Dimensional Reduction Formulas for Branched Polymer Correlation Functions

David C. Brydges
Department of Mathematics
The University of British Columbia
Room 121, 1984 Mathematics Road
Vancouver, B.C., Canada V6T 1Z2
db5d@math.ubc.ca
and
Department of Mathematics
University of Virginia
Charlottesville VA 22904-4137

John Z. Imbrie
Department of Mathematics
University of Virginia
Charlottesville VA 22904-4137
ji2k@virginia.edu

Abstract

In [BI01] we have proven that the generating function for self-avoiding branched polymers in $D+2$ continuum dimensions is proportional to the pressure of the hard-core continuum gas at negative activity in D dimensions. This result explains why the critical behavior of branched polymers should be the same as that of the $i\varphi^3$ (or Yang-Lee edge) field theory in two fewer dimensions (as proposed by Parisi and Sourlas in 1981).

In this article we review and generalize the results of [BI01]. We show that the generating functions for several branched polymers are proportional to correlation functions of the hard-core gas. We derive Ward identities for certain branched polymer correlations. We give reduction formulae for multi-species branched polymers and the corresponding repulsive gases. Finally, we derive the massive scaling limit for the 2-point function of the one-dimensional hard-core gas, and thereby obtain the scaling form of the 2-point function for branched polymers in three dimensions.

1 Introduction and Main Results

We define the generating function for branched polymers (mod translations) to be

$$Z_{\mathrm{BP}}(\mathrm{z}) = \sum_{N=1}^{\infty} \frac{\mathrm{z}^N}{N!} \sum_T \int_{\mathbb{R}^{(D+2)(N-1)}} dy_2 \ldots dy_N \prod_{ij \in T} [2U'(|y_i - y_j|^2)] \prod_{ij \notin T} U(|y_i - y_j|^2). \tag{1.1}$$

Here $y_1 = 0, y_2, \dots, y_N$ are the positions of the monomers, and we sum over all tree graphs T on $\{1, \dots, N\}$. We assume that $U(t)$ is a positive weight function which tends to 1 as $t \to \infty$, and that $U'(|y|^2)$ is a positive, integrable function of $y \in \mathbb{R}^{D+2}$. By taking limits, we may take $U(t) = \theta(t-1)$, where θ is the Heaviside step function. In this case, $2U'(|y_i - y_j|^2) = \delta(|y_i - y_j| - 1)$, and we obtain our standard model of hard spheres such that spheres i and j are required to touch if $ij \in T$.

The above definition is a direct translation to the continuum of the familiar model of lattice branched polymers. On the lattice $\mathbb{Z}^{D+2}$, a branched polymer is a finite connected set of nearest-neighbor bonds with no cycles [Sla99]. An N-vertex branched polymer is a subset $\{y_1, \dots, y_N\}$ of $\mathbb{Z}^{D+2}$, together with a tree graph on $\{y_1, \dots, y_N\}$ such that for every $\{y_i, y_j\} \in T$, $|y_i - y_j| = 1$. One defines c_N to be the number of N-vertex branched polymers mod translations. Then, as in [Frö86], the generating function $Z_{\mathrm{BP}}(\mathrm{z}) = \sum_N \mathrm{z}^N c_N$ can be written as

$$Z_{\mathrm{BP}}(\mathrm{z}) = \sum_{N=1}^{\infty} \frac{\mathrm{z}^N}{N!} \sum_T \sum_{y_2, \dots, y_N} \prod_{ij \in T} [2U'_{ij}] \prod_{ij \notin T} U_{ij}, \tag{1.2}$$

where $2U'_{ij} = \delta_{|y_i - y_j|,1}$ and $U_{ij} = 1 - \delta_{y_i, y_j}$ enforce the adjacency and loop-free conditions, respectively. For example, $c_3 = 6$ in $\mathbb{Z}^2$, which is correctly accounted for in (1.2) as there are 3 trees, 4 possibilities for y_2, and then 3 for y_3. For more details, see [BI01].

