OPTIMALLY DENSE PACKINGS OF HYPERBOLIC SPACE

by

Lewis Bowen * and Charles Radin **

Mathematics Department, University of Texas at Austin

Abstract

In previous work a probabilistic approach to controlling difficulties of density in hyperbolic space led to a workable notion of optimal density for packings of bodies. In this paper we extend an ergodic theorem of Nevo to provide an appropriate notion of optimally dense packings. Examples are given to illustrate various aspects of the density problem.

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0. Introduction

One of the main themes in discrete geometry has always been the study of optimally dense packings of bodies, in regions of finite and, especially, infinite volume – the latter being the situation with which we will be mainly concerned. There is a large literature for packing in Euclidean spaces $\mathbb{E}^n$, but much less for packing in hyperbolic spaces $\mathbb{H}^n$; see [GrW] and references therein, as well as the classics [Fe5, Rog]. The difficulties in hyperbolic spaces have been documented in many papers dating back at least to the early 50's (see [Fe1-5], [Bo1-2], [BoF], [FeK], [FKK], [Kup]), and are well understood to be related to the phenomenon of the exponential rate of growth of the volume of a sphere with respect to radius.

In general it is difficult, even in $\mathbb{E}^n$, to actually determine the optimal packing density even for simple shapes; for instance this is unknown even for spheres for dimension $n \geq 4$. For certain polyhedra such optima are occasionally computable, and one outgrowth of this, since the late 60's, has been the subject sometimes referred to as "aperiodic tiling"; see [Rad] and references therein. That subject is concerned with the geometric symmetries of those packings (tilings) which achieve optimal density, symmetries of unusual form, which are studied through probabilistic techniques.

In hyperbolic space the difficulties in analyzing optimal density are, as hinted above, much worse; not only is it hard to determine the optimal density, say for spheres, it has even been hard to decide whether this is a meaningful question.

Finding a meaningful, and computationally useful, way to analyze optimal density in $\mathbb{H}^n$ was the subject of [BoR, Bow], in which some of the probabilistic methods of aperiodic tiling were employed. The papers [BoR, Bow] did not, however, convincingly address the question of optimally dense packings themselves; that is, a formalism for analyzing the density was found, but not a way to address the packings that achieve that optimal density. That is one of the main aims of this paper.

The other major objective of this paper is to present a series of examples, some new and some which are variants of historically important examples, adjusted to reveal significant connections. Although one source of difficulty in the subject is well understood, the rate of growth of volume with diameter, these examples point to a very different source, related to structures in the space of packings.

I. Some troublesome packings

We will be analyzing the density of certain subsets of Euclidean n-dimensional space $\mathbb{E}^n$ or hyperbolic n-dimensional space $\mathbb{H}^n$ of curvature -1; we let $\mathbb{S}$ stand for any of these spaces. The subsets of $\mathbb{S}$ whose density we will consider will generally be "packings" of (infinitely many) "bodies" β_j, where a packing is a collection of bodies with pairwise disjoint interiors, and a body is a compact, connected set which is the closure of its interior. One of the features of our analysis will be an emphasis on distinguishing between the density of the packing versus the density of the set which is the union of the bodies in the packing; that is, it will be significant to maintain the individuality of each of those bodies.

Our main focus will be on the "densest" packings possible by the given bodies, and this requires examination of the primitive notion of density. If we were packing a region S of *finite* volume by the bodies β_j, the density of such a packing would be unambiguous – the fraction of the volume of S covered by the bodies – but density must be defined more subtly for packings of a region, such as $\mathbb{S}$, of infinite volume. The most widely accepted [FeK] primitive notion is that the density of a packing $\mathcal{P}$ of $\mathbb{S}$ should be obtainable by choosing a family of finite volume regions S_k, with $S_k \subset S_{k+1}$ and $\cup_k S_k = \mathbb{S}$, and the density of $\mathcal{P}$ should be

$$\lim_{k\to\infty} \frac{vol(\mathcal{P} \cap S_k)}{vol(S_k)}, \tag{1}$$

where $vol(\cdot)$ denotes volume in $\mathbb{S}$ and $\mathcal{P} \cap S_k$ denotes the portion of S_k covered by bodies in $\mathcal{P}$. We would want the density to be reasonably independent of the family S_k.

It is worth noting that the limit in (1) can easily fail to exist. Consider the sequence $\{D_j \mid j \geq 1\}$ of closed disks in $\mathbb{E}^2$, D_j of radius 2^j and centered at the origin. Let P_j be the annulus D_j/D_{j-1} between successive disks, and let S be the union of those P_j with $j \geq 2$ even. If we try to define the density of S using the expanding regions $S_k = D_k$, the sequence of local or approximate densities $vol(S \cap S_k)/vol(S_k)$ would not have a limit as $k \to \infty$, due to oscillation. (We could get the same qualitative result by replacing our region S by its intersection with some simple packing of disks, such as the packing of unit diameter disks whose centers have integer coordinates.)

Even though there are packings without a well defined density there is no real difficulty in defining *optimal* density of packings in Euclidean space. In fact we now show how to construct densest packings of Euclidean space. Let $\mathbb{S} = \mathbb{E}^n$ and let S_k be a cube centered at the origin, with edges of length k aligned with the axes. For any $k > 0$, let $\mathcal{P}_k$ be a packing by (congruent copies of the bodies in) $\mathcal{B} \equiv \{\beta_j\}$ such that all bodies in $\mathcal{P}_k$ intersect S_k and $vol(\mathcal{P}_k \cap S_k)$ is optimally large. (Such a packing is easily shown to exist by a simple compactness argument [GrS; p. 154].) For any packing $\mathcal{P}$ in $\mathbb{S}$ we define

$$d_k(\mathcal{P}) = \frac{vol(S_k \cap \mathcal{P})}{vol(S_k)} \tag{2}$$

$$d_k = \max_{\mathcal{P}} d_k(\mathcal{P}) \tag{3}$$

$$d = \limsup_{k\to\infty} d_k. \tag{4}$$

($\lim_{k\to\infty} d_k$ exists but we do not need this fact.)

At this point it is convenient to have a space $\tilde{\Sigma}_{\mathcal{B}}$ of all possible packings of $\mathbb{S}$ by the bodies β_j, equipped with a metric topology such that a sequence of packings converges if and only if it converges uniformly on compact subsets of $\mathbb{S}$. We will spell this out in section II b, but assume for now such a space makes sense and is in fact compact. Then we let $\mathcal{P}_\infty$ be an accumulation point of $\{\mathcal{P}_k\}$.

The following is a simple observation.

Lemma 1. $d_k(g\mathcal{P}_\infty) \to d$ as $k \to \infty$ for every fixed rigid motion g.

Proof. The main estimates needed are the simple facts, for $k' > k$:

$$vol(\mathcal{P}_k \cap S_k) \geq vol(\mathcal{P}_{k'} \cap S_k) \tag{5}$$

$$vol(\mathcal{P}_{k'} \cap S_k) \geq vol(\mathcal{P}_k \cap S_k) - [k^n - (k - C)^n], \tag{6}$$

where C is larger than the diameter of any body in $\mathcal{B}$. The latter holds because if it did not one could arrive at a contradiction by altering $\mathcal{P}_{k'}$ as follows. First replace the bodies of $\mathcal{P}_{k'}$ that are completely contained in S_k by the bodies of $\mathcal{P}_k$ that do not overlap the other bodies of $P_{k'}$ (i.e. that do not overlap any body of $P_{k'}$ that overlaps the complement of S_k). Note that the volume of bodies of P_k that we have introduced is at least as large as the right hand side of (6). Since $vol(\mathcal{P}_{k'})$ is as large as possible, this operation could not have increased its volume. This proves equation 6. Since $\mathcal{P}_\infty$ is a limit of $\mathcal{P}_k$ we get that (6) holds if $\mathcal{P}_{k'}$ is replaced by $\mathcal{P}_\infty$.

Finally, if k_m is a sequence such that $d_{k_m} \to d$ as $m \to \infty$:

$$\begin{aligned} |d - d_{k_m}(g\mathcal{P}_\infty)| &\leq |d_{k_m} - d| + |d_{k_m} - d_{k_m}(g\mathcal{P}_\infty)| \\ &= |d_{k_m} - d| + |d_{k_m}(\mathcal{P}_{k_m}) - d_{k_m}(g\mathcal{P}_\infty)| \end{aligned} \tag{7}$$

and $|d_{k_m}(\mathcal{P}_{k_m}) - d_{k_m}(g\mathcal{P}_\infty)| \to 0$ as $m \to \infty$ from (6). ■

Thus, in Euclidean space optimally dense packings $\mathcal{P}$ exist for any collection $\mathcal{B}$ in the sense that their density defined by (1) exists, is independent of the center of expansion of (1), and is as large as that for any packing.

As we shall see, the above technique does not extend to $\mathbb{S} = \mathbb{H}^n$ and therefore some other method must be used to define optimal density in $\mathbb{H}^n$. Before exhibiting such a method, we present some examples to highlight some differences between hyperbolic and Euclidean packings.

Example 1 (half-space).

Consider the half space region S, defined, in the upper half plane model of the hyperbolic plane, as the set of points (x, y) with $x \geq 0$. If we try to define the density of this region by circles all expanding about a common center c, it is easy to see that the density would depend on c, with any value strictly between 0 and 1 being obtainable for appropriate c. This suggests that we will want the origin, used for the expanding regions in (1), to be arbitrary.

Example 2 (stripe model).

