QUANTUM ELECTRODYNAMICS OF RELATIVISTIC BOUND STATES WITH CUTOFFS. II

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Abstract. In this note we consider an Hamiltonian, with ultraviolet and infrared cutoffs, describing the interaction of relativistic electrons and positrons in a Coulomb potential with photons in Coulomb gauge. The interaction includes both interaction of the current-density with transversal photons and the Wick ordering corrected Coulomb interaction of charge-density with itself.

We prove that the Hamiltonian is self-adjoint and has a ground state for a sufficiently small coupling constant.

1. Introduction

In [DG] an Hamiltonian with cuttofs describing relativistic electrons and positrons in a Coulomb potential interacting with transversal photons in Coulomb gauge is studied. In this note we add the Wick ordering corrected Coulomb interaction of the charge-density with itself to the Hamiltonian described in [DG].

We are studying an Hamiltonian with ultraviolet and infrared cutoff functions with respect to the momenta of photons, but also with respect to the momenta of electrons and positrons. This ensure that the total Hamiltonian in the Fock space of electrons, positrons and photons is well defined in the Furry picture.

In this note we announce results concerning self-adjointness of this Hamiltonian and the existence of a ground state when the coupling constant is sufficiently small. In [BFS1], Bach, Fröhlich and Sigal proved the existence of a ground state for the Pauli-Fierz Hamiltonian with an ultraviolet cutoff for photons, and for sufficiently small values of the fine structure constant, without introducing an infrared cutoff. Their result has been extended by Griesemer, Lieb and Loss [GLL] under the binding condition. For related results see [AH1, AH2, AH3, G, S].

No-pair Hamiltonians for relativistic electrons in QED have been recently considered in [FGS, LL, M].

The case of relativistic electrons in classical magnetic fields was studied earlier in [LSS] and [GT]. There, it was proven instability for the Brown and Ravenhall model in the free picture. In [GT] it is even deduced from this result that instability also holds in QED context, i.e. for the Brown and Ravenhall model in the free picture, coupled to the second quantized radiation field with or without cutoff.

In our case, working in the Furry picture and imposing both electronic and photonic cutoff prevents from instability.

Our methods follow those of [A, BFS1, BFS2, BFS3], in which the spectral theory of the spin-boson and Pauli-Fierz Hamiltonians is studied.

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2. The Hamiltonian

From now on we use the same notations as in [DG]. Let $\mathcal{F}_D$ be the Fock space of relativistic electrons and positrons in a Coulomb potential. We work here in the Furry picture, i.e., the one-electron space (resp. the one-positron space) is the positive (resp. negative) spectral subspace of the Dirac-Coulomb operator. Let $b_{\gamma,+}(p)$ (resp. $b_{\gamma,n}$, $b_{\gamma,-}(p)$) be the annihilation operators corresponding to the electrons in the continuum (resp. the bound states, the positrons). See [DG], [BDG], [T] and [K] for details. Here γ is a triplet of angular momentum and spin-orbit quantum numbers. Moreover $n \in \mathbb{N}$ and $p \in \mathbb{R}_+$. The associated creation operators are denoted by $b^*_{\gamma,+}(p)$, $b^*_{\gamma,n}$ and $b^*_{\gamma,-}(p)$.

The Hamiltonian of the quantized Dirac-Coulomb field is given by:

$$d\Gamma(H_D) = \sum_{\gamma,n} E_{\gamma,n}\, b^*_{\gamma,n}\, b_{\gamma,n} + \sum_{\epsilon=+,-} \sum_{\gamma} \int_{\mathbb{R}+} dp\, \omega(p)\, b^*_{\gamma,\epsilon}(p)\, b_{\gamma,\epsilon}(p)$$

Here $(E_{\gamma,n})_{\gamma,n}$ are the eigenvalues of the Dirac operator H_D for a relativistic electron in the Coulomb potential generated by a nucleus of charge $Z \leq 118$. We have $0 < E_0 = \inf E_{\gamma,n}$ and $\omega(p) = (p^2 + m^2c^4)^{\frac{1}{2}}$, $p \in \mathbb{R}_+$. Here m is the mass of the electrons and c is the speed of light. The operator $d\Gamma(H_D)$ is a self-adjoint operator in $\mathcal{F}_D$ and its point spectrum contains the one of H_D.

Let $\mathcal{F}_{\rm ph}$ be the Fock space of photons in Coulomb gauge. Let $a_\mu(k)$ (resp. $a^*_\mu(k)$) denotes the annihilation (resp. creation) operator associated with a field $\varepsilon_\mu(k)$, $\mu = 1,2$, of transversal polarizations for the photons (see [GR]). The Hamiltonian of the quantized electromagnetic field, denoted by $H_{\rm ph}$, is

$$H_{\rm ph} = \sum_{\mu=1,2} \int_{\mathbb{R}^3} d^3k\ \omega(k)\, a^*_\mu(k)\, a_\mu(k)\ ,$$

where $\omega(k) = c|k|$. The operator $H_{\rm ph}$ is self-adjoint in $\mathcal{F}_{\rm ph}$ and its spectrum is $[0,\infty)$.