Returning to the continuum, we define the partition function for the repulsive gas in a box $\Lambda \subset \mathbb{R}^D$:

$$Z_{\mathrm{HC}}(\mathrm{z}) = \sum_{N=0}^{\infty} \frac{\mathrm{z}^N}{N!} \int_{\Lambda^N} dx_1 \dots dx_N \prod_{1 \le i < j \le N} U(|x_i - x_j|^2). \tag{1.3}$$

The main result of [BI01] is that the identity

$$\lim_{\Lambda \nearrow \mathbb{R}^D} \frac{1}{|\Lambda|} \log Z_{\mathrm{HC}}(\mathrm{z}) = -2\pi Z_{\mathrm{BP}} \left(-\frac{\mathrm{z}}{2\pi} \right) \tag{1.4}$$

holds for all z such that the right-hand side converges absolutely. The left-hand side of (1.4) is $1/(kT)$ times the pressure of the repulsive gas. Evidently, its leading singularity $\sim (\mathrm{z} - \mathrm{z}_c)^{2-\alpha_{\mathrm{HC}}}$ is identical to the leading singularity $\sim (\mathrm{z} - \mathrm{z}_c)^{2-\gamma_{\mathrm{BP}}}$ of Z_{BP}, where z_c is the closest singularity to the origin. Hence

$$\alpha_{\mathrm{HC}}(D) = \gamma_{\mathrm{BP}}(D+2). \tag{1.5}$$

If one can define θ from the asymptotic form $c_N \sim \mathrm{z}_c^{-N} N^{-\theta}$, then $\theta = 3 - \gamma_{\mathrm{HC}}$ by an Abelian theorem. Furthermore, one expects that σ, the Yang-Lee edge exponent, is equal to $1 - \alpha_{\mathrm{HC}}$ [PF99, eqn. 7], which with (1.5) leads to the Parisi-Sourlas relation [PS81]

$$\theta(D+2) = \sigma(D) + 2. \tag{1.6}$$

One can also see that the exponents $\nu_{\rm BP}$, $\eta_{\rm BP}$ are equal to their hard-core counterparts in two fewer dimensions (see Section 3.

If one takes

$$U(|x_i - x_j|^2) = e^{-w(x_i - x_j)}, \tag{1.7}$$

with $\hat{w}(k) > 0$, then by the sine-Gordon transformation, (1.3) can be written as

$$Z_{\rm HC}({\rm z}) = \int \exp\left(\int_\Lambda dx \hat{\rm z} e^{i\varphi(x)}\right) d\mu_w(\varphi), \tag{1.8}$$

where $d\mu_w$ is the Gaussian measure with covariance w, and $\hat{\rm z} = {\rm z}e^{w(0)/2}$. Thus certain branched polymer models can be written as $-\hat{\rm z}e^{i\varphi}$ field theories in two fewer dimensions. Taking into account an effective mass term $\sim \varphi(x)^2$ from $d\mu_w$, one finds a critical value for φ on the imaginary axis, and at the critical z the interaction is $i\varphi^3+$ higher order. Thus one expects that these theories are in the same universality class as the Yang-Lee edge ($i\varphi^3$) theory [Fis78] (see [LF95], [PF99] for a more complete investigation of the hypothesis that the repulsive-core singularity is in the Yang-Lee class).

Cardy has argued recently [Car01] (see also his contribution to this issue) that the crossover from area-weighted self-avoiding loops to ordinary self-avoiding loops in two dimensions is governed by a scaling function related to the Airy function. Part of his argument is the reduction of two-dimensional branched polymers to zero-dimensional $i\varphi^3$ theory. This is, in essence, the content of equations (1.4) and (1.8).