We now give a simple example of a region S in the hyperbolic plane such that, when we try to define the density of S relative to a sequence of circles expanding about some point, we get the kind of oscillation we found in the Euclidean annulus example. We define the "stripe model" in the (upper half plane model of the) hyperbolic plane, where the stripes are the regions separated by the horocycles h_j, $j \in \mathbb{Z}$, defined by $y = y_j \equiv e^{(j+1/2)W}$, where fixed $W >> 1$ is to be specified. These curves are equidistant by W in the hyperbolic metric. We call those stripes separated by h_{2j} and h_{2j+1} "black", and the others white, and we declare the region S of interest to be the union of the black stripes.

Consider the circle with hyperbolic center $c = (0, 1)$ and hyperbolic radius $R = (N + 1/2)W$, where $N >> 1$ is to be specified. We will use the following relations between the hyperbolic center (H, K) and hyperbolic radius R of a given circle and its Euclidean center (h, k) and Euclidean radius r:

$$h = H, \quad k^2 - r^2 = K^2, \quad r = k \tanh(R). \tag{8}$$

So our circle has Euclidean center $(0, \cosh[R])$ and Euclidean radius $\sinh(R)$.

We will show that, if N is even, the area inside the circle, of the black stripes is larger than that of the white stripes; in particular, each black stripe, between h_j and h_{j+1}, $j \le N - 3$, is larger (by a factor 2) than that of the neighboring white stripe above it (between h_{j+1} and h_{j+2}), and therefore the area of the circle is at least 2/3 black.

For $-N - 1 \le j \le N - 1$, the area A_j of the stripe between h_j and h_{j+1} is:

$$\begin{aligned} A_j &= \int_{y_j}^{y_{j+1}} \int_{-[2y\cosh(R)-1-y^2]^{\frac{1}{2}}}^{[2y\cosh(R)-1-y^2]^{\frac{1}{2}}} \frac{1}{y^2}\, dx\, dy \\ &= \int_{y_j}^{y_{j+1}} \frac{2[2y\cosh(R) - 1 - y^2]^{\frac{1}{2}}}{y^2}\, dy. \end{aligned} \tag{9}$$

For $-N \le j \le N - 2$ the leading behavior as $N,\ W \to \infty$ (and recalling that $R = [N + 1/2]W$), is

$$\begin{aligned} A_j &\sim \int_{y_j}^{y_{j+1}} \frac{2y^{1/2} e^{R/2}}{y^2}\, dy \\ &\sim 4e^{[R/2-(j+1/2)W/2]} \end{aligned} \tag{10}$$

where $a \sim b$ means $\dfrac{a}{b} \to 1$ as $N,\ W \to \infty$. So

$$\frac{A_j}{A_{j+1}} \sim e^{W/2}. \tag{11}$$

For $j = -N - 1$ we have:

$$
\begin{aligned}
A_{-N-1} = \sim \int_{e^{-R}}^{e^{-R+W}} \frac{2(e^R y-1)^{1/2}}{y^2}\, dy \\
\sim 2e^R \int_1^{e^W} \frac{(z-1)^{1/2}}{z^2}\, dz \\
\gtrsim 2e^R \int_2^{e^W} \frac{1}{z^2}\, dz \\
\gtrsim e^R
\end{aligned}
\tag{12}
$$

so

$$
\frac{A_{-N-1}}{A_{-N}} \gtrsim \frac{1}{4} e^{W/2}. \tag{13}
$$

Finally we note that $\frac{1}{4}e^{W/2}$ can be made as large as desired, in particular larger than 2, which completes the argument that the relative densities of the set S of black stripes does not have a well defined limit.

The example of the stripe model in the hyperbolic plane, where the stripes are all of equal "width", is more unsettling than the example of annuli in Euclidean space discussed above, where in a sense the oscillation was more obviously built in. We will see below that this stripe model is only a simple version of a well known disk packing.

There has been another common way to compute or estimate the density of packings in Euclidean spaces, using tilings associated with the packings, and the relative densities of the bodies in the tiles. (A tile is a homeomorphic image of the closed unit ball, and a tiling is a packing by tiles for which the union of the tiles is the full space $\mathbb{S}$.) We emphasize that this is an attempt to reduce the intuitive global idea of density, which involves taking a limit of approximate densities in expanding regions of finite volume, to a more local notion. As a significant example of this approach we note an elegant proof [Fe1, Rog] of the optimal density for packings of equal disks in the Euclidean plane. The proof uses the Vorononi cells of the bodies of a packing, where the cell for a body β is the set of all points $p \in \mathbb{S}$ as close to β as to any body of the packing. The proof shows that the relative density in its Voronoi cell of any disk of any packing is bounded above by that of any of the Voronoi cells in the obvious hexagonal packing. This argument was extended to sphere packings in $\mathbb{S}$ by K. Böröczky, who showed [Bo2] that the relative density of any sphere of any packing of $\mathbb{S}$ in its Voronoi cell is bounded above by the relative density associated with that of a regular simplex. (See [FeK] for details.) Such relative densities in tiles of associated tilings have remained an important tool in analyzing optimal densities of sphere packings in Euclidean spaces [FeK, Bez].

Example 3 (tight radius packings)

In hyperbolic space, particularly the plane $\mathbb{H}^2$, the above method of estimating or computing a density of sphere packings through an associated tiling has been used convincingly for the special case of disks of "tight" radius. The radius r of a sphere in $\mathbb{S}$ is called tight if the regular simplex of side length $2r$ admits a (full-face to full-face) tiling of $\mathbb{S}$. In $\mathbb{H}^2$ this is the case if and only if the equilateral triangle of edge length $2r$ has angles of the form $2\pi/n$ for some $m \geq 7$, in which case

$$2r = 2r_m = \cosh^{-1}[\cot(\frac{\pi}{m})\cot(\frac{2\pi}{m})] \tag{14}$$

and clearly $r_m \to \infty$ as $m \to \infty$. For disks with tight radius r_m the obvious "periodic" packing, in which each disk is surrounded by m disks touching it, has a well defined density in the sense that, besides the method using Voronoi tilings, *any* reasonable way to compute the density would give the same value (namely $[3\csc(\pi/m) - 6]/[m-6]$ [Fe5]), in particular any limit of the form (1) [BoR].

Example 4 (Böröczky's packing).

There is an influential example due to Böröczky [Bo1] which points out a difficulty in using relative density in tiles to define the density of at least some packings in hyperbolic space, even some which are rather symmetric. Place disks in the upper half plane model of the hyperbolic plane with Euclidean centers at those points with coordinates

$$\{(e^{2j+\frac{1}{2}}(k+1/2),\ e^{2j+\frac{1}{2}}) \mid j,k \in \mathbb{Z}\}. \tag{15}$$

The connection between this and the stripe model is simple: we are placing the disks equally spaced in the black stripes (and we are taking the value $W = 1$ for the width of the stripes). See Figure 1 for a picture of the packing, which includes some horocycles and geodesics to help understand the structure.

In Figure 2 we see the same packing with two congruent tiles in dark outline. For each tile consider the tiling of the plane made by congruent copies of the tile, as follows. First produce copies of the tile by the congruences: $(x,y) \to (x + mw, y)$, $m \in \mathbb{Z}$, where w is the Euclidean width of the body. This fills out a black and white stripe. Then produce, from these, more copies of the tile by the congruences: $(x,y) \to (e^{2m}x, e^{2m}y)$, $m \in \mathbb{Z}$. Together these copies of the original tile will cover the whole plane. The two tilings made this way, one from each of the tiles in Figure 2, are both simply related to the same packing of disks. The punchline is, the tiling made by starting with the tile on the left in Figure 2 would suggest assigning a "density" of the packing of disks twice the value suggested by the tiling made by starting with the tile on the right! We repeat the point that using a tiling to compute the density of some packing, thus making the computation more local, is useful in Euclidean spaces but is less convincing in hyperbolic spaces.

We now return to the question of a definition of optimally dense packings of

$\mathbb{H}^n$. As we say above, for packings of Euclidean space the notion of densest packings is easy to clarify, and one way to understand this is through the computation of the ratio $f(\rho, a)$ of volumes of concentric spheres of radii ρ and $\rho + a$.

Note that:

i) in $\mathbb{E}^n$ $f(\rho, a) \equiv \dfrac{\rho^n}{(\rho + a)^n}$, so for fixed $a > 0$ and n, $f(\rho, a) \to 1$ as $\rho \to \infty$;

ii) in $\mathbb{E}^n$, for fixed $a > 0$ and ρ, $f(\rho, a) \to 0$ as $n \to \infty$;

iii) in $\mathbb{H}^n$, for fixed $a > 0$ and n, $f(\rho, a) \to e^{-ca}$ as $\rho \to \infty$, for some constant $c > 0$.

To see why these phenomena interfere with a generalization to hyperbolic space of the method used earlier for Euclidean packings, consider the packings $\mathcal{P}_\rho$ of the hyperbolic plane, by disks of fixed radius R, defined for each $\rho >> 0$ as follows. For each sufficiently large radius $\rho >> R$, place disks of radius R on the circumference of a circle C_ρ of radius ρ, so that: they cover all but perhaps one arc of the circumference; there are as many disks as possible without overlap; disks intersect only at points of the circumference. We now show that by taking R (and therefore ρ) large enough we can ensure that the fraction of the area of C_ρ covered by the disks is as close to 1 as desired.

The fraction of the area of C_ρ which is in the annulus between C_ρ and the concentric $C_{\rho'}$ for $\rho > \rho'$ is of the order $1 - e^{\rho' - \rho}$ for large ρ, ρ', and by taking $0 << \rho - \rho' << R << \rho' << \rho$ we can ensure that most of this area is inside the disks of radius R – all except those regions outside pairs of touching disks of radius R and outside the circle $C_{\rho'}$, plus the region near any uncovered arc of C_ρ. But using the convexity of circles, the former regions are each contained in triangles of the form TUV (see Figure 7), so have negligible area, and another simple triangle argument applies to the region near any uncovered arc of C_ρ.

So by choosing R appropriately we could get almost all the area of C_ρ to lie outside C'_ρ.

