LARGE DEVIATIONS AND A FLUCTUATION SYMMETRY FOR CHAOTIC HOMEOMORPHISMS

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Abstract. We consider expansive homeomorphisms with the specification property. We give a new simple proof of a large deviation principle for Gibbs measures corresponding to a regular potential and we establish a general symmetry of the rate function for the large deviations of the antisymmetric part, under time-reversal, of the potential. This generalizes the Gallavotti-Cohen fluctuation theorem to a larger class of chaotic systems.

1. Introduction

Since the beginning of statistical mechanics, there has been an ongoing exchange of ideas between the theory of heat and the theory of dynamical systems. The thermodynamic formalism has become a standard chapter for studies in dynamical systems and, ever since Clausius, heat is understood as motion.

More recently, there has been a fruitful revival of connecting the two theories. In particular, programs are running for understanding the effect of nonlinearities on transport coefficients and for defining nonequilibrium ensembles in terms of Sinai-Ruelle-Bowen (SRB) measures, [3, 13, 5]. It was also argued that the role of entropy production is, in strongly chaotic dynamical systems, played by the phase space contraction. Gallavotti and Cohen went on to prove a fluctuation symmetry for the distribution of the time-averages of the phase space contraction rate and they hypothesized that this symmetry is much more general and relevant also for the construction of nonequilibrium statistical mechanics. In [9] it was emphasized that this symmetry results from the Gibbs formalism and in the more recent [10] the general relation between entropy production and phase space contraction was investigated. It was found that phase space contraction obtains its formal analogy with the physical entropy production as source term in the potential for time-reversal breaking. For Anosov diffeomorphisms f, as were considered by Gallavotti-Cohen, this can be seen as follows: the SRB distribution is a Gibbs measure for the potential

$$\varphi(x) = -\log ||Df|_{E^u(x)}||$$

and

$$\tilde{\varphi}_1(x) = -\log ||Df|_{E^s(x)}|| = -\varphi(i \circ f(x))$$

is its time-reversal. Here $E^u(x)$ and $E^s(x)$ are the unstable and stable subspaces of the tangent space at the point x and i is the time-reversal for which f is dynamically reversible: $i \circ f = f^{-1} \circ i$. The unstable and stable subspaces are not orthogonal but for Anosov systems, the angle between these spaces is uniformly bounded away from

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0. Therefore there exists a constant $C > 0$ such that

$$\Big|\sum_{k=0}^{n-1} \sigma(f^k(x)) - \sum_{k=0}^{n-1} (\varphi + \tilde{\varphi}_1)(f^k(x))\Big| \leq C$$

for all x and $n \in \mathbb{N}$ with

$$\sigma(x) = -\log ||Df(x)||$$

the phase space contraction rate. While, from [9, 10], the most natural analogue of entropy production rate is given by the antisymmetric part of the potential under time-reversal, that is

$$\psi_0(x) = \varphi(x) - \varphi(i(x))$$

for the purpose of computing the time-average and its large deviations, for Anosov systems, no distinction can be made between $\psi_0, \psi_1 = \varphi + \tilde{\varphi}_1$ and σ. Within the set-up of Gallavotti and Cohen, the ergodic averages of ψ_0 and of ψ_1 and of σ have exactly the same large deviation rate function. Once this is recognized, it is natural to generalize the fluctuation theorem of [5] to other dynamical systems. We believe this is interesting because it takes us away from uniform hyperbolic behavior, which is not typical for real physical systems. Moreover, physically relevant dynamical systems such as billiards or systems of hard balls have singularities, i.e., they are not everywhere differentiable. Our proof however of the fluctuation symmetry for expansive homeomorphisms with the specification property relies on the corresponding thermodynamic formalism established by Ruelle and Haydn [12, 6]. These results are valid for homeomorphisms and hence do not require differentiable systems.

In sections 2 and 3 we start with some basic definitions and results. We do this in order to make the text, as much as possible, self-contained. Section 4 provides a new proof of the large deviation principle for expansive homeomorphisms with the specification property. It is based on directly checking the conditions of the Gärtner-Ellis large deviation theorem. Section 5 has the proof of the fluctuation symmetry of the time-averages of what is denoted above by ψ_1 (or, equivalently, ψ_0). We end with some further remarks.

2. Expansive Homeomorphisms with Specification

The following definitions and basic results can be found in [2, 6, 12, 7].

We will always assume (X, d) to be a compact metric space.

Definition 2.1. *A homeomorphism $f : X \to X$ is called* **expansive** *if there exists a constant $\gamma > 0$ such that if*

$$d(f^n(x), f^n(y)) < \gamma \ \textit{ for all } n \in \mathbb{Z} \quad \textit{then} \quad x = y. \tag{2.1}$$

The largest $\gamma > 0$ is called the **expansivity constant** *of f.*

Another important property is the following.