Correlation Functions

We define first the basic n-point density correlations for branched polymers and for repulsive gases. Let

$$\rho(\tilde{x}) = \sum_{i=1}^N \delta(\tilde{x} - x_i), \qquad \rho(\tilde{y}) = \sum_{i=1}^N \delta(\tilde{y} - y_i), \tag{1.9}$$

where $\tilde{x}, x_i \in \mathbb{R}^D$ and $\tilde{y}, y_i \in \mathbb{R}^{D+2}$. Then we put

$$\begin{aligned} G^{(n)}_{\rm BP}(\tilde{y}_1, \ldots, \tilde{y}_n; {\rm z}) &= \sum_{N=1}^\infty \frac{{\rm z}^N}{N!} \sum_T \int_{\mathbb{R}^{(D+2)N}} dy_1 \ldots dy_N \prod_{i=1}^n \rho(\tilde{y}_i) \prod_{ij \in T} [2U'_{ij}] \prod_{ij \notin T} U_{ij} \\ G^{(n)}_{\rm HC}(\tilde{x}_1, \ldots, \tilde{x}_n; {\rm z}) &= \lim_{\Lambda \nearrow \mathbb{R}^D} \left\langle \prod_{i=1}^n \rho(\tilde{x}_i) \right\rangle_{{\rm HC},\Lambda}. \end{aligned} \tag{1.10}$$

Here $U'_{ij} := U'(|y_i - y_j|^2)$, $U_{ij} := U(|y_i - y_j|^2)$, and $\langle \cdot \rangle_{{\rm HC},\Lambda}$ is the expectation in the measure for which $Z_{\rm HC}({\rm z})$ is the normalizing constant. We also write $G^{(n),{\rm T}}_{\rm HC}$ for the corresponding truncated expectation.

If $y_1, \dots, y_n$ are distinct points, then we have

$$G^{(n)}_{\mathrm{BP}}(y_1, \dots, y_n; \mathrm{z}) = \sum_{M=0}^{\infty} \frac{\mathrm{z}^M}{M!} \sum_{T \text{ on } \{1,\dots,n+M\}} \int dy_{n+1} \dots dy_{n+M} \prod_{ij \in T} [2U'_{ij}] \prod_{ij \notin T} U_{ij}. \tag{1.11}$$

Thus, for distinct points $G^{(n)}_{\mathrm{BP}}$ is a sum/integral over branched polymers whose vertices include $y_1, \dots, y_n$. When points are not distinct, $G^{(n)}_{\mathrm{BP}}$ and $G^{(n)}_{\mathrm{HC}}$ are understood by smearing each $\rho(\tilde{y})$ or $\rho(\tilde{x})$ by test functions. Thus in general, $G^{(n)}_{\mathrm{BP}}$ and $G^{(n)}_{\mathrm{HC}}$ are distributions which contain δ-function singularities at coinciding points. In addition, if U'_{ij} is not smooth (for example in the hard sphere case $U(t) = \theta(t-1)$), then $G^{(n)}_{\mathrm{BP}}$ will inherit singularities from U'_{ij}.

The density correlations $G^{(n)}_{\mathrm{HC}}$ arise naturally when taking an order n variational derivative of Z_{HC} with respect to an external field. However, we will need a different set of Green's functions for the repulsive gas. Stripping $G^{(n)}_{\mathrm{HC}}$ of its singularities at coinciding points, we write

$$g^{(n)}_{\mathrm{HC}}(x_1, \dots, x_n; \mathrm{z}) = Z_{\mathrm{HC}}(\mathrm{z})^{-1} \sum_{m=0}^{\infty} \frac{\mathrm{z}^{n+m}}{m!} \int_{\Lambda^m} dx_{n+1} \dots dx_{n+m} \prod_{1 \le i < j \le n+m} U_{ij}. \tag{1.12}$$