Where in the Euclidean argument we used larger and larger cubes, in hyperbolic space we would use fundamental domains of cocompact subgroups of the isometry group $\mathcal{G}$ of $\mathbb{H}^n$. But we needed the fact, in Euclidean space, that the volume of the portion of a packing near the boundary of the fundamental domain would be negligible, while we see now that for large fundamental domains and large bodies, this is far from the case. In summary, where we used i) to show the existence of optimal packings in Euclidean space, in hyperbolic space we have instead iii), which for large spheres is approximately ii). This is the intuitive reason why there has been difficulty defining optimally dense packings in hyperbolic space for so long.

II. Some responses to the problem

We have summarized above the arguments that, for at least some packings in hyperbolic space, there seems to be no reasonable notion of density. We now consider how one might proceed with an analysis of optimal density.

Two avenues of response that come to mind are: to replace the essentially global definition of density with something more local; or to find a way to define density for those packings where it is reasonable, together with a convincing argument for excluding the others.

a. Completely Saturated Packings

A packing is called completely saturated [FeK, FKK, Kup] if it is not possible to replace a finite number of bodies of the packing with a greater total volume of bodies and still remain a packing. Intuitively, we think of a completely saturated packing as one that is locally densest. In [FKK], it was proven that any convex body of Euclidean space admits a completely saturated packing (and more generally any body with the strict nested similarity property) (see also [Kup]). In [Bow], it is proven that completely saturated packings exist for all bodies β in either Euclidean of hyperbolic space. The argument given there extends easily to finite collections $\mathcal{B}$.

Example 5 (a completely saturated packing with low density).

As pointed out in [FKK], a completely saturated packing of Euclidean space is a densest packing. This is not true in hyperbolic space. In this example, we construct a pair of bodies β_1, β_2 in $\mathbb{H}^2$ such that two completely saturated periodic packings by $\{\beta_1, \beta_2\}$ exist that have different densities. The reason, as you will see, is due to the fact that the length of the boundary of a region in the hyperbolic plane is comparable to its area. Let β_1 be the tile shown in Figure 8. It is a regular octagon with all interior angles equal to $2\pi/8$. Let $\mathcal{T}_1$ be the unique periodic tiling by β_1. Let β_2' be the tile shown in Figure 9. It is formed from β_1 by adding "protrusions" to some edges and "identations" to others. We will assume that these protrusions and indentations are made so that they fit together but are narrow enough so that there is a region of finite area C_1 in each indentation that cannot be occupied by a nonoverlapping copy of β_2' unless it is occupied by a protrusion. Also we assume that each protrusion fits into a unique indentation.

β_2' admits a unique periodic tiling $\mathcal{T}_2$. Let β_2 be equal to β_2' with a small hole removed from its interior. Let $\mathcal{P}$ be the obvious periodic packing by β_2 (i.e. the one that comes from $\mathcal{T}_2$ by removing a small hole from the interior of each tile). Since β_1 admits a periodic tiling, it is clear that the optimal density of $\{\beta_1, \beta_2\}$ is one. Just as clear, is the fact that the density of $\mathcal{P}$ is $area(\beta_2)/area(\beta_1) < 1$. We will show that $\mathcal{P}$ is completely saturated (if the hole in β_2 is small enough).

It is a standard fact of hyperbolic geometry that there exists a constant $C_2 > 0$ depending only on the symmetry group of $\mathcal{T}_1$ (and the fact that β_1 contains a fundamental domain for this group) such that for all finite subtilings $\mathcal{T}'$ of β_1, $|\partial\mathcal{T}'| \geq C_2|\mathcal{T}'|$ (by $|\partial\mathcal{T}'|$ we mean the number of edges contained in exactly one tile of $\mathcal{T}'$ and by $|\mathcal{T}'|$ we mean the number of tiles in $\mathcal{T}'$). Since the hole in the interior of β_2' can be made as small as we like, we may assume that $area(\beta_2) > area(\beta_1) - C_1C_2/2$.

Suppose for a contradiction that $\mathcal{P}$ is not completely saturated. Then there exists a finite subpacking $\mathcal{P}' \subset \mathcal{P}$ and another finite packing $\mathcal{P}''$ such that $(\mathcal{P} - \mathcal{P}') \cup \mathcal{P}''$ is a packing and $area(\mathcal{P}'') > area(\mathcal{P}')$. We may assume without loss of generality that $\mathcal{P} \cap \mathcal{P}'' = \emptyset$.

We claim that the number of edges of $\mathcal{P}'$ that have protrusions on them coming from bodies of $\mathcal{P}'$ is at least $|\partial\mathcal{P}'|/2$. So let e be any edge on the boundary of $\mathcal{P}'$. Let $e = e_0, e_1, .., e_n$ be the sequence of edges defined by for $1 \le i < n$, e_{i+1} and e_i are on a body of $\mathcal{P}'$ and e_{i+1} is the "opposite side" of e_i in the sense that if e_i has a protrusion on it (relative to the body containing both e_i and e_{i+1}) then e_{i+1} is its corresponding indentation and vice versa. This sequence is uniquely defined and ends in an edge e_n on the boundary of $\mathcal{P}'$. It is easy to see that if e_0 corresponds to an indentation of $\mathcal{P}'$ (i.e. e_0 has an indentation on it coming from a body of $\mathcal{P}'$) then e_n corresponds to a protrusion and vice versa. Thus the claim is proven.

Note that it is not possible for any body of $\mathcal{P}''$ to fill completely any indentation on the boundary of $\mathcal{P} - \mathcal{P}'$ (in fact a region of area at least C_1 is always unfilled). Hence the total area of $\mathcal{P}''$ is at most

$$\begin{aligned} area(\mathcal{P}'') \le& |\mathcal{P}'| area(\beta_1) - (C_1/2)|\partial\mathcal{P}'| \\ \le& |\mathcal{P}'|[area(\beta_1) - C_1 C_2/2] \\ <& |\mathcal{P}'| area(\beta_2) \\ =& area(\mathcal{P}'). \end{aligned} \tag{16}$$

This contradicts the choice of $\mathcal{P}''$. So $\mathcal{P}$ is completely saturated. The moral is that, in hyperbolic space, locally densest does not imply globally densest.

b. Controlling pathological packings

We now discuss an approach to density specifically aimed at controlling those packings, such as the above example of Böröczky, which pose difficulty in computing a reliable density. Even though the methods are also applicable to Euclidean space, the interests of this article make it natural to specialize the discussion from now on to $\mathbb{S} = \mathbb{H}^n$.

The key idea is to use a pointwise ergodic theorem of Nevo ([Nev, Thm. 1] for dimension $n \ge 3$; [NeS, Thm. 3] for $n \ge 2$), the conclusion of which is the existence of limits of the type (1) in the intuitive definition of density. The fact that such theorems only prove existence of the limit "almost everywhere" is not a defect, it is a feature, necessitated by examples such as that of Böröczky.

We begin by reproducing some notation and results from [BoR]. Let $d(\cdot, \cdot)$ be the usual metric on $\mathbb{S}$, and let $\mathcal{O}$ be a distinguished origin. We suppose given a finite collection $\mathcal{B}$ of bodies β_j in $\mathbb{S}$. Let $\Sigma_{\mathcal{B}}$ be the space of all "relatively-dense" packings of $\mathbb{S}$ by congruent copies of the β_j, that is, packings $\mathcal{P}$ with the property that any congruent copy of a body in $\mathcal{B}$ intersects a body of $\mathcal{P}$. On $\Sigma_{\mathcal{B}}$ we put the following metric, corresponding to uniform convergence on compact subsets of $\mathbb{S}$:

$$d_{\mathcal{B}}(\mathcal{P}_1, \mathcal{P}_2) = \sup_{k \ge 1} \frac{1}{k} h(B_k \cap \mathcal{P}_1, B_k \cap \mathcal{P}_2), \tag{17}$$

where B_k denotes the closed ball of radius k centered at the origin, and for compact sets A and C we use the Hausdorff metric

$$h(A,C) \equiv \max\{\sup_{a\in A}\inf_{c\in C} d(a,c), \sup_{c\in C}\inf_{a\in A} d(a,c)\}. \tag{18}$$

It is not hard to see [RaW] that $\Sigma_{\mathcal{B}}$ is compact in this metric topology, and that the natural action: $(g,\mathcal{P}) \in \mathcal{G}\times\Sigma_{\mathcal{B}} \longrightarrow g(\mathcal{P}) \in \Sigma_{\mathcal{B}}$ of the isometry group $\mathcal{G}$ of $\mathbb{S}$ on $\Sigma_{\mathcal{B}}$ is (jointly) continuous. Let $\mathcal{M}(\mathcal{B})$ be the family of Borel probability measures on $\Sigma_{\mathcal{B}}$. We call a measure $\mu \in \mathcal{M}(\mathcal{B})$ "invariant" if for any Borel subset $E \subset \Sigma_{\mathcal{B}}$ and any $g \in \mathcal{G}$, $\mu(gE) = \mu(E)$. Let $\mathcal{M}_I(\mathcal{B})$ be the subset of invariant measures and $\mathcal{M}_I^e(\mathcal{B})$ the convex extreme ("ergodic") points of $\mathcal{M}_I(\mathcal{B})$, all in their weak* topology, in which $\mathcal{M}(\mathcal{B})$ and $\mathcal{M}_I(\mathcal{B})$ are compact.

We will study these ergodic measures as a substitute for studying individual packings. As we will see, for any ergodic measure $\mu \in \mathcal{M}_I(\mathcal{B})$ there is a set of packings Z of full μ-measure such that for each $\mathcal{P} \in Z$, the orbit of $\mathcal{P}$ is dense in the support of μ. So studying μ is a lot like studying a packing in Z. We will make this relationship more clear in what follows but first some examples.

