0 & \text{if } s\mu \geq i\nu \\ \frac{1}{h_{i\nu}} - \frac{1}{h_{s\mu}} & \text{if } s\mu < i\nu \end{cases} \tag{35}$$

where $s\mu$ and $i\nu$ are shorthand for $s\mu_{\mathsf{g},q-1}(b, b')$ and $i\nu_{\mathsf{g},q-1}(b, b')$ respectively. Note that $s\mu_{\mathsf{g},q-1}(b, b') = s\mu_{\mathsf{g}}(b, b') \leq q - 1$ and $i\nu_{\mathsf{g},q-1}(b, b') = i\nu_{\mathsf{g}}(b, b') \leq q - 1$.

• If b, $b' \in W(\Gamma_{q-1,\mathsf{g}})$, then we let

$$\omega(\mathsf{g}, \mathbf{h}, q) \stackrel{\mathrm{def}}{=} -\frac{1}{h_{s\mu}} \tag{36}$$

where, again, $s\mu$ is shorthand for $s\mu_{\mathsf{g},q-1}(b, b')$. Note again that $s\mu_{\mathsf{g},q-1}(b, b') = s\mu_{\mathsf{g}}(b, b') \leq q - 1$.

• Finally, in every other case for b and b', we let

$$\omega(\mathsf{g}, \mathbf{h}, q) \stackrel{\mathrm{def}}{=} 0 \ . \tag{37}$$

Now let $l = \{b, b'\}$ be an unordered pair of elements of $\mathcal{L}$, such that $b = (\Delta, k)$ and $b' = (\Delta', k')$; we then introduce the functional derivation operator:

$$D_l \stackrel{\mathrm{def}}{=} \int_\Delta dx \int_{\Delta'} dx'\ C(x, x') \frac{\delta}{\delta\Phi(x, k)} \frac{\delta}{\delta\Phi(x', k')} \tag{38}$$

We also introduce

$$\mathcal{R}(\mathsf{g}, \mathbf{h}) \stackrel{\mathrm{def}}{=} \int d\mu_{\mathcal{C}[\mathcal{M}_{\mathsf{g},\mathbf{h}}]}(\Phi) \prod_{q=1}^{p} \left(\omega(\mathsf{g}, \mathbf{h}, q) D_{l_q}\right)$$
$$\prod_{i=1}^{n} \Phi(x_i, 0) \exp\left(-\lambda \sum_{(\Delta,k)\in\mathcal{L}_{\Lambda,N}} \int_\Delta P(\Phi(x, k)) dx\right) \tag{39}$$

the functional derivations acting on any factor to their right. We are now ready to state the main lemma for our expansion scheme.

Lemma 3 *For any $m \geq 1$,*

$$H_{\Lambda,N}(x_1, \ldots, x_n) =$$
$$\sum_{0\leq p<m} \sum_{\mathsf{g}=(l_1,\ldots,l_p)} \int_{1>h_1>\cdots>h_p>0} dh_1 \ldots dh_p\ \mathcal{R}(\mathsf{g}, (h_1, \ldots, h_p, 0))$$
$$+ \sum_{\mathsf{g}=(l_1,\ldots,l_m)} \int_{1>h_1>\cdots>h_m>0} dh_1 \ldots dh_m\ \mathcal{R}(\mathsf{g}, (h_1, \ldots, h_m, h_m)) \ . \tag{40}$$

The sums on g *are on all cluster-graphs with the prescribed length.*

Proof : We first prove the lemma for $m = 1$. For that we notice, according to equation (29), that

$$H_{\Lambda,N}(x_1, \ldots, x_n) = \mathcal{R}(\mathsf{g}, \mathbf{h}) \tag{41}$$

where $\mathsf{g} = \emptyset$ is the empty graph and $\mathbf{h} = (h_1)$ with $h_1 = 1$. We then simply write

$$H_{\Lambda,N}(x_1, \ldots, x_n) = \mathcal{R}(\emptyset, (0)) + \int_0^1 dh_1 \frac{d}{dh_1} \mathcal{R}(\emptyset, (h_1)) \quad . \tag{42}$$

The covariance matrix appearing in $\mathcal{R}(\emptyset, (h_1))$ is

$$\mathcal{M}_{\emptyset,(h_1)} = h_1 \mathcal{M}_\emptyset + (1 - h_1) T_{\Gamma_0}[\mathcal{M}_\emptyset] \quad . \tag{43}$$

Therefore, the derivation with respect to h_1, produces a functional derivation operator acting on the integrand, associated to a matrix element of $\mathcal{M}_\emptyset - T_{\Gamma_0}[\mathcal{M}_\emptyset]$ (this is obvious by Wick's theorem for polynomial integrands, then true for our smooth decreasing integrand by an easy limiting argument, see [8]). That is we get a sum over $l_1 = \{b, b'\}$ and a factor $(\mathcal{M}_\emptyset - T_{\Gamma_0}[\mathcal{M}_\emptyset])(b, b') D_{l_1}$ in the functional integral defining $\mathcal{R}(\emptyset, (h_1))$. It is a simple check to verify, with our previous definitions, that

$$\begin{aligned} &(\mathcal{M}_\emptyset - T_{\Gamma_0}[\mathcal{M}_\emptyset])(b, b') \\ &\quad = \omega(l_1, (h_1, h_1), 1) \qquad (44) \\ &\quad = \begin{cases} 1 & \text{if } b \in \Gamma_0, b' \in W(\Gamma_0) \cap \mathcal{L}_0 \\ -1 & \text{if } b \neq b' \in W(\Gamma_0) \text{ and } \{b, b'\} \not\subset \mathcal{L}_0 \ . \end{cases} \qquad (45) \end{aligned}$$

Besides, the covariance matrix $\mathcal{M}_{\emptyset,(h_1)}$ involved in the functional integral can be rewritten, according to (17), as $\mathcal{M}_{(l_1),(h_1,h_1)}$. Therefore

$$H_{\Lambda,N}(x_1, \ldots, x_n) = \mathcal{R}(\emptyset, (0)) + \sum_{l_1} \int_0^1 dh_1 \ \mathcal{R}((l_1), (h_1, h_1)) \tag{46}$$

which is the wanted result for $m = 1$.

We now prove the induction step from $m \geq 1$ to $m + 1$. For this, we simply have to show that, given a cluster-graph $\mathsf{g} = (l_1, \ldots, l_m)$ of length m and parameters $1 > h_1 > \cdots > h_m > 0$,

$$\begin{aligned} \mathcal{R}(\mathsf{g}, (h_1, \ldots, h_m, h_m)) &= \mathcal{R}(\mathsf{g}, (h_1, \ldots, h_m, 0)) \\ &+ \sum_{l_{m+1}} \int_0^{h_m} dh_{m+1} \ \mathcal{R}((\mathsf{g}, l_{m+1}), (h_1, \ldots, h_m, h_{m+1}, h_{m+1})) \end{aligned} \tag{47}$$

which is proven in the same way as for the $m=1$ case. Indeed, we write

$$\begin{aligned}&\mathcal{R}(\mathsf{g},(h_1,\ldots,h_m,h_m)) = \\ &\quad \mathcal{R}(\mathsf{g},(h_1,\ldots,h_m,0))\int_0^{h_m} dh_{m+1}\frac{d}{dh_{m+1}}\mathcal{R}(\mathsf{g},(h_1,\ldots,h_m,h_{m+1}))\end{aligned} \tag{48}$$

and use (17) to explicit the dependence on h_{m+1} of the covariance matrix:

$$\mathcal{M}_{\mathsf{g},(h_1,\ldots,h_{m+1})} = \frac{h_{m+1}}{h_m}\mathcal{M}_{\mathsf{g}',(h_1,\ldots,h_m)} + \left(1-\frac{h_{m+1}}{h_m}\right)T_{\Gamma_{m,\mathsf{g}}}[\mathcal{M}_{\mathsf{g}',(h_1,\ldots,h_m)}] \tag{49}$$

where $\mathsf{g}'=(l_1,\ldots,l_{m-1})$. Derivation with respect to h_{m+1} again introduces a sum over a new link $l_{m+1}=\{b,b'\}$, with a corresponding functional derivation operator $D_{l_{m+1}}$ times a factor

$$\frac{1}{h_m}\left(\mathcal{M}_{\mathsf{g}',(h_1,\ldots,h_m)} - T_{\Gamma_{m,\mathsf{g}}}[\mathcal{M}_{\mathsf{g}',(h_1,\ldots,h_m)}]\right)(b,b') \tag{50}$$

which is easily checked to be equal to

$$\omega((\mathsf{g},l_{m+1}),(h_1,\ldots,h_{m+1},h_{m+1}),m+1) \quad . \tag{51}$$

