Curvature-induced bound states for a δ interaction supported by a curve in $\mathbb{R}^3$

P. Exner and S. Kondej

Abstract. We study the Laplacian in $L^2(\mathbb{R}^3)$ perturbed on an infinite curve Γ by a δ interaction defined through boundary conditions which relate the corresponding generalized boundary values. We show that if Γ is smooth and not a straight line but it is asymptotically straight in a suitable sense, and if the interaction does not vary along the curve, the perturbed operator has at least one isolated eigenvalue below the threshold of the essential spectrum.

1 Introduction

Relations between the geometry and spectral properties are one of the vintage topics of mathematical physics. In the last decade they attracted attention also in the context of quantum mechanics. A prominent example is the curvature-induced binding in infinite tube like regions [EŠ, GJ, DE, RB]. This effect appears to be a robust one: it has been demonstrated recently that bends can produce localized states not only if the transverse confinement is hard, i.e. realized by a Dirichlet condition, but also when it is weaker corresponding to a potential well or a δ interaction [EI].

The result is appealing, not only because it concerns an interesting mathematical problem, but also in view of applications in mesoscopic physics where such operators are used as a natural model for semiconductor "quantum wires". Since in the latter electrons are trapped due to interfaces between two different materials representing finite potential jumps, by tunneling effect they can be found outside the wire, albeit not too far because the exterior is (for the energies in question) the classically forbidden region.

The main result of the paper [EI] concerns nontriviality of the discrete spectrum for a class of operators in $L^2(\mathbb{R}^2)$ which can be formally written as

$-\Delta - \alpha\delta(x-\Gamma)$ with $\alpha > 0$, where Γ is a curve which is not a straight line but it is asymptotically straight in a suitable sense. A question naturally arises whether a similar result is valid for a curve in $\mathbb{R}^3$. Such an extension is not trivial, because the argument in [EI] relies on the resolvent formula of [BEKŠ] representing in a sense a generalization of the Birman-Schwinger theory. The said formula is valid for singular perturbations of the Laplacian which can be treated by means of a quadratic-form sum, i.e. as long as the codimension of the manifold supporting the perturbation is one.

Thus if we want to address the stated question, we are forced to look for other tools. One possibility is to employ the resolvent formula for a curve in $\mathbb{R}^3$ derived in [Ku]. However, since it uses rather strong regularity hypotheses about the curve we take another route and begin instead with an abstract formula for strongly singular perturbations due to A. Posilicano [Po]. When it is specified to our particular case, it contains again an embedding operator into a space of functions supported on the curve Γ, however, this time it is not the "naive" L^2 but rather a suitable element from the scale of Sobolev spaces. Of course, one can regard it as a generalization of Krein's formula; recall that such a way of expressing the resolvent can be used not only to describe δ interaction perturbations but also more general dynamics supported by zero measure sets [Ka, KK, Ko].

Another aspect of the absence of a description in terms of the quadratic-form sum concerns the very definition of the operator we want to study. We have to employ boundary conditions which relate the corresponding generalized boundary values in the normal plane to the curve modeled after the usual two-dimensional δ interaction [AGHH], which requires us to impose stronger regularity conditions on Γ. Furthermore, a modification of the Birman-Schwinger technique used in [EI] demands stronger restrictions on the regularity of the curve. On the other hand, apart of these technical hypotheses our main result – stated in Theorem 5.6 below – is analogous to that of [EI], namely that for any curve which is asymptotically straight but not a straight line the corresponding operator has at least one isolated eigenvalue. This conclusion is by no means obvious having in mind how different are the point interactions in one and two dimensions.

2 The resolvent formula

As a preliminary let us show how self-adjoint extensions of symmetric operators can be characterized in terms of a Krein-type formula derived in [Po]; we refer to this paper for the proof and a more detailed discussion. With a later purpose on mind we do not strive for generality and restrict ourselves to the case of the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^3) \equiv L^2$ and the Laplace operator, $-\Delta : D(\Delta) \to L^2$, which is well known to be self-adjoint on the domain $D(\Delta)$ which coincides with the usual Sobolev space $H^2(\mathbb{R}^3) \equiv H^2$.

For any z belonging to the resolvent set $\varrho(-\Delta) = \mathbb{C}\backslash[0,\infty)$ we define the resolvent as the bounded operator $R^z := (-\Delta - z)^{-1} : L^2 \to H^2$. Consider a bounded operator

$$\tau : H^2 \to \mathcal{X}$$

into a complex Banach space $\mathcal{X}$ and its adjoint in the dual space $\mathcal{X}'$. Recall that for a closed linear operator $A : \mathcal{X} \to \mathcal{Y}$ the adjoint is defined by $(A^* l)(x) = l(Ax)$ for all $x \in D(A)$ and $l \in D(A^*) \subseteq \mathcal{Y}'$. Then we can introduce the operators

$$R^z_\tau = \tau R^z : L^2 \to \mathcal{X}\,, \quad \breve{R}^z_\tau = (R^{\bar{z}}_\tau)^* : \mathcal{X}' \to L^2\,,$$

which are obviously bounded too. Let Z be an open subset of $\varrho(-\Delta)$ symmetric w.r.t. the real axis, i.e. such that $z \in Z$ implies $\bar{z} \in Z$. Suppose that for any $z \in Z$ there exists a closed operator $Q^z : D \subseteq \mathcal{X}' \to \mathcal{X}$ satisfying the following conditions,

$$Q^z - Q^w = (z - w) R^w_\tau \breve{R}^z_\tau\,, \tag{2.1}$$

$$\forall l_1, l_2 \in D\,, \quad l_1(Q^{\bar{z}} l_2) = \overline{l_2(Q^z l_1)}\,. \tag{2.2}$$