Suppose $\mathcal{P}$ is a "periodic" packing, i.e. the symmetry group $\Gamma_{\mathcal{P}}$ of $\mathcal{P}$ is cocompact in $\mathcal{G}$. We will construct a measure $\mu_{\mathcal{P}} \in \mathcal{M}_I^e(\mathcal{B})$ whose support is contained in the orbit $O(\mathcal{P}) \equiv \{g\mathcal{P}\,|\, g\in\mathcal{G}\} \subset \Sigma_{\mathcal{B}}$ of $\mathcal{P}$. $O(\mathcal{P})$ is naturally homeomorphic to the (metrizable) space $\mathcal{G}/\Gamma_{\mathcal{P}}$ of left cosets by the homeomorphism $q_{\mathcal{P}} : O(\mathcal{P}) \to \mathcal{G}/\Gamma_{\mathcal{P}}$ with $q_{\mathcal{P}}(g\mathcal{P}) = g\Gamma_{\mathcal{P}}$. There is a natural probability measure on $\mathcal{G}/\Gamma_{\mathcal{P}}$ induced by Haar measure on $\mathcal{G}$ by the projection map $\pi_{\mathcal{P}} : \mathcal{G} \to \mathcal{G}/\Gamma_{\mathcal{P}}$. (Aside from an overall normalization the measure on $\mathcal{G}/\Gamma_{\mathcal{P}}$ can be defined on sufficiently small open balls $B \subset \mathcal{G}/\Gamma_{\mathcal{P}}$ as the Haar measure of any of the components of $\pi_{\mathcal{P}}^{-1}(B)$.) Hence $q_{\mathcal{P}}$ induces a probability measure $\hat{\mu}_{\mathcal{P}}$ on $O(\mathcal{P})$. This measure can then be extended to all of $\Sigma_{\mathcal{B}}$ in the following way: $\mu_{\mathcal{P}}(E) = \hat{\mu}_{\mathcal{P}}[E \cap O(\mathcal{P})]$ for any Borel set $E \subseteq \Sigma_{\mathcal{B}}$. We will use the term "periodic measure" to denote any measure in $M_I(\mathcal{B})$ associated in this way with the orbit of a periodic packing. It is not hard to prove from the uniqueness of Haar measure on $\mathcal{G}$ that there is only one probability measure, with support in the orbit of a periodic packing, which is invariant under $\mathcal{G}$.

Next, we define the density of an invariant measure. After the definition, we will show how the density of an invariant measure relates to the density of packings in its support.

For any $p \in \mathbb{H}^n$ we define the real valued function F_p on $\Sigma_{\mathcal{B}}$ as the indicator function of the set of all packings $\mathcal{P}$ such that p is contained in a body of $\mathcal{P}$. (The latter condition will sometimes be expressed as $p \in \mathcal{P}$.)

Definition 1. For any invariant measure $\mu \in \mathcal{M}_I(\mathcal{B})$, the "average density" $D(\mu)$ is defined as $\int_{\Sigma_{\mathcal{B}}} F_p(y)\, d\mu(y)$.

Note: the average density $D(\mu)$ is independent of the choice of p, because of the invariance of the measure, so p is not needed in the notation. For convenience we sometimes use $p = \mathcal{O}$.

If $\mathcal{P}_\mu$ is a random packing with distribution μ then the above definition states that the density of μ is the probability that the origin is contained in a body of $\mathcal{P}_\mu$.

For periodic packings $\mathcal{P}$ there is an obvious notion of density using a fundamental domain of $\Gamma_{\mathcal{P}}$. The above definition of density coincides with this intuitive notion for such special $\mathcal{P}$.

Proposition 1 [BoR]. If $\mathcal{P}$ is a periodic packing, $D(\mu_x)$ is the relative volume of any fundamental domain for Γ_x taken up by the bodies of x.

We need the following notation. As usual we let $\mathcal{G}$ denote the group of orientation preserving isometries of hyperbolic n-space $\mathbb{H}^n$ (for some fixed $n \geq 2$). Let $\pi : \mathcal{G} \to \mathbb{H}^n$ be the projection map $g \to g\mathcal{O}$ where $\mathcal{O}$ is some fixed point in $\mathbb{H}^n$. Then we let $\tilde{B}_k$ denote the inverse image under π of the closed ball of radius k centered at $\mathcal{O}$. Finally let $\lambda_{\mathcal{G}}$ denote a Haar measure on $\mathcal{G}$, normalized so that $\lambda_{\mathcal{G}}(\tilde{B}_k)$ is the volume of the k-ball in $\mathbb{H}^n$.

We will use the following special case of Theorem 3 in [NeS] to relate the density of an ergodic measure to the density of (almost every) packing in its support.

Theorem 1 [Nevo]. If $\mathcal{G}$ acts continuously on a compact metric space X such that there is a Borel probability measure μ on X that is invariant and ergodic under this action, then for every function $f \in L^p(X, \mu)$ $(1 < p < \infty)$ there is a set Z of full μ measure such that for every $z \in Z$,

$$\int_X f d\mu = \lim_{n \to \infty} \frac{1}{\lambda_{\mathcal{G}}(\tilde{B}_n)} \int_{\tilde{B}_n} f(gz)\, d\lambda_{\mathcal{G}}(g). \tag{19}$$

Actually we will use the following extension of this result.

Theorem 2. Under the same hypotheses as the above theorem, the set Z may be taken to be invariant under G.

We will prove this result in the next section. Applying Theorem 2 to the function F_p we get

Theorem 3. If $\mu \in M_I^e(\mathcal{B})$ then there exists a set Z of full μ-measure such that for all $p \in \mathbb{H}^n$ and all $\mathcal{P} \in Z$,

$$\lim_{r \to \infty} \frac{vol[\mathcal{P} \cap B_p(r)]}{vol[B_p(r)]} = D(\mu). \tag{20}$$

Note that this implies that the orbit of the stripe model has measure zero with respect to every invariant measure. We will give another explanation for this fact in a later section.

From example 4 we concluded that it is not possible, in general, to compute the density of a hyperbolic packing using a tiling associated to it. In spite of this we will show that it is possible to compute the density of an invariant measure using an associated space a tilings.

Let Σ be a (compact, invariant) space of packings of $\mathbb{H}^n$. Let μ be a $Isom^+(\mathbb{H}^n)$ invariant measure on Σ. Suppose that Θ is a space of tilings (of $\mathbb{H}^n$) and that there is an equivariant map $\phi : \Sigma \to \Theta$. For example, Θ may be the space of Voronoi (or Delone) tilings [FeK] corresponding to Σ. For $\mathcal{P} \in \Sigma$ such that the origin is contained in a tile of $\phi(\mathcal{P})$, let $\tau_p(\mathcal{P})$ denote the tile of $\phi(\mathcal{P})$ containing the point $p \in \mathbb{H}^n$. We claim that for any p:

$$D(\mu) = \int_\Sigma \frac{vol[\mathcal{P} \cap \tau_p(\mathcal{P})]}{vol[\tau_p(\mathcal{P})]} \, d\mu(\mathcal{P}). \tag{21}$$

Define a function $f : \mathbb{H}^n \times \mathbb{H}^n \times \Sigma \to \mathbb{R}$ by $f(p, q, \mathcal{P}) = 1/vol[\tau_p(\mathcal{P})]$ if p and q are in the same Voronoi cell of $\mathcal{P}$ and p is in a body of $\mathcal{P}$ (otherwise $f(p, q, \mathcal{P}) = 0$). Define a measure ν on $\mathbb{H}^n \times \mathbb{H}^n$ by

$$\nu(E \times F) = \int_\Sigma \int_E \int_F f(p, q, \mathcal{P}) \, dvol(p) \, dvol(q) \, d\mu(\mathcal{P}). \tag{22}$$

Since μ is invariant, it is easy to check that for all $g \in Isom^+(\mathbb{H}^n)$, $\nu(gE \times gF) = \nu(E \times F)$. The mass-transport principle [BeS] implies that $\nu(E \times \mathbb{H}^n) = \nu(\mathbb{H}^n \times E)$ for any measureable $E \subset \mathbb{H}^n$. But it can easily be checked that $\nu(E \times \mathbb{H}^n) = vol(E)D(\mu)$ and $\nu(\mathbb{H}^n \times E) = vol(E) \int_\Sigma vol[\mathcal{P} \cap \tau_p(\mathcal{P})]/vol[\tau_p(\mathcal{P})] \, d\mu(\mathcal{P})$ (for any p). This proves the claim. For emphasis, we repeat that when μ is an invariant measure we can compute its density with respect to local structures such as the Voronoi or Delone tilings. If $\mu_{\mathcal{P}}$ is a periodic measure and Θ is the space of tilings by a fundamental domain of $\Gamma_{\mathcal{P}}$ then there is a natural equivariant map from the orbit of $\mathcal{P}$ to Θ. The above result then yields Proposition 1.

We now define optimality through measures.

Definition 2. $D(\mathcal{B}) \equiv \sup_{\mu \in \mathcal{M}_I^e(\mathcal{B})} D(\mu)$ will be called the "optimal density for $\mathcal{B}$", and any ergodic measure $\tilde{\mu} \in \mathcal{M}_I^e(\mathcal{B})$ will be called "optimally dense (for $\mathcal{B}$)" if $D(\tilde{\mu}) = D(\mathcal{B})$. We define "optimally dense packings" a little differently than in [BoR]. We say that a packing $\mathcal{P}$ is optimally dense if there is an optimally dense measure μ such that the orbit of $\mathcal{P}$ is dense in the support of μ and for every $p \in \mathbb{H}^n$ $D(\mu)$ is equal to the limit of the relative fraction of volume in expanding sphere centered at p taken up by bodies of $\mathcal{P}$.