The Fock space for electrons, positrons and photons is the following Hilbert space

$$\mathcal{F} = \mathcal{F}_D \otimes \mathcal{F}_{\rm ph}.$$

The free Hamiltonian for electrons, positrons and photons, denoted by H_0, is the following operator in $\mathcal{F}$

$$H_0 = d\Gamma(H_D) \otimes \mathbf{1}_{\rm ph} + \mathbf{1}_D \otimes H_{\rm ph}.$$

defined on

$$\mathcal{D}\left(d\Gamma(H_D) \otimes \mathbf{1}_{\rm ph}\right) \cap \mathcal{D}\left(\mathbf{1}_D \otimes H_{\rm ph}\right).$$

H_0 is a positive self-adjoint operator which has the same point spectrum as $d\Gamma(H_D)$. Its continuous spectrum covers the half line.

The interaction between electrons, positrons and photons that we consider includes two terms. The first one is the interaction of the current-density with transversal photons already considered in [DG]. The second one is the Coulomb interaction of the charge-density with itself, with a Wick ordering correction.

We have to introduce several cutoff functions in the Dirac-Coulomb field and the electromagnetic vector potential in order to get a well defined total Hamiltonian in the Fock space.

The first term in the interaction, denoted by $H_I^{(1)}$, is given by:

$$\begin{aligned}
H_I^{(1)} = & \\
& \sum_{\gamma,\gamma',n,\ell}\sum_{\mu=1,2}\int d^3k\left(G^{\mu}_{d,\gamma,\gamma',n,\ell}(k)b^*_{\gamma,n}b_{\gamma',\ell}a^*_{\mu}(k)+\text{h.c.}\right)\\
& +\sum_{\epsilon=+,-}\sum_{\gamma,\gamma',n}\sum_{\mu=1,2}\int d^3k\,dp\left(G^{\mu}_{d,\epsilon,\gamma,\gamma',n}(p;k)\Big(b^*_{\gamma,n}b_{\gamma',\epsilon}(p)\right.\\
& \qquad\qquad\left.+b^*_{\gamma,\epsilon}(p)b_{\gamma',n}\Big)a^*_{\mu}(k)+\text{h.c.}\right)\\
& +\sum_{\gamma,\gamma'}\sum_{\mu=1,2}\int d^3k\,dp\,dp'\left(G^{\mu}_{+,-,\gamma,\gamma'}(p,p';k)\Big(b^*_{\gamma,+}(p)b^*_{\gamma',-}(p')\right.\\
& \qquad\qquad\left.+b_{\gamma,-}(p)b_{\gamma',+}(p')\Big)a^*_{\mu}(k)+\text{h.c.}\right)\\
& +\sum_{\epsilon=+,-}\sum_{\gamma,\gamma'}\sum_{\mu=1,2}\int d^3k\,dp\,dp'\left(G^{\mu}_{\epsilon,\epsilon,\gamma,\gamma'}(p,p';k)b^*_{\gamma,\epsilon}(p)b_{\gamma',\epsilon}(p')a^*_{\mu}(k)+\text{h.c.}\right)
\end{aligned}$$

For the second term let us introduce some notations. In the case of electrons, ξ will be equal to (γ,p) and (γ,n) with $\int d\xi=\sum_{\gamma}\int dp+\sum_{\gamma,n}$. In the case of positrons, ξ will be equal to (γ,p) with $\int d\xi=\sum_{\gamma}\int dp$. The second term of the interaction, denoted by $H_I^{(2)}$, is then given by