It will be used to construct a family of self-adjoint operators which coincide with $-\Delta$ when restricted to $\ker\tau$. They can be parametrized by symmetric operators $\Theta : D(\Theta) \subseteq \mathcal{X}' \to \mathcal{X}$. To this end, we define

$$\begin{aligned} &Q^z_\Theta = \Theta + Q^z : D(\Theta) \cap D \subseteq \mathcal{X}' \to \mathcal{X}\,, \\ &Z_\Theta := \{\, z \in \rho(-\Delta) : \ (Q^z_\Theta)^{-1},\ (Q^{\bar{z}}_\Theta)^{-1} \text{ exist and are bounded} \,\}\,. \end{aligned}$$

With this notation we can state the result we want to borrow from [Po].

Theorem 2.1 *Assume that the conditions*

$$Z_\Theta \neq \emptyset \tag{2.3}$$

and

$$\operatorname{Ran}\tau^* \cap L^2 = \{0\} \tag{2.4}$$

are satisfied. Then the bounded operator

$$R^z_{\tau,\Theta} := R^z - \breve{R}^z_\tau (Q^z_\Theta)^{-1} R^z_\tau\,, \quad z \in Z_\Theta\,,$$

is the resolvent of the self-adjoint operator $-\Delta_{\tau,\Theta}$ defined by

$$\begin{aligned} &D(\Delta_{\tau,\Theta}) = \{\, f \in L^2 : f = f_z - \breve{R}^z_\tau (Q^z_\Theta)^{-1} \tau f_z,\; f_z \in D(\Delta)\,\}\,,\\ &(-\Delta_{\tau,\Theta} - z)f := (-\Delta - z) f_z\,, \end{aligned}$$

which coincides with $-\Delta$ on the $\ker\tau$*.*

3 Singular perturbation on a curve in $\mathbb{R}^3$

Henceforth, we will be interested in a specific class of perturbations of the Laplacian on $\mathcal{H} = L^2(\mathbb{R}^3)$. The free resolvent

$$R^z = (-\Delta - z)^{-1} :\; L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3)\,, \quad z \in \varrho(-\Delta)\,,$$

is an integral operator with the kernel

$$G^z(x-y) = \frac{\mathrm{e}^{i\sqrt{z}|x-y|}}{4\pi\, |x-y|}\,.$$

Let $\Gamma \subset \mathbb{R}^3$ be a curve defined as a graph of a continuous function which is assumed to be piecewise C^1. Recall that Γ admits a natural parametrization by the arc length which is unique up to a choice if the reference point; we denote the parameter as s and use the symbol $\gamma(s) \,:\, \mathbb{R} \to \mathbb{R}^3$ for the corresponding function. Then we have

$$|\gamma(s) - \gamma(s')| \le |s - s'|\,. \tag{3.1}$$

To specify further the family of curves which we will consider, we introduce for any $\tilde\omega \in (0,1)$ and $\tilde\varepsilon > 0$ the set

$$\begin{aligned} S_{\tilde\omega,\tilde\varepsilon} := \Big\{ (s,s') :\; &\tilde\omega < \frac{s}{s'} < \tilde\omega^{-1} \quad \text{if} \quad |s+s'| > \xi(\tilde\omega)\tilde\varepsilon\,,\\ &\text{and}\;\; |s-s'| < \tilde\varepsilon \quad \text{if} \quad |s+s'| < \xi(\tilde\omega)\tilde\varepsilon \Big\}\,, \end{aligned}$$

where $\xi(\tilde\omega) := \frac{1+\tilde\omega}{1-\tilde\omega}$. We adopt the following assumptions:

(a1) there exists a $c \in (0,1)$ such that $|\gamma(s)-\gamma(s')| \geq c\,|s-s'|$,

(a2) there are $\omega \in (0,1)$, $\mu \geq 0$ and positive ε, d such that the inequality

$$1 - \frac{|\gamma(s)-\gamma(s')|}{|s-s'|} \leq d\, \frac{|s-s'|}{(|s-s'|+1)(1+(s^2+s'^2)^\mu)^{1/2}}$$

holds for all $(s,s') \in S_{\omega,\varepsilon}$.

The first condition means, in particular, that Γ has no cusps and self-intersections. The second assumption is basically a requirement of asymptotic straightness (see Remark 5.7), but in contrast to [EI] it restricts also the behaviour of $|\gamma(s)-\gamma(s')|$ at small distances; it is straightforward to check that the bound cannot be satisfied unless Γ is C^1-smooth.

To make use of Theorem 2.1 we take $\mathcal{X} = L^2(\mathbb{R})$ and denote the corresponding scalar product by $(\cdot,\cdot)_l$ (see also Remark 3.1 below). The operator $\tau : H^2(\mathbb{R}^3) \to L^2(\mathbb{R})$ which we will employ in our construction is a trace map defined in the following way:

$$\tau\phi(s) := \phi(\gamma(s))\,;$$

it is a standard matter to check that the definition makes sense and the operator τ is bounded [BN]. The adjoint operator $\tau^* : L^2(\mathbb{R}) \to H^{-2}(\mathbb{R}^3)$ is determined by the relation

$$\langle \tau^* h, \omega \rangle = (h, \tau\omega)_l\,, \quad h \in L^2(\mathbb{R})\,, \quad \omega \in H^{-2}(\mathbb{R}^3)\,,$$

where $\langle\cdot,\cdot\rangle$ stands for the duality between $H^{-2}(\mathbb{R}^3)$ and $H^2(\mathbb{R}^3)$, in other words, we can write

$$\tau^* h = h\delta_\Gamma\,,$$

where δ_Γ is the Dirac measure supported by Γ. Since $\delta_\Gamma \notin L^2(\mathbb{R}^3)$ we get

$$\operatorname{Ran}\tau^* \cap L^2(\mathbb{R}^3) = \{0\}\,,$$

so condition (2.4) is satisfied.