One of the main results of [BoR] asserts the existence of optimally dense measures. Also, it was proven that for every $\mu \in M_I^e(\mathcal{B})$ there exists a set Z of full μ-measure such that for every $\mathcal{P} \in Z$, the orbit of $\mathcal{P}$ is dense in the support of μ. Using Theorem 3 this implies

Theorem 4. For any finite collection $\mathcal{B}$ of bodies there exists an optimally dense measure μ on $\Sigma_{\mathcal{B}}$, and a subset of the support of μ, of full μ-measure, of optimally dense packings.

Note: There may be many optimally dense measures for a given $\mathcal{B}$.

In [Bow] it is proven that the set of completely saturated packings has full-measure with respect to any optimally dense measure μ. In other words, if $\mathcal{P}_\mu$ is a random packing with optimally dense distribution μ then $\mathcal{P}_\mu$ is completely saturated almost surely. Example 5 shows that the converse is false.

A major result of [BoR] was that the set of all radii r such that there exists an optimally dense *periodic* measure for the sphere of radius r (in hyperbolic space) is at most countable. Thus most optimally dense sphere packings are complicated.

c. What Invariant Measures Avoid

Some packings, such as the Böröczky example, do not have a well-defined density. We claim this is due to the fact that the closure of the orbit of such a packing has measure zero with respect to every invariant measure μ. In this section we prove this statement and show other examples of packings that are not "seen" by invariant measures.

Example 6 (Penrose's binary tilings)

Perhaps the most relevant to the discussion in section I is the $\mathcal{B}$ consisting of the body β shown in Figure 3. (This is a minor variation on the tile in [Pen], and a special case of tiles in [MaM].) We know copies of this body can tile $\mathbb{H}^2$, and since limits in $\Sigma_{\mathcal{B}}$ of tilings will again be tilings, if there were any invariant measure $\mu \in \mathcal{M}_I(K)$ with support in the orbit closure of such a tiling it would clearly have density 1. However we can see there is no such measure as follows. First consider the slightly simpler, and better known, example of the natural action of the isometry group $\mathcal{G}$ of $\mathbb{H}^2$ (namely $\mathcal{G} = PSL_2(\mathbb{R})$) on the boundary Δ of $\mathbb{H}^2$. Assume there is a measure μ on Δ invariant under $\mathcal{G}$. Any hyperbolic element $g_h \in \mathcal{G}$ has 2 fixed points in Δ, p_1, p_2, and moves all other points towards one and away from the other. From its invariance under g_h, $\mu(\{p_1, p_2\}) = 1$. Then considering that any elliptic element $g_e \in \mathcal{G}$ has no fixed points in Δ, and μ must also be invariant under g_e, we get a contradiction. So there are no probability measures on Δ invariant under $\mathcal{G}$. Going back to our space $\Theta_{\mathcal{B}}$ of tilings by our body β, consider the function f from $\Theta_{\mathcal{B}}$ to Δ, which takes each tiling to the point "pointed to" by the protrusion on each body in the tiling. f is obviously continuous. If there were a probability measure μ on the space $\Theta_{\mathcal{B}}$ of tilings, invariant under the action of $\mathcal{G}$, we could define a corresponding measure μ_f on Δ by $\mu_f(E) = \mu(f^{-1}[E])$. Since no such μ_f exists, this proves no such μ exists.

Now assume the optimal density for $\mathcal{B}$, $D(\mathcal{B})$, is 1, with an optimal measure μ. For each $R > 0$ consider the function on $\Sigma_{\mathcal{B}}$

$$f_R(\mathcal{P}) = \frac{1}{vol(B_R)} \int_{\tilde{B}_R} F_{\mathcal{O}}(g\mathcal{P})\, d\lambda_{\mathcal{G}}(g), \tag{23}$$

which gives the relative area of the ball B_R covered by the disks of $\mathcal{P}$. From the invariance of of μ,

$$\int_{\Sigma_{\mathcal{B}}} f_R(\mathcal{P})\, d\mu(\mathcal{P}) = \int_{\Sigma_{\mathcal{B}}} F_{\mathcal{O}}(\mathcal{P})\, d\mu(\mathcal{P}) = D(\mu) = 1, \tag{24}$$

so $f_R(\mathcal{P}) = 1$ for μ-almost every $\mathcal{P} \in \Sigma_{\mathcal{B}}$. Letting R run through the positive integers, and intersecting the sets of full measure we get for each such R, we see there is a set of packings of full measure which are tilings. Since the closure of a set of tilings can only contain tilings, and the support of μ must be invariant under $\mathcal{G}$, that support is contained in the set of all tilings of β. But we saw above that there can be no such measure as μ, and this proves that $D(\mathcal{B}) \neq 1$.

Using modifications of the above example, it can be shown [Bow] that for every $\epsilon > 0$, there exists a body β that admits a tiling of $\mathbb{H}^n$ but $D(\beta) < \epsilon$.

The formalism above leads one to assert that the densest packings of the body β have density bounded away from 1, even though one can tile $\mathbb{H}^2$ with copies of β. The "reason" for this is that there are no invariant measures which can "see" the tilings; they are a set of measure zero for every invariant measure on the space of all packings by β. We explore the consequences of this using some of the examples we discussed earlier.

Consider the tile β shown in Figure 3. Congruent copies of β can tile the plane, in only one way up to an overall rigid motion, as in Figure 4 (in which the little bumps on the tiles are not shown.) Construct the tile $\bar{\beta}$ of Figure 5 out of three abutting copies of β. Now drill a hole in $\bar{\beta}$, producing the body $\bar{\beta}_0$, as shown in Figure 6. Note that the packings of the plane by $\bar{\beta}_0$ obtained in the obvious way from the tilings by $\bar{\beta}$, are precisely the complements of the disk packings of Böröczky discussed above. The point is, although it might seem reasonable to assign a density of 1 to the tiling of Figure 4, that would seem to imply a well defined density to the packing of Figure 2, which we know is misleading. In other words, the meaningfulness of the density of the tiling of Figure 4 is unstable under arbitrarily small perturbations (drilling arbitrarily small holes). Notice that when we drill these small holes we turn the tiling into a mere packing, forcing us to give up the "simplicity" of the tiling, as a global object with seemingly obvious density, and leaving us to find some meaningful way to assign a density to the resulting packing. As we will see below, the difficulty in assigning a density to a packing, for instance congruent copies of a single body β, can derive from the complexity of the set of rigid motions of β that define the packing. And in this sense a tiling is no simpler; treating it as a global object with an "obvious" density simply avoids coming to grips with the essential nature of the assignment of density for packings.

In other words, the phenomenon whereby the "optimal" density can be less (even far less) than 1 for a body which can tile space, can be understood as related to the instability of the meaningfulness of the density of the tilings under removal of small holes in the tiles. This suggest that even for tilings one needs to keep track of the individuality of the tiles. In this example that amounts to noting the various sets of congruences used in producing the tilings; in some sense those sets of congruences are too complicated to be analyzed through the formalism.

We have shown that if $\mathcal{T}$ is a tiling whose orbit closure in the space of packings factors onto the space at infinity of the hyperbolic plane, then there are no invariant measures on the orbit closure of $\mathcal{T}$. All of our examples of strange behaviour in the hyperbolic plane have, so far, been constructed using this principle. Could this be the only way of constructing such examples?

Example 7.

Let τ be the tile shown in Figure 10. It is an all right octagon whose sides come in two different lengths and there is a protrusion on one of the short sides and an indentation on each of the other three short sides. Two copies of τ can be joined along their long sides to produce an X-tile (see Figure 11. Many copies of this X tile can be glued together to form a noncompact surface S without boundary. There is in fact, only one way to do this. S is homeomorphic to the boundary of a regular neighborhood of the standard Cayley graph for the free group on two generators embedded in $\mathbb{R}^3$. The hyperbolic plane isometrically covers S and therefore the tiling $\mathcal{T}_S$ of S by copies of τ lifts to a tiling $\mathcal{T}$ of $\mathbb{H}^2$ by copies of τ.

Note that the free group F_2 on 2 generators $\{a, b\}$ acts naturally and isometrically on S. This action lifts to an isometric action of $\mathbb{H}^2$ via the covering map. Though not relevant to what follows, note that this lift is unique up to postcomposition by rigid motions of $\mathbb{H}^2$.

If we start a walk in $\mathcal{T}_S$ from some initial tile and follow the protrusions we get "closer" to a point on the ideal boundary of S. It is not too hard to see that this point does not depend on the initial tile chosen but only on the tiling $\mathcal{T}_S$. Therefore, there is a map from the space of tilings of S by X (defined similar to the same way $\Sigma_{\mathcal{B}}$ is defined) to the ideal boundary of S that commutes with the action of F_2. Since the ideal boundary does not admit an invariant Borel probability measure (for practically the same reason that Δ does not admit an invariant measure), neither does the space of tilings of S by X.

Suppose that there exists an invariant measure μ whose support is contained in the orbit closure of $\mathcal{T}$. Then this measure pushes forward via the covering map to a measure μ_S on the space of tilings on S by τ. This measure μ_S is invariant under the action of F_2 but this contradicts the previous paragraph.

Now suppose that there is an equivariant map ϕ from the orbit closure $O(\bar{\mathcal{T}})$ of $\mathcal{T}$ in Σ_τ to Δ. Let $p = \phi(\mathcal{T})$. Since ϕ is equivariant, the stabilizer of $\mathcal{T}$ must be contained in the stabilizer of p. However, the stabilizer of $\mathcal{T}$ is noncyclic (since it contains an isomorphic copy of the fundamental group of S which is noncyclic). By the theory of fuchsian groups, $\mathcal{T}$ does not fix any point at infinity. This contradiction shows that ϕ cannot exist.

