On the Long Time Behavior of Infinitely Extended Systems of Particles Interacting via Kac Potentials

Paolo Buttà* Emanuele Caglioti* Carlo Marchioro*

Abstract

We analyze the long time behavior of an infinitely extended system of particles in one dimension, evolving according to the Newton laws and interacting via a non-negative superstable Kac potential $\phi_\gamma(x) = \gamma\phi(\gamma x)$, $\gamma \in (0,1]$. We first prove that the velocity of a particle grows at most linearly in time, with rate of order γ. We next study the motion of a fast particle interacting with a background of slow particles, and we prove that its velocity remains almost unchanged for a very long time (at least proportional to γ^{-1} times the velocity itself). Finally we shortly discuss the so called "Vlasov limit", when time and space are scaled by a factor γ.

Key words: Infinite particle system, Kac potential, Vlasov limit.
Running title: Long time behavior for Kac particle systems.

*Dipartimento di Matematica, Università di Roma 'La Sapienza', P.le Aldo Moro 2, 00185 Roma, Italy. E-mail: `butta@mat.uniroma1.it`, `caglioti@mat.uniroma1.it`, `marchior@mat.uniroma1.it`. Fax: +39–06–44701007.

1. Introduction

The thermodynamic equilibrium structure of an infinitely extended particle system interacting via Kac potentials has been largely studied in the literature. As well known, in a suitable limit, the phase diagram converges, for any temperature, to the van der Waals phase diagram, comprehensive of the Maxwell equal area rule [9, 13]. Among the most recent results we mention [11, 12], where the existence of the liquid-vapor branch for point particles in $\mathbb{R}^d$, interacting via a potential with long but finite range, has been rigorously proved. Furthermore, the stochastic time evolutions of these systems, motivated by the analysis of non-equilibrium phenomena like metastability and phase segregation, have been addressed by many papers in the last decade, see e.g. [8] and references therein.

In the present paper we investigate the dynamical behavior of infinitely many particles in one dimension evolving according to the usual Newton laws and interacting via a Kac potential (see below). We discuss two type of results; firstly, some sharp estimates that remain valid for very long time; secondly, the relation between the particle system and the Vlasov equation. More precisely, in Section 2 we obtain a bound of the growth of the velocity of a particle which remains significant for long time. In Section 3 we consider a fast particle interacting with a background of slow ones. We show that the background cannot slow down rapidly the fast particle, which thus almost preserves its velocity for a long time. Recently, problems similar to that studied in Sections 2 and 3 have been investigated for a short range interaction [2]. Here we prove that, as expected, the time involved is longer by a factor γ^{-1}, where $\gamma \in (0,1]$ is the inverse of the range of the Kac potential. This is not a trivial change of scale, because the initial particle distribution does not depend on γ. Part of the proofs are similar to the ones in [2], with some improvements that will be outlined in the sequel. Here we only remark the main difficulty of the new proof. In [2] the presence of fast particles was excluded by energy conservation, while in the present case the latter only implies an upper bound on the number of fast particles. However we show that this does not affect the asymptotic behavior of the motion. The estimates depend on the dimension d and do not catch the right behavior for $d > 1$: this is the reason for which we restricted our analysis to dimension $d = 1$.

We finally discuss, in Section 4, the Vlasov limit, when space and time are scaled by a factor γ: by assuming convergence of the initial data, the solution of the Newton system converges (in the local weak topology) to the solution of the Vlasov equation. This statement is well known for systems with finite total mass; in the present paper we extend the result to the case of infinite total mass.

2. Bounds on the growth of the velocity of a particle

We consider an infinite particle system in one dimension evolving according to the Newton laws and interacting by means of a non-negative, two-body Kac potential $\phi_\gamma(x) = \gamma\phi(\gamma x)$, $x \in \mathbb{R}$, $\gamma \in (0,1]$. The function $\phi(x)$ is symmetric, twice differentiable, strictly positive at the origin and short-range: without loss of generality we assume

$$\phi(x) = 0 \qquad \text{if} \qquad |x| \geq 1. \tag{2.1}$$

We point out that the potential is assumed non-negative for technical reasons: the extension to a generic superstable potential requires different arguments.

The state of the system is determined by the infinite sequence $X = \{x_i, v_i\}_{i\in\mathbb{N}}$ of positions and velocities of the particles. The state X is assumed to have a locally finite density and energy. We define, for any $\mu \in \mathbb{R}$ and $R > 0$,

$$Q_\gamma(X;\mu,R) \doteq \sum_i \chi_i(\mu,R)\left\{\frac{v_i^2}{2} + \frac{1}{2}\sum_{j:j\neq i}\phi_\gamma(x_i - x_j) + 1\right\}, \tag{2.2}$$

where $\chi_i(\mu,R) = \chi(|x_i - \mu| \leq R)$ and $\chi(A)$ denotes the characteristic function of the set A.

In order to consider configurations which are typical for the thermodynamic states, we allow initial data with logarithmic divergences in the velocities and local densities. More precisely, by defining

$$Q_\gamma(X) \doteq \sup_{\mu} \sup_{R:R>\log(e+|\mu|)} \frac{Q_\gamma(X;\mu,R)}{2R}, \tag{2.3}$$

the set of all configurations for which $Q_\gamma(X) < +\infty$ has a full measure w.r.t. any Gibbs state associated to the potential ϕ_γ [7].

In fact the condition $Q_\gamma(X) < +\infty$ does not depend on the scaling parameter $\gamma \in (0,1]$. Indeed, since $\phi(x)$ is non-negative and $\phi(0) > 0$, the interaction $\phi_\gamma(x)$ is superstable [15]. More precisely, there are constants $B_1 > 0$ and $B_2 \geq 0$ such that, for any $\gamma \in (0,1]$ and for any finite configuration of particles $\{x_1, \dots, x_n\}$, $n \in \mathbb{N}$,

$$\sum_{i<j} \phi_\gamma(x_i - x_j) \geq B_1 \gamma \sum_{k\in\mathbb{Z}} n_k^2 - B_2 \gamma n, \tag{2.4}$$

where n_k is the number of particles in the interval $[k\gamma^{-1}, (k+1)\gamma^{-1})$. By using (2.4) we shall prove in the Appendix that there is a constant $B_3 \geq 1$ such that, for any $\gamma \in (0,1]$ and for any particle configuration X,

$$\frac{\gamma}{B_3} Q_1(X) \leq Q_\gamma(X) \leq B_3 Q_1(X), \tag{2.5}$$

which implies in particular that $Q_\gamma(X) < +\infty$ if and only if $Q_1(X) < +\infty$. With this in mind we introduce the set $\mathcal{X} \doteq \{X : Q_1(X) < +\infty\}$.

The time evolution $t \mapsto X(t)$ is defined by the solutions of the Newton equations:

$$\ddot{x}_i(t) = \sum_{\substack{j\in\mathbb{N}\\ j\neq i}} F_\gamma(x_i(t) - x_j(t)), \qquad i \in \mathbb{N}, \tag{2.6}$$

where $F_\gamma(x) = -\nabla\phi_\gamma(x) = -\gamma^2 \nabla\phi(\gamma x)$.

The Cauchy problem for this system of infinite equations is well posed when the initial condition is chosen in the set $\mathcal{X}$, and the solution is constructed by means of the following limiting procedure.