Remark 3.1 Notice that the map τ as introduced above is not surjective. Indeed, since $\gamma(s)$ is a Lipschitz function we have $\operatorname{Ran}\tau = H^1(\mathbb{R})$ – cf. [BN]. However, we lose nothing by keeping $\mathcal{X} = L^2(\mathbb{R})$ in the further discussion.

The problem at hand is to define an operator $Q^z : D \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ satisfying the conditions (2.1) and (2.2). To this end some preliminaries are needed. Since our considerations concern spectral properties at the negative halfline, it suffices for further discussion to restrict ourselves to $z = -\kappa^2$ with $\kappa > 0$. In such a case it is convenient to modify slightly the used notation by introducing

$$\mathbf{Q}^\kappa := Q^{-\kappa^2}, \quad \mathbf{R}_\tau^\kappa := R_\tau^{-\kappa^2}, \quad \breve{\mathbf{R}}_\tau^\kappa := \breve{R}_\tau^{-\kappa^2}.$$

and similarly

$$\mathbf{G}^\kappa(s-s') := \frac{\mathrm{e}^{-\kappa|s-s'|}}{4\pi\,|s-s'|}, \quad \mathbf{G}^\kappa(\gamma(s)-\gamma(s')) = \frac{\mathrm{e}^{-\kappa|\gamma(s)-\gamma(s')|}}{4\pi\,|\gamma(s)-\gamma(s')|}.$$

The difference of these two kernels,

$$B_\kappa(s,s') = \mathbf{G}^\kappa(\gamma(s)-\gamma(s')) - \mathbf{G}^\kappa(s-s'),$$

defines the integral operator $B_\kappa : D(B_\kappa) \to L^2(\mathbb{R})$ with the domain $D(B_\kappa) = \{f \in L^2(\mathbb{R}) : B_\kappa f \in L^2(\mathbb{R})\}$. A key observation is that this operator has a definite sign: in view of (3.1) and of the fact that the function $\xi \mapsto \frac{\mathrm{e}^{-\kappa\xi}}{\xi}$ decreases monotonically for κ, ξ positive, we have

$$B_\kappa(s,s') \geq 0. \tag{3.2}$$

The operator B_κ is related obviously with the deviation of Γ from a straight line; below we shall demonstrate that properties for a curve satisfying the assumptions (a1) and (a2) with any $\mu \geq 0$ is bounded (see Remark 5.4).

Next we need to show how the free resolvent kernel behaves when one of the three dimensions is integrated out. By a direct computation one can show that for all $\kappa, \kappa' > 0$ and $f_1, f_2 \in L^2(\mathbb{R})$ the following relation,

$$\begin{aligned} &\int_{\mathbb{R}^2} f_1(s)\overline{f_2(s')}\,[\mathbf{G}^\kappa(s-s') - \mathbf{G}^{\kappa'}(s-s')]\,ds\,ds' \\ &\qquad = \int_{\mathbb{R}^2} f_1(s)\overline{f_2(s')}\,[\breve{T}_\kappa(s-s') - \breve{T}_{\kappa'}(s-s')]\,ds\,ds', \end{aligned}$$

is valid, where

$$\breve{T}_\kappa(s-s') := -\frac{1}{(2\pi)^2}\int_{\mathbb{R}} \ln\left(p^2+\kappa^2\right)^{1/2} \mathrm{e}^{ip(s-s')}\,dp. \tag{3.3}$$

This result means, in particular, that

$$\int_{\mathbb{R}^2} f_1(s)\overline{f_2(s')}\,\mathbf{G}^\kappa(s-s')\,ds\,ds' - \int_{\mathbb{R}^2} f_1(s)\overline{f_2(s')}\,\check{T}_\kappa(s-s')\,ds\,ds' \tag{3.4}$$

is κ-independent. Let $T_\kappa : D(T_\kappa) \to L^2(\mathbb{R})$ be the integral operator with the domain $D(T_\kappa) = \{f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} \check{T}_\kappa(s-s')f(s')\,ds' \in L^2(\mathbb{R})\}$ and the kernel $T_\kappa(s-s') := \check{T}_\kappa(s-s') + \frac{1}{2\pi}(\ln 2 + \psi(1))$ where $-\psi(1) \approx 0.577$ is Euler's number. Then T_k is self-adjoint and we can define the operator

$$\mathbf{Q}^\kappa f = (T_\kappa + B_\kappa)f :\ D \equiv D(T_\kappa) \to L^2(\mathbb{R})\,,$$

which is also self-adjoint and has the needed properties:

Lemma 3.2 *The operators $Q^{-\kappa^2} \equiv \mathbf{Q}^\kappa$ satisfy the conditions (2.1), (2.2).*