Definition 2.2. *We say that $f : X \to X$ is a homeomorphism with the specification property (abbreviated to "a homeomorphism with specification") if for each $\delta > 0$ there exists an integer $p = p(\delta)$ such that the following holds:*
if

a) *$I_1, \ldots, I_n$ are intervals of integers, $I_j \subseteq [a, b]$ for some $a, b \in \mathbb{Z}$ and all j,*

b) *dist*$(I_j, I_{j'}) \geq p(\delta)$ *for* $j \neq j'$,
then for arbitrary $x_1, \ldots, x_n \in X$ *there exists a point* $x \in X$ *such that for every* j

$$d(f^k(x), f^k(x_j)) < \delta \quad \textit{for all} \quad k \in I_j.$$

Homeomorphisms that are expansive and satisfy the specification property, form a general class of "chaotic" dynamical systems.

For example, the following is an immediate corollary of Theorem 18.3.9 in [7].

Theorem 2.3. [7, Theorem 18.3.9] *If* $f : X \to X$ *is a transitive Anosov diffeomorphism, then* f *is expansive and satisfies the specification property.*

3. Topological pressure, Regular Potentials and Gibbs Distributions

Definition 3.1. *For every* $n \in \mathbb{N}$ *and* $x, y \in X$ *define a new metric*

$$d_n(x, y) = \max_{j=0,\ldots,n-1} d(f^j(x), f^j(y)),$$

and let $\mathcal{B}_n(x, \varepsilon) = \{y \in X : \ d_n(x, y) < \varepsilon\}$ *for* $\varepsilon > 0$.

The set $E \subset X$ *is said to be* (n, ε)*-separated if for every* $x, y \in E$ *such that* $x \neq y$ *we have* $d_n(x, y) > \varepsilon$.

The set $F \subset X$ *is said to be* (n, ε)*-spanning if for every* $y \in X$ *there exist* $x \in F$ *such that* $d_n(x, y) < \varepsilon$.

For a function $\varphi : X \to \mathbb{R}$ (to be called **potential**) and $x \in X$ put

$$(S_n\varphi)(x) = \sum_{k=0}^{n-1} \varphi(f^k(x)).$$

The **topological pressure** is defined on the space $C(X)$ of all continuous functions on (X, d).

Definition 3.2. *For* $n \in \mathbb{N}$ *and* $\varepsilon > 0$ *define*

$$Z_n(\varphi, \varepsilon) = \sup_E \left\{ \sum_{x \in E} \exp\Big((S_n\varphi)(x)\Big) \right\}, \tag{3.1}$$

where the supremum is taken over all (n, ε)*-separated sets* E. *The pressure is then defined as*

$$P(\varphi) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Z_n(\varphi, \varepsilon). \tag{3.2}$$

The topological entropy of f, denoted by $h_{top}(f)$, is by definition the topological pressure of $\varphi \equiv 0$. The topological entropy of an expansive homeomorphism on a compact metric space is always finite, so is, the topological pressure of any continuous function. The topological pressure $P : C(X) \to \mathbb{R}$ is a continuous and convex function.

The following statement is known as the Variational Principle [15].

Theorem 3.3. *Denote by* $\mathcal{M}(X, f)$ *the set of all* f*-invariant Borel probability measures on* X. *Let* $\varphi \in C(X)$. *Then*

$$P(\varphi) = \sup_{\mu \in \mathcal{M}(X,f)} \left(h_\mu(f) + \int \varphi d\mu \right).$$

This result inspires the following definition.

Definition 3.4. *An element μ of $\mathcal{M}(X,f)$ is called an equilibrium measure for the potential φ if*

$$P(\varphi) = h_\mu(f) + \int \varphi \, d\mu.$$

The equilibrium measure for $\varphi \equiv 0$ (if it exists) is called a measure of maximal entropy.

We impose additional conditions on the class of potentials under consideration. As we shall see later (Theorem 3.8), the corresponding equilibrium measures will then be **Gibbs measures**.

Definition 3.5. *A continuous function φ is called* **regular** *if for every sufficiently small $\varepsilon > 0$ there exists $K = K(\varepsilon) > 0$ such that for all $n \in \mathbb{N}$*

$$d(f^k(x), f^k(y)) < \varepsilon \text{ for } k = 0, \ldots, n-1 \;\Rightarrow\; \Big|(S_n\varphi)(x) - (S_n\varphi)(y)\Big| < K.$$

The set of all regular functions is denoted by $\mathcal{V}_f(X)$.

Example. For a hyperbolic diffeomorphism f, any Hölder continuous function φ is in $\mathcal{V}_f(X)$ [7, Prop.20.2.6].