Given $X \in \mathcal{X}$ and $n \in \mathbb{N}$, let $I_n \doteq \{i \in \mathbb{N} : x_i \in B(0,n)\}$, where $B(\mu, R) \doteq \{y \in \mathbb{R} : |y - \mu| \leq R\}$. We define the n-partial dynamics $t \mapsto X^{(n)}(t) = \{x_i^{(n)}(t), v_i^{(n)}(t)\}_{i\in I_n}$ as the solution of the differential system:

$$\begin{cases} \ddot{x}_i^{(n)}(t) = \displaystyle\sum_{\substack{j\in I_n\\ j\neq i}} F_\gamma(x_i^{(n)}(t) - x_j^{(n)}(t)) \\ x_i^{(n)}(0) = x_i, \quad v_i^{(n)}(0) = v_i, \qquad i \in I_n. \end{cases} \tag{2.7}$$

Notice that the set I_n is determined by the initial conditions X. Then (see [2] and references therein):

Theorem 2.1. For any $X \in \mathcal{X}$ there exists a unique flow $t \mapsto X(t) = \{x_i(t), v_i(t)\}_{i\in\mathbb{N}} \in \mathcal{X}$ satisfying (2.6) with initial data $X(0) = X$. Moreover, for any $t \geq 0$ and $i \in \mathbb{N}$,

$$\lim_{n\to+\infty} x_i^{(n)}(t) = x_i(t), \qquad \lim_{n\to+\infty} v_i^{(n)}(t) = v_i(t). \tag{2.8}$$

Observe that, in order to simplify notation, we omit the explicit dependence on γ in $X(t)$ and $X^{(n)}(t)$. We also remark that this dependence is only due to the forces: the initial condition $X \in \mathcal{X}$ is chosen independently of this parameter.

We now state the main result of the present section.

Theorem 2.2. For any $X \in \mathcal{X}$ there are two positive constants C_1 and C_2 such that for any $\gamma \in (0,1]$ the following holds. Let $t \mapsto X(t)$ be the (unique) solution of (2.6) with initial data $X(0) = X$. Then, for any $i \in \mathbb{N}$ and $t \geq 0$,

$$|v_i(t)| \leq C_1\sqrt{\log(e + |x_i| + \gamma^{-1})} + C_2\gamma t. \tag{2.9}$$

Remarks. Here we are interested in the long time behavior of the velocities. Indeed the bound (2.9) is not good for t and $|x_i|$ small, as follows by observing that, from definitions (2.2) and (2.3),

$$|v_i| \leq 2\sqrt{Q_1(X)\log(e + |x_i|)} \qquad \forall\, i \in \mathbb{N}. \tag{2.10}$$

On the other hand, for $t \leq \tau = \gamma^{-1}|\log\gamma|^{-1/2}$ and $|x_i| \leq a\gamma^{-1}$, $a > 0$, the bound (2.9) implies that the maximal velocity of the i-th particle is bounded by a constant multiple of $\sqrt{\log[e + (1+a)\gamma^{-1}]}$, so that its maximal displacement is bounded by a constant multiple of γ^{-1}. Therefore for any $r > 0$ there is $c(r)$ such that, during the time $[0,\tau]$, the force acting on a single particle initially located in $B(0, r\gamma^{-1})$ is bounded by $c(r)\gamma$, and hence $|v_i(t)| \leq |v_i| + c(r)\gamma t$ for $t \leq \tau$ (which is an improvement of (2.9)).

We finally observe that the estimate (2.9) for $\gamma = 1$ improves the analogous one in [2] where the dependence on the initial position of the particle was not explicitly determined.

To prove the theorem we first obtain a bound on the velocities of the particles for the n-partial dynamics, which is the content of Proposition

2.3 below; we then get (2.9) from this bound by evaluating the difference between the partial and the infinite dynamics.

Proposition 2.3. For any $X \in \mathcal{X}$ there are two positive constants C_3 and C_4 such that for any $\gamma \in (0,1]$ and $n \in \mathbb{N}$ the following holds. Let $t \mapsto X^{(n)}(t)$ be the solution of (2.7). Then, for any $i \in I_n$ and $t \geq 0$,

$$|v_i^{(n)}(t)| \leq C_3 \sqrt{\log(e+n)} + C_4 \gamma t. \tag{2.11}$$

Remark. The proof of the proposition is completely different from the one in [2] for the analogous statement with $\gamma = 1$. The argument used in [2], which is based on energy estimates, is not sufficient to get the correct dependence on γ as in (2.11). We thus need a more deeper analysis of the motion. More precisely, we notice that the force acting on a fast particle can be written as the sum of its interaction with the slow particles and with the other equally fast particles. The first contribution is small because the time of collision is very short. The second contribution is controlled by using the conservation of energy, which implies an upper bound on the total number of fast particles at each time. An extra argument is finally required to control the re-collisions. As a result we are able to prove that fast particles should be necessarily present also at time zero, and this is excluded by our assumptions on the distribution of the initial data.

Proof of Proposition 2.3. We define

$$V_n(t) \doteq \max_{i \in I_n} \sup_{s \in [0,t]} |v_i^{(n)}(s)| \tag{2.12}$$

and, for C_3, $C_4 > 0$ to be fixed later,

$$U_n(t) \doteq C_3 \sqrt{\log(e+n)} + C_4 \gamma t. \tag{2.13}$$

Let

$$T_n \doteq \sup\{t \geq 0 \,:\, V_n(s) \leq U_n(s) \;\; \forall\, s \in [0,t]\}, \tag{2.14}$$

setting $T_n = 0$ if the above set is empty (i.e. when $V_n(t) > U_n(t)$ for all $t \geq 0$). The proposition is proved if we show that, for C_3 and C_4 sufficiently large, $T_n = +\infty$ for all $n \in \mathbb{N}$ and $\gamma \in (0,1]$.

We first assume $C_3 \geq 4\sqrt{Q_1(X)}$ so that, by (2.10), $V_n(0) \leq U_n(0)/2$. By continuity, this implies $T_n > 0$ for all $n \in \mathbb{N}$. We next proceed by contradiction. We fix $n \in \mathbb{N}$ and assume $0 < T_n < +\infty$, which implies $V_n(T_n) = U_n(T_n)$. Since $V_n(0) < 3U_n(0)/4$, there exists $i \in I_n$ such that $|v_i^{(n)}(t_1)| = 3U_n(T_n)/4$ for some $t_1 \in (0, T_n)$. Setting

$$t_0 \doteq \inf\left\{t \in [0, t_1] \, : \, \left| |v_i^{(n)}(t)| - \frac{3U_n(T_n)}{4} \right| \leq \frac{U_n(T_n)}{5} \right\},$$

and observing $|v_i^{(n)}(t_0)| > U_n(0)/2 \geq V_n(0)$, we get an absurd if we show that $t_0 = 0$.

Clearly $t_0 < t_1$ and, from (2.1), (2.7), and the definition of F_γ,

$$\left| |v_i^{(n)}(t_0)| - \frac{3U_n(T_n)}{4} \right| \leq \|\nabla\phi\|_\infty \gamma^2 \int_{t_0}^{t_1} ds \sum_{\substack{j \in I_n \\ j \neq i}} \chi_{i,j}^{(n)}(s) \tag{2.15}$$

where $\chi_{i,j}^{(n)}(s) \doteq \chi(|x_i^{(n)}(s) - x_j^{(n)}(s)| \leq \gamma^{-1})$, i.e. the characteristic function of the set $\{s \in [t_0, t_1] : |x_i^{(n)}(s) - x_j^{(n)}(s)| \leq \gamma^{-1}\}$. We look for an upper bound to the right hand side of (2.15): if we show it can be done smaller than e.g. $U_n(T_n)/10$, then $t_0 = 0$. We decompose

$$\{s \in [t_0, t_1] : |x_i^{(n)}(s) - x_j^{(n)}(s)| \leq \gamma^{-1}\} = \Delta_{i,j}^+ \cup \Delta_{i,j}^-$$

with

$$\Delta_{i,j}^+ \doteq \left\{ s \in [t_0, t_1] \, : \, |v_j^{(n)}(s)| > \frac{U_n(T_n)}{2}, \quad |x_i^{(n)}(s) - x_j^{(n)}(s)| \leq \gamma^{-1} \right\},$$

$$\Delta_{i,j}^- \doteq \left\{ s \in [t_0, t_1] \, : \, |v_j^{(n)}(s)| \leq \frac{U_n(T_n)}{2}, \quad |x_i^{(n)}(s) - x_j^{(n)}(s)| \leq \gamma^{-1} \right\},$$

so that $\chi_{i,j}^{(n)}(s) = \chi(\Delta_{i,j}^+)(s) + \chi(\Delta_{i,j}^-)(s)$. We note that

$$|v_i^{(n)}(s)| - |v_j^{(n)}(s)| \geq \frac{U_n(T_n)}{20} \qquad \forall\, s \in \Delta_{i,j}^-$$

and, since $|v_j^{(n)}(s)| \leq U_n(T_n)$ for any $s \in [0, T_n]$,

$$\left| |v_i^{(n)}(s)| - |v_j^{(n)}(s)| \right| \leq \frac{9U_n(T_n)}{20} \qquad \forall\, s \in \Delta_{i,j}^+.$$