Here n particles are forced to be at $x_1, \dots, x_n$, and if these are distinct points, then $g^{(n)}_{\mathrm{HC}} = G^{(n)}_{\mathrm{HC}}$. In general, $G^{(n)}_{\mathrm{HC}}$ is a sum of terms, each with some $g^{(j)}_{\mathrm{HC}}$, $j \le n$ multiplied by δ-functions in some of the x_i's. For example,

$$G^{(2)}_{\mathrm{HC}}(x_1, x_2) = g^{(2)}_{\mathrm{HC}}(x_1, x_2) + g^{(1)}_{\mathrm{HC}}(x_1)\delta(x_1 - x_2). \tag{1.13}$$

We shall see that $g^{(n)}_{\mathrm{HC}}$ can be related to a certain n-tree branched polymer correlation function:

$$g^{(n)}_{\mathrm{BP}}(y_1, \dots, y_n; \mathrm{z}) = \sum_{p=0}^{\infty} \frac{\mathrm{z}^{p+n}}{p!} \sum_{F^{(n)}} \int_{\mathbb{R}^{(D+2)p}} dy_{n+1} \dots dy_{n+p} \prod_{ij \in F^{(n)}} [2U'_{ij}] \prod_{ij \notin F^{(n)}} U_{ij}. \tag{1.14}$$

Here $F^{(n)}$ is a loop-free graph or *forest* on $\{1, \dots, n+p\}$ which consists of n connected components or *trees*, each of which contains one of $y_1, \dots, y_n$.

Theorem 1.1. *If* z *is in the interior of the domain of convergence at* Z_{BP}*, then in the limit* $\Lambda \nearrow \mathbb{R}^D$*,*

$$g^{(n)}_{\mathrm{HC}}(x_1, \dots, x_n; \mathrm{z}) = (-2\pi)^n g^{(n)}_{\mathrm{BP}}\left(x_1, \dots, x_n; -\frac{\mathrm{z}}{2\pi}\right), \tag{1.15}$$

and

$$G^{(n),\mathrm{T}}_{\mathrm{HC}}(x_1, \dots, x_n; \mathrm{z}) = (-2\pi) \int_{\mathbb{C}^{n-1}} dz_2 \dots dz_n G^{(n)}_{\mathrm{BP}}\left(x_1, y_2, \dots, y_n; -\frac{\mathrm{z}}{2\pi}\right). \tag{1.16}$$

Here $x_i \in \mathbb{R}^D$ *and* $y_i = (x_i, z_i) \in \mathbb{R}^{D+2}$*.*

The relation (1.16) was proven in [BI01] by differentiating (1.4) with respect to sources. We prove (1.15) in Section 2. As $g_{\mathrm{HC}}^{(n)}$ and $G_{\mathrm{HC}}^{(n)}$ agree at non-coinciding points, (1.15) and (1.16) combine to give relations between $g_{\mathrm{BP}}^{(n)}$ and $G_{\mathrm{BP}}^{(n)}$. In particular, we show that the two-point functions obey a Ward identity

$$\frac{d}{d(r^2)}\, g_{\mathrm{BP}}^{(2)} = \frac{1}{2} G_{\mathrm{BP}}^{(2)}, \tag{1.17}$$

where by rotation and translation invariance $g_{\mathrm{BP}}^{(2)}$ and $G_{\mathrm{BP}}^{(2)}$ can be thought of as functions of $r^2 = |y_1 - y_2|^2$ only.

In an appendix, we generalize the above results to repulsive gases and branched polymers with more than one species of particle/monomer and species-dependent interactions. Examples include the Widom-Rowlinson model of penetrable hard spheres [WR70]. As with the models discussed above, dimensional reduction is actually a consequence of an underlying supersymmetry of the branched polymer model. This requires that the attractive interaction between neighboring monomers be related to the repulsive interaction.