Proof. Let $f_1, f_2 \in D$, then a direct computation yields

$$\begin{aligned}&(\kappa^2 - \kappa'^2)(f_1, \mathbf{R}_\tau^{\kappa'}\breve{\mathbf{R}}_\tau^\kappa f_2)_l \\ &\quad = \int_{\mathbb{R}^2} f_1(s)\overline{f_2(s')}\,[\mathbf{G}^\kappa(\gamma(s)-\gamma(s')) - \mathbf{G}^{\kappa'}(\gamma(s)-\gamma(s'))]\,ds\,ds'\,.\end{aligned}$$

On the other hand, by definition of $\mathbf{Q}^\kappa$ and the κ-independence of the expression (3.4) we find that $(f_1, (\mathbf{Q}^\kappa - \mathbf{Q}^{\kappa'})f_2)_l$ is also given by the right-hand side of the last formula, which proves (2.1). Since $\mathbf{Q}^\kappa$ is self-adjoint, the condition (2.2) is satisfied too. ■

The operator $\Theta : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ appearing in Theorem 2.1 will be identified here with the multiplication by a real number, $\Theta f = -\alpha f$ with $\alpha \in \mathbb{R}$ and the sign convention made with a later purpose on mind. Then the operator

$$\mathbf{Q}^\kappa_\Theta = \Theta + \mathbf{Q}^\kappa :\ D \to L^2(\mathbb{R})$$

is closed and by Proposition 1 of Ref. [Po] we conclude that (2.3) is satisfied.

For simplicity we identify in the following the symbols of the operators τ, Θ with γ, α, respectively. In this notation Theorem 2.1 says the following: if $\kappa \in Z_\alpha$, i.e. if the operator $(\mathbf{Q}^\kappa_\alpha)^{-1} = (\mathbf{Q}^\kappa - \alpha)^{-1} :\ L^2(\mathbb{R}) \to L^2(\mathbb{R})$ exists and is bounded, then

$$\mathbf{R}^\kappa_{\gamma,\alpha} = \mathbf{R}^\kappa - \breve{\mathbf{R}}^\kappa_\gamma(\mathbf{Q}^\kappa - \alpha)^{-1}\mathbf{R}^\kappa_\gamma \tag{3.5}$$

is the resolvent of a self-adjoint operator which we denote as $-\Delta_{\gamma,\alpha}$. It coincides with $-\Delta$ on $\ker\tau = \{g \in H^2(\mathbb{R}^3) : g(x) = 0,\ x \in \Gamma\}$ and

$$\begin{aligned}&D(-\Delta_{\gamma,\alpha}) = \{f \in L^2 :\ f = f_\kappa - \breve{\mathbf{R}}^\kappa_\gamma(\mathbf{Q}^\kappa - \alpha)^{-1}\tau f_k,\ f_\kappa \in D(\Delta)\}\,, \\ &(-\Delta_{\gamma,\alpha} + \kappa^2)f = (-\Delta + \kappa^2)f_\kappa\,.\end{aligned}$$

4 The interaction in terms of boundary conditions

To proceed further we have to impose slightly stronger regularity requirement on the curve Γ . Specifically, we assume that it is given by a function $\gamma(s) : \mathbb{R} \to \mathbb{R}^3$ which is C^1 everywhere and piecewise C^2, and satisfies the condition (a1). Then we can introduce, apart of a discrete set, the Frenet's frame for Γ, i.e. the triple $(t(s), b(s), n(s))$ of the tangent, binormal and normal vectors, which are by assumption piecewise continuous functions of s. Given $\xi, \eta \in \mathbb{R}$ we denote $r = (\xi^2+\eta^2)^{1/2}$ and define the set the "shifted" curve

$$\Gamma_r \equiv \Gamma_r^{\xi\eta} := \{\, \gamma_r(s) \equiv \gamma_r^{\xi\eta}(s) := \gamma(s) + \xi b(s) + \eta n(s) \,\} \,.$$

It follows from the smoothness of γ in combination with (a1) that there exists an $r_0 > 0$ such that $\Gamma_r \cap \Gamma = \emptyset$ holds for each $r < r_0$.

Since any function $f \in H^2_{loc}(\mathbb{R}^3 \setminus \Gamma)$ is continuous on $\mathbb{R}^3 \setminus \Gamma$ its restriction to Γ_r, $r < r_0$ is well defined; we denote it as $f \restriction_{\Gamma_r}(s)$. In fact, we can regard $f \restriction_{\Gamma_r}(s)$ as a distribution from $D'(\mathbb{R})$ with the parameter r. We shall say that a function $f \in H^2_{loc}(\mathbb{R}^3 \setminus \Gamma) \cap L^2(\mathbb{R}^3)$ belongs to Υ if the following limits

$$\begin{aligned} \Xi(f)(s) &:= -\lim_{r\to 0} \frac{1}{\ln r} f \restriction_{\Gamma_r}(s) \,, \\ \Omega(f)(s) &:= \lim_{r\to 0} \left[f \restriction_{\Gamma_r}(s) + \Xi(f)(s) \ln r \right] \,, \end{aligned}$$

exist a.e. in $\mathbb{R}$, are independent of the direction $\frac{1}{r}(\xi, \eta)$, and define functions from $L^2(\mathbb{R})$. The limits here are understood in the sense of the $D'(\mathbb{R})$ topology. With these prerequisites we are able now to characterize the operator $-\Delta_{\gamma,\alpha}$ discussed above in terms of (generalized) boundary conditions, postponing the proof to the appendix.