Indeed, if $b\neq b'\in W(\Gamma_{m,\mathsf{g}})$, (50) is equal to

$$\frac{1}{h_m}\left(\mathcal{M}_{\mathsf{g}',(h_1,\ldots,h_m)}(b,b')-1\right) = \frac{1}{h_m}\left(h_m\left(\frac{1}{h_m}-\frac{1}{h_{s\mu_{\mathsf{g}'}(b,b')}}\right)-1\right) \tag{52}$$

since $\nu_{\mathsf{g}'}(b)=\nu_{\mathsf{g}'}(b')=m$. The situation $b\in\Gamma_{m,\mathsf{g}}$, $b'\in W(\Gamma_{m,\mathsf{g}})$ can be checked in the same way.

Finally the involved covariance matrix can be rewritten, thanks to (17), as

$$\mathcal{M}_{\mathsf{g},(h_1,\ldots,h_{m+1})} = \mathcal{M}_{(\mathsf{g},l_{m+1}),(h_1,\ldots,h_{m+1},h_{m+1})} \tag{53}$$

which proves (47). ■

The easy proof that the cluster-graphs that are summed over in lemma 3 satisfy the conditions (i) and (ii) stated earlier, is left to the reader. We are now ready to move on to the proof of theorem 1.

We first notice that, if $\mathsf{g}=(l_1,\ldots,l_p)$ is cluster-graph, then $\#(\Gamma_{p,\mathsf{g}})\geq p$; besides, the contribution of g in (40) vanishes if $\Gamma_{p,\mathsf{g}}$ is not contained in $\mathcal{D}_{\Lambda,N}$ since a functional derivation $\frac{\delta}{\delta\Phi(x,k)}$ would have nothing to contract to. As a result, $p > \#(\mathcal{D}_{\Lambda,N})$ implies that $\mathtt{g} = (l_1, \ldots, l_p)$ gives a zero contribution; it is then straight-forward to take the limit $m \to +\infty$ in (40) to write

$$H_{\Lambda,N}(x_1, \ldots, x_n) = \sum_{p=0}^{+\infty} \sum_{\substack{\mathtt{g}=(l_1,\ldots,l_p) \\ \Gamma_{p,\mathtt{g}} \subset \mathcal{D}_{\Lambda,N}}} \int_{1>h_1>\cdots>h_p>0} dh_1 \ldots dh_p \ \mathcal{R}(\mathtt{g}, (h_1, \ldots, h_p, 0)) \ . \qquad (54)$$

We can now write an expression for the normalized Schwinger functions since:

$$S_\Lambda(x_1, \ldots, x_n) = \frac{H_{\Lambda,N}(x_1, \ldots, x_n)}{Z(\Lambda) \times Z_0^{N|\Lambda|}} \qquad (55)$$

$$= \sum_{p=0}^{+\infty} \sum_{\substack{\mathtt{g}=(l_1,\ldots,l_p) \\ \Gamma_{p,\mathtt{g}} \subset \mathcal{D}_{\Lambda,N}}} \mathcal{A}(\mathtt{g}, \Lambda, N) \qquad (56)$$

where

$$\mathcal{A}(\mathtt{g}, \Lambda, N) = \frac{\mathcal{A}_0(\mathtt{g})}{Z_0^{\#(\Gamma_{p,\mathtt{g}})}} \times \frac{Z(Y_\mathtt{g}) \cdot Z_0^{\#(\Lambda) - \#(Y_\mathtt{g})}}{Z(\Lambda)} \qquad (57)$$

with the following notations.

- First, $Y_\mathtt{g} \stackrel{\text{def}}{=} \{\Delta \in \Lambda | h_{\Gamma_{p,\mathtt{g}}}(\Delta) \geq N\}$.
- Next, $Z(Y_\mathtt{g})$ is defined as in (3) by

$$Z(Y_\mathtt{g}) \stackrel{\text{def}}{=} \int d\mu_C(\Phi) \exp\left(-\lambda \sum_{\Delta \in Y_\mathtt{g}} \int_\Delta P(\phi(x)) dx\right) \qquad (58)$$

with a free boundary condition covariance.

- Finally, $\mathcal{A}_0(\mathtt{g})$ is defined, <u>independently of Λ and N</u>, by

$$\mathcal{A}_0(\mathtt{g}) = \int_{1>h_1>\cdots>h_p>0} dh_1 \ldots dh_p \int d\mu_{\mathcal{C}[\overline{\mathcal{M}}_{\mathtt{g},(h_1,\ldots,h_p)}]}(\Phi)$$
$$\prod_{q=1}^{p} \left(\omega(\mathtt{g}, (\mathbf{h}, 0), q) D_{l_q}\right)$$
$$\prod_{i=1}^{n} \Phi(x_i, 0) \exp\left(-\lambda \sum_{(\Delta,k) \in \Gamma_{p,\mathtt{g}}} \int_\Delta P(\Phi(x,k)) dx\right) \qquad (59)$$

where

$$\overline{\mathcal{M}}_{\mathtt{g},(h_1,\ldots,h_p)}(b, b') \stackrel{\text{def}}{=} \begin{cases} \mathcal{M}_{\mathtt{g},(h_1,\ldots,h_p,0)}(b, b') & \text{if } b, b' \in \Gamma_{p,\mathtt{g}} \\ 0 & \text{otherwise.} \end{cases} \qquad (60)$$

The factorization (57) stems from the fact that the parameter vectors involved in (54) have a null last component, and therefore the corresponding covariance matrix is

$$\mathcal{M}_{\mathsf{g},(h_1,\ldots,h_p,0)} = T_{\Gamma_{p,\mathsf{g}}}[\mathcal{M}_{\mathsf{g},(h_1,\ldots,h_p,0)}] \tag{61}$$

which completely couples together the cubes of $W(\Gamma_{p,\mathsf{g}})$ and decouples them from the rest of $\mathcal{L}$. This accounts for the factor $Z(Y_\mathsf{g})$ which might be different from $Z(\Lambda)$, in case $\Gamma_{p,\mathsf{g}}$ reaches the highest cubes of $\mathcal{L}_{\Lambda,N}$ which contain all interaction terms of the form $\exp(-\lambda\int_\Delta P(\Phi(x,k))dx)$. For a given g, $\mathcal{A}(\mathsf{g},\Lambda,N) = \frac{\mathcal{A}_0(\mathsf{g})}{Z_0^{\#(\Gamma_{p,\mathsf{g}})}}$ as soon as $N > \max\{h_{\Gamma_{p,\mathsf{g}}}(\Delta)|\delta \in \mathcal{D}\}$ which is finite. Besides, the only dependence in Λ is embodied in the condition $\Gamma_{p,\mathsf{g}} \subset \mathcal{L}_{\Lambda,N}$.

We will then show in the next section that there exists a positive function $\mathcal{B}(\mathsf{g})$ of cluster-graphs g, depending on λ, such that, for small λ,

$$\sum_{\mathsf{g}} \mathcal{B}(\mathsf{g}) < +\infty \tag{62}$$

where the sum is without restriction on g, and such that

$$|\mathcal{A}(\mathsf{g},\Lambda,N)| \le \mathcal{B}(\mathsf{g}) \tag{63}$$

for any g, Λ, and N satisfying $\Gamma_{p,\mathsf{g}} \subset \mathcal{L}_{\Lambda,N}$ and $N \ge \#(\Lambda)$.