Then, setting $t_{i,j} = \sup\{s : s \in \Delta_{i,j}^-\}$, the previous bounds imply that

$$|x_i^{(n)}(t_{i,j}) - x_j^{(n)}(t_{i,j})| \geq \frac{U_n(T_n)}{20}|\Delta_{i,j}^-| - \frac{9U_n(T_n)}{20}|\Delta_{i,j}^+|.$$

But $|x_i^{(n)}(t_{i,j}) - x_j^{(n)}(t_{i,j})| \le \gamma^{-1}$, so that

$$|\Delta^-_{i,j}| \le \frac{20\gamma^{-1}}{U_n(T_n)} + 9|\Delta^+_{i,j}|.$$

It follows there is a decomposition $\Delta^-_{i,j} = \Delta^*_{i,j} \cup \Delta^\dagger_{i,j}$ with

$$\Delta^*_{i,j} \cap \Delta^\dagger_{i,j} = \emptyset, \qquad |\Delta^*_{i,j}| \le \frac{20\gamma^{-1}}{U_n(T_n)}, \qquad |\Delta^\dagger_{i,j}| \le 9|\Delta^+_{i,j}|.$$

Then $\chi(\Delta^-_{i,j})(s) = \chi(\Delta^*_{i,j})(s) + \chi(\Delta^\dagger_{i,j})(s)$ and therefore:

$$\begin{aligned} \int_{t_0}^{t_1} ds\, \chi^{(n)}_{i,j}(s) &\le \int_{t_0}^{t_1} ds\, \chi(\Delta^*_{i,j})(s) + 10|\Delta^+_{i,j}| \\ &= \int_{t_0}^{t_1} ds\, \left[\chi(\Delta^*_{i,j})(s) + 10\chi(\Delta^+_{i,j})(s)\right]. \end{aligned} \tag{2.16}$$

From (2.15) and (2.16) we obtain the following estimate:

$$\left| |v_i^{(n)}(t_0)| - \frac{3U_n(T_n)}{4} \right| \le \|\nabla\phi\|_\infty \gamma^2 (A_1 + A_2), \tag{2.17}$$

with

$$A_1 = \int_{t_0}^{t_1} ds \sum_{\substack{j\in I_n \\ j\neq i}} \chi(\Delta^*_{i,j})(s) \le \frac{20\gamma^{-1}}{U_n(T_n)} \bar{N}_n, \tag{2.18}$$

where $\bar{N}_n$ denotes the number of particles which can interact with the i-th particle during the time $[0, T_n]$, and

$$A_2 = 10 \int_{t_0}^{t_1} ds \sum_{\substack{j\in I_n \\ j\neq i}} \chi(\Delta^+_{i,j})(s) \le 10 T_n \sup_{s\in[0,T_n]} N_n^>(s), \tag{2.19}$$

where $N_n^>(s)$ denotes the number of particles of $X^{(n)}(s)$ which are contained in $B(x_i^{(n)}(s), \gamma^{-1})$ and whose velocity is bigger than $U_n(T_n)/2$.

Now we have to bound the quantities $\bar{N}_n$ and $N_n^>(s)$. For the latter we shall need the following lemma, whose proof is given in the Appendix.

Lemma 2.4. For any $X \in \mathcal{X}$ there is $C_5 > 0$ such that, for any $\gamma \in (0,1]$, $n \in \mathbb{N}$, and $t \ge 0$,

$$\sup_\mu Q_\gamma(X^{(n)}(t); \mu, R_n(t)) \le C_5 R_n(t), \tag{2.20}$$

where

$$R_n(t) \doteq \log(e+n) + 2\gamma^{-1} + \int_0^t ds\, V_n(s) \tag{2.21}$$

and $V_n(\cdot)$ is defined in (2.12).

We observe that the i-th particle can interact, during the time $[0, T_n]$, only with particles which are initially in $B(x_i, 2R_n(T_n))$. Then, denoting by $N(X; \mu, R)$ the number of particles of X which are in $B(\mu, R)$, and recalling the definition (2.3), we have:

$$\begin{aligned} \bar{N}_n &\leq N(X; x_i, 2R_n(T_n)) \leq 4Q_1(X)R_n(T_n) \\ &\leq 4Q_1(X)[\log(e+n) + 2\gamma^{-1} + U_n(T_n)T_n]. \end{aligned} \tag{2.22}$$

From (2.20) and recalling the definition (2.2), we have, for all $s \geq 0$,

$$\begin{aligned} \frac{U_n(T_n)^2}{4} N_n^{>}(s) &\leq Q_\gamma(X^{(n)}(s); x_i^{(n)}(s), \gamma^{-1}) \\ &\leq \sup_\mu Q_\gamma(X^{(n)}(s); \mu, R_n(s)) \leq C_5 R_n(s), \end{aligned}$$

so that

$$\sup_{s\in[0,T_n]} N_n^{>}(s) \leq 4C_5 \frac{\log(e+n) + 2\gamma^{-1} + U_n(T_n)T_n}{U_n(T_n)^2}. \tag{2.23}$$

From (2.18), (2.22), by using the definition (2.13) of $U_n(\cdot)$, and neglecting some positive terms, it follows that

$$A_1 \leq 80Q_1(X) \left(\frac{3}{C_3^2} + \frac{1}{C_4} \right) \gamma^{-2} U_n(T_n). \tag{2.24}$$

Analogously, from (2.19) and (2.23),

$$A_2 \leq \frac{40C_5}{C_4} \left(\frac{3}{C_3^2} + \frac{1}{C_4} \right) \gamma^{-2} U_n(T_n). \tag{2.25}$$

From (2.17), (2.24), and (2.25), by choosing C_3 and C_4 large enough we obtain $\left| |v_i^{(n)}(t_0)| - 3U_n(T_n)/4 \right| \leq U_n(T_n)/10$, hence $t_0 = 0$ and the proposition is proved. ■

Proof of Theorem 2.2. Let

$$\delta_i(n,t) \doteq |x_i^{(n)}(t) - x_i^{(n-1)}(t)| + |v_i^{(n)}(t) - v_i^{(n-1)}(t)|. \tag{2.26}$$

From the equations of motion in integral form we have:

$$
\begin{cases}
v_i^{(n)}(t) = v_i + \displaystyle\int_0^t ds \sum_{\substack{j\in I_n\\ j\neq i}} F_\gamma(x_i^{(n)}(s) - x_j^{(n)}(s)) \\
\\
x_i^{(n)}(t) = x_i + v_i t + \displaystyle\int_0^t ds\,(t-s) \sum_{\substack{j\in I_n\\ j\neq i}} F_\gamma(x_i^{(n)}(s) - x_j^{(n)}(s)).
\end{cases}
\tag{2.27}
$$

By (2.11) each particle $i \in I_n$ may interact, during the time $[0,t]$, only with the particles initially contained in $B(x_i, p_n(t))$, where

$$p_n(t) \doteq 2\gamma^{-1} + 2[C_3\sqrt{\log(e+n)} + C_4\gamma t]t. \tag{2.28}$$

Therefore, by definitions (2.2) and (2.3), for any $s \in [0,t]$,

$$
\begin{aligned}
N(X^{(n)}(s); x_i^{(n)}(s), \gamma^{-1}) &\leq N(X; x_i, p_n(t)) \leq Q_1(X; x_i, p_n(t)) \\
&\leq 2Q_1(X)[\log(e+n) + p_n(t)].
\end{aligned}
\tag{2.29}
$$

Now fix $k \in \mathbb{N}$ and define

$$n(k) \doteq \min\{m \in \mathbb{N} : n > 1 + k + p_n(t) \quad \forall n \geq m\}. \tag{2.30}$$