Theorem 3.6. [2, 11, 7] *If f is an expansive homeomorphism with specification and $\varphi \in \mathcal{V}_f(X)$ then there exists a unique equilibrium measure μ_φ, i.e., μ_φ is the unique element of $\mathcal{M}(X,f)$ such that*

$$P(\varphi) = h_{\mu_\varphi}(f) + \int \varphi d\mu_\varphi.$$

Moreover, μ_φ is ergodic, positive on open sets and mixing.

This equilibrium measure μ_φ can be constructed from the measures concentrated on periodic points in the following way. For every $n \geq 1$ define a probability measure $\mu_{\varphi,n}$ supported on the set of periodic points $\mathrm{Fix}(f^n) = \{x \in X : \; f^n(x) = x\}$ as follows

$$\mu_{\varphi,n} = \frac{1}{Z(f,\varphi,n)} \sum_{x \in \mathrm{Fix}(f^n)} e^{(S_n\varphi)(x)} \delta_x, \tag{3.3}$$

where δ_x is a Dirac measure at x and $Z(f,\varphi,n) = \sum_{x \in \mathrm{Fix}(f^n)} e^{(S_n\varphi)(x)}$ is a normalizing constant.

Theorem 3.7. [2, 7] *The equilibrium measure μ_φ is a weak* limit of the sequence $\{\mu_{\varphi,n}\}$, i.e., for every $h \in C(X)$*

$$\int h(x) d\mu_{\varphi,n} \to \int h(x) d\mu_\varphi \quad \text{as} \quad n \to \infty.$$

The next result gives a "local" (i.e., Gibbs) description of equilibrium measures for regular potentials, see [6] for a detailed discussion.

Theorem 3.8. [6, Proposition 2.1],[7, Theorem 20.3.4] *Let f be an expansive homeomorphism with the specification property. Let $\varphi \in \mathcal{V}_f(X)$ and denote its unique equilibrium measure by μ_φ. Then, for sufficiently small $\varepsilon > 0$, there exist constants $A_\varepsilon, B_\varepsilon > 0$ such that for all $x \in X$ and $n \geq 1$*

$$A_\varepsilon \leq \frac{\mu_\varphi\big(\{y \in X :\ d(f^k(x), f^k(y)) < \varepsilon \text{ for } k = 0, \ldots, n-1\}\big)}{\exp\left(-nP(\varphi) + (S_n\varphi)(x)\right)} \leq B_\varepsilon. \tag{3.4}$$

We have seen that for every $\varphi \in \mathcal{V}_f(X)$ there exists a unique equilibrium measure. Using (3.3) and (3.4) we are able to give necessary and sufficient conditions on potentials $\varphi_1, \varphi_2 \in \mathcal{V}_f(X)$ to have the same equilibrium measures $\mu_1 = \mu_{\varphi_1} = \mu_{\varphi_2} = \mu_2$.

Theorem 3.9. *Let f be an expansive homeomorphism with specification. The equilibrium measures μ_1 and μ_2 corresponding to the potentials $\varphi_1, \varphi_2 \in \mathcal{V}_f(X)$ coincide if and only if there exists a constant $c \in \mathbb{R}$ such that*

$$(S_n\varphi_1)(x) = (S_n\varphi_2)(x) + nc \tag{3.5}$$

for all $x \in Fix(f^n)$ and all n.

Proof. If (3.5) holds for all $x \in \mathrm{Fix}(f^n)$ and n, then, by (3.3), one has $\mu_{1,n} = \mu_{2,n}$ for all n. Thus $\mu_1 = \mu_2$.

Suppose that μ_1 and μ_2 coincide, and let $\mu = \mu_1 = \mu_2$. Consider "adjusted" potentials $\widehat{\varphi}_1 = \varphi_1 - P(\varphi_1)$ and $\widehat{\varphi}_2 = \varphi_2 - P(\varphi_2)$. Let $x \in \mathrm{Fix}(f^n)$ for some $n \in \mathbb{N}$. Applying (3.4) for sufficiently small $\varepsilon > 0$ we conclude that

$$A^1_\varepsilon \exp\big(\,(S_n\widehat{\varphi}_1)(x)\,\big) \leq \mu(\mathcal{B}_n(x,\varepsilon)) \leq B^2_\varepsilon \exp\big(\,(S_n\widehat{\varphi}_2)(x)\,\big).$$

This implies that $(S_n\widehat{\varphi}_1)(x) \leq (S_n\widehat{\varphi}_2)(x) + C'$ for some constant C' independent of x and n. Since $x \in \mathrm{Fix}(f^{kn})$ for all $k \in \mathbb{N}$ we have that

$$(S_n\widehat{\varphi}_1)(x) = \lim_{k\to\infty} \frac{(S_{kn}\widehat{\varphi}_1)(x)}{k} \leq \lim_{k\to\infty} \frac{(S_{kn}\widehat{\varphi}_2)(x)}{k} = (S_n\widehat{\varphi}_2)(x).$$

By symmetry we obtain the opposite inequality. Hence

$$(S_n\widehat{\varphi}_1)(x) = (S_n\widehat{\varphi}_2)(x)$$

for all $x \in \mathrm{Fix}(f^n)$ and $n \in \mathbb{N}$. This implies (3.5) with $c = P(\varphi_1) - P(\varphi_2)$. □

We now recall some properties of the pressure for expansive homeomorphisms with specification. These facts will be used later in the proof of the main results.