The discrete version of the Lebesgue dominated convergence theorem will thus allow us to first take the limit $N \to +\infty$ and then the limit $\Lambda \nearrow \mathbb{R}^d$ in (56) thereby proving theorem 1. The next section is devoted to finding a uniform estimate $\mathcal{B}(\mathsf{g})$ which does the job.

IV The uniform estimates

We first use a very coarse bound for the "parasite" factors in (57).

Lemma 4

$$0 < \frac{1}{Z_0^{\#(\Gamma_{p,\mathsf{g}})}} \times \frac{Z(Y_\mathsf{g})\cdot Z_0^{\#(\Lambda)-\#(Y_\mathsf{g})}}{Z(\Lambda)} \le \exp\left(2K_3\lambda\#(\Gamma_{p,\mathsf{g}})\right) \tag{1}$$

where

$$K_3 \stackrel{\text{def}}{=} K_2\left(1+\frac{(2m)!}{2^m m!}C(0,0)\right) \quad . \tag{2}$$

Proof : Indeed as we derived in section 3 a lower bound for $Z(\Lambda)$, it is easy to do the same with Z_0 and $Z(Y_{\mathsf{g}})$, from which we obtain the three estimates

$$1 \geq Z(\Lambda) \geq \exp(-K_3\lambda\#(\Lambda)) \tag{3}$$

$$1 \geq Z(Y_{\mathsf{g}}) \geq \exp(-K_3\lambda\#(Y_{\mathsf{g}})) \tag{4}$$

and

$$1 \geq Z_0 \geq \exp(-K_3\lambda) \quad . \tag{5}$$

Now given g, Λ and N, with $N \geq \#(\Lambda)$, we have two possible situations:
1st case: $Y_{\mathsf{g}} = \Lambda$.

Then

$$\frac{1}{Z_0^{\#(\Gamma_{p,\mathsf{g}})}} \times \frac{Z(Y_{\mathsf{g}}) \cdot Z_0^{\#(\Lambda)-\#(Y_{\mathsf{g}})}}{Z(\Lambda)} = Z_0^{-\#(\Gamma_{p,\mathsf{g}})} \leq \exp\left(K_3\lambda\#(\Gamma_{p,\mathsf{g}})\right) \quad . \tag{6}$$

2nd case: $Y_{\mathsf{g}} \subset \Lambda$ and $Y_{\mathsf{g}} \neq \Lambda$.

Then $N \leq \max\{h_{\Gamma_{p,\mathsf{g}}}(\Delta) | \delta \in \mathcal{D}\}$ from the remarks at the end of section 3. But $\#(\Lambda) \leq N$ and $\max\{h_{\Gamma_{p,\mathsf{g}}}(\Delta) | \delta \in \mathcal{D}\} \leq \#(\Gamma_{p,\mathsf{g}})$ so that $\#(\Lambda) \leq \#(\Gamma_{p,\mathsf{g}})$ and thus

$$\begin{aligned} \frac{1}{Z_0^{\#(\Gamma_{p,\mathsf{g}})}} \times \frac{Z(Y_{\mathsf{g}}) \cdot Z_0^{\#(\Lambda)-\#(Y_{\mathsf{g}})}}{Z(\Lambda)} &\leq Z_0^{-\#(\Gamma_{p,\mathsf{g}})} \cdot Z(\Lambda)^{-1} && (7) \\ &\leq \exp\left(2K_3\lambda\#(\Gamma_{p,\mathsf{g}})\right) \quad . && (8) \end{aligned}$$

■

We now need a few lemmas to bound $\mathcal{A}_0(\mathsf{g})$.

Lemma 5 *If $b = (\Delta, k) \in \Gamma_{p,\mathsf{g}}$, and $\Delta' \in \mathcal{D}$, then*

$$\sum_{k' \geq 0} \overline{\mathcal{M}}_{\mathsf{g},(h_1,\ldots,h_p)}(b, (\Delta', k')) \leq 1 \tag{9}$$

Proof : Let us denote $b' = (\Delta', k')$. Now only $b' \in \Gamma_{p,\mathsf{g}}$ contributes. Besides, either $b = b'$ or $s\mu_{\mathsf{g}}(b, b') < i\nu_{\mathsf{g}}(b, b')$ is needed for $\overline{\mathcal{M}}_{\mathsf{g},(h_1,\ldots,h_p)}(b, b') \neq 0$. Now remark that, for any $c \in \Gamma_{p,\mathsf{g}}$, $\mu_{\mathsf{g}}(c) \leq j < \nu_{\mathsf{g}}(c)$ is equivalent to $c \in W(\Gamma_{i,\mathsf{g}})$. Therefore $s\mu_{\mathsf{g}}(b, b') < i\nu_{\mathsf{g}}(b, b')$ means that there is i, $-1 \leq i \leq p$, such that both b and b' belong to $W(\Gamma_{i,\mathsf{g}})$.
1st case: $\Delta = \Delta'$.

Since $W(\Gamma)$ has a unique cube with a given Δ, whatever is the cluster Γ, the only contribution comes from $k' = k$ which gives 1 and satisfies the inequality.

2nd case: $\Delta \neq \Delta'$.

Let $[k'_1, k'_2] \stackrel{\text{def}}{=} \{k' | \exists i, \mu_{\mathsf{g}}(b) \leq i < \nu_{\mathsf{g}}(b), (\Delta, k') \in W(\Gamma_{i,\mathsf{g}})\}$. Let us first suppose that $k'_2 \geq k'_1 + 1$. We let $\mu_{k'} \stackrel{\text{def}}{=} \mu_{\mathsf{g}}((\Delta', k'))$ and $\nu_{k'} \stackrel{\text{def}}{=} \nu_{\mathsf{g}}((\Delta', k'))$. If $b' = (\Delta', k')$ with $k'_1 < k' < k'_2$, it follows from the definition of a cluster-graph like g that we have $\nu_{k'} = \mu_{k'+1}$, $\mu_{k'} > \mu_{\mathsf{g}}(b)$ and $\nu_{k'} < \nu_{\mathsf{g}}(b)$. Therefore

$$\sum_{k'_1 < k' < k'_2} \overline{\mathcal{M}}_{\mathsf{g},(h_1,\ldots,h_p)}(b, (\Delta', k')) = \sum_{k'_1 < k' < k'_2} h_{\nu_{\mathsf{g}}(b)} \left(\frac{1}{h_{\mu_{k'+1}}} - \frac{1}{h_{\mu_{k'}}} \right) \tag{10}$$

$$= h_{\nu_{\mathsf{g}}(b)} \left(\frac{1}{h_{\mu_{k'_2}}} - \frac{1}{h_{\mu_{k'_1+1}}} \right) \quad . \tag{11}$$

One also checks easily that the contribution of $k' = k'_1$ is

$$h_{\nu_{\mathsf{g}}(b)} \left(\frac{1}{h_{\mu_{k'_1+1}}} - \frac{1}{h_{\mu_{\mathsf{g}}(b)}} \right) \tag{12}$$

and that of $k' = k'_2$ is

$$h_{\nu_{k'_2}} \left(\frac{1}{h_{\nu_{\mathsf{g}}(b)}} - \frac{1}{h_{\mu_{k'_2}}} \right) \quad . \tag{13}$$

Therefore

$$\sum_{k' \geq 0} \overline{\mathcal{M}}_{\mathsf{g},(h_1,\ldots,h_p)}(b, (\Delta', k')) = \frac{h_{\nu_{\mathsf{g}}(b)}}{h_{\mu_{k'_2}}} - \frac{h_{\nu_{\mathsf{g}}(b)}}{h_{\mu_{\mathsf{g}}(b)}} + \frac{h_{\nu_{k'_2}}}{h_{\nu_{\mathsf{g}}(b)}} - \frac{h_{\nu_{k'_2}}}{h_{\mu_{k'_2}}} \quad . \tag{14}$$