In Section 3 we focus on the case $D = 1$ and derive a number of results for the standard hard-core gas using the method of Laplace transforms. We give fairly explicit formulas for $G_{\mathrm{HC}}^{(2),\mathrm{T}}$ and derive the values $\alpha_{\mathrm{HC}} = \frac{3}{2}$, $\nu_{\mathrm{HC}} = \frac{1}{2}$, $\eta_{\mathrm{HC}} = -1$, thereby obtaining the same values for γ_{BP}, ν_{BP}, η_{BP} in three dimensions. (Note that for $D = 0$, $\log Z_{\mathrm{HC}} = \log(1 + z)$, so that the two-dimensional $Z_{\mathrm{BP}}(\mathrm{z})$ has a logarithmic singularity at $\mathrm{z} = 2\pi$, which implies that $\gamma_{\mathrm{BP}} = \alpha_{\mathrm{HC}} = 2$. Unfortunately, dimensional reduction gives no information on ν_{BP}, η_{BP} in this case.) We derive the scaling form of the two-point function near $\mathrm{z}_c = -e^{-1}$:

$$G_{\mathrm{HC}}^{(2),\mathrm{T}}(0, x; \mathrm{z}) \sim |x|^{-(D-2+\eta_{\mathrm{HC}})} K_{\mathrm{HC}}(x/\xi), \tag{1.18}$$

with

$$K_{\mathrm{HC}}(\hat{x}) = -\frac{4}{\hat{x}^2}\, e^{-\hat{x}}, \tag{1.19}$$

which implies

$$G_{\mathrm{BP}}^{(2)}(0, y; \mathrm{z}) \sim |x|^{-d-2+\eta_{\mathrm{BP}}} K_{\mathrm{BP}}(x/\xi), \tag{1.20}$$

with

$$K_{\mathrm{BP}}(\hat{x}) = \frac{1}{\pi^2 \hat{x}}\, e^{-\hat{x}}. \tag{1.21}$$

The form of $K_{\mathrm{HC}}(\hat{x})$ is the same as that of the one-dimensional Ising model near the Yang-Lee edge [Fis80]. The form of $K_{\mathrm{BP}}(\hat{x})$ agrees with the prediction of Miller [Mil91].

2 The Forest-Root Formula and Dimensional Reduction

We wish to derive relationships between the hard-core Green's functions in D dimensions and the branched polymer Green's functions in $D+2$ dimensions. The key is the Forest-Root formula, proven in [BI01]. Let $f(\mathbf{t})$ be any smooth function of variables

$$t_{ij} = |z_i - z_j|^2, \qquad 1 \le i < j \le N \quad \text{and} \quad t_i = |z_i|^2, \qquad 1 \le i \le N,$$

where each $\mathrm{z}_i \in \mathbb{C}$. Assume that f, when regarded as a function of $\mathrm{z}_1, \ldots, \mathrm{z}_N$, has compact support. Any subset of the bonds $\{ij | 1 \le i < j \le N\}$ forms a graph on the vertices $\{1, \ldots, N\}$. A subset R of vertices is called a set of *roots*. A *forest* F is a graph that has no loops. The connected components of a forest are *trees*. We are declaring that a graph with no bonds and just one vertex is also a tree. See Fig. 1.