Lemma 3.10. *Suppose $f : X \to X$ is an expansive homeomorphism with specification. Let $\varphi, \psi \in \mathcal{V}_f(X)$. Then the function $P(\varphi + q\psi)$, $q \in \mathbb{R}$, is continuously differentiable with respect to q and its derivative is given by*

$$\frac{dP(\varphi + q\psi)}{dq} = \int \psi d\mu_q,$$

where μ_q is the equilibrium measure corresponding to the potential $\varphi + q\psi$.

Moreover, $P(\varphi + q\psi)$ is a strictly convex function of q provided the equilibrium measure for the potential ψ is not the measure of maximal entropy.

When the equilibrium measure for the potential ψ is the measure of maximal entropy one has

$$P(\varphi + q\psi) = P(\varphi) + q\int \psi d\mu_\varphi$$

for all $q \in \mathbb{R}$, where μ_φ is the equilibrium measure for φ.

The proof of this lemma is almost identical to the proof of Lemma 4.1 in [14], and relies on the results of Walters [16], who showed that for expansive dynamical systems differentiability of the pressure function $P(\cdot)$ at φ is equivalent to the uniqueness of equilibrium measures for φ. Since the specification condition together with the regularity condition on φ imply uniqueness of equilibrium measures, we obtain the desired differentiability of the pressure function.

In order to prove the Large Deviations result for expansive homeomorphism with specification, we will have to use some results on the convergence

$$\frac{1}{n}\log Z_n(\varphi,\varepsilon) \to P(\varphi)$$

in (3.2).

Definition 3.11. *We say that E is a maximal (n,ε)-separated set if it cannot be enlarged by adding new points and preserving the separation condition.*

Thus every maximal (n,ε)-separated set E is (n,ε)-spanning as well.

The following estimates from [6] will be used later.

Lemma 3.12. *Let f be an expansive homeomorphisms and $\gamma > 0$ be its expansivity constant. Let $\varphi \in \mathcal{V}_f(X)$. For every finite set E put*

$$Z_n(\varphi,E) = \sum_{x\in E} \exp\Big((S_n\varphi)(x)\Big).$$

(1) *If $\varepsilon, \varepsilon' < \gamma/2$ and E, E' are the maximal (n,ε)- and (n,ε')-separated sets respectively, then one has*

$$Z_n(\varphi,E) \le C Z_n(\varphi,E'),$$

where the constant $C = C(\varepsilon,\varepsilon')$ is independent of n. In particular,

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n}\log Z_n(\varphi,E_n), \tag{3.6}$$

where E_n are the arbitrary maximal (n,ε)-separated sets.

(2) *If f satisfies the specification property and $\varepsilon < \gamma/2$ then there exists a constant $D = D(\varphi,\varepsilon) > 0$ such that*

$$|\log Z_n(\varphi,E_n) - nP(\varphi)| < D \tag{3.7}$$

for all n and all maximal (n,ε)-separated sets.

(3) *Suppose f is expansive and satisfies the specification property, then*

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n}\log Z_n(\varphi,\mathrm{Fix}(f^n)) \tag{3.8}$$

4. LARGE DEVIATIONS

In this section we establish the Large Deviation Principle for expansive homeomorphisms with specification and Gibbs measures corresponding to regular potentials. In fact, one can deduce this from more general results of Young [17] or Kifer [8]. However, for our class of dynamical systems one can easily check the conditions of the Gärtner–Ellis theorem. For the sake of completeness and to stand on it in the next section, we provide the details here.

Suppose, $f : X \to X$ is an expansive homeomorphism with specification, and φ, ψ are regular functions, i.e., $\varphi, \psi \in \mathcal{V}_f(X)$.

Let $\mu = \mu_\varphi$ be an equilibrium measure for the potential φ. We will study the distribution of ergodic averages of ψ with respect to μ_φ. Namely, we will establish that the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\varphi\Big(\Big\{x \in X : \ \frac{1}{n}\sum_{k=0}^{n-1} \psi(f^k(x)) \in A\Big\}\Big)$$

exists for the appropriate intervals $A \subset \mathbb{R}$.

Our first step will be the study of the so-called free energy function. For every $q \in \mathbb{R}$ and $n \in \mathbb{N}$ define

$$c_n(q) = \frac{1}{n} \log \int \exp\Big(q(S_n\psi)(x)\Big) d\mu_\varphi.$$

We have to prove that $c_n(q)$ converges for every q, and that the limiting function $c(q)$ is finite and sufficiently smooth in q. This is done in the next lemma.