But, from $\mu_{\mathsf{g}}(b) < \mu_{k'_2} < \nu_{\mathsf{g}}(b) \leq \nu_{k'_2}$, it follows that there is α, β, $\gamma \in [0, 1]$ such that $h_{\nu_{k'_2}} = \alpha h_{\nu_{\mathsf{g}}}(b)$, $h_{\nu_{\mathsf{g}}}(b) = \beta h_{\mu_{k'_2}}$ and $h_{\mu_{k'_2}} = \gamma h_{\mu_{k'_2}}$. Thus

$$\sum_{k' \geq 0} \overline{\mathcal{M}}_{\mathsf{g},(h_1,\ldots,h_p)}(b, (\Delta', k')) = \beta - \beta\gamma + \alpha - \alpha\beta \tag{15}$$

$$\leq \beta + \alpha - \alpha\beta \tag{16}$$

$$\leq 1 - (1 - \alpha)(1 - \beta) \tag{17}$$

$$\leq 1 \tag{18}$$

which proves the assertion. ■

As a consequence of this lemma we have a bound

$$\left| \mathcal{C}[\overline{\mathcal{M}}_{\mathsf{g},(h_1,\ldots,h_p)}](x, k; x', k') \right| \leq G((\Delta(x), k); (\Delta(x'), k')) \tag{19}$$

where the function $G(b,b')$ on $\mathcal{L}^2$ satisfies

$$\forall b \in \mathcal{L}, \quad \sum_{b' \in \mathcal{L}} G(b,b') \le K_4 \tag{20}$$

for some constant K_4. Indeed,

$$\begin{aligned} &G((\Delta,k);(\Delta',k')) \stackrel{\text{def}}{=} \\ &\overline{\mathcal{M}}_{\mathsf{g},(h_1,\ldots,h_p)}((\Delta,k);(\Delta',k')) \times K_1(d+1) \times (1+d(\Delta,\Delta'))^{-(d+1)} \end{aligned} \tag{21}$$

with $d(\Delta,\Delta') \stackrel{\text{def}}{=} \min\{|x-y| \mid x \in \Delta, y \in \Delta'\}$ works, since the sum over k', by lemma 5, is no greater than 1, and the sum over Δ' is bounded by the rapid decay (1) of the propagator. Note that K_4, unlike $G(b,b')$, is independent of g and $(h_1,\ldots,h_p)$.

Lemma 6 *(The principle of local factorials)*
We have the bound:

$$\left| \int d\mu_{\mathcal{C}[\overline{\mathcal{M}}_{\mathsf{g},(h_1,\ldots,h_p)}]}(\Phi)\ \Phi(z_1,k_1)\cdots\Phi(z_r,k_r) \right| \le K_5^5 \times \prod_{b\in\mathcal{L}} \sqrt{n(b)!} \tag{22}$$

where $n(b) \stackrel{\text{def}}{=} \#(\{j|1 \le j \le r, (\Delta(z_j),k_j) = b_j\})$ *and* K_5 *is a constant.*

Proof : Using Wick's theorem, the functional integral can be computed as a sum over contractions c of the fields $\Phi(z_j,k_j)$, with the propagator of $\mathcal{C}[\overline{\mathcal{M}}_{\mathsf{g},(h_1,\ldots,h_p)}]$. c is simply an involution without fixed points of the set $J = \{1,\ldots,r\}$. We get

$$\begin{aligned} &\left| \int d\mu_{\mathcal{C}[\overline{\mathcal{M}}_{\mathsf{g},(h_1,\ldots,h_p)}]}(\Phi)\ \Phi(z_1,k_1)\cdots\Phi(z_r,k_r) \right| \\ &\quad = \left| \sum_c \prod_{\substack{\{j,j'\}\subset J \\ j'=c(j)}} \mathcal{C}[\overline{\mathcal{M}}_{\mathsf{g},(h_1,\ldots,h_p)}](x,j;x_{c(j)},k_{c(j)}) \right| && (23) \\ &\quad \le \sum_c \prod_{\substack{\{j,j'\}\subset J \\ j'=c(j)}} G(b_j,b_{c(j)}) && (24) \end{aligned}$$

where b_j denotes $(\Delta(x_j),k_j) \in \mathcal{L}$. Suppose we have ordered J as $\{j_1,\ldots,j_s\}$ such that $n(b_{j_1}) \ge n(b_{j_2}) \ge \cdots \ge n(b_{j_s})$. To sum over $c(j_1)$, we first sum over $b_{c(j_1)}$, then over $c(j_1)$ knowing $b_{c(j_1)}$. The sum over $b_{c(j_1)}$ is bounded by K_4. The sum over $c(j_1)$ knowing $b_{c(j_1)}$ costs a factor $n(b_{c(j_1)}) \le \sqrt{n(b_{j_1})n(b_{c(j_1)})}$ because of the ordering of J. We now pick the element j with the smallest label in $J\backslash\{j_1, c(j_1)\}$, and sum over $c(j)$ in the same way, thus getting a factor $K_4\sqrt{n(b_j)n(b_{c(j)})}$, and so on. Since $\sqrt{n(b_j)}$ will appear exactly once by definition of a contraction c, we obtain a bound

$$K_4^{\frac{r}{2}} \times \prod_{j\in J} \sqrt{n(b_j)} = K_4^{\frac{r}{2}} \times \prod_{\substack{b\in\mathcal{L} \\ n(b)\neq 0}} \sqrt{n(b)^{n(b)}} \tag{25}$$

$$\leq K_5^r \prod_{b\in\mathcal{L}} \sqrt{n(b)!} \tag{26}$$

with $K_5 \stackrel{\text{def}}{=} \sqrt{eK_4}$. ■

We now explain the bound on $\mathcal{A}_0(\mathsf{g})$. First note that $\mathcal{A}_0(\mathsf{g})$ decomposes as

$$\mathcal{A}_0(\mathsf{g}) = \sum_\rho \mathcal{A}_0(\mathsf{g}, \rho) \tag{27}$$

where ρ is a derivation procedure for the operators D_{l_q} and $\mathcal{A}_0(\mathsf{g}, \rho)$ is the contribution of ρ in the expansion that computes the action of $\prod_{q=1}^p D_{l_q}$ on the integrand

$$\prod_{i=1}^n \Phi(x_i, 0) \exp\left(-\lambda \sum_{(\Delta,k)\in\Gamma_{p,\mathsf{g}}} \int_\Delta P(\Phi(x,k))dx\right) . \tag{28}$$

When considering the expression for $\mathcal{A}_0(\mathsf{g}, \rho)$, we take out of the functional integral all the $\omega(\mathsf{g}, (\mathbf{h}, 0), q)$ factors, as well as the $C(x, x')$ factors coming from $\prod_{q=1}^p D_{l_q}$, and also the spatial integrations $\int_\Delta dx$ that come from the D_{l_q}, as well as all numerical factors such as λ or the coefficients of the polynomial P.

The resulting expression is a functional integral of the form:

$$\mathcal{I} = \int d\mu_{\tilde{\mathcal{C}}}(\Phi)\ \Phi(z_1, k_1)\ldots\Phi(z_r, k_r)$$
$$\exp\left(-\lambda \sum_{(\Delta,k)\in\Gamma_{p,\mathsf{g}}} \int_\Delta P(\Phi(x,k))dx\right) \tag{29}$$

where $\tilde{\mathcal{C}}$ denotes $\mathcal{C}[\overline{\mathcal{M}}_{\mathsf{g},(h_1,\ldots,h_p)}]$. We bound it using

$$|\mathcal{I}| \leq \int d\mu_{\tilde{\mathcal{C}}}(\Phi)\ |\Phi(z_1, k_1)\ldots\Phi(z_r, k_r)| \exp(\lambda K_6 \#(\Gamma_{p,\mathsf{g}})) \tag{30}$$

where $K_6 = \min\{P(x)|x \in \mathbb{R}\}$. Then by the Cauchy-Schwartz inequality,

$$|\mathcal{I}| \leq \exp(\lambda K_6 \#(\Gamma_{p,\mathsf{g}}))\sqrt{\int d\mu_{\tilde{C}}(\Phi)\ \Phi(z_1,k_1)^2 \ldots \Phi(z_r,k_r)^2}\ . \tag{31}$$

Now we bound the functional integral in the last inequality using lemma 6 thus obtaining:

$$|\mathcal{I}| \leq \exp(\lambda K_6 \#(\Gamma_{p,\mathsf{g}})) \times K_5^r \times \prod_{b\in\mathcal{L}} (2n_{\mathsf{g},\rho}(b))!^{\frac{1}{4}} \tag{32}$$

where $n_{\mathsf{g},\rho}(b) \stackrel{\text{def}}{=} \#(\{j|1 \leq j \leq r, (\Delta(z_j),k_j) = b\})$.