Theorem 4.1 *With the assumption stated above we have*

$$\begin{aligned} D(-\Delta_{\gamma,\alpha}) &= \Upsilon_\alpha := \{\, g \in \Upsilon :\; 2\pi\alpha\Xi(g)(s) = \Omega(g)(s) \,\} \,, \\ -\Delta_{\gamma,\alpha} f &= -\Delta f \quad \text{for} \quad x \in \mathbb{R}^3 \setminus \Gamma \,. \end{aligned} \tag{4.1}$$

5 Curvature-induced bound states

Let us first find the spectrum of $-\Delta_{\gamma_0,\alpha}$ where γ_0 is a linear function describing a straight line. Since $B_\kappa = 0$ holds in this case we have $\mathbf{Q}^\kappa = T_\kappa$. Then the resolvent formula (3.5) yields

$$\sigma(-\Delta_{\gamma_0,\alpha}) = \{\, -\kappa^2 :\ \alpha \in \sigma(T_\kappa) = \sigma_{\rm ac}(T_\kappa)\,\}\,.$$

Using the momentum representation of T_κ we immediately get

$$\sigma_{\rm ac}(T_\kappa) = (-\infty, s_\kappa]\,,$$

where $s_\kappa := \frac{1}{2\pi}(\psi(1) - \ln(\kappa/2))$. Hence the spectrum of $-\Delta_{\gamma_0,\alpha}$ is given by

$$\sigma(-\Delta_{\gamma_0,\alpha}) = \sigma_{\rm ac}(-\Delta_{\gamma_0,\alpha}) = [\zeta_0, \infty)\,,$$

where $\zeta_0 = -4{\rm e}^{2(-2\pi\alpha+\psi(1))}$ as we expect with the spectrum of a two-dimensional δ interaction [AGHH] and the natural separation of variables in mind.

To find the spectrum of $-\Delta_{\gamma,\alpha}$ for a non-straight curve we treat the respective operator $\mathbf{Q}^\kappa$ as a perturbation of the one corresponding to a straight line. First we have to localize the essential spectrum. Following step by step the argument given in the proof of Proposition 1 of Ref. [EI] we get

Lemma 5.1 *Let Γ be a curve given by a function $\gamma(s)$ satisfying (a1) and (a2) with $\mu > 1/2$. Then $\sigma_{\rm ess}(-\Delta_{\gamma,\alpha}) = [\zeta_0, \infty)$.*

Next we observe that a nontrivial bending pushes the upper bound of the spectrum of $\mathbf{Q}^\kappa$ up.

Lemma 5.2 *If Γ is not a straight line we have*

$$\sup \sigma(\mathbf{Q}^\kappa) > s_\kappa\,. \tag{5.1}$$

Proof. Let ϕ be a positive function from $C_0^\infty(\mathbb{R}^3)$. Given $\lambda > 0$ we set $\phi_\lambda(s) := \lambda^{1/2}\phi(\lambda s)$. To show (5.1) it suffices to check the following inequality

$$(\mathbf{Q}^\kappa \phi_\lambda, \phi_\lambda)_l - s_\kappa(\phi_\lambda, \phi_\lambda)_l > 0\,,$$

which is easily seen to be equivalent to

$$-\frac{1}{2\pi}\int_{\mathbb{R}} \ln\left(1+\left(\frac{\lambda u}{\kappa}\right)^2\right)^{1/2} \left|\hat\phi(u)\right|^2 du + \lambda\int_{\mathbb{R}^2} B_\kappa(s,s')\phi(\lambda s)\phi(\lambda s')\, ds\, ds' > 0\,, \tag{5.2}$$

where $\hat{\phi}$ stands for the Fourier transform of ϕ. The first term in the last expression can expanded as

$$-\frac{1}{4\pi}\left(\frac{\lambda}{\kappa}\right)^2 \int_{\mathbb{R}} u^2 \left|\hat{\phi}(u)\right|^2 du + \mathcal{O}(\lambda^4)\,.$$

Since Γ is not straight by assumption the inequality (3.2) is sharp in an open subset of $\mathbb{R}^2$, so there is $D > 0$ such that $\lambda \int_{\mathbb{R}^2} B_\kappa(s,s')\phi(\lambda s)\phi(\lambda s')\, ds\, ds' \geq D\lambda$ as $\lambda \to 0$. Consequently, for all sufficiently small λ the inequality (5.2) is satisfied. ■

On the other hand, the part of the spectrum in (s_κ, ∞) added in this way is at most discrete provided the curve has the asymptotic straightness properties expressed by the assumption (a2) with μ large enough.

Lemma 5.3 *If $\mu > 1/2$ then B_k are Hilbert-Schmidt operators. Moreover, norms $\|B_\kappa\|_{HS}$ are uniformly bounded with respect $\kappa \geq \kappa_0 = |\zeta_0|^{1/2}$.*

Proof. Denote $\rho \equiv \rho(s,s') := |\gamma(s) - \gamma(s')|$ and $\sigma \equiv \sigma(s,s') := |s - s'|$. In this notation the assumptions (a1), (a2) can be written as

(a1) there is a $c \in (0,1)$ such that $\rho(s,s') \geq c\sigma(s,s')$,

(a2) there are $\omega \in (0,1)$, $\mu \geq 0$ and $\varepsilon, d > 0$ s.t. for all $(s,s') \in S_{\omega,\varepsilon}$ we have

$$1 - \frac{\rho(s,s')}{\sigma(s,s')} \leq \frac{d\sigma(s,s')}{(\sigma(s,s')+1)(1+(s^2+s'^2)^\mu)^{1/2}}\,.$$