Lemma 4.1. *For every $q \in \mathbb{R}$ the following limit exists*

$$c(q) = \lim_{n\to\infty} c_n(q).$$

The limit $c(q)$ is also given by

$$c(q) = P(\varphi + q\psi) - P(\varphi), \tag{4.1}$$

where $P(\cdot)$ is the topological pressure.

The free energy $c(q)$ is finite, differentiable and convex for every q. It is strictly convex provided the equilibrium measure for φ is not the measure of maximal entropy.

Proof. We start by giving estimates for $c_n(q)$, which will lead to (4.1). All other properties of the free energy $c(q)$ will follow from the corresponding properties of the topological pressure.

Let $\varepsilon > 0$ be sufficiently small. Let $E_n = \{x_i\}$ be any maximal (n, ε)-separated set. Since E_n is a maximal separated set, for every $x \in X$ there exists $x_i \in E_n$ such that $x \in \mathcal{B}_n(x_i, \varepsilon)$. Since ψ is a regular function, for $x \in \mathcal{B}_n(x_i, \varepsilon)$ (i.e., $d_n(x, x_i) < \varepsilon$) one has

$$\Big|(S_n\psi)(x) - (S_n\psi)(x_i)\Big| \le K(\psi, \varepsilon)$$

for some constant $K(\psi,\varepsilon)$. Therefore,

$$\begin{aligned}
\exp(nc_n(q)) &= \int \exp\big(q(S_n\psi)(x)\big)\,d\mu_\varphi \\
&\le \sum_{x_i\in E_n}\int_{\mathcal{B}_n(x_i,\varepsilon)} \exp\big(q(S_n\psi)(x)\big)\,d\mu_\varphi \\
&\le \sum_{x_i\in E_n} \exp\big(|q|K(\psi,\varepsilon)+q(S_n\psi)(x_i)\big)\mu_\varphi(\mathcal{B}_n(x_i,\varepsilon)) \\
&\le C\exp(-nP(\varphi))\sum_{x_i\in E_n}\exp\big(q(S_n\psi)(x_i)+(S_n\varphi)(x_i)\big) \\
&= C\exp(-nP(\varphi))Z_n(\varphi+q\psi,E_n),
\end{aligned}$$

where $C=B_\varepsilon\exp(|q|K(\psi,\varepsilon))$, and where we have used an upper estimate from (3.4) on the measures of balls $\mathcal{B}_n(x_i,\varepsilon)$.

To prove a similar lower estimate we use that for two different points x_i,x_j from an (n,ε)-separated set E_n the intersection $\mathcal{B}_n(x_i,\varepsilon/2)\cap\mathcal{B}_n(x_j,\varepsilon/2)$ is empty. Hence,

$$\begin{aligned}
\exp(nc_n(q)) &= \int \exp\big(q(S_n\psi)(x)\big)\,d\mu_\varphi \\
&\ge \sum_{x_i\in E_n}\int_{\mathcal{B}_n(x_i,\varepsilon/2)} \exp\big(q(S_n\psi)(x)\big)\,d\mu_\varphi \\
&\ge \sum_{x_i\in E_n} \exp\big(-|q|K(\psi,\varepsilon)+q(S_n\psi)(x_i)\big)\mu_\varphi(\mathcal{B}_n(x_i,\varepsilon/2)) \\
&\ge C'\exp(-nP(\varphi))\sum_{x_i\in E_n}\exp\big(q(S_n\psi)(x_i)+(S_n\varphi)(x_i)\big) \\
&= C'\exp(-nP(\varphi))Z_n(\varphi+q\psi,E_n),
\end{aligned}$$

where $C'=A_{\varepsilon/2}\exp(-|q|K(\psi,\varepsilon))$, and we have used the lower estimate from (3.4). Combining together the upper and the lower estimates on $c_n(q)$ we obtain

$$\frac{1}{n}\log Z_n(\varphi+q\psi,E_n)-P(\varphi)+\frac{C_*}{n}\le c_n(q)\le \frac{1}{n}\log Z_n(\varphi+q\psi,E_n)-P(\varphi)+\frac{C^*}{n}$$

for some constants C_*,C^*. Since for a sufficiently small $\varepsilon>0$ the limit

$$\lim_{n\to\infty}\frac{1}{n}\log Z_n(E_n,\varphi+q\psi)$$

exists (Lemma 3.12) and is equal to $P(\varphi+q\psi)$ we obtain the first part of the statement.