We now explain the bound on the sum over the derivation procedures ρ that act on

$$\prod_{i=1}^{n} \Phi(x_i,0) \exp\left(-\lambda \sum_{(\Delta,k)\in\Gamma_{p,\mathsf{g}}} \int_\Delta P(\Phi(x,k))dx\right)\ . \tag{33}$$

First we bound the propagators $C(x,x')$ corresponding to a D_{l_q} with $l_q = \{(\Delta,k),(\Delta',k')\}$ by $K_1(r)(1+d(\Delta,\Delta'))^{-r}$. The exponent r will be adjusted later. We also bound the spatial integrations $\int_\Delta dx$ by 1. Since each $(\Delta,k) \in \Gamma_{p,\mathsf{g}}\backslash\Gamma_0$ belongs to an l_q, there is at least a $\frac{\delta}{\delta\Phi}$ that acts on the corresponding interaction term $\exp(-\lambda\int_\Delta P(\Phi(x,k))dx)$; therefore there is at least $\lambda^{\#(\Gamma_{p,\mathsf{g}})-\#(\Gamma_0)}$ in factor and eventually some more factors λ that we bound by 1 as we assume from now on that $\lambda \leq 1$.

We also introduce the notation $||P||$ for the maximum of absolute value of the coefficients of the polynomial P. Note that each $\frac{\delta}{\delta\Phi(x,k)}$ can derive an interaction term, and thus generate a coefficient of P. We therefore globally bound these factors by $(1+||P||)^{2p}$. We let $n_{\mathsf{g}} \stackrel{\text{def}}{=} \#(\{q|1 \leq q \leq p, b \in l_q\})$, i.e. the coordinance of b with respect to the graph g, for any $b \in \Gamma_{p,\mathsf{g}}$. We also let $s(b) \stackrel{\text{def}}{=} \#(\{i|1 \leq i \leq n, b = (\Delta(x_i),0)\})$ that counts the sources located in b.

Choose an arbitrary order to perform the functional derivations. Let $\frac{\delta}{\delta(x,k)}$ be the one performed last. It is located in $b = (\Delta(x),k)$, and can either derive one of the sources, which gives $s(b)$ possibilities. It can also derive a new vertex from the interaction $\exp(-\lambda\int_{\Delta(x)} P(\Phi(y,k))dy)$, we then have to choose the derived monomial in P, and the field in the monomial which gives at most $(2m)^2$ new possibilities. Finally it can rederive a vertex that was derived for the first time by a previously performed functional derivation $\frac{\delta}{\delta\Phi(x',k)}$ that is also located in b. This gives a total number of possibilities, for $\frac{\delta}{\delta\Phi(x,k)}$, that is bounded by $s(b) + 4m^2 n_{\mathsf{g}}(b)$.

We then do the same sum over the ways of computing the before last functional derivation, and so on. It follows that the number of derivation procedures ρ is bounded by

$$\prod_{b\in\Gamma_{p,\mathrm{g}}} \left(s(b)+4m^2 n_{\mathrm{g}}(b)\right)^{n_{\mathrm{g}}(b)} \tag{34}$$

since there is $n_{\mathrm{g}}(b)$ functional derivations in each b. We write for convenience

$$\begin{aligned}\prod_{b\in\Gamma_{p,\mathrm{g}}} \left(s(b)+4m^2 n_{\mathrm{g}}(b)\right)^{n_{\mathrm{g}}(b)} &\leq \prod_{b\in\Gamma_{p,\mathrm{g}}} \left(n_{\mathrm{g}}(b)! e^{s(b)+4m^2 n_{\mathrm{g}}(b)}\right) &(35)\\ &\leq e^{n+8m^2 p} \prod_{b\in\Gamma_{p,\mathrm{g}}} n_{\mathrm{g}}(b)! \quad . &(36)\end{aligned}$$

Now note that in (32), $r \leq n+4mp$, and for each b, $n_{\mathrm{g},\rho} \leq s(b)+2mn_{\mathrm{g}}(b)$. As a result, the previous bound on $\mathcal{I}$ becomes

$$\begin{aligned}|\mathcal{I}| &\leq \exp(K_6\#(\Gamma_{p,\mathrm{g}})) \times (1+K_5)^{n+4mp} \prod_{b\in\Gamma_{p,\mathrm{g}}} (2s(b)+4mn_{\mathrm{g}}(b))!^{\frac{1}{4}} &(37)\\ &\leq \exp(K_6\#(\Gamma_{p,\mathrm{g}})) \times (1+K_5)^{n+4mp}\\ &\quad\times \prod_{b\in\Gamma_{p,\mathrm{g}}} \left(\sqrt{s(b)!} \times (n_{\mathrm{g}}(b)!)^m \times \exp(3ms(b)+6m^2 n_{\mathrm{g}}(b))\right) &(38)\\ &\leq \exp(K_6\#(\Gamma_{p,\mathrm{g}})) \times (1+K_5)^{n+4mp}\\ &\quad\times\sqrt{n!} \times e^{3mn+12m^2 p} \times \prod_{b\in\Gamma_{p,\mathrm{g}}} (n_{\mathrm{g}}(b)!)^m \quad . &(39)\end{aligned}$$

We are now able to write a raw bound on $\mathcal{A}_0(\mathrm{g})$ as:

$$\begin{aligned}|\mathcal{A}_0(\mathrm{g})| \leq\ & \lambda^{\#(\Gamma_{p,\mathrm{g}})-\#(\Gamma_0)} \times \prod_{q=1}^{p} \left(K_1(r)(1+d(\Delta_q,\Delta'_q))^{-r}\right)\\ &\times \int_{1>h_1>\cdots>h_p>0} dh_1 \ldots dh_p \prod_{q=1}^{p} |\omega(\mathrm{g},(\mathbf{h},0),q)|\\ &\times (1+||P||)^{2p} \times \exp(K_6\#(\Gamma_{p,\mathrm{g}})) \times (1+K_5)^{n+4mp}\\ &\times\sqrt{n!} \times e^{(3m+1)n+20m^2 p} \times \prod_{b\in\Gamma_{p,\mathrm{g}}} (n_{\mathrm{g}}(b)!)^{m+1} \end{aligned} \tag{40}$$

where Δ_q, Δ'_q are such that $l_q = \{(\Delta_q,k_q),(\Delta'_q,k'_q)\}$, for some k_q and k'_q.

The right-hand side is not quite $\mathcal{B}(\mathrm{g})$, we need first to get rid of the local factorials $n_{\mathrm{g}}(b)!$. This requires a volume argument and the next two lemmas.

Lemma 7 *If* $\mathsf{g} = (l_1, \ldots, l_p)$ *is a cluster-graph with* $\mathcal{A}_0(\mathsf{g}) \neq 0$, *and* $l_{q_\alpha} = \{b_\alpha, b'_\alpha\}$, $1 \leq \alpha \leq 3$, *are three links in* g *such that* $q_1 < q_2 < q_3$ *and* $b_1 = b_2 = b_3$*; then* b'_1, b'_2 *and* b'_3 *cannot all be of the form* (Δ', k'_α) *with the same* $\Delta' \in \mathcal{D}$.