For $n \geq n(k)$ each particle $i \in I_k$ cannot interact (during the time $[0,t]$) with the particles $j \in I_n \setminus I_{n-1}$. Then, from (2.26) and (2.27), for any $i \in I_k$ and $n \geq n(k)$, we have that

$$\delta_i(n,t) \leq \|\Delta\phi\|_\infty \gamma^3 (1+t) \int_0^t ds \sum_j{}^{*} \left(\delta_i(n,s) + \delta_j(n,s)\right),$$

where $\sum_j^*$ means the sum restricted to all the particles $j \in I_{n-1}$ closer than γ^{-1} to $x_i^{(n)}(s)$ or $x_i^{(n-1)}(s)$. The number of these particles is bounded in (2.29). Hence, defining

$$u_k(n,t) \doteq \sup_{i\in I_k} \delta_i(n,t), \tag{2.31}$$

we get

$$u_k(n,t) \leq g_n(t) \int_0^t ds\, u_{k_1}(n,s), \tag{2.32}$$

where $k_1 = \mathrm{Int}\,[k + p_n(t)] + 1$ ($\mathrm{Int}\,[\cdot]$ denotes the integer part of $\cdot$) and

$$g_n(t) \doteq 2\|\Delta\phi\|_\infty \gamma^3 (1+t) Q_1(X)[\log(e+n) + p_n(t)]. \tag{2.33}$$

Setting $k_q = \mathrm{Int}\,[k_{q-1} + p_n(t)] + 1$, $q \in \mathbb{N}$, and $k_0 = k$, we can iterate the inequality (2.32) ℓ times, with

$$\ell = \mathrm{Int}\left[\frac{n-k-1}{1+p_n(t)}\right] \tag{2.34}$$

(which ensures $n > n(k_{\ell-1})$). Since $u_k(n,t) \le a_n(t)$ with

$$a_n(t) \doteq 2(1+t)[C_3\sqrt{\log(e+n)} + C_4\gamma t], \tag{2.35}$$

we finally obtain the bound:

$$u_k(n,t) \le a_n(t)\frac{[g_n(t)t]^\ell}{\ell!}. \tag{2.36}$$

We can now prove (2.9). We choose $k = \mathrm{Int}\,[|x_i|] + 1$ and we consider the n^*-dynamics, where

$$n^* = \mathrm{Int}\left[\alpha(k^2 + \gamma^{-2})e^{\gamma t}\right], \tag{2.37}$$

with $\alpha > 1$ to be fixed later. From (2.11) and (2.37) $v_i^{(n^*)}(t)$ satisfies a bound like (2.9) with suitable positive constants C_1^* and C_2^*. On the other hand, by (2.31),

$$|v_i(t) - v_i^{(n^*)}(t)| \le \sum_{n \ge n^*} u_k(n,t). \tag{2.38}$$

From definitions (2.30) and (2.37) it is easy to check that there exists α_0 such that if $\alpha \ge \alpha_0$ then $n^* \ge n(k)$ for all $k \ge 1$, $\gamma \in (0,1]$, and $t \ge 0$. We can then use (2.36) to bound each term in the sum on the right hand side of (2.38). Moreover, from (2.37) and recalling definitions (2.28), (2.33), (2.34), and (2.35), there is $C_6 > 1$ such that, for any $\alpha \ge \alpha_0$ and $n \ge n^*$, the following bounds hold:

$$\begin{array}{lcllcl} t & \le & \gamma^{-1}\log(e+n), & p_n(t) & \le & C_6\gamma^{-1}\log^2(e+n), \\ g_n(t) & \le & C_6\gamma\log^3(e+n), & a_n(t) & \le & C_6\gamma^{-1}\log^2(e+n), \\ \ell & \ge & \dfrac{\gamma(n-k-1)}{2C_6\log^2(e+n)}. & & & \end{array} \tag{2.39}$$

Inserting the bounds above in (2.36) and using Stirling formula we get:

$$u_k(n,t) \le C_6 \gamma^{-1} \exp\left[-\ell \log \frac{\gamma(n-k-1)}{2eC_6^2 \log^8(e+n)}\right]. \tag{2.40}$$

Since $n^* \ge \alpha(k^2+\gamma^{-2})$, there is $\alpha_1 \ge \alpha_0$ such that the log in the square brackets on the right hand side of (2.40) is not smaller than 1 for all $k \in \mathbb{N}$, $\alpha \ge \alpha_1$, and $n \ge n^*$. Hence, from (2.38), the last bound in (2.39), and (2.40) we obtain, for all $\alpha \ge \alpha_1$,

$$|v_i(t) - v_i^{(n^*)}(t)| \le C_6 \gamma^{-1} \sum_{n \ge \alpha(k^2+\gamma^{-2})} \exp\left[-\frac{\gamma(n-k-1)}{2C_6 \log^2(e+n)}\right].$$

By choosing α large enough, the right hand side is bounded uniformly in $k \in \mathbb{N}$ and $\gamma \in (0,1]$. The theorem is proved. ■

3. Interaction of a fast particle with a background of slow particles

In this section we study a system composed by a tagged particle of position and velocity $(\hat{x}, \hat{v})$ and mass M coupled with an infinite particle system like the one discussed in the previous section. The tagged particle interacts with the other particles via a Kac potential $\hat{\phi}_\gamma(x) = \gamma \hat{\phi}(\gamma x)$, $x \in \mathbb{R}$, $\gamma \in (0,1]$. The function $\hat{\phi}(x)$ is symmetric, twice differentiable, and short-range: without loss of generality we assume $\hat{\phi}(x) = 0$ if $|x| \ge 1$. Setting $\hat{F}_\gamma(x) = -\gamma^2 \nabla \hat{\phi}(\gamma x)$, the equations of motion are:

$$\begin{cases} \ddot{\hat{x}}(t) = M^{-1} \sum_{j \in \mathbb{N}} \hat{F}_\gamma(\hat{x}(t) - x_j(t)), \\ \\ \ddot{x}_i(t) = \sum_{\substack{j \in \mathbb{N} \\ j \ne i}} F_\gamma(x_i(t) - x_j(t)) + \hat{F}_\gamma(x_i(t) - \hat{x}(t)), \qquad i \in \mathbb{N}, \\ \\ \hat{x}(0) = \hat{x}_0, \quad \hat{v}(0) = \hat{v}_0, \quad X(0) = X. \end{cases} \tag{3.1}$$

Without loss of generality we assume $\hat{x}_0 = 0$ and $\hat{v}_0 \ge 0$. For $X \in \mathcal{X}$ the existence and uniqueness of the solutions for the Cauchy problem (3.1) is a trivial generalization of Theorem 2.1. The main result of the present section is the following theorem.

Theorem 3.1. For any $X \in \mathcal{X}$ there are three positive constants C^*, $\hat{C}$, and $\bar{C}$ such that, for any $\gamma \in (0,1]$ and $\hat{v}_0 \geq C^* \sqrt{\log(e+\gamma^{-1})}$,

$$|\hat{v}(t) - \hat{v}_0| \leq \bar{C} \qquad \forall\, t \in [0, \hat{C}\gamma^{-1}\hat{v}_0]. \tag{3.2}$$

Proof. We shall prove the theorem by showing an analogous result for the n-partial dynamics which is uniform in n. Let $(\hat{x}^{(n)}, \hat{v}^{(n)})$ be the position and velocity of the tagged particle w.r.t. the n-partial dynamics. From now on we assume $\hat{v}_0 \geq C^* \sqrt{\log(e+\gamma^{-1})}$, with C^* a positive constant to be fixed independently of n. Let

$$\hat{V}_n(t) \doteq \max_{i \in I_n} G\left(\inf_{s\in[0,t]} |x_i^{(n)}(s)| - 2\hat{v}_0 t - \gamma^{-1}\right) \sup_{s\in[0,t]} |v_i^{(n)}(s)|, \tag{3.3}$$

where $G \in C(\mathbb{R})$ is not increasing and satisfying: $G(x) = 1$ for $x \leq 0$, $G(x) = 0$ for $x \geq 1$. We next define

$$\hat{T}_n \doteq \sup\left\{ t \geq 0 \,:\, \hat{V}_n(t) \leq \frac{\hat{v}_0}{2}, \quad \sup_{s\in[0,t]} |\hat{v}^{(n)}(s) - \hat{v}_0| \leq \frac{\hat{v}_0}{10} \right\}, \tag{3.4}$$

setting $\hat{T}_n = 0$ if the above set is empty. Observe that for any $t \in [0, \hat{T}_n]$ the i-th particle can interact with the tagged one during the time $[0,t]$ only if $i \in A_n(t)$, where

$$A_n(t) \doteq \left\{ i \in I_n \,:\, \inf_{s\in[0,t]} |x_i^{(n)}(s)| \leq 2\hat{v}_0 t + \gamma^{-1} \right\}.$$