Next we notice that the perturbation kernel is monotonous with respect to the spectral parameter,

$$B_\kappa(s,s') \leq B_{\kappa'}(s,s') \quad \text{for} \quad \kappa' < \kappa\,,$$

thus to prove lemma it suffices to show that B_{κ_0} is a Hilbert-Schmidt operator. Since the function $v \mapsto \frac{e^{-\kappa_0 v}}{v}$ is strictly decreasing and convex in $(0,\infty)$, we have the following estimate,

$$0 \leq \frac{e^{-\kappa_0 \rho}}{\rho} - \frac{e^{-\kappa_0 \sigma}}{\sigma} \leq -\left[\frac{e^{-\kappa_0 \sigma_c}}{\sigma_c}\right]'(\sigma - \rho)\,,$$

where $\sigma_c := c\sigma$ and c is the constant appearing in (a1). Thus we get

$$0 \le \frac{\mathrm{e}^{-\kappa_0\rho}}{\rho} - \frac{\mathrm{e}^{-\kappa_0\sigma}}{\sigma} \le (\kappa_0\sigma_c + 1)\frac{\sigma-\rho}{\sigma_c^2}\,\mathrm{e}^{-\kappa_0\sigma_c}\,, \tag{5.3}$$

and moreover, the assumption (a1) gives the bound

$$\frac{\sigma-\rho}{\sigma} \le 1-c\,.$$

In view of (a2), there exists a positive $\tilde{c}$ such that

$$\sigma(s,s') \ge \tilde{c}\,.$$

holds for any $(s,s') \in \mathbb{R}^2 \setminus S_{\omega,\varepsilon}$. Combining the last three inequalities we have in $\mathbb{R}^2 \setminus S_{\omega,\varepsilon}$ the estimate

$$\frac{1}{4\pi}\left[\frac{\mathrm{e}^{-\kappa_0\rho}}{\rho} - \frac{\mathrm{e}^{-\kappa_0\sigma}}{\sigma}\right] \le M_1\mathrm{e}^{-\kappa_0\sigma_c} \tag{5.4}$$

with $M_1 := (4\pi)^{-1}(1-c)\,c^{-2}(\kappa_0 c+\tilde{c}^{-1})$. On the other hand using (5.3) and (a2) we get

$$\frac{1}{4\pi}\left[\frac{\mathrm{e}^{-\kappa_0\rho}}{\rho} - \frac{\mathrm{e}^{-\kappa_0\sigma}}{\sigma}\right] \le M_2\mathrm{e}^{-\kappa_0\sigma_c}\frac{1}{(1+(s^2+s'^2)^\mu)^{1/2}} \tag{5.5}$$

for $(s,s') \in S_{\omega,\varepsilon}$, where $M_2 := (4\pi)^{-1}dc^{-2}\max\{1,\kappa_0 c\}$. Putting now the estimates (5.4), (5.5) together we find

$$\begin{aligned}
&\int_{\mathbb{R}^2} B_k(s,s')^2 ds ds' \\
&\quad\le M_1^2\int_{\mathbb{R}^2\setminus S_{\omega,\varepsilon}} \mathrm{e}^{-2\kappa_0 c|s-s'|}\,ds\,ds' + M_2^2\int_{S_{\omega,\varepsilon}} \frac{\mathrm{e}^{-2\kappa_0 c|s-s'|}}{1+(s^2+s'^2)^\mu}\,ds\,ds' \\
&\quad\le \frac{1}{2}M_1^2\,\frac{1+\omega}{1-\omega}\int_0^\infty \mathrm{e}^{-\kappa_0 cu}\,u\,du + M_2^2\int_{S_{\omega,\varepsilon}} \frac{\mathrm{e}^{-2\kappa_0 c|s-s'|}}{1+(s^2+s'^2)^\mu}\,ds\,ds' < \infty\,,
\end{aligned}$$

which proves the result because the last integral converges for $\mu > 1/2$. ■

Remark 5.4 As we have said, the assumption (a2) includes a decay of the quantity characterizing the non-straightness at large distances within $S_{\omega,\varepsilon}$ as well as a restriction for s close to s'. The latter (which is independent of μ) ensures the boundedness of B_κ uniformly w.r.t. κ. As in the proof of the above lemma the uniformity is easy; it suffices to check that B_{κ_0} is bounded. To this end we employ the Schur-Holmgren bound: we have $\|B_{\kappa_0}\|_l \le \|B_{\kappa_0}\|_{SH}$, where the right-hand side of the last inequality is for integral operators with symmetric positive kernels defined as

$$\|B_{\kappa_0}\|_{SH} = \sup_{s\in\mathbb{R}} \int_{\mathbb{R}} B_{\kappa_0}(s,s')\,ds'\,.$$

Let us use the notation from the previous proof. If $\sigma \le \varepsilon$, then by assumption (a2) there exists for any $\mu \ge 0$ a $C_1 > 0$ such that

$$B_{\kappa_0}(s,s') \le C_1.$$

On the other hand, if $\sigma > \varepsilon$ then by (5.3) we can find $C_2 > 0$ such that

$$B_{\kappa_0}(s,s') \le C_2\,\mathrm{e}^{-\kappa_0\sigma_c}.$$

Combining the above two inequalities we get the following estimate,

$$\int_{\mathbb{R}} B_{\kappa_0}(s,s')ds' \le C_1\int_{s-\varepsilon}^{s+\varepsilon} ds' + 2C_2\int_{s+\varepsilon}^{\infty} \mathrm{e}^{-\kappa_0 c|s-s'|}ds' = 2\left(C_1\varepsilon + C_2\frac{\mathrm{e}^{-\kappa_0 c\varepsilon}}{\kappa_0 c}\right),$$

which shows that $\|B_{\kappa_0}\|_{SH}$ is finite.