Proof : *Ad absurdum.* Let $b = b_1 = b_2 = b_3 = (\Delta, k)$, and $b'_\alpha = (\Delta', k'_\alpha)$, $1 \leq \alpha \leq 3$. Since $l_q \not\subset \Gamma_{q-1,\mathsf{g}}$ for any q, and since $q_1 < q_2 < q_3$ we have that k'_1, k'_2 and k'_3 are distinct. We even have $k'_1 < k'_2 < k'_3$. Indeed, if for instance $k'_2 < k'_1$, since $l_{q_1} = \{b, (\Delta', k'_1)\} \subset \Gamma_{q_1,\mathsf{g}}$ and $\Gamma_{q_1,\mathsf{g}}$ is a cluster, it would follow that $(\Delta', k'_2) \in \Gamma_{q_1,\mathsf{g}}$ and thus $l_{q_2} \subset \Gamma_{q_1,\mathsf{g}} \subset \Gamma_{q_2-1,\mathsf{g}}$ which is not allowed.

Now if we only consider l_{q_1} and l_{q_2}, since $b \in \Gamma_{q_2-1,\mathsf{g}}$, l_{q_2} can only be of type cluster-roof, and $\omega(\mathsf{g}, (\mathbf{h}, 0), q_2) \neq 0$ implies $s\mu_{\mathsf{g},q_2-1}(b, b'_2) < i\nu_{\mathsf{g},q_2-1}(b, b'_2)$. That is, there exists $q < q_2$ such that b, $b'_2 \in W(\Gamma_{q,\mathsf{g}})$. Thus $b \notin \Gamma_{q,\mathsf{g}}$ and therefore $q < q_1$. Besides, $b'_2 \in W(\Gamma_{q,\mathsf{g}})$ and $k'_2 > k'_1$ implies $b'_1 \in \Gamma_{q,\mathsf{g}} \subset \Gamma_{q_1-1,\mathsf{g}}$. But $l_{q_1} \not\subset \Gamma_{q_1-1,\mathsf{g}}$, therefore $b \notin \Gamma_{q_1-1,\mathsf{g}}$. As a result, $\mu_{\mathsf{g}}(b'_1) < \mu_{\mathsf{g}}(b) = q_1 - 1$.

We can now do the same reasoning, considering l_{q_2} and l_{q_3} this time, to conclude $\mu_{\mathsf{g}}(b'_2) < \mu_{\mathsf{g}}(b) = q_2 - 1$ as well, which gives a different value for $\mu_{\mathsf{g}}(b)$ and proves a contradiction. ■

Lemma 8 *(The volume argument)*

We have, with the notations of (40),

$$\prod_{b \in \Gamma_{p,\mathsf{g}}} (n_{\mathsf{g}}(b)!)^{m+1} \times \prod_{q=1}^{p} (1 + d(\Delta_q, \Delta'_q))^{-r_1} \leq K_7^p \tag{41}$$

for some constants r_1 *and* K_7 *that only depend on the dimension* d *and the degree* $2m$ *of the interaction.*

Proof : We let $r_1 = 4d(m+2)$. We now write

$$\prod_{b \in \Gamma_{p,\mathsf{g}}} (n_{\mathsf{g}}(b)!)^{m+1} \times \prod_{q=1}^{p} (1 + d(\Delta_q, \Delta'_q))^{-r_1} = \prod_{\substack{b \in \Gamma_{p,\mathsf{g}} \\ n_{\mathsf{g}}(b) \geq 1}} \xi(b) \tag{42}$$

with

$$\xi(b) \stackrel{\text{def}}{=} n_{\mathsf{g}}(b)! \times \prod_{b' \text{ linked to } b} (1 + d(\Delta(b), \Delta(b')))^{-\frac{r_1}{2}} \tag{43}$$

where the product is over all $b' \in \Gamma_{p,\mathsf{g}}$ such that $\{b, b'\}$ is a link of g, and $\Delta(b)$ denotes the first projection on $\mathcal{D}$ of the pair $b \in \mathcal{L}$. Now it follows from lemma 7 that there cannot be more than two cubes b', with the same $\Delta(b')$, linked to b. Remark that there is a constant K such that for δ big enough

$$\#(\{\Delta' \in \mathcal{D} | d(\Delta(b), \Delta') \leq \delta\}) \leq K\delta^d \quad . \tag{44}$$

Therefore

$$\#(\{b' \in \Gamma_{p,\mathsf{g}} |\ b' \text{ linked to } b, d(\Delta(b), \Delta(b')) \leq \delta\}) \leq 2K\delta^d \quad . \tag{45}$$

If $n_{\mathsf{g}}(b)$ is big enough and if we set $\delta = (\frac{n}{4K})^{\frac{1}{d}}$, it follows that at least $\frac{n_{\mathsf{g}}(b)}{2}$ cubes b' that are linked to b satisfy $d(\Delta(b), \Delta(b')) > \delta$. As a result:

$$\xi(b) \leq (n_{\mathsf{g}}(b)!)^{m+1} \times (1+\delta)^{-\frac{r_1 n_{\mathsf{g}}(b)}{4}} \tag{46}$$

$$\leq n_{\mathsf{g}}(b)^{(m+1)n_{\mathsf{g}}(b)} \times \left(\frac{n_{\mathsf{g}}(b)}{4K}\right)^{-\frac{r_1 n_{\mathsf{g}}(b)}{4d}} \tag{47}$$

$$\leq n_{\mathsf{g}}(b)^{-n_{\mathsf{g}}(b)} \times (4K)^{\frac{r_1 n_{\mathsf{g}}(b)}{4d}} \tag{48}$$

because of our choice for r_1. It easily follows that $\xi(b) \leq K'$ for some constant $K' \geq 1$, for any value of $n_{\mathsf{g}}(b)$. Taking $K_7 \stackrel{\text{def}}{=} K'^2$ concludes the proof of the lemma. ■

We now return to (40) and proceed to define the bounding term $\mathcal{B}(\mathsf{g})$. First we choose $r = r_1 + d + 1$. Next we note that $\#(\Gamma_{p,\mathsf{g}}) - \#(\Gamma_0) \geq p$ and $\#(\Gamma_{p,\mathsf{g}}) \leq 2p + n$. Combining lemma 4, (40) and lemma 8, we now easily obtain a bound

$$|\mathcal{A}(\mathsf{g}, \Lambda, N)| \leq K_8(n) K_9^p \lambda^p \times \int_{1>h_1>\cdots>h_p>0} dh_1 \ldots dh_p$$

$$\prod_{q=1}^{p} \left(|\omega(\mathsf{g}, (\mathbf{h}, 0), q)|(1 + d(\Delta_q, \Delta_q'))^{-(d+1)}\right) \tag{49}$$

where $K_8(n)$ and K_9 are independent of g, Λ and N. We let $\mathcal{B}(\mathsf{g})$ be the righthand side of (49). The proof of theorem 1 will be complete when we prove the following result.

Proposition 1 *There exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0]$,*

$$\sum_{\mathsf{g}} \mathcal{B}(\mathsf{g}) < +\infty \tag{50}$$

where the cluster-graph g is summed without any restriction of volume in $\mathcal{L}$.

Proof : For any cluster-graph g with nonzero contribution, we define the following function $\sigma_{\mathsf{g}} : \{1, \dots, p\} \to \{0, \dots, p-1\}$. Let q, $1 \le q \le p$, and $l_q = \{b_q, b'_q\}$, and let $\overline{b}_q$ and $\overline{b}'_q$ be the two elements of $W(\Gamma_{q-1,\mathsf{g}})$ with the same first projection on $\mathcal{D}$ as b_q and b'_q respectively. We pose, by definition,

$$\sigma_{\mathsf{g}}(q) \stackrel{\text{def}}{=} \max(\mu_{\mathsf{g}}(\overline{b}_q), \mu_{\mathsf{g}}(\overline{b}'_q)) < q \quad . \tag{51}$$

Note that, indeed, $\sigma_{\mathsf{g}}(q) \ge 0$, otherwise we would have $\overline{b}_q, \overline{b}'_q \in W_{-1} = \mathcal{L}_0$ and therefore also b_q, $b'_q \in W_{-1}$, which would give $\omega(\mathsf{g}, (\mathbf{h}, 0), q) = 0$ and a zero contribution for g. We will first bound the conditional sum on g, knowing σ_{g}.