III. Proof of Theorem 2

Our main tool in proving Theorem 2 is an estimate on the volume of the symmetric difference of two balls of radius R whose centers are some distance r apart in hyperbolic space. We need the following well-known fact [AVS]:

Lemma 1. There exist positive real constants c_1 and c_2 such that

$$\lim_{r\to\infty} vol[B_p(R)]e^{-c_1 r} = c_2. \tag{25}$$

Lemma 2. If the distance between points p and q in $\mathbb{H}^n$ is r then

$$F(r) \equiv \lim_{R\to\infty} \frac{vol[B_p(R) - B_q(R)]}{vol(B_r)} \tag{26}$$

exists and $\lim_{r\searrow 0} F(r) = 0$.

Proof of Lemma 2. Let r be fixed for now. Let M_R (for middle) denote the convex hull of p, q and $S_p(R)\cap S_q(R)$ (where $S_p(R)$ denotes the sphere of radius R centered at p). For example, in 2-dimensions, M_R is a quadrilateral. Let $O_p(R)$ (for obtuse cone) denote the component of $B_p(R) - M_R$ whose closure contains p. Similarly let $O_q(R)$ denote the component of $B_q(R) - M_R$ whose closure contains q. Let $A_q(R)$ (for acute cone) denote $B_q(R) - O_q(R)$. Since $B_p(R) - B_q(R) = O_p(R) - A_q(R)$ and $A_q(R) - O_p(R) = M_R$, we have $vol[B_p(R) - B_R(q)] = vol[O_p(R)] - vol[A_q(R)] + vol(M_R)$. Thus it suffices to show that

$$\lim_{R\to\infty} \frac{vol[O_p(R)] - vol[A_q(R)]}{vol(B_R)} \tag{27}$$

exists and goes to 0 as $r \to 0$ and

$$\lim_{R\to\infty} \frac{vol(M_R)}{vol(B_R)} = 0. \tag{28}$$

We prove the second claim first. Let H be a two dimensional geodesic plane containing p and q. Let $x_R \in H \cap B_p(R) \cap B_R(q)$. Note (from two-dimensional geometry) that the angle $\angle px_Rq$ goes to 0 as $R \to 0$. In fact we can compute from the law of cosines,

$$\cos(\angle px_Rq) = \frac{\cosh^2(R) - \cosh(r)}{\sinh^2(R)} \to 1 \text{ as } R \to \infty. \tag{29}$$

Let $W >> 0$. Let $\{y_R, z_R\} = H \cap \partial M_R \cap B_p(R - W)$. Then $\angle px_Rq = \angle y_Rx_Rz_R$. Let g_R be a hyperbolic isometry taking H to the upper-half plane $\{(x_1, 0, 0, .., 0, x_n)|\, x_1 \in \mathbb{R},\ x_n > 0\}$ (in the upper-half space model) and such that $g_Ry_R = (0, 0, ..., 0, e^W)$ and $g_Rx_R = (0, 0, .., 1)$. Note that $g_Rp \to \infty$ and the sphere $S_p(R)$ limits onto the horocycle defined by $x_n = 1$. So the nth coordinate of g_Rz_R goes to e^W as $R \to \infty$. Since the angle $\angle y_Rx_Rz_R$ goes to zero as R goes to infinity, the distance between z_R and y_R goes to 0 as $R \to \infty$. Now it should be clear that the area of the triangle $x_Ry_Rz_R$ goes to zero as R goes to infinity. Note that this triangle contains $H \cap M_R \cap [B_p(R) - B_p(R - W)]$.

We wish to show that the relative fraction of volume of $B_p(R)$ taken up by $M_R \cap [B_p(R) - B_p(R - W)]$ goes to zero as $R \to \infty$. Note that $M_R \cap [B_p(R) - B_p(R - W)]$ is obtained by revolving $H \cap M_R \cap [B_p(R) - B_p(R - W)]$ about the geodesic γ that passes through p and q. Thus it suffices to show that if T_R is any region in $H \cap B_p(R)$ such that the area of T_R goes to 0 as $R \to \infty$ then the relative fraction of volume in $B_p(R)$ taken up by the body obtained by revolving T_R about γ goes to zero as $R \to \infty$. For this, consider the "worst case scenario" when T_R is as far as possible from γ (and thus the body of revolution has as much as volume as possible). Then there exists $\epsilon_R > 0$ such that $\epsilon_R \to 0$ as $R \to \infty$ and T_R is contained in $H \cap [B_p(R) - B_p(R - \epsilon_R)]$. But then the body obtained from revolving T_R about γ is contained in $B_p(R) - B_p(R - \epsilon_R)$. By lemma 1,

$$\frac{vol[B_p(R) - B_p(R - \epsilon_R)]}{vol(B_p(R)} \to 1 - e^{-c_1 \epsilon_R} \to 0$$

as $R \to \infty$. We have now shown

$$\lim_{R \to \infty} \frac{vol(M_R \cap [B_p(R) - B_p(R - W)])}{vol[B_p(R)]} = 0. \tag{30}$$

Since

$$\lim_{R \to \infty} \frac{vol[B_p(R - W)]}{vol[B_p(R)]} = e^{-c_1 W}, \tag{31}$$

$$\lim_{R \to \infty} \frac{vol(M_R)}{vol[B_p(R)]} = \lim_{R \to \infty} \frac{vol[M_R \cap B_p(R - W)]}{vol[B_p(R)]} \le e^{-c_1 W}. \tag{32}$$

Since W is arbitrary,

$$\frac{vol(M_R)}{vol[B_p(R)]} \to 0 \text{ as } R \to \infty, \tag{33}$$

which proves the second claim.

To prove the first claim, let K be the half space containing q whose boundary is the perpendicular bisector of the seqment with endpoints p and q. Let Q be the convex hull of p and K. Then

$$\lim_{R \to \infty} \frac{vol[O_p(R)]}{vol[B_p(R)]} = \frac{vol[B_p(R) - Q]}{vol[B_p(R)]} \tag{34}$$

where the R on the right hand side is arbitrary. In fact this limit converges montonically. Thus, as $r \to \infty$, the above limit equals 1/2. Since $vol[A_q(R)] = vol[B_p(R)] - vol[O_p(R)]$, this finishes the proof of Lemma 2. ■

For $f \in L^\infty(\mathcal{G})$, let $A_-(f)$ and $A_+(f) \in L^\infty(\mathcal{G})$ be defined by

$$A_-(f)(g) = \lim \inf_{R \to \infty} \frac{1}{vol(B_R)} \int_{\tilde{B}_R} f(g'g)\, d\lambda_{\mathcal{G}}(g') \tag{35}$$

$$A_+(f)(g) = \lim \sup_{R \to \infty} \frac{1}{vol(B_R)} \int_{\tilde{B}_R} f(g'g)\, d\lambda_{\mathcal{G}}(g'). \tag{36}$$

Lemma 3. If f is any nonnegative L^∞ function on $\mathcal{G}$ and $g_1, g_2 \in \mathbb{H}^n$ are such that the distance between $g_1\mathcal{O}$ and $g_2\mathcal{O}$ is r then

$$|A_+(f)(g_1) - A_+(f)(g_2)| \le F(r)||f||_\infty \tag{37}$$

and

$$|A_-(f)(g_1) - A_-(f)(g_2)| \le F(r)||f||_\infty. \tag{38}$$

In particular, $A_-(f)$ and $A_+(f)$ are uniformly continuous.

Proof. We will only prove the first inequality as the second can be proved similarly.

$$
\begin{aligned}
A_+(f)(g_1) =& \lim_{R\to\infty} \sup \frac{1}{vol(B_R)} \int_{\tilde{B}_R} f(g'g_1)\, d\lambda_G(g') \\
=& \lim_{R\to\infty} \sup \frac{1}{vol(B_R)} \int_{\tilde{B}_R g_1^{-1}} f(g')\, d\lambda_G(g') \\
=& \lim_{R\to\infty} \sup \frac{1}{vol(B_R)} \int_{\pi^{-1}[B_R(g_1\mathcal{O})]} f(g')\, d\lambda_G(g') \\
\geq& \lim_{R\to\infty} \sup \frac{1}{vol(B_R)} \int_{\pi^{-1}[B_R(g_1\mathcal{O})]\cap\pi^{-1}[B_R(g_2\mathcal{O})]} f(g')\, d\lambda_G(g') \\
\geq& A_+(f)(g_2) - \lim_{R\to\infty} \frac{1}{vol(B_R)} \int_{\pi^{-1}[B_R(g_2\mathcal{O})]-\pi^{-1}[B_R(g_1\mathcal{O})]} ||f||_\infty\, d\lambda_G \\
=& A_+(f)(g_2) - \lim_{R\to\infty} \frac{vol[B_R(g_2\mathcal{O}) - B_R(g_1\mathcal{O})]}{vol(B_R)} ||f||_\infty \\
=& A_+(f)(g_2) - F(r)||f||_\infty.
\end{aligned}
\tag{39}
$$

Since g_1 and g_2 are arbitrary, the last equation proves the lemma. ■

We are now ready to prove Theorem 2.