Observe also that $\hat{V}_n(\cdot)$ is a continuous and non decreasing function such that

$$\max_{i\in A_n(t)} \sup_{s\in[0,t]} |v_i^{(n)}(s)| \leq \hat{V}_n(t) \leq \max_{i\in \bar{A}_n(t)} \sup_{s\in[0,t]} |v_i^{(n)}(s)|, \tag{3.5}$$

where

$$\bar{A}_n(t) \doteq \left\{ i \in I_n \,:\, \inf_{s\in[0,t]} |x_i^{(n)}(s)| \leq 2\hat{v}_0 t + \gamma^{-1} + 1 \right\}.$$

We finally define

$$\hat{T} \doteq \inf_{n\in\mathbb{N}} \hat{T}_n, \qquad T^* \doteq \min\{\hat{T}; \hat{C}\gamma^{-1}\hat{v}_0\}, \tag{3.6}$$

with $\hat{C}$ a positive constant to be fixed later. By continuity, Eq. (2.10) and the existence of the infinite dynamics imply that if $C^* > 4\sqrt{Q_1(X)}$ then

$\hat{T} > 0$. We shall prove that there are C^* large enough and $\hat{C}$ small enough such that, for all $n \in \mathbb{N}$,

$$\hat{V}_n(T^*) \le \frac{\hat{v}_0}{4}, \qquad \sup_{t\in[0,T^*]} |\hat{v}^{(n)}(t) - \hat{v}_0| \le \frac{\hat{v}_0}{20}. \tag{3.7}$$

By continuity, Eq. (3.7) implies that $\hat{T} > \hat{C}\gamma^{-1}\hat{v}_0$. Moreover, we will show that $|\hat{v}^{(n)}(t) - \hat{v}_0|$ is actually bounded by a constant for $t \in [0, \hat{C}\gamma^{-1}\hat{v}_0]$, which is the statement of the theorem.

In the Appendix the Lemma 2.4 is generalized to the n-partial dynamics $t \mapsto (\hat{x}^{(n)}(t), X^{(n)}(t))$ by proving that

$$\sup_{\mu} Q_\gamma(X^{(n)}(t); \mu, R_n(t)) \le C_7 R_n(t) \qquad \forall\, t \in [0, \hat{T}_n), \tag{3.8}$$

with C_7 not depending on $\hat{v}_0$. Then, by proceeding as in Section 2, we can show that, for any $i \in I_n$,

$$|v_i^{(n)}(t)| \le C_8 \sqrt{\log(e+n)} + C_9 \gamma t \qquad \forall\, t \in [0, \hat{T}_n), \tag{3.9}$$

with C_8 and C_9 not depending on $\hat{v}_0$. Following the same steps of the proof of Theorem 2.2 and substituting the definition (2.26) by

$$\begin{aligned}\delta_i(n,t) \doteq\; & |x_i^{(n)}(t) - x_i^{(n-1)}(t)| + |v_i^{(n)}(t) - v_i^{(n-1)}(t)| \\ & + |\hat{x}^{(n)}(t) - \hat{x}^{(n-1)}(t)| + |\hat{v}^{(n)}(t) - \hat{v}^{(n-1)}(t)|,\end{aligned}$$

from (3.9) we can prove that the infinite dynamics differs from the partial dynamics by a negligible quantity for $t \in [0, \hat{T})$. More precisely, there are positive constants C_{10} and C_{11}, not depending on $\hat{v}_0$, such that, for any $i \in \mathbb{N}$, $n \ge |x_i|$, and $\gamma \in (0,1]$,

$$|v_i^{(n)}(t)| \le C_{10} \sqrt{\log(e + |x_i| + \gamma^{-1}\hat{v}_0)} + C_{11}\gamma t \qquad \forall\, t \in [0, \hat{T}). \tag{3.10}$$

Recalling the definition (3.6), from (3.10) we get

$$\sup_{t\in[0,T^*]} |v_i^{(n)}(t)| \le C_{10} \sqrt{\log(e + |x_i| + \gamma^{-1}\hat{v}_0)} + C_{11}\hat{C}\hat{v}_0.$$

Hence, if $5C_{11}\hat{C} < 1$ and C^* is large enough, all the particles $i \in \bar{A}_n(T^*)$ are initially in $B(0, 4\hat{C}\gamma^{-1}\hat{v}_0^2)$, and moreover their velocity is smaller than $\hat{v}_0/4$ during the time $[0, T^*]$. By the second inequality in (3.5), this proves the first bound in (3.7).

We are left with the proof of the second bound in (3.7). We shall use the same strategy of [2]. The key point is the existence, up to the time $\hat{T}$, of a gap between the velocity of the tagged particle and the velocities of the other particles: this allows to study the problem by means of a first order perturbation theory.

Let $n_0 = \mathrm{Int}\,[\hat{C}\hat{v}_0^2\gamma^{-2}\exp(\hat{C}\hat{v}_0^2)]$; by arguing as in the proof of Theorem 2.2 we can prove that if C^* is large enough then

$$\sup_{t\in[0,T^*]} \left|\hat{v}^{(n)}(t) - \hat{v}^{(n_0)}(t)\right| \le C_{12} \qquad \forall\, n > n_0, \tag{3.11}$$

with C_{12} not depending on $\hat{v}_0$. Let us consider the case $n \le n_0$. From here to the end of the section we use the shortened notation $(\hat{x}(t), X(t))$ for $(\hat{x}^{(n)}(t), X^{(n)}(t))$. For $t \in [0, T^*]$ we define

$$\hat{p}(t) \doteq \hat{v}(t) + \sum_i \frac{\hat{\phi}_\gamma(\hat{x}(t) - x_i(t))}{M(\hat{v}(t) - v_i(t))}. \tag{3.12}$$

From the equations of motion we have:

$$\begin{aligned}\dot{\hat{p}}(t) \quad &= \quad \sum_i \frac{\hat{\phi}_\gamma(\hat{x}(t) - x_i(t))}{M(\hat{v}(t) - v_i(t))^2} \\ &\quad \times \left[\hat{F}_\gamma(x_i(t) - \hat{x}(t)) + \sum_{j:j\ne i} F_\gamma(x_i(t) - x_j(t))\right] \\ &\quad - \sum_{i,j} \frac{\hat{\phi}_\gamma(\hat{x}(t) - x_i(t))}{M^2(\hat{v}(t) - v_i(t))^2} \hat{F}_\gamma(\hat{x}(t) - x_j(t)).\end{aligned} \tag{3.13}$$

Recalling the definition of $A_n(t)$, for $t \in [0, T^*]$ a particle can contribute in the sums on the right hand side of (3.13) only if it belongs to $A_n(T^*)$. Since $T^* \le \hat{T}$, by the first inequality in (3.5), if $i \in A_n(T^*)$ then $|\hat{v}(t) - v_i(t)| > 2\hat{v}_0/5$, so that

$$|\dot{\hat{p}}(t)| \le \frac{C_{13}\gamma^3}{\hat{v}_0^2} N^2(t),$$

where $N(t) \doteq N(X(t); \hat{x}(t), 2\gamma^{-1})$. Hence:

$$\sup_{t\in[0,T^*]} |\hat{p}(t) - \hat{p}(0)| \le \frac{C_{13}\gamma^3}{\hat{v}_0^2} \int_0^{T^*} ds\, N^2(s). \tag{3.14}$$

By (2.4) there is a constant $B_4 > 0$ such that

$$N(X; \mu, R)^2 \le B_4\gamma^{-1} \max\{1; \gamma R\} Q_\gamma(X; \mu, R).$$

Then, recalling $R_n(t) > 2\gamma^{-1}$, by (3.8),

$$
\begin{aligned}
N^2(t) &\le 2B_4\gamma^{-1}Q_\gamma(X(t);\hat{x}(t),2\gamma^{-1}) \\
&\le 2B_4\gamma^{-1}C_7R_n(t) \le C_{14}\hat{C}\gamma^{-2}\hat{v}_0^2,
\end{aligned} \tag{3.15}
$$

where we used that $R_n(t) \le C_{15}\hat{C}\gamma^{-1}\hat{v}_0^2$ when $t \in [0,T^*]$ and $n \le n_0$.