The properties of the pressure function $P(\psi+q\varphi)$ are given by Lemma 3.10. This finishes the proof. □

The rate function $I(p)$ is obtained from the free energy $c(q)$ by a Legendre transform: for $p\in\mathbb{R}$ put

$$I(p)=\sup_q(qp-c(q)).$$

Since $c(q)$ is differentiable, we can introduce the following quantities

$$\overline{p}=\sup_q c'(q)=\lim_{q\to+\infty}c'(q),\quad \underline{p}=\inf_q c'(q)=\lim_{q\to-\infty}c'(q). \tag{4.2}$$

Existence of the limits follows from the convexity of the free energy $c(q)$. Standard arguments of convex analysis show that

$$I(p) \text{ is finite for } p \in (\underline{p}, \overline{p})$$

and $I(p) = +\infty$ for $p \notin [\underline{p}, \overline{p}]$. Moreover, since $c(q)$ is smooth and convex, $I(p)$ is also a smooth and convex function of p on $(\underline{p}, \overline{p})$.

Now all the conditions of the Gärtner-Ellis theorem [4] are satisfied and we obtain a Large Deviations result for expansive homeomorphisms with specification.

Theorem 4.2 (Large deviations). *Let $f : X \to X$ be an expansive homeomorphism with specification. Let $\varphi, \psi \in \mathcal{V}_f(X)$, and let μ_φ be the Gibbs measure for φ. Assume that μ_ψ is not the measure of maximal entropy. Then there exists a smooth real convex function I on the open interval $(\underline{p}, \overline{p})$ such that, for every interval J with $J \cap (\underline{p}, \overline{p}) \neq \emptyset$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\varphi\Big(\Big\{x : \frac{1}{n}(S_n\psi) \in J\Big\}\Big) = -\inf_{p\in J\cap(\underline{p},\overline{p})} I(p).$$

5. FLUCTUATION SYMMETRY

Choose some regular potential $\varphi \in \mathcal{V}_f(X)$ and let μ_φ be the corresponding Gibbs measure. As always, $f : X \to X$ is an expansive homeomorphism with specification on a compact metric space (X, d).

We make two assumptions:

A) Reversibility. There exist a homeomorphism $i : X \to X$, preserving the metric d such that

$$i \circ f \circ i = f^{-1} \quad \text{and} \quad i^2 = \textit{identity}.$$

Fix an integer k and define $\tilde{\varphi}_k$ and ψ_k as follows

$$\tilde{\varphi}_k(x) = -\varphi(i \circ f^k(x)), \quad \psi_k(x) = \varphi(x) + \tilde{\varphi}_k(x).$$

From the point of statistical mechanics, the most natural choice is for $k = 0$. Yet, to connect with phase space contraction for Anosov diffeomorphisms it is natural to take $k = 1$. While the proofs remain valid and unchanged for all choices of k, we choose to present the rest for $k = 1$ and we simply write $\tilde{\varphi} = \tilde{\varphi}_1, \psi = \psi_1$. Note that $\tilde{\varphi}$ and ψ are also regular potentials.

B) Dissipativity. Assume that the equilibrium measure for $\psi \in \mathcal{V}_f(X)$ is not the measure of maximal entropy.

The assumption B can be viewed as a generalization of the corresponding dissipation condition of [5] or [13]: If the equilibrium measure for $\psi = \varphi + \tilde{\varphi}$ is not the measure of maximal entropy, then

$$\int \psi \, d\mu_\varphi > 0$$

which expresses the breaking of time-reversal symmetry. In fact these conditions are equivalent as we will show in Theorem 5.2 below. Assumption B is quite natural because only under this assumption can one talk about a non-trivial fluctuation symmetry.

To understand the role of assumption A, it is instructive to make the following calculation. Recall the definition of the approximants $\mu_{\varphi,n}$ of (3.3), see Theorem 3.7. Denote by

$$\mathbb{E}_n(g) = \frac{1}{Z(f,\varphi,n)} \sum_{x\in \mathrm{Fix}(f^n)} e^{(S_n\varphi)(x)} g(x)$$

the expectation of a function g with respect to $\mu_{\varphi,n}$. Let $g(x) = G(x, f(x), \ldots, f^{n-1}(x))$ for some G on X^n and define $g^\star(x) = G(i\circ f^{n-1}(x), \ldots, i\circ f(x), i(x)) = g(i\circ f^{n-1}(x))$. Then, by a change of variables that leaves the set $\mathrm{Fix}(f^n)$ globally invariant (under assumption A),

$$\mathbb{E}_n(g^\star) = \frac{1}{Z(f,\varphi,n)} \sum_{x\in \mathrm{Fix}(f^n)} e^{(S_n\varphi)(f^{1-n}\circ i(x))} g(x)$$