$$\begin{aligned}
H_I^{(2)} = & \\
& \int d\xi_1 d\xi_2 d\xi_3 d\xi_4\, F^{(1)}(\xi_1,\xi_2,\xi_3,\xi_4)b^*_{+}(\xi_1)b^*_{-}(\xi_2)b_{+}(\xi_3)b_{-}(\xi_4)\\
& +\sum_{\epsilon=+,-}\int d\xi_1 d\xi_2 d\xi_3 d\xi_4\, F^{(2)}_{\epsilon}(\xi_1,\xi_2,\xi_3,\xi_4)b^*_{\epsilon}(\xi_1)b^*_{\epsilon}(\xi_2)b_{\epsilon}(\xi_3)b_{\epsilon}(\xi_4)\\
& +\sum_{\substack{\epsilon,\epsilon'=+,-,\\ \epsilon+\epsilon'=0}}\int d\xi_1 d\xi_2 d\xi_3 d\xi_4\left(F^{(3)}_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4)b^*_{\epsilon}(\xi_1)b_{\epsilon}(\xi_2)b_{\epsilon'}(\xi_3)b_{\epsilon}(\xi_4)\right.\\
& \qquad\qquad\left.-F^{(3)}_{\epsilon,\epsilon'}(\xi_4,\xi_2,\xi_3,\xi_1)b^*_{\epsilon}(\xi_1)b^*_{\epsilon}(\xi_2)b^*_{\epsilon'}(\xi_3)b_{\epsilon}(\xi_4)\right)\\
& +\sum_{\substack{\epsilon,\epsilon'=+,-,\\ \epsilon+\epsilon'=0}}\int d\xi_1 d\xi_2 d\xi_3 d\xi_4\left(F^{(4)}_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4)b_{\epsilon}(\xi_1)b_{\epsilon}(\xi_2)b_{\epsilon'}(\xi_3)b_{\epsilon'}(\xi_4)\right.\\
& \qquad\qquad\left.+F^{(4)}_{\epsilon,\epsilon'}(\xi_4,\xi_2,\xi_3,\xi_1)b^*_{\epsilon}(\xi_1)b^*_{\epsilon}(\xi_2)b^*_{\epsilon'}(\xi_3)b^*_{\epsilon'}(\xi_4)\right).
\end{aligned}$$

Furthermore, we suppose that

$$F^{(1)}(\xi_1,\xi_2,\xi_3,\xi_4)=\overline{F^{(1)}(\xi_3,\xi_4,\xi_1,\xi_2)}$$

$$F^{(2)}_{\epsilon}(\xi_1,\xi_2,\xi_3,\xi_4)=\overline{F^{(2)}_{\epsilon}(\xi_4,\xi_3,\xi_2,\xi_1)}$$

Our Hamiltonian for relativistic electrons and positrons in a Coulomb potential interacting with photons in Coulomb gauge is finally given by

$$H(g)=H_0+g\,H_I^{(1)}+g^2H_I^{(2)},$$

where g is a real coupling constant.

3. Main results

It is easy to show that $H(g)$ is a symmetric operator in $\mathcal{F}$ as soon as the kernels $F^{(i)}$'s and $G^{(i)}$'s are square integrable. With stronger conditions on the $F^{(i)}$'s and $G^{(i)}$'s, we recover a self-adjoint operator with a ground state, as stated in Theorem 1 and 2 below.

Let, for $\beta = 0, 1, 2$,

$$C_\beta = \sum_{\mu=1,2}\left(\sum_{\gamma,\gamma',n,\ell}\int_{\mathbb{R}^3}|G^{\mu}_{d,\gamma,\gamma',n,\ell}(k)|^2\omega(k)^{-\beta}d^3k\right)^{1/2}$$
$$+\sum_{\epsilon=+,-}\sum_{\mu=1,2}\left(\sum_{\gamma,\gamma',n}\int_{\mathbb{R}^3\times\mathbb{R}+}|G^{\mu}_{d,\epsilon,\gamma,\gamma',n}(p;k)|^2\omega(k)^{-\beta}dp\,d^3k\right)^{1/2}$$
$$+\sum_{r}\sum_{\mu=1,2}\left(\sum_{\gamma,\gamma'}\int_{\mathbb{R}^3\times\mathbb{R}+\times\mathbb{R}+}|G^{\mu}_{r,\gamma,\gamma'}(p,p';k)|^2\omega(k)^{-\beta}dp\,dp'\,d^3k\right)^{1/2},$$

where $r = \{+,+\},\{+,-\},\{-,-\}$.

We now state our main results :

Theorem 1. *We assume that $F^{(j)} \in L^2$, $j = 1, 2, 3, 4$. Furthermore, we suppose that $C_0 < \infty$ and $\frac{|g|}{E_0}C_1 + \frac{g^2}{E_0}\|F^{(1)}\|_{L^2} < 1$. Then, $H(g)$ is self-adjoint on the domain $D(H_0)$.*

Theorem 2. *We assume that $F^{(j)} \in L^2$, $j = 1, 2, 3, 4$. Furthermore, we suppose that $C_0 < \infty$, $\frac{|g|}{E_0}C_1 + \frac{g^2}{E_0}\|F^{(1)}\|_{L^2} < 1$ and $C_2 < \infty$. Then, there exists $g_0 > 0$ such that, for every $g \in [-g_0, g_0]$, the self-adjoint operator $H(g)$ has a ground state.*

Recall that E_0 is the ground state energy of the Dirac-Coulomb operator H_D.