Moreover, during the time $[0,T^*]$, the tagged particle can interact with another one for a time not bigger than $5(\gamma\hat{v}_0)^{-1}/2$, and we know that the particles in $A_n(T^*)$ are initially in $B(0,4\hat{C}\gamma^{-1}\hat{v}_0^2)$. Then:

$$
\int_0^{T^*} ds\, N(s) \le \frac{5(\gamma\hat{v}_0)^{-1}}{2} N(X;0,4\hat{C}\gamma^{-1}\hat{v}_0^2) \le C_{16}\hat{C}\gamma^{-2}\hat{v}_0. \tag{3.16}
$$

By (3.15) and (3.16),

$$
\int_0^{T^*} ds\, N^2(s) \le \sup_{t\in[0,T^*]} N(t) \int_0^{T^*} ds\, N(s) \le C_{17}\hat{C}^{3/2}\gamma^{-3}\hat{v}_0^2,
$$

so that, from (3.14),

$$
\sup_{t\in[0,T^*]} |\hat{p}(t) - \hat{p}(0)| \le C_{18}\hat{C}^{3/2}.
$$

Then, for $t \in [0,T^*]$,

$$
|\hat{v}(t) - \hat{v}_0| \le C_{18}\hat{C}^{3/2} + |\hat{p}(t) - \hat{v}(t)| + |\hat{p}(0) - \hat{v}_0|. \tag{3.17}
$$

By definition (3.12) and (3.15), the last two terms on the right hand side of (3.17) are bounded by $C_{19}\sqrt{\hat{C}}$. By (3.11) and (3.17) the difference $|\hat{v}^{(n)}(t) - \hat{v}_0|$ is thus bounded by a constant for $t \in [0,T^*]$ and $n \in \mathbb{N}$. In particular, if C^* is large enough, also the second bound in (3.7) is true. ■

4. On the Vlasov limit

We consider the system (2.6) after rescaling of space and time by a factor γ, i.e.

$$
\ddot{x}_i(t) = \gamma \sum_{\substack{j\in\mathbb{N}\\ j\neq i}} F(x_i(t) - x_j(t)), \qquad i \in \mathbb{N}, \tag{4.1}
$$

where $F = -\nabla\phi$ and ϕ as in Section 2. Eqs. (4.1) can also be interpreted as the equations of motion of a system of particles of mass γ and force

$\gamma^2 F$. Assume that the initial data are chosen in such a way that, in the limit $\gamma \to 0$, the particles are distributed with a mass density $f_0(x,v)$ in the one particle phase space (x,v). If at later times the system can still be described by a mass density $f(x,v;t)$, then the latter should be a solution (at least formally) of the Vlasov equation, which reads (in one dimension):

$$(\partial_t + v\,\partial_x + E(x;t)\,\partial_v)\,f(x,v;t) = 0 \qquad x,v \in \mathbb{R}, \tag{4.2}$$

where

$$E(x;t) = \int dx'\,dv'\,f(x',v';t)\,F(x-x') \tag{4.3}$$

and

$$f(x,v;0) = f_0(x,v). \tag{4.4}$$

This equation, introduced by Vlasov many years ago to study the plasma physics [16], describes the evolution of a system of many particles in the mean field limit. It is useful to introduce a weak version of the previous equation by using the characteristics. Problem (4.2)–(4.4) reduces to find a pair of functions,

$$(x,v) \mapsto \big(X(x,v;t), V(x,v;t)\big), \quad (x,v) \mapsto f(x,v;t), \tag{4.5}$$

that satisfy the following evolution equations:

$$\dot X(x,v;t) = V(x,v;t), \tag{4.6}$$

$$\dot V(x,v;t) = E(X(x,v;t);t), \tag{4.7}$$

$$X(x,v;0) = x, \quad V(x,v;0) = v, \tag{4.8}$$

$$f(X(x,v;t),V(x,v;t);t) = f_0(x,v). \tag{4.9}$$

We assume that the initial distribution $f_0(x,v)$ is measurable and satisfies, for some $\lambda_1, \lambda_2 > 0$,

$$0 \le f_0(x,v) \le \lambda_1 e^{-\lambda_2 v^2}. \tag{4.10}$$

We notice that this time evolution is "Hamiltonian" and preserves the measure $dx\,dv$, so that the Jacobian $|J(X,V|x,v)|$ is equal to one.

The existence and uniqueness of solutions for these equations have been proved in many papers. For finite total mass, i.e.

$$\int dx\,dv\,f(x,v;t) < \infty,$$

the first result in one dimension was obtained in [10]; since then, many other papers studied the problem in higher dimension and for a large class of interactions (see for instance [14]). For infinite total mass the problem has been recently solved in three (or lower) dimensions for positive and bounded interactions [4], and in two dimensions for singular Coulomb-like interactions [5].

A natural problem arises from the Hamiltonian nature of the Vlasov equation: does a particle system well approximating this equation exist? In the case of finite total mass the answer is positive and it has been given many years ago, see [1, 6, 14]. Here we shortly discuss the case of unbounded total mass, by considering the one dimensional case of system (4.1), but we believe that the same considerations may apply whenever the solution of the Vlasov equation exists (and it is continuous with respect to the initial data).

We first observe that the initial data discussed in Sections 2 and 3 arise in a natural way in problems of statistical mechanics, while in the framework of the Vlasov equation we have to consider initial data satisfying (4.10).

The main assumption is that initially the particle system "well approximates" (as $\gamma \to 0$) the initial data of the Vlasov equation. There are slightly different ways to approximate an unbounded Vlasov equation by an infinite particle system. We consider one of such ways without entering into the topic of possible generalizations.

Let $\mathcal{M}$ be the space of Radon measures in $\mathbb{R}^2$; we introduce the "local bounded Lipschitz distance" ρ on $\mathcal{M}$ defined by

$$\rho(\mu,\nu) \doteq \sup_{\xi_0\in\mathbb{R}^2} \sup_{\varphi\in D} \left| \int_{B(\xi_0)} \mu(d\xi)\,\varphi(\xi) - \int_{B(\xi_0)} \nu(d\xi)\,\varphi(\xi) \right|, \tag{4.11}$$

where $B(\xi_0)$ denotes the unit ball centered in ξ_0 and

$$D = \left\{\varphi : \mathbb{R}^2 \to [0,1] \,:\, |\varphi(\xi) - \varphi(\xi')| \le |\xi - \xi'| \quad \forall\, \xi, \xi' \in \mathbb{R}^2 \right\}.$$

Finally, for $t \mapsto \{x_i(t), v_i(t)\}_{i\in\mathbb{N}}$ being a solution of Eqs. (4.1), we define the empirical measure $t \mapsto \omega_t^{(\gamma)} \in \mathcal{M}$ by setting ($d\xi = dx\,dv$ below)

$$\omega_t^{(\gamma)}(d\xi) \doteq \gamma \sum_{i\in\mathbb{N}} \delta_{x_i(t)}(dx)\,\delta_{v_i(t)}(dv). \tag{4.12}$$

Then:

Theorem 4.1. We consider an initial condition for the Vlasov equation $f_0(x,v)$ satisfying (4.10), and an infinite particle system evolving via the Newton laws (4.1) with initial data such that

$$\lim_{\gamma\to 0} \rho\big(\omega_0^{(\gamma)}, \mu_0\big) = 0, \tag{4.13}$$

where $\mu_0(dx\,dv) \doteq f_0(x,v)\,dx\,dv$. Then, for any $T>0$,

$$\lim_{\gamma\to 0} \sup_{t\in[0,T]} \rho\big(\omega_t^{(\gamma)}, \mu_t\big) = 0 \tag{4.14}$$

where $\mu_t(dx\,dv) \doteq f(x,v;t)\,dx\,dv$.