Proof of Theorem 2. Let $\mathcal{G}_0$ be a countable dense subset of $\mathcal{G}$. Let Z_0 be as in Theorem 1. Let $Z = \bigcap_{g\in\mathcal{G}_0} g^{-1}Z_0$. Since Z is a countable intersection of sets of measure 1, $\mu(Z) = 1$. By definition, for all $z \in Z$ and for all $g \in \mathcal{G}_0$, $gz \in Z_0$. For $x \in X$, define $h_x : \mathcal{G} \to \mathbb{R}$ by $h_x(g) = f(gx)$. Since f is bounded, so is h_x. For $z \in Z$ and $g \in \mathcal{G}_0$, we have

$$
\begin{aligned}
A_+(h_z)(g) =& \lim_{R\to\infty} \sup \frac{1}{vol(B_R)} \int_{\tilde{B}_R} h_z(g'g)\, d\lambda_G(g') \\
=& \lim_{R\to\infty} \sup \frac{1}{vol(B_R)} \int_{\tilde{B}_R} f(g'gz)\, d\lambda_G(g') \\
=& \int_X f d\mu.
\end{aligned}
\tag{40}
$$

The last equation holds since $gz \in Z_0$. Since $A_+(h_z)$ is continuous, we have that the above equations hold for all $g \in \mathcal{G}$. Similarly, we get that $A_-(h_z)(g) = \int_X f d\mu$ for all $g \in \mathcal{G}$ and $z \in Z$. This concludes the proof of the theorem. ■

IV. Summary

In [BoR] the notion of "optimal density" was defined for packings in hyperbolic space $\mathbb{H}^n$, as above, through the use of probability measures, on a space of packings, invariant under the congruence group of $\mathbb{H}^n$. The notion of an optimally dense packing was also introduced in [BoR], but not very successfully; the justification for that term was not well connected to a limit (1). The main advance in this paper is an extension of Nevo's ergodic theorem which allows us to prove the existence of "optimally dense packings" which indeed have a density relative to a family of expanding spheres of the form (1), independent of the center of expansion. We also contrasted our approach to density with earlier approaches, and compared some key examples.

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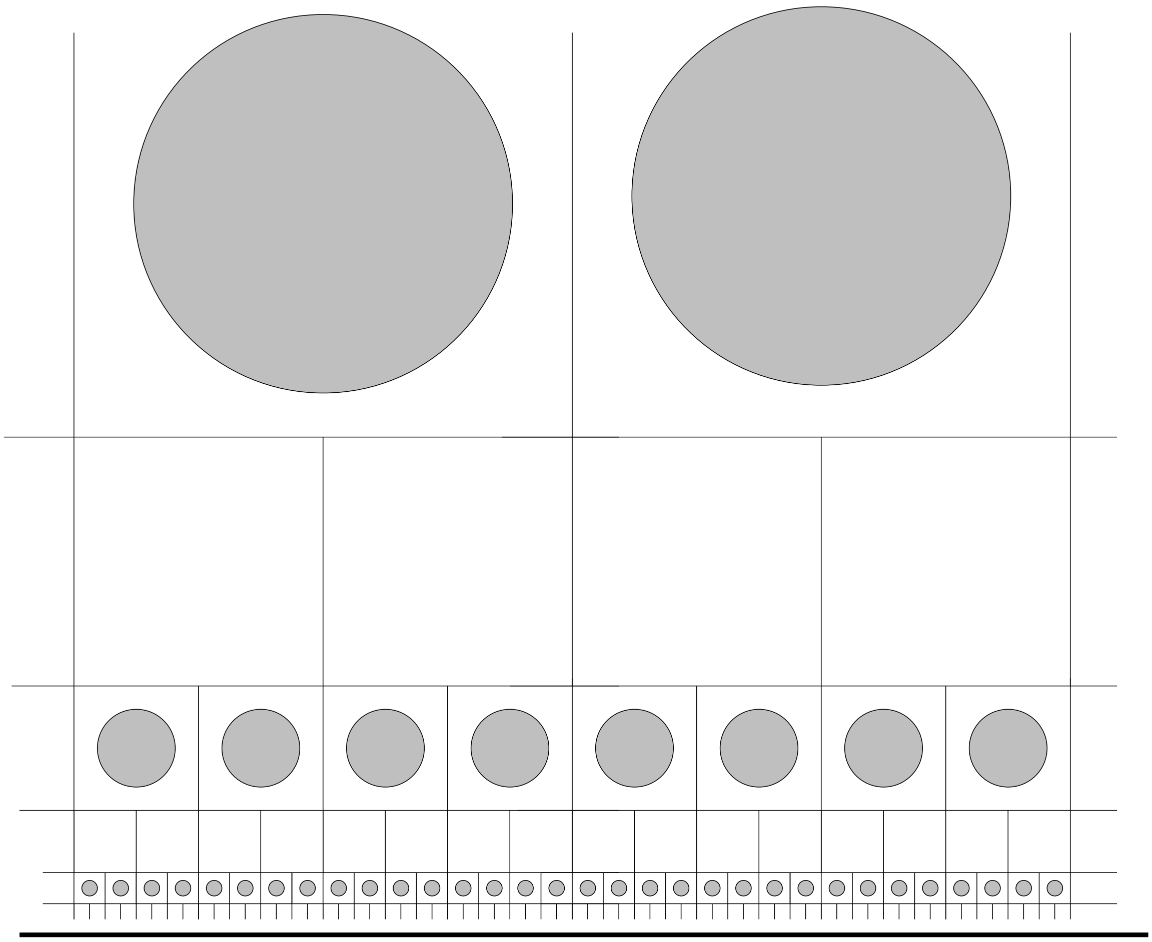