Lemma 5.5 *Let Γ be defined as before. Then the function $\kappa \to \mathbf{Q}^\kappa$ is continuous in the norm operator in (κ_0,∞), and moreover,*

$$\lim_{\kappa\to\infty} \sup\sigma(\mathbf{Q}^\kappa) = -\infty\,. \tag{5.6}$$

Proof. First we observe that the function $\kappa \to T_\kappa$ is continuous in the norm operator. Indeed, for any $f \in D$ we have

$$\begin{aligned} \|(T_\kappa - T_{\kappa'})f\|_l &= \frac{1}{4(2\pi)^3}\int_{\mathbb{R}}\left(\ln\frac{p^2+\kappa^2}{p^2+\kappa'^2}\right)^2 |\hat f(p)|^2\,dp \\ &\le \frac{1}{4(2\pi)^3}\left(\ln\frac{\kappa}{\kappa'}\right)^2 \|f\|_l^2 \to 0. \end{aligned} \tag{5.7}$$

as $\kappa' \to \kappa$. On the other hand, in analogy with [EI] we can estimate

$$|(B_\kappa - B_{\kappa'})(s,s')|^2 \le 2(B_\kappa(s,s')^2 + B_{\kappa'}(s,s')^2) \le 4B_{\tilde\kappa}(s,s')^2\,,$$

where $\tilde{\kappa} := \min\{\kappa, \kappa'\}$ arriving therefore at

$$\lim_{\kappa' \to \kappa} \|B_\kappa - B_{\kappa'}\|_{HS} \to 0\,; \tag{5.8}$$

from (5.7) and (5.8) we get the norm-operator continuity. Let further $f \in D$. The limiting relation (5.6) follows directly from the bound

$$\begin{aligned}(\mathbf{Q}^\kappa f, f)_l &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}} \Big(-\ln \left(p^2 + \kappa^2\right)^{1/2} + \ln 2 + \psi(1) \Big) \, |\hat{f}(p)|^2 \, dp \\ +(B_\kappa f, f)_l &\le \frac{1}{(2\pi)^{3/2}} \left(-\ln \frac{\kappa}{2} + \psi(1)\right) \|f\|_l^2 + S\,\|f\|_l^2 \,,\end{aligned}$$

where $S := \sup_{\kappa \ge \kappa_0} \|B_k\|_l < \infty$. ■

Now we are in position to state and prove our main result.

Theorem 5.6 *Let Γ be a curve determined by a function $\gamma : \mathbb{R} \to \mathbb{R}^3$ which is C^1 and piecewise C^2, and satisfies the conditions (a1), (a2) with $\mu > 1/2$. Then the operator $-\Delta_{\gamma,\alpha}$ has at least one isolated eigenvalue in $(-\infty, \zeta_0)$.*

Proof. By Lemma 5.2 we have $\sup \sigma(\mathbf{Q}^\kappa) > s_\kappa$, while by Lemma 5.3 this operator has only isolated eigenvalues of a finite multiplicity in (s_κ, ∞). Let $\lambda(\kappa)$ be such an eigenvalue of $\mathbf{Q}^\kappa$. Using then Lemma 5.5 we conclude that the function $\lambda(\cdot)$ is continuous and $\lambda(\kappa) \to -\infty$ as $\kappa \to \infty$. Consequently, there is $\tilde{\kappa} > |\zeta_0|^{1/2}$ such that $\lambda(\tilde{\kappa}) = \alpha$. From the resolvent formula (3.5) we then infer that $-\tilde{\kappa}^2 \in (-\infty, \zeta_0)$ is an eigenvalue of $-\Delta_{\gamma,\alpha}$. ■

Remarks 5.7 (a) It is clear that the claim holds without the C^2 assumption, however, the latter is needed if we want to interpret the δ interaction on the curve in the spirit of Theorem 4.1. Furthermore, we see that any deviation from a straight Γ pushes the spectrum threshold below the value ζ_0 but without the assumption (a2) we cannot be sure about the nature of this added part of the spectrum.

(b) One may ask what the requirement of asymptotic straightness expressed by (a2) means. Suppose that γ is C^2 smooth. Then the curvature of Γ is everywhere defined and can expressed as $k(s) = \left(\sum_{i=1}^3 k_i(s)^2\right)^{1/2}$, where $k_i(s) := \varepsilon_{ijk}\gamma_j'(s)\gamma_k''(s)$ with the summation convention for the indices of the