Again by reversibility and by the definition of $\tilde{\varphi}$,

$$\sum_{k=0}^{n-1} \varphi(f^{k+1-n}\circ i(x)) = -\sum_{k=0}^{n-1} \tilde{\varphi}(f^{n-k-2}(x))$$

so that for $x \in \mathrm{Fix}(f^n)$, $S_n\varphi(f^{1-n}\circ i(x)) - S_n\varphi(x) = -S_n\psi(x)$. We conclude that

$$\mathbb{E}_n(g^\star) = \mathbb{E}_n(ge^{-S_n\psi})$$

or, $S_n\psi$ can be viewed as the logarithmic ratio of the probability of a trajectory and the probability of the corresponding time-reversed trajectory, see [9, 10]. These basic identities drive the following

Theorem 5.1 (Fluctuation Symmetry). *Assume A)-B). There exists $p^* > 0$ such that, if $|p| < p^*$, then*

$$\lim_{\delta\to 0}\lim_{n\to\infty} \frac{1}{n} \log \frac{\mu_\varphi(\{x : \sigma_n(x) \in (p-\delta, p+\delta)\})}{\mu_\varphi(\{x : \sigma_n(x) \in (-p-\delta, -p+\delta)\})} = p,$$

where

$$\sigma_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} \psi\big(f^k(x)\big).$$

Proof. The fluctuation symmetry can be expressed in terms of the rate function I of Theorem 4.2; we must show it has the symmetry

$$I(-p) - I(p) = p \quad \text{for} \quad |p| < p^*. \tag{5.1}$$

Since $I(p)$ is the Legendre transform of $c(q) = P(\varphi + q\psi) - P(\varphi)$, the symmetry (5.1) must be reflected in a certain symmetry of the free energy $c(q)$. We claim that

$$c(q) = c(-1-q) \quad \text{for all} \quad q \in \mathbb{R}. \tag{5.2}$$

Assume for a moment that (5.2) is true. Then from (4.2) we conclude that

$$\underline{p} = -\overline{p},$$

Since under the assumptions of the theorem $c(q)$ is not identically equal to a constant, we obtain $\underline{p} \neq \overline{p}$. Hence, the domain of the rate function $I(p)$ is a symmetric interval containing zero. Let $p^* = \overline{p} = -\underline{p}$. For every p in $(-p^*, p^*)$, $I(p)$ is finite and satisfies (5.1):

$$\begin{aligned} I(-p) &= \sup_{q\in\mathbb{R}}\Big((-p)q - c(q)\Big) = \sup_{q\in\mathbb{R}}\Big(p(-q) - c(q)\Big) \\ &= \sup_{q\in\mathbb{R}}\Big(pq - c(-q)\Big) = \sup_{q\in\mathbb{R}}\Big(pq - c(-1+q)\Big) \qquad\qquad \text{(by (5.2))} \\ &= \sup_{q\in\mathbb{R}}\Big(p(-1+q) - c(-1+q)\Big) + p = \sup_{q\in\mathbb{R}}\Big(pq - c(q)\Big) + p \\ &= I(p) + p. \end{aligned}$$

Now let us prove (5.2), or, what amounts to the same thing (see Lemma 4.1),

$$P\big((q+1)\varphi + q\tilde{\varphi}\big) = P\big(-q\varphi - (1+q)\tilde{\varphi}\big), \quad \forall q \in \mathbb{R}, \tag{5.3}$$

where the topological pressure $P(\cdot)$ is obtained from (3.8).

The 'time reversing' homeomorphism i maps the set $\mathrm{Fix}(f^n)$ to itself. For $x \in \mathrm{Fix}(f^n)$ one has

$$\begin{aligned} \sum_{k=0}^{n-1}(1+q)\varphi(f^k(x)) + q\tilde{\varphi}(f^k(x)) &= \sum_{k=0}^{n-1} -(1+q)\tilde{\varphi}(i\circ f^{k+1}(x)) - q\varphi(i\circ f^{k+1}(x)) \\ &= \sum_{k=0}^{n-1} -(1+q)\tilde{\varphi}(f^n\circ i\circ f^{k+1}(x)) - q\varphi(f^n\circ i\circ f^{k+1}(x)) \\ &= \sum_{k=0}^{n-1} -(1+q)\tilde{\varphi}(f^{n-k-1}\circ i(x)) - q\varphi(f^{n-k-1}\circ i(x)) \\ &= \sum_{m=0}^{n-1} -(1+q)\tilde{\varphi}(f^m\circ i(x)) - q\varphi(f^m\circ i(x)). \end{aligned}$$

Now, taking into account that i is a bijection on $\mathrm{Fix}(f^n)$, we obtain that

$$\sum_{x\in\mathrm{Fix}(f^n)} \exp\big(\sum_{k=0}^{n-1}\varphi(f^k(x)) + q\psi(f^k(x))\big) = \sum_{x\in\mathrm{Fix}(f^n)} \exp\big(\sum_{k=0}^{n-1}\varphi(f^k(x)) - (1+q)\psi(f^k(x))\big),$$

implying $c(q) = c(-1-q)$ and finishing the proof. $\square$

Theorem 5.2 (Dissipativity conditions). *Let f, i, φ, and $\tilde{\varphi}$ be as above. Then the following conditions are equivalent:*

1) *the equilibrium measure for $\psi = \varphi + \tilde{\varphi}$ is not the measure of maximal entropy;*
2) *the equilibrium measure μ_φ for φ is not the equilibrium measure for $-\tilde{\varphi}$;*
3) $\displaystyle\int \psi \, d\mu_\varphi > 0$.