We start by summing over the last link l_p knowing $\mathsf{g}' = (l_1, \dots, l_{p-1})$ and σ_{g}. We first perform the sum over $l_p = \{b_p, b'_p\}$ with $b_p = (\Delta_p, k)$ and $b'_p = (\Delta'_p, k')$, knowing Δ_p and Δ'_p. This is done thanks to the factor $|\omega(\mathsf{g}, (\mathbf{h}, 0), p)|$ as in lemma 5. Note that there are three cases.

1st case: l_p is a roof-roof link.

In this situation b_p, $b'_p \in W(\Gamma_{p-1,\mathsf{g}})$ and thus $b_p = \overline{b}_p$, $b'_p = \overline{b}'_p$ and

$$|\omega(\mathsf{g}, (\mathbf{h}, 0), p)| = \left| -\frac{1}{h_{s\mu_{\mathsf{g}}(b_p, b'_p)}} \right| = \frac{1}{h_{\sigma_{\mathsf{g}}(p)}} \quad . \tag{52}$$

2nd case: l_p is cluster-roof, with $b_p \in W(\Gamma_{p-1,\mathsf{g}})$ and $b'_p \in \Gamma_{p-1,\mathsf{g}}$.

Then $b_p = \overline{b}_p$ is unique, and we have to sum over the second projection k' of b'_p, $0 \le k' \le h_{\Gamma_{p-1,\mathsf{g}}}(\Delta'_p)$, with the condition that $s\mu_{\mathsf{g}}(b_p, b'_p) < i\nu_{\mathsf{g}}(b_p, b'_p)$. We obtain, the previous condition being implicit in the following sums,

$$\sum_{k'} |\omega(\mathsf{g}, (\mathbf{h}, 0), p)| = \sum_{k'} \left(\frac{1}{h_{i\nu_{\mathsf{g}}(b_p, b'_p)}} - \frac{1}{h_{s\mu_{\mathsf{g}}(b_p, b'_p)}} \right) \quad . \tag{53}$$

Note that $\Delta_p \ne \Delta'_p$ as no link is vertical. We let

$$\begin{aligned}[k'_1, k'_2] \stackrel{\text{def}}{=} \{k' | 0 \le k' \le h_{\Gamma_{p-1,\mathsf{g}}}(\Delta'_p) \\ \text{and } \exists i, \mu_{\mathsf{g}}(b_p) \le i \le p-1, (\Delta'_p, k') \in W(\Gamma_{i,\mathsf{g}})\} \ .\end{aligned} \tag{54}$$

With the notation $\mu_{k'} = \mu_{\mathsf{g}}((\Delta'_p, k'))$ and $\nu_{k'} = \nu_{\mathsf{g}}((\Delta'_p, k'))$, we have that for any k', $k'_1 \le k' < k'_2$, $\mu_{k'+1} = \nu_{k'}$. Note also that $\nu_{k'_2} = \mu_{\mathsf{g}}(\overline{b}'_p)$. Therefore

$$\sum_{k'} \left(\frac{1}{h_{i\nu_{\mathsf{g}}(b_p, b'_p)}} - \frac{1}{h_{s\mu_{\mathsf{g}}(b_p, b'_p)}} \right) =$$

$$\sum_{k'_1<k'<k'_2}\left(\frac{1}{h_{\mu_{k'+1}}}-\frac{1}{h_{\mu_{k'}}}\right)+\left(\frac{1}{h_{\mu_{k'_1+1}}}-\frac{1}{h_{\mu_{\mathtt{g}}(b_p)}}\right)$$

$$+\left(\frac{1}{h_{\mu_{\mathtt{g}}(\overline{b}'_p)}}-\frac{1}{h_{\mu_{k'_2}}}\right) \tag{55}$$

$$=\frac{1}{h_{\mu_{\mathtt{g}}(\overline{b}'_p)}}-\frac{1}{h_{\mu_{\mathtt{g}}(b_p)}} \tag{56}$$

which is positive; since $\mu_{\mathtt{g}}(\overline{b}'_p) \geq \mu_{\mathtt{g}}(b_p)$ is necessary for the existence of cluster-roof links $\{b_p, b'_p\}$ with b'_p under $\overline{b}'_p$. Finally

$$\sum_{k'}|\omega(\mathtt{g},(\mathbf{h},0),p)| \leq \frac{1}{h_{\mu_{\mathtt{g}}(\overline{b}'_p)}}=\frac{1}{h_{\sigma_{\mathtt{g}}(p)}} \quad . \tag{57}$$

3rd case: l_p is cluster-roof, with $b'_p \in W(\Gamma_{p-1,\mathtt{g}})$ and $b_p \in \Gamma_{p-1,\mathtt{g}}$.

The symmetric of the 2nd case is treated in the same way, giving a bound of $\frac{1}{h_{\sigma_{\mathtt{g}}(p)}}$ again.

So summing on l_p, knowing Δ_p and Δ'_p, gives a bound of $\frac{3}{h_{\sigma_{\mathtt{g}}(p)}}$.

We then need to sum over the unordered pair $\{\Delta_p, \Delta'_p\}$, knowing $\mathtt{g}' = (l_1, \ldots, l_{p-1})$ and $\sigma_{\mathtt{g}}$. Note that one of the cubes $\overline{b}_p$ and $\overline{b}'_p$ has a $\mu_{\mathtt{g}}$ equal to $\sigma_{\mathtt{g}}(p)$. Assume it is $\overline{b}_p$ for instance. Since $\sigma_{\mathtt{g}}(p) = \mu_{\mathtt{g}}(\overline{b}_p) \geq 0$, we have that $\overline{b}_p \notin \mathcal{L}_0$. There is then a unique box $\underline{b}$ just under $\overline{b}_p$, i.e. such that $\underline{b} = (\Delta_p, k-1)$ if $\overline{b}_p = (\Delta_p, k)$. We then have $\nu_{\mathtt{g}}(\underline{b}) = \mu_{\mathtt{g}}(\overline{b}_p) = \sigma_{\mathtt{g}}(p)$.

Either $\sigma_{\mathtt{g}}(p) = 0$, in this case $\underline{b} \in \Gamma_0$, for which there is at most $\#(\Gamma_0) \leq n$ possibilities. Or $\sigma_{\mathtt{g}}(p) > 0$; in that case $\underline{b} \in l_{\sigma_{\mathtt{g}}(p)} \backslash \Gamma_{\sigma_{\mathtt{g}}(p)-1,\mathtt{g}}$ which leaves two possibilities. Once we know $\underline{b}$, we know one of the elements of $\{\Delta_p, \Delta'_p\}$. The sum over the other one is done thanks to the factor $(1 + d(\Delta_p, \Delta'_p))^{-(d+1)}$, and is bounded by some constant. As a result

$$\sum_{l_p}|\omega(\mathtt{g},(\mathbf{h},0),p)|(1+d(\Delta_p,\Delta'_p))^{-(d+1)}$$

$$\leq \frac{K_{10}}{h_{\sigma_{\mathtt{g}}(p)}}\left(1\!\!1_{\{\sigma_{\mathtt{g}}(p)>0\}}+n1\!\!1_{\{\sigma_{\mathtt{g}}(p)=0\}}\right) \tag{58}$$

for some constant K_{10}, the sum being over l_p knowing $(l_1, \ldots, l_{p-1})$ and the full map $\sigma_{\mathtt{g}}$. $1\!\!1_{\{\ldots\}}$ denotes the characteristic function of the event between braces.