The proofs are partly based on the following lemmas (details can be found in[DG] and [BDG]). For a sake of simplicity, we reduce the discussion here to the case of electrons in the continuum and positrons only.

Set $\xi = (\gamma, p)$, with $\int d\xi = \sum_\gamma \int dp$. Let $G(\xi, \xi') = G(\gamma, p, \gamma', p')$, with

$$\|G(\cdot,\cdot)\|^2_{L^2} = \sum_{\gamma,\gamma'}\int dp\,dp'|G(\gamma,p,\gamma',p')|^2.$$

For G such that $\|G\|_{L^2} < \infty$ we define

$$\begin{aligned}
A_\pm &= \iint d\xi d\xi'\,\overline{G(\xi,\xi')}b_\pm(\xi)b_\pm(\xi')\ ,\\
A_{+-} &= \iint d\xi d\xi'\,\overline{G(\xi,\xi')}b_+(\xi)b_-(\xi')\ ,\\
A_{-+} &= \iint d\xi d\xi'\,\overline{G(\xi,\xi')}b_-(\xi)b_+(\xi')\ ,\\
B_\pm &= \iint d\xi d\xi'\,\overline{G(\xi,\xi')}b^*_\pm(\xi)b_\pm(\xi')\ .
\end{aligned}$$

Lemma 1. *We have the following norm estimates*

$$\begin{aligned}
\|A_\pm\| &= \|A^*_\pm\| \le \|G_a\|_{L^2}\ ,\\
\|A_{+-}\| &= \|A^*_{+-}\| \le \|G_a\|_{L^2}\ ,\\
\|A_{-+}\| &= \|A^*_{-+}\| \le \|G_a\|_{L^2}\ ,
\end{aligned}$$

where $G_a(\xi,\xi') = G(\xi,\xi') - G(\xi',\xi)$, and

$$\begin{aligned}\|B_\pm(N_D+1)^{-1/2}\| &\leq \|G\|_{L^2}\ ,\\ \|(N_D+1)^{-1/2}B_\pm\| &\leq \|G\|_{L^2}\ ,\end{aligned}$$

where $N_D = \sum_{\gamma,n} b^*_{\gamma,n}b_{\gamma,n} + \sum_{\epsilon=+,-}\int d\xi\, b^*_\epsilon(\xi)b_\epsilon(\xi)$ *is the number operator for electrons and positrons.*

For the quartic terms in $b^\#$ we have the following results

Lemma 2. *For every* $\psi \in \mathcal{D}(N_D)$ *we have*

$$\begin{aligned}&\|\int d\xi_1 d\xi_2 d\xi_3 d\xi_4\, F^{(1)}(\xi_1,\xi_2,\xi_3,\xi_4)b^*_+(\xi_1)b^*_-(\xi_2)b_+(\xi_3)b_-(\xi_4)\psi\|^2\\ &\qquad\leq \|F^{(1)}\|^2_{L^2}\|(N_D+1)\psi\|^2 + C\|(N_D+1)^{1/2}\psi\|^2 + C\|\psi\|^2\end{aligned}$$

where C *is a nonnegative constant.*

Lemma 3. *For every* $\psi \in \mathcal{D}(N_D)$ *we have*

$$\begin{aligned}&\|\sum_{\epsilon=+,-}\int d\xi_1 d\xi_2 d\xi_3 d\xi_4\, F^{(2)}_\epsilon(\xi_1,\xi_2,\xi_3,\xi_4)b^*_\epsilon(\xi_1)b^*_\epsilon(\xi_2)b_\epsilon(\xi_3)b_\epsilon(\xi_4)\psi\|^2\\ &\qquad\leq C\left(\sum_{\epsilon=+,-}\|F^{(2)}_\epsilon\|_{L^2}\right)\|(N_D+1)^{\frac12}\psi\|^2,\end{aligned}$$

where C *is a nonnegative constant.*

Similar estimates hold for the term including $F^{(3)}_{\epsilon,\epsilon'}$ *in* $H^{(2)}_I$.

Lemma 4. *The operator*

$$\begin{aligned}\sum_{\substack{\epsilon,\epsilon'=+,-,\\ \epsilon+\epsilon'=0}}\int d\xi_1 d\xi_2 d\xi_3 d\xi_4 \Big(&F^{(4)}_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4)b_\epsilon(\xi_1)b_\epsilon(\xi_2)b_{\epsilon'}(\xi_3)b_{\epsilon'}(\xi_4)\\ &+F^{(4)}_{\epsilon,\epsilon'}(\xi_4,\xi_2,\xi_3,\xi_1)b^*_\epsilon(\xi_1)b^*_\epsilon(\xi_2)b^*_{\epsilon'}(\xi_3)b^*_{\epsilon'}(\xi_4)\Big)\ ,\end{aligned}$$

is bounded.

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