Figure 1. Boroczky's packing of disks

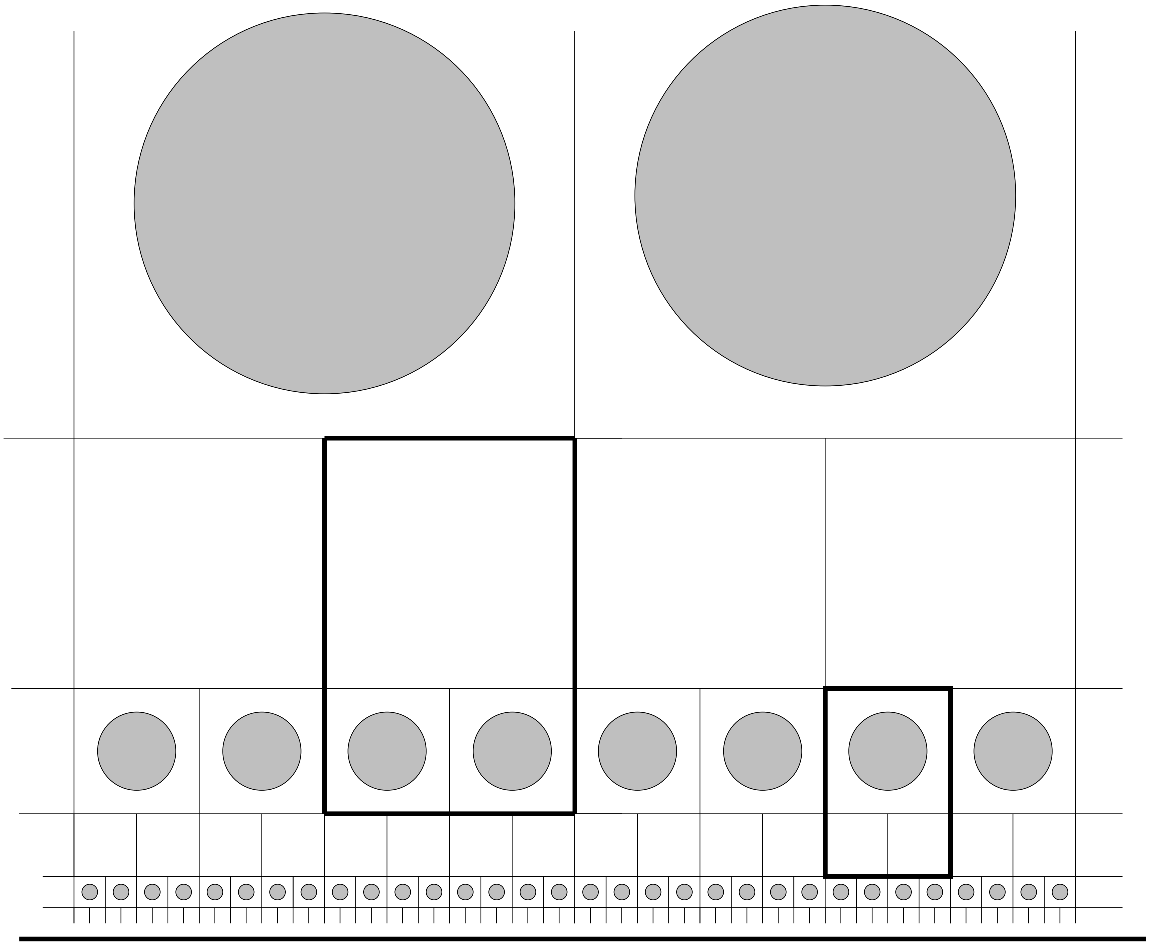

Figure 2. Boroczky's packing with two tiles in dark outline

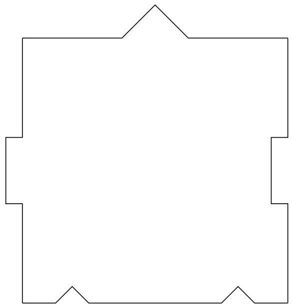

Figure 3. A tile

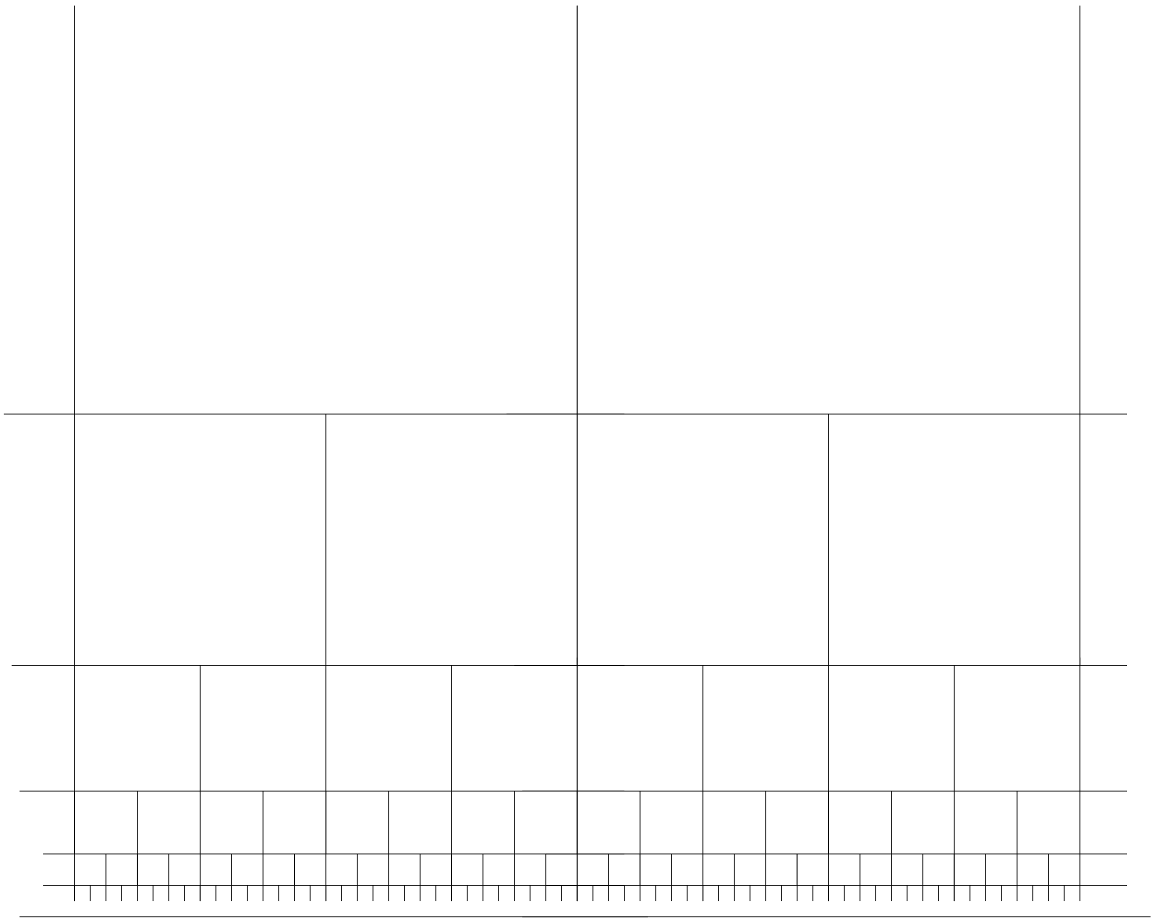

Figure 4. A tiling

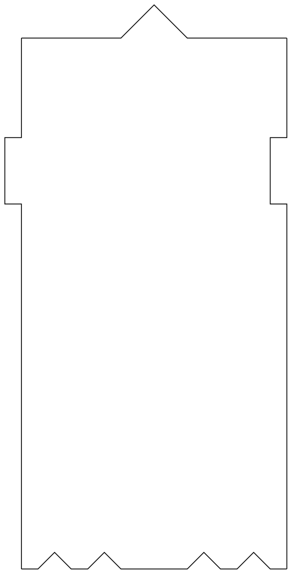

Figure 5. A tile

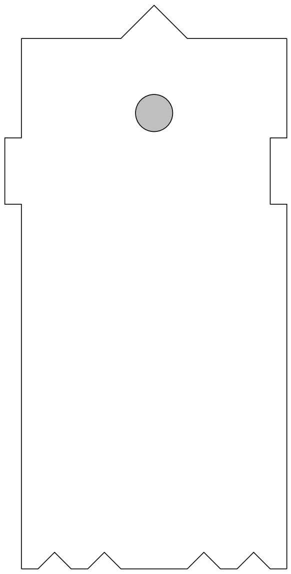

Figure 6. A body

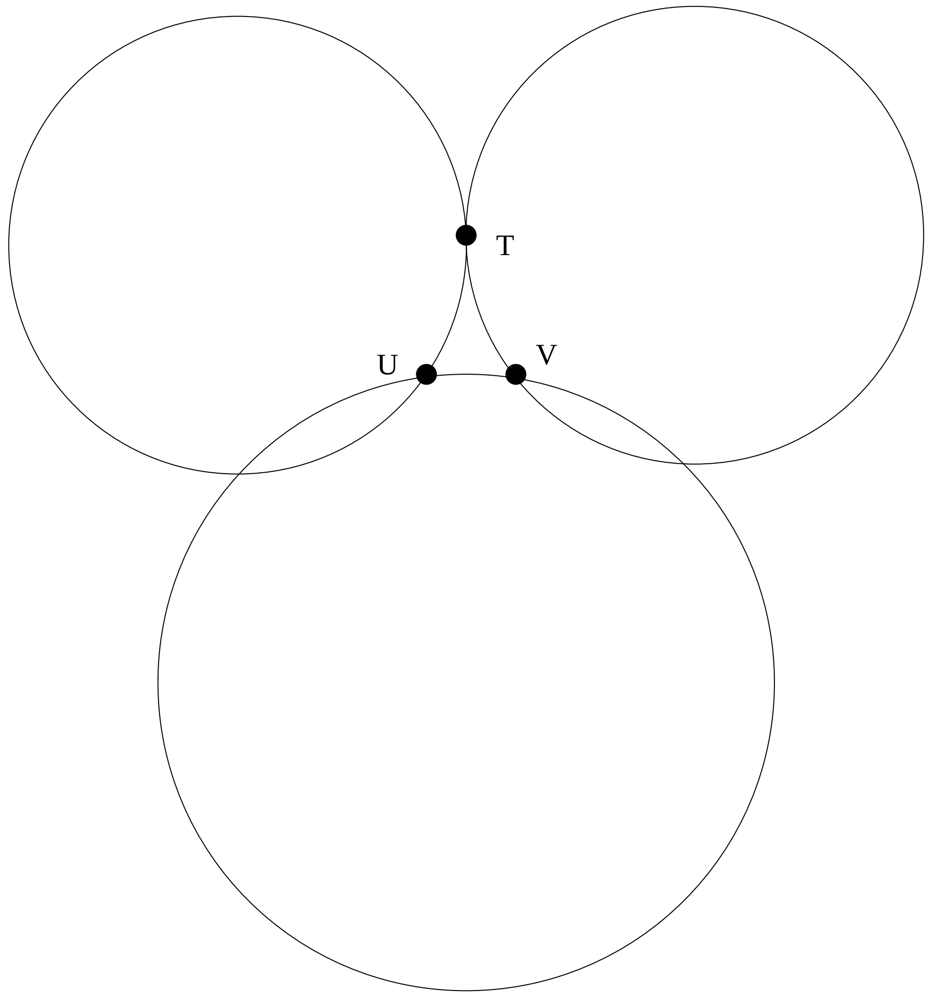

Figure 7. Uncovered regions

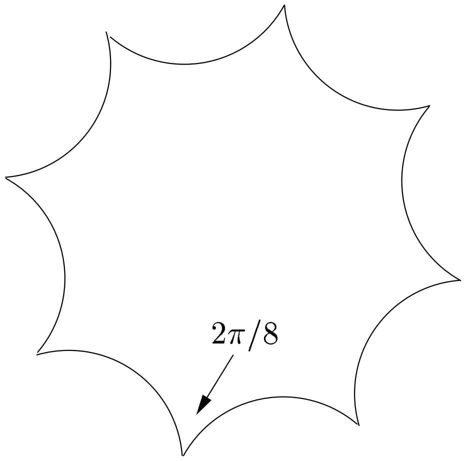

Figure 8. β_1

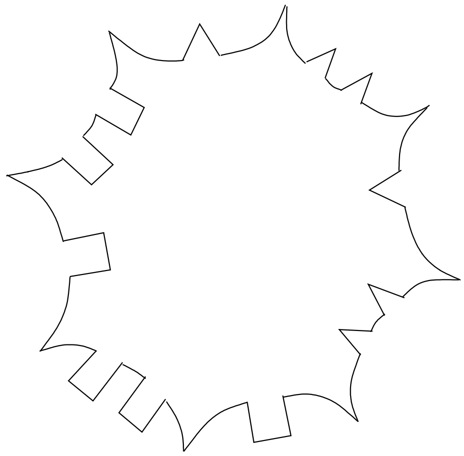

Figure 9. β_2'

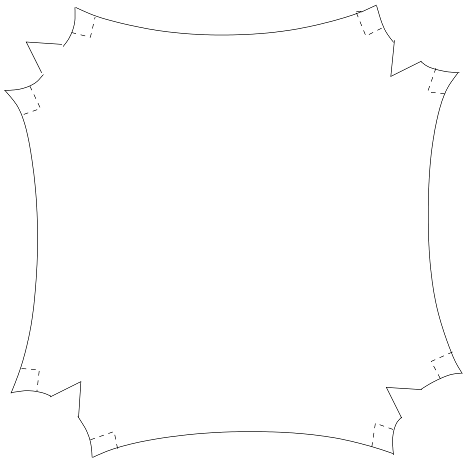

Figure 10. τ

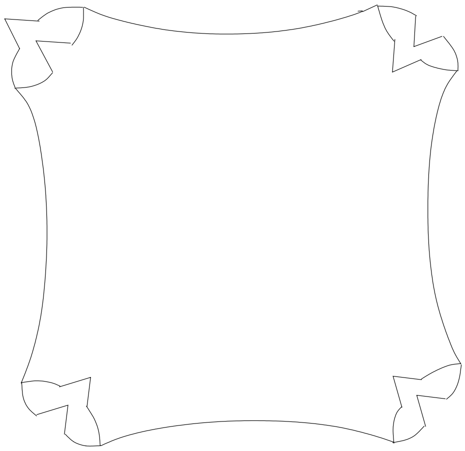

Figure 11. X