We can now repeat the operation and sum over l_{p-1} knowing $(l_1, \ldots, l_{p-2})$ and $\sigma_{\mathtt{g}}$; and so on. We then get

$$\sum_{\substack{\mathtt{g}\ \text{of} \\ \text{length}\ p}} \mathcal{B}(\mathtt{g}) \leq \sum_{\sigma} K_8(n) K_9^p \lambda^p K_{10}^p \times \int_{1>h_1>\cdots>h_p>0} dh_1 \ldots dh_p$$
$$\prod_{q=1}^{p} \frac{\mathbb{1}_{\{\sigma(p)>0\}} + n \mathbb{1}_{\{\sigma(p)=0\}}}{h_{\sigma(p)}} \tag{59}$$

where the sum is over all maps $\sigma : \{1, \ldots, p\} \to \{0, \ldots, p-1\}$ such that $\sigma(q) < q$ for any q, $1 \leq q \leq p$. The last step relies on the following lemma.

Lemma 9 *For any $p \geq 1$, any $J = \{j_1, \ldots, j_\alpha\} \subset \{1, \ldots, p\}$ with $j_1 < \cdots < j_\alpha$, we have*

$$\sum_{\sigma|J} \int_{1>h_1>\cdots>h_p>0} dh_1 \ldots dh_p \ \prod_{q=1}^{p} \frac{1}{h_{\sigma(q)}} \leq \frac{e^p}{\alpha!} \tag{60}$$

where the sum is over maps $\sigma : \{1, \ldots, p\} \to \{0, \ldots, p-1\}$ such that for any $q \in J$, $\sigma(q) = 0$ and for any $q \notin J$, $1 \leq \sigma(q) < q$.

Proof of the lemma : We perform a change of variables by letting $h_q = s_1 s_2 \ldots s_q$, $1 \leq q \leq p$, so that

$$\int_{1>h_1>\cdots>h_p>0} dh_1 \ldots dh_p \prod_{q=1}^{p} \frac{1}{h_{\sigma(q)}} = \int_0^1 ds_1 \ldots \int_0^1 ds_p \prod_{q=1}^{p} \left(\prod_{\sigma(q)<j<q} s_j \right) \tag{61}$$

and

$$\sum_{\sigma|J} \int_{1>h_1>\cdots>h_p>0} dh_1 \ldots dh_p \prod_{q=1}^{p} \frac{1}{h_{\sigma(q)}} = \int_0^1 ds_1 \ldots \int_0^1 ds_p \prod_{q=1}^{p} P_q(s) \tag{62}$$

where

$$P_q(s) \stackrel{\text{def}}{=} \begin{cases} s_1 s_2 \ldots s_{q-1} & \text{if } q \in J \\ 1 + s_{q-1} + s_{q-1}s_{q-2} + \cdots + s_{q-1}s_{q-2} \ldots s_2 & \text{if } q \notin J \ . \end{cases} \tag{63}$$

Suppose $q \notin J$ and $q+1 \in J$. The product of the corresponding factors is then

$$\begin{aligned} &(1 + s_{q-1} + s_{q-1}s_{q-2} + \cdots + s_{q-1}s_{q-2} \ldots s_2) s_1 s_2 \ldots s_q \\ &\quad \leq (1 + s_{q-1} + s_{q-1}s_{q-2} + \cdots + s_{q-1}s_{q-2} \ldots s_2) s_1 s_2 \ldots s_q \\ &\qquad + s_1 s_2 \ldots s_{q-1} &(64) \\ &\quad = s_1 s_2 \ldots s_{q-1} (1 + s_q + s_q s_{q-1} + \cdots + s_q s_{q-1} \ldots s_2) &(65) \end{aligned}$$

which is the product we would get if the opposite situation occurred that is $q \in J$ and $q+1 \notin J$. Therefore, if we lower the elements of J, one by one, in $\{1,\dots,p\}$ we maximize the righthand side of (62), and we only need to prove the bound for

$$\int_0^1 ds_1 \dots \int_0^1 ds_p \prod_{q=1}^{\alpha} (s_1 s_2 \dots s_{q-1}) \\ \times \prod_{\alpha+1}^{p} (1 + s_{q-1} + s_{q-1}s_{q-2} + \dots + s_{q-1}s_{q-2}\dots s_2) \ . \tag{66}$$

Now for given $s_1,\dots,s_\alpha$ we compute

$$\int_0^1 ds_{\alpha+1} \dots \int_0^1 ds_p \prod_{\alpha+1}^{p} (1 + s_{q-1} + s_{q-1}s_{q-2} + \dots + s_{q-1}s_{q-2}\dots s_2) \tag{67}$$

by changing to the variables $y_{\alpha+1},\dots,y_p$ defined by

$$y_q \stackrel{\text{def}}{=} s_q(1 + s_{q-1} + s_{q-1}s_{q-2} + \dots + s_{q-1}s_{q-2}\dots s_2) \tag{68}$$

for $\alpha+1 \le q \le p$. We then obtain, with $y_\alpha \stackrel{\text{def}}{=} s_\alpha + s_\alpha s_{\alpha-1} + \dots + s_\alpha s_{\alpha-1}\dots s_2 \le \alpha - 1$

$$\int_0^{1+y_\alpha} dy_{\alpha+1} \int_0^{1+y_{\alpha+1}} dy_{\alpha+2} \dots \int_0^{1+y_{p-1}} dy_p$$

$$\le \int_0^{1+y_\alpha} dy_{\alpha+1} \int_0^{1+y_{\alpha+1}} dy_{\alpha+2} \dots \int_0^{1+y_{p-2}} e^{y_{p-1}} dy_{p-1} \tag{69}$$

$$\le \int_0^{1+y_\alpha} dy_{\alpha+1} \int_0^{1+y_{\alpha+1}} dy_{\alpha+2} \dots \int_0^{1+y_{p-3}} (e^{1+y_{p-2}} - 1) dy_{p-2} \tag{70}$$

$$\le e \int_0^{1+y_\alpha} dy_{\alpha+1} \int_0^{1+y_{\alpha+1}} dy_{\alpha+2} \dots \int_0^{1+y_{p-3}} e^{y_{p-2}} dy_{p-2} \tag{71}$$

and, by repeating the argument leading from (69) to (71), we get the inequality

$$\int_0^1 ds_{\alpha+1} \dots \int_0^1 ds_p \prod_{\alpha+1}^{p} \left(\sum_{j=2}^{q} \prod_{j \le k \le q-1} s_k \right)$$

$$\le e^{y_\alpha} \cdot e^{p-1-\alpha} \tag{72}$$

$$\le e^{\alpha-1} \cdot e^{p-1-\alpha} = e^{p-2} \ . \tag{73}$$

Therefore

$$
\begin{aligned}
&\sum_{\sigma|J}\int_{1>h_1>\cdots>h_p>0} dh_1\dots dh_p\ \prod_{q=1}^{p}\frac{1}{h_{\sigma(q)}} \\
&\quad\le e^{p-2}\times\int_0^1 ds_1\dots\int_0^1 ds_\alpha\prod_{q=1}^{\alpha}(s_1s_2\dots s_{q_1}) && (74)\\
&\quad\le \frac{e^{p-2}}{\alpha!} && (75)
\end{aligned}
$$

which proves the lemma. ■

Now the end of the proof of convergence is trivial:

$$
\begin{aligned}
\sum_{g}\mathcal{B}(g) &\le \sum_{p\ge 0}\ \sum_{J\subset\{1,\dots,p\}}\ \sum_{\sigma|J} K_8(n)K_9^p\lambda^pK_{10}^pn^{\#(J)} \\
&\qquad\times\int_{1>h_1>\cdots>h_p>0} dh_1\dots dh_p\prod_{q=1}^{p}\frac{1}{h_{\sigma(q)}} && (76)\\
&\le \sum_{p\ge 0}\ \sum_{0\le j\le p}\binom{p}{j}K_8(n)K_9^p\lambda^pK_{10}^pe^p\frac{n^j}{j!} && (77)\\
&\le K_8(n)e^n\sum_{p\ge 0}(2eK_9K_{10}\lambda)^p<+\infty && (78)
\end{aligned}
$$

for λ small enough. ■

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