Proof. We first show the equivalence of conditions 1) and 2). According to Theorem 3.9, the equilibrium measure for ψ is the measure of maximal entropy if and only if there exists a constant c_1 such that

$$\sum_{k=0}^{n-1}\psi(f^k(x)) = nc_1 \tag{5.4}$$

for all $n \in \mathbb{N}$ and every $x \in \mathrm{Fix}(f^n)$. Similarly, φ and $-\tilde{\varphi}$ have the same equilibrium measure if and only if for some constant c_2 one has

$$\sum_{k=0}^{n-1} \varphi(f^k(x)) = -\sum_{k=0}^{n-1} \tilde{\varphi}(f^k(x)) + nc_2, \tag{5.5}$$

again for all $n \in \mathbb{N}$ and every $x \in \mathrm{Fix}(f^n)$. Clearly, since $\psi = \varphi + \tilde{\varphi}$, (5.4) and (5.5) are equivalent.

To show that the second and the third condition are equivalent, first recall that in the proof of Theorem 5.1 we have established (5.3), and in particular, the following equality

$$P(\varphi) = P(-\tilde{\varphi}). \tag{5.6}$$

Now, since μ_φ is an equilibrium measure for φ, one has

$$P(\varphi) = h_{\mu_\varphi}(f) + \int \varphi d\mu_\varphi.$$

On the other hand, applying the Variational Principle to $-\tilde{\varphi}$, we conclude that

$$P(-\tilde{\varphi}) \geq h_{\mu_\varphi}(f) - \int \tilde{\varphi} d\mu_\varphi,$$

with equality if and only if μ_φ is the equilibrium measure for $-\tilde{\varphi}$.

Therefore

$$\int \psi \, d\mu_\varphi \geq P(\varphi) - P(-\tilde{\varphi}) = 0.$$

with equality if and only if μ_φ is the equilibrium measure for $-\tilde{\varphi}$. This finishes the proof. □

6. Concluding remarks.

1) As mentioned already in the introduction, the Fluctuation Theorem of Gallavotti and Cohen says that the large deviation rate function $I(p)$ for the time-averages of the phase space contraction in the SRB measure for dissipative and reversible Anosov diffeomorphism has the symmetry:

$$I(-p) - I(p) = p.$$

Our results are valid under a greater generality: not only for SRB measures, but also for the Gibbs measures for expansive homeomorphisms with the specification property. Phase space contraction then gets replaced by the antisymmetric part of the potential under time-reversal which is essentially the same as the phase space contraction rate for Anosov systems. The original proof of Gallavotti and Cohen used Markov partitions (symbolic dynamics). Clearly, this is not an option for us. On the other hand, Ruelle in [13] gave a proof, again for Anosov systems, based on shadowing and one can check that the argument from [13] goes through without many substantial modifications in the case of expansive homeomorphisms with specification. Our approach is still different and even simpler: it is different, physically, because we concentrate on the antisymmetric part of the potential under time-reversal and mathematically, we obtain a symmetry of the rate function $I(p)$ directly from the properties of the topological pressure. It is important that in this generalization, we can also treat continuous (therefore, not necessarily differentiable) transformations.

2) Transitive Anosov systems are expansive and do satisfy the specification property. One can find examples of smooth expansive dynamical systems with the specification property, which are not Anosov, see e.g. [1]. Unfortunately, we were not able to find any interesting reversible examples. However, this is quite a typical situation in the field: there are also not very many examples of reversible dissipative Anosov systems. Nevertheless, we think that the validity of the fluctuation symmetry for a larger class of dynamical systems is a step forward. The main reason was already mentioned: uniform hyperbolic behavior and everywhere differentiability is not typical for real physical systems. Secondly, the definition of regular potentials can be extended to include discontinuous functions which satisfy the key property

$$d(f^k(x), f^k(y)) < \varepsilon \text{ for } k = 0, \ldots, n-1 \Rightarrow \left|\sum_{k=0}^{n-1} \varphi(f^k x) - \sum_{k=0}^{n-1} \varphi(f^k y)\right| < K.$$

It is important to understand whether hyperbolic systems with singularities, such as hard ball systems or billiards, satisfy this condition for natural potentials. If this is indeed the case, then one immediately obtains the fluctuation symmetry for such systems as well.

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