Proof. The proof is achieved in three steps.

Step 1. Fix two quantities N and $L(N)$, a point $\xi_0 = (x_0, v_0)$, and consider an approximate problem in which the initial condition of the Vlasov equation is 0 if $|x-x_0| > N$ and $|v| > L(N)$, and $f_0(x,v)$ otherwise. It can be shown that, for N sufficiently large, there is a choice of $L(N)$ for which the solution of the infinite Vlasov equation can be approximated, in the ball $B(\xi_0)$, by the solution of this finite mass equation with an error $g_1(N)$, independent of ξ_0 and vanishing as $N\to\infty$. This property is equivalent to the existence of the solution of the Vlasov equation with infinite mass, discussed in detail in [4] in the more difficult three dimensional case. Here the proof is very simple since the boundedness of the force field implies the boundedness of the displacement of each fluid particle, so that an iterative procedure similar to the one of Section 2 can be easily constructed.

Step 2. We compare the solution of the finite mass Vlasov equation with the solution of a particle system evolving via a n-partial dynamics (relative to Eqs. (4.1)) with $n = \text{Int}\,[N]$; the latter is obtained by considering only the particles which are initially contained in the ball of radius n and center x_0, and whose initial velocity is not bigger that $L(N)$. We are now in the hypothesis discussed in the literature [1, 6, 14], and the error can be shown to be smaller than a quantity $g_2(\gamma, N)$, not depending on ξ_0 and vanishing as $\gamma\to 0$ for any $N>0$.

Step 3. It remains to prove that the n-partial dynamics (relative to Eqs. (4.1)) converges in $B(\xi_0)$ to the infinite dynamics, *uniformly* in γ (and ξ_0), as $n \to \infty$. This uniformity can be seen by writing explicitly the converging procedure. We omit the details of the proof.

The rapidity of convergence, the case of more dimension and/or more general interactions are out of the purposes of the present section. We finally remark that it would be reasonable to obtain results analogous to those of Sections 2 and 3 directly for the Vlasov equation.

Appendix

Warning. In the sequel we shall denote by C a generic positive constant whose numerical value may change from line to line.

Proof of Eq. (2.5). Let $N_k(X)$, $k \in \mathbb{Z}$, be the number of particles of the configuration X which are located in the interval $[k, k+1)$. Since $\phi_\gamma(x) = \gamma\phi(\gamma x)$, from (2.1) it follows that, for any $\mu \in \mathbb{R}$ and $R > 1$,

$$\sum_{i<j} \chi_i(\mu, R)\, \phi_\gamma(x_i - x_j) \le C\gamma \sum_{\langle k,k'\rangle} N_k(X)\, N_{k'}(X),$$

where $\langle k, k'\rangle$ means that the sum is restricted to all the pairs $k, k' \in \mathbb{Z}$ for which $|k-\mu| \le R+1$ and $|k'-k| \le \gamma^{-1}+1$. Then, since $N_k(X)N_{k'}(X) \le N_k(X)^2 + N_{k'}(X)^2$,

$$\begin{aligned}\sum_{i<j} \chi_i(\mu, R)\, \phi_\gamma(x_i - x_j) &\le C \sum_{k\in\mathbb{Z}} \chi(|k-\mu| \le R+1)\, N_k(X)^2 \\ &+ C\min\{1; \gamma R\} \sum_{k'\in\mathbb{Z}} \chi(|k'-\mu| \le R+\gamma^{-1}+2)\, N_{k'}(X)^2. \qquad \text{(A.1)}\end{aligned}$$

Recalling the definition (2.2), from (2.4) with $\gamma = 1$ and (A.1),

$$\begin{aligned}Q_\gamma(X;\mu,R) \le C\big[&Q_1(X;\mu,R+1) \\ &+ \min\{1;\gamma R\}Q_1(X;\mu,R+\gamma^{-1}+2)\big],\end{aligned}$$

which implies the second bound in (2.5). Analogously, from (A.1) with $\gamma = 1$ and (2.4),

$$Q_1(X;\mu,R) \le C\gamma^{-1}\left[Q_\gamma(X;\mu,R+1) + Q_\gamma(X;\mu,R+3)\right],$$

which implies the first bound in (2.5). ■

Proof of Lemma 2.4 and Eq. (3.8). We shall prove (3.8); the proof of (2.20) is recovered by putting $\hat{\phi} = 0$, in which case we can assume $\hat{T}_n = +\infty$. We introduce a mollified version of $Q_\gamma(X;\mu,R)$ by defining

$$W_\gamma(X;\mu,R) \doteq \sum_i f_i^{\mu,R}\left\{\frac{v_i^2}{2} + \frac{1}{2}\sum_{j:j\neq i}\phi_\gamma(x_i - x_j) + 1\right\}, \tag{A.2}$$

where

$$f_i^{\mu,R} = f\left(\frac{|x_i - \mu|}{R}\right) \tag{A.3}$$

and $f \in C^\infty(\mathbb{R}_+)$ is not increasing and satisfies: $f(x) = 1$ for $x \in [0,1]$, $f(x) = 0$ for $x \geq 2$, and $|f'(x)| \leq 2$. Clearly:

$$Q_\gamma(X;\mu,R) \leq W_\gamma(X;\mu,R) \leq Q_\gamma(X;\mu,2R). \tag{A.4}$$

For $0 \leq s \leq t < \hat{T}_n$, we define

$$R_n(t,s) \doteq \log(e+n) + 2\gamma^{-1} + \int_0^t d\tau\, V_n(\tau) + \int_s^t d\tau\, V_n(\tau) \tag{A.5}$$

(note that $R_n(t,t) = R_n(t)$ and $R_n(t,0) \leq 2R_n(t)$) and compute

$$\partial_s W_\gamma(X^{(n)}(s);\mu,R_n(t,s)) = \sum_i \left[\kappa_i(t,s)E_i(s) + f_i^{\mu,R_n(t,s)}\dot{E}_i(s)\right], \tag{A.6}$$

where, denoting by $\hat{x}_i^\mu(s)$ the sign of $x_i(s) - \mu$,

$$\begin{aligned}
\kappa_i(t,s) &= f'\left(\frac{|x_i(s)-\mu|}{R_n(t,s)}\right)\left[\frac{\hat{x}_i^\mu(s)v_i(s)}{R_n(t,s)} - \frac{\partial_s R_n(t,s)}{R_n(t,s)^2}|x_i(s)-\mu|\right], \\
E_i(s) &= \frac{v_i^2(s)}{2} + \frac{1}{2}\sum_{j:j\neq i}\phi_\gamma\big(x_i(s) - x_j(s)\big) + 1,
\end{aligned}$$

and, to simplify notation, we have omitted the explicit dependence on n of x_i, v_i, κ_i, and E_i.

Since $f'(|y|) \leq 0$, $f'(|y|) = 0$ if $|y| \leq 1$, $\partial_s R_n(t,s) = -V_n(s)$, and $|v_i(s)| \leq V_n(s)$, then $\kappa_i(t,s) \leq 0$. On the other hand, from the equations of motion,

$$\dot{E}_i(s) = v_i(s)\hat{F}_\gamma\big(x_i(s) - \hat{x}(s)\big) + \sum_{j:j\neq i}\frac{v_i(s)+v_j(s)}{2}F_\gamma\big(x_i(s) - x_j(s)\big).$$

Then, by (A.6) and using $F_\gamma(\cdot)$ is odd,

$$\begin{aligned}\partial_s W_\gamma(X^{(n)}(s);\mu,R_n(t,s)) \;&\leq\; \sum_i f_i^{\mu,R_n(t,s)}\hat F_\gamma\big(x_i(s)-\hat x(s)\big)v_i(s)\\ &+\;\frac12\sum_{i\neq j}\big(f_i^{\mu,R_n(t,s)}-f_j^{\mu,R_n(t,s)}\big)F_\gamma\big(x_i(s)-x_j(s)\big)v_i(s).\end{aligned} \tag{A.7}$$