Levi-Civita tensor. It allows us estimate the distance between $\gamma(s)$ and $\gamma(s')$ in the following way,

$$\begin{aligned} |\gamma(s)-\gamma(s')| &= \left[\sum_{\nu=0}^{1}\left(\sum\nolimits_{i=1}^{3}\int_{s'}^{s}\cos\left(\int_{s'}^{s_1}k_i(s_2)\,ds_2+\frac{\pi\nu}{2}\right)\,ds_1\right)^2\right]^{1/2} \\ &\geq \sum\nolimits_{i=1}^{3}\int_{s'}^{s}\left(1-\frac{1}{2}\left(\int_{s'}^{s_1}k_i(s_2)\,ds_2\right)^2\right)\,ds_1\,, \end{aligned}$$

where we assume without loss of generality that $s > s'$. Suppose that there are positive β, c_i such that $|k_i(s)| \leq c_i\,|s|^{-\beta}$. Then $|k(s)| \leq 3c\,|s|^{-\beta}$, where $c = \max_i\{c_i\}$ and one can estimate

$$\begin{aligned} 1-\frac{|\gamma(s)-\gamma(s')|}{|s-s'|} &\leq \frac{1}{2\,|s-s'|}\int_{s'}^{s}\left[\int_{s'}^{s_1}k(s_2)\,ds_2\right]^2 ds_1 \\ &\leq \frac{3c^2}{2\,|s-s'|}\,\frac{1}{|s'|^{2\beta}}\int_{s'}^{s}|s'-s_1|^2\,ds_1 \leq \frac{c^2}{2}\,\frac{|s'-s|^2}{|s'|^{2\beta}} \leq \frac{c^2}{2}\,\frac{1}{|s'|^{2\beta-2}}\,. \end{aligned}$$

Thus the conclusion is the same as in the two-dimensional case discussed in [EI]: the assumption (a2) with $\mu > 1/2$ is satisfied if $\beta > 5/4$.

Appendix: proof of Theorem 4.1

First we check the inclusion $D(-\Delta_{\gamma,\alpha}) \subseteq \Upsilon_\alpha$. Suppose that $f \in D(-\Delta_{\gamma,\alpha})$, i.e. that there is $f_\kappa \in D(\Delta)$ such that

$$f = f_\kappa - \breve{\mathbf{R}}^\kappa_\gamma(\mathbf{Q}^\kappa-\alpha)^{-1}\tau f_\kappa\,. \tag{A.1}$$

Denote $h := (\mathbf{Q}^\kappa-\alpha)^{-1}\tau f_\kappa \in L^2(\mathbb{R})$, so $f = f_\kappa - \mathbf{R}^\kappa\tau^* h$. Since $f_\kappa \in H^2(\mathbb{R}^3)$ and $\tau^* h \in H^{-2}(\mathbb{R}^3)$ is a measure supported by Γ we can conclude that $f \in H^2_{loc}(\mathbb{R}^3 \setminus \Gamma)$ – see [RS]. Using properties of the Macdonald function $K_0(\varsigma)$ and the following relation

$$\frac{1}{4\pi}\frac{\mathrm{e}^{-\kappa(r^2+(s-s')^2)^{1/2}}}{(r^2+(s-s')^2)^{1/2}} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}} K_0((p_1^2+\kappa^2)^{1/2}r)\,\mathrm{e}^{ip_1(s-s')}dp_1$$

we can check that

$$\lim_{r\to 0}\left[\frac{1}{4\pi}\int_{\mathbb{R}}\frac{\mathrm{e}^{-\kappa(r^2+(s-s')^2)^{1/2}}}{(r^2+(s-s')^2)^{1/2}}h(s')ds' + \frac{1}{2\pi}\ln r h(s)\right] = T_\kappa h(s).$$

Now it is easy to demonstarte that the function

$$(\breve{\mathbf{R}}^\kappa_\gamma h)(x) = \frac{1}{4\pi}\int_{\mathbb{R}} \frac{\mathrm{e}^{-\kappa|x-\gamma(s)|}}{|x-\gamma(s)|} h(s)\, ds$$

satisfies the limiting relation

$$\lim_{r\to 0}\left[(\breve{\mathbf{R}}^\kappa h)\restriction_{\Gamma_r}(s) + \frac{1}{2\pi}\ln r h(s)\right] = T_\kappa h(s) + B_\kappa h(s) \tag{A.2}$$

with respect to families of "shifted" curves described in Sec. 4. The above limits are understood in distributional sense. It follows from (A.1) and (A.2) that

$$\Xi(f)(s) = -\frac{1}{2\pi} h(s). \tag{A.3}$$

On the other hand, since $f_\kappa \in D(\Delta) = H^2(\mathbb{R}^3)$ the same relations (A.1) and (A.2) yield

$$\Omega(f)(s) = (\tau f_\kappa)(s) - (\mathbf{Q}_\kappa h)(s) = -\alpha h(s). \tag{A.4}$$

Combining (A.3) and (A.4) we obtain that $f \in \Upsilon$ and $2\pi\alpha\Xi(f)(s) = \Omega(f)(s)$. Conversely, one can show by analogous considerations that any function from Υ_α can be represented in the form $f = f_\kappa - \breve{\mathbf{R}}^\kappa_\gamma(\mathbf{Q}^\kappa - \alpha)^{-1}\tau f_\kappa$ with $f_\kappa \in D(\Delta)$, so $D(-\Delta_{\gamma,\alpha}) = \Upsilon_\alpha$. Moreover, since

$$(-\Delta_{\gamma,\alpha} + \kappa^2) f = (-\Delta + \kappa^2) f_\kappa$$

and $\tau^* h \in H^{-2}(\mathbb{R}^3)$ is a measure supported by Γ we infer that

$$-\Delta_{\gamma,\alpha} f(x) = -\Delta f(x), \quad x \in \mathbb{R}^3 \backslash \Gamma\,.$$

This completes the proof.

Acknowledgments

The work was supported by GAAS under the contract #1048101.

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Department of Theoretical Physics
Nuclear Physics Institute
Academy of Sciences
25068 Řež near Prague, Czech Republic
E-mail: exner@ujf.cas.cz, kondej@ujf.cas.cz