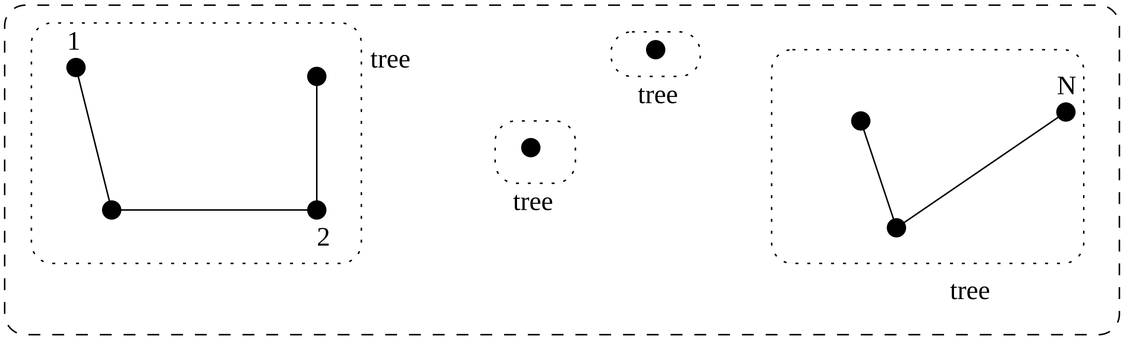

Figure 1: Example of a forest

Theorem 2.1. *(Forest-Root Formula)*

$$f(\mathbf{0}) = \sum_{(F,R)} \int_{\mathbb{C}^N} f^{(F,R)}(\mathbf{t}) \left(\frac{d^2 z}{-\pi} \right)^N, \tag{2.1}$$

where $f^{(F,R)}(\mathbf{t})$ denotes the derivative with respect to the variables t_{ij} with $ij \in F$ and t_i with $i \in R$. The sum is over all forests F and all sets R of roots with the property that each tree in F contains exactly one root from R, and $d^2 z = du\ dv$, where $z = u + iv$.

Proof of Theorem 1.1 (1.15). In order to examine $g^{(n)}_{\mathrm{HC}}$, we set $N = n + m$ and put

$$f(\mathbf{t}) = \prod_{1 \le i < j \le N} U(|x_{ij}|^2 + t_{ij}) \prod_{i=1}^{m} g(\epsilon t_i) \prod_{j=1}^{n} g(t_j/\epsilon), \tag{2.2}$$

where $x_{ij} = x_i - x_j$, $\epsilon > 0$, and where g is a smooth, decreasing function with compact support such that $g(0) = 1$. Working in a finite box Λ, we write

$$g^{(n)}_{\mathrm{HC}}(x_1, \dots, x_n) Z_{\mathrm{HC}}(\mathrm{z}) = \sum_{m=0}^{\infty} \frac{\mathrm{z}^{m+n}}{m!} \int_{\Lambda^n} dx_{n+1} \dots dx_{n+m} f(\mathbf{0}), \tag{2.3}$$

and insert (2.1). With $y_i = (x_i, z_i)$, this becomes

$$\sum_{m=0}^{\infty} \frac{\mathrm{z}^{m+n}}{m!} \int_{\Lambda^n} dx_{n+1} \dots dx_{n+m} \sum_{(F,R)} \int_{\mathbb{C}^N} \left(\frac{d^2 z}{-\pi}\right) \prod_{ij \in F} U'_{ij} \prod_{ij \notin F} U_{ij} \prod_{i \in R} \frac{dg}{dt_i} \prod_{i \notin R} g. \tag{2.4}$$

The forest F may be divided into $F^{(n)}$ (which consists of all the trees containing any of the vertices $1, \dots, n$), and the rest, $\tilde{F}$. If any of the vertices $1, \dots, n$ are not roots, then the corresponding factor $g(t/\epsilon)$ is not differentiated, and so lacks a factor ϵ^{-1} to compensate for the $O(\epsilon)$ integration volume. Therefore, neglecting $O(\epsilon)$ terms, $F^{(n)} = T_1 \cup \dots \cup T_n$ with $x_i \in T_i$ for $i = 1, \dots, n$, and T_i disjoint. Let $m = p + M$, where M is the number of vertices in $\tilde{F}$, and rewrite (2.4) as

$$\begin{aligned}
&\sum_{p=0}^{\infty} \frac{z^{p+n}}{p!} \int_{\Lambda^p} dx_{n+1} \dots dx_{n+p} \sum_{F^{(n)}} \int_{\mathbb{C}^p} \left(\frac{d^2 z}{-\pi}\right)^p \prod_{ij \in F^{(n)}} U'_{ij} \\
&\cdot \sum_{M=0}^{\infty} \frac{\mathrm{z}^