By the inequality

$$\big|f_i^{\mu,R}-f_j^{\mu,R}\big|\leq 2\,\frac{|x_i-x_j|}{R}\big[\chi_i(\mu,2R)+\chi_j(\mu,2R)\big],$$

and since $R_n(t,s)>\gamma^{-1}$, the modulus of the double sum in the right hand side of (A.7) can be bounded from above by

$$-\,2\|\nabla\phi\|_\infty\gamma\frac{\partial_s R_n(t,s)}{R_n(t,s)}\sum_{i\neq j}\chi_i(\mu,4R_n(t,s))\chi_j(\mu,4R_n(t,s))\chi_{i,j}(s) \tag{A.8}$$

where we shortened $\chi_{i,j}(s)=\chi\big(|x_i(s)-x_j(s)|\leq\gamma^{-1}\big)$. From (2.4) and arguing as in the proof of Eq. (2.15) in [3], the double sum in the right hand side of (A.8) can be bounded by $C\gamma^{-1}W_\gamma(X^{(n)}(s);\mu,4R_n(t,s))$; moreover, setting

$$W_\gamma(X;R)\doteq\sup_\mu W_\gamma(X;\mu,R), \tag{A.9}$$

it can be proved that

$$W_\gamma(X;\mu,2R)\leq CW_\gamma(X;R) \tag{A.10}$$

(see e.g. [3]), and hence, by (A.7),

$$\begin{aligned}\partial_s W_\gamma(X^{(n)}(s);\mu,R_n(t,s)) \;&\leq\; \sum_i f_i^{\mu,R_n(t,s)}\big|\hat F_\gamma\big(x_i(s)-\hat x(s)\big)v_i(s)\big|\\ &-\;C\,\frac{\partial_s R_n(t,s)}{R_n(t,s)}\,W_\gamma(X^{(n)}(s);R_n(t,s))\end{aligned}$$

from which, by integrating and taking the supremum on μ,

$$\begin{aligned}W_\gamma(X^{(n)}(s);R_n(t,s)) &\leq W_\gamma(X^{(n)}(0);R_n(t,0))\\ &+\sup_\mu\int_0^s d\tau\,\sum_i f_i^{\mu,R_n(t,\tau)}\big|\hat F_\gamma\big(x_i(\tau)-\hat x(\tau)\big)v_i(\tau)\big|\\ &-\,C\int_0^s d\tau\,\frac{\partial_s R_n(t,\tau)}{R_n(t,\tau)}\,W_\gamma(X^{(n)}(\tau);R_n(t,\tau)).\end{aligned} \tag{A.11}$$

In the sum on the right hand side of (A.11), only the particles which are initially in $B(\mu, 4R_n(t,0))$ can contribute; the number of these particles is bounded by $W_\gamma(X^{(n)}(0); 4R_n(t,0))$. Moreover, since $s \in [0, \hat{T}_n)$, the tagged particle interacts with the i-th particle for a time not bigger than $5(\gamma\hat{v}_0)^{-1}/2$ and $|v_i(\tau)| \le \hat{v}_0/2$. Then:

$$\begin{aligned}
&\int_0^s d\tau \sum_i f_i^{\mu, R_n(t,\tau)} \left| \hat{F}_\gamma\big(x_i(\tau) - \hat{x}(\tau)\big) v_i(\tau) \right| \\
&\qquad \le \frac{5\gamma \|\nabla\hat{\phi}\|_\infty}{4} W_\gamma(X^{(n)}(0); \mu, 4R_n(t,0)) \\
&\qquad \le C W_\gamma(X^{(n)}(0); R_n(t,0)), \qquad \text{(A.12)}
\end{aligned}$$

where in the last inequality we have used (A.10). Inserting (A.12) in (A.11) we obtain a differential inequality which can be solved getting (for some positive constants c_1 and c_2)

$$W_\gamma(X^{(n)}(s); R_n(t,s)) \le c_1 W_\gamma(X^{(n)}(0); R_n(t,0)) \left(\frac{R_n(t,0)}{R_n(t,s)}\right)^{c_2}.$$

Setting $s = t$ and using that $R_n(t,0) \le 2R_n(t,t) = 2R_n(t)$,

$$W_\gamma(X^{(n)}(t); R_n(t)) \le C W_\gamma(X^{(n)}(0); R_n(t)).$$

Then, from (2.5), (A.4), and (A.9) we conclude that, for any $t \in [0, \hat{T}_n)$,

$$\begin{aligned}
Q_\gamma(X^{(n)}(t); \mu, R_n(t)) &\le C W_\gamma(X^{(n)}(0); R_n(t)) \\
&\le C \sup_\mu Q_\gamma(X^{(n)}(0); \mu, 2R_n(t)) \\
&\le 4B_3 C Q_1(X) R_n(t)
\end{aligned}$$

which proves (3.8). ■

Acknowledgments

Work performed under the auspices of the GNFM-INDAM and the Italian Ministry of the University (MURST). We thank Silvia Caprino for useful discussions on the Vlasov limit.

References

[1] W. Braun and K. Hepp, The Vlasov dynamics and its fluctuations in the $1/N$ limit of interacting classical particles, *Commun. Math. Phys.* **56**:101–11 (1977).

[2] E. Caglioti and C. Marchioro, On the long time behavior of a particle in an infinitely extended system in one dimension, *J. Stat. Phys.* **106**:663–680 (2002).

[3] E. Caglioti, C. Marchioro, and M. Pulvirenti, Non-equilibrium dynamics of three-dimensional infinite particle systems, *Commun. Math. Phys.* **215**:25–43 (2000).

[4] E. Caglioti, S. Caprino, C. Marchioro, and M. Pulvirenti, The Vlasov equation with infinite mass, *Arch. Rat. Mech. Anal.* **159**:85–108 (2001).

[5] S. Caprino, C. Marchioro, and M. Pulvirenti, On the two-dimensional Vlasov-Helmholtz equation with infinite mass, to appear on *Comm. Partial Differential Equations* (2002).

[6] R.L. Dobrushin, Vlasov equations, *Soviet Journ. Functional Anal.* **13**:115–123 (1979).

[7] R.L. Dobrushin and J. Fritz, Non-equilibrium dynamics of one-dimensional infinite particle systems with hard-core interaction, *Commun. Math. Phys.* **55**:275–292 (1977).

[8] A. De Masi, E. Orlandi, E. Presutti, and L. Triolo, Glauber evolution with Kac potentials. I. Mesoscopic and macroscopic limits, interface dynamics, *Nonlinearity* **7**:1–6 (1994).

[9] M. Kac, G. Uhlenbeck, and P.C. Hemmer, On the van der Waals theory of vapor-liquid equilibrium. I. Discussion of a one dimensional model, *J. Math. Phys.* **4**:216–228 (1963), II. Discussion of the distribution functions, *J. Math. Phys.* **4**:229–247 (1963), III. Discussion of the critical region, *J. Math. Phys.* **5**:60–74 (1964).

[10] S.V. Iordanski, The Cauchy problem for the kinetic equation of plasma, *Transl. Ser* **35** *Amer. Math. Soc.* Providence (1964).

[11] J.L. Lebowitz, A. Mazel, and E. Presutti, Rigorous proof of a liquid-vapor phase transition in a continuum particle system, *Phys. Rev. Letters* **80**:4701 (1998).

[12] J.L. Lebowitz, A. Mazel, and E. Presutti, Liquid-vapor phase transitions for systems with finite-range interactions, *J. Stat. Phys.* **94**:955–1025 (1999).

[13] J.L. Lebowitz and O. Penrose, Rigorous treatment of the van der Waals Maxwell theory of the liquid vapour transition, *J. Math. Phys.* **7**:98–113 (1966).

[14] H. Neunzert, An introduction to the nonlinear Boltzmann-Vlasov equation, Kinetic theories and the Boltzmann equation (Montecatini, 1981), 60–110. *Lecture Notes in Math.* **1048**, Springer, Berlin, 1984.

[15] D. Ruelle, *Statistical Mechanics. Rigorous Results.* W.A. Benjamin, Inc., New York-Amsterdam 1969.

[16] A. Vlasov, On the kinetic theory of an assembly of particles with collective interaction, *Acad. Sci. USSR J. Phys.* **9**:25–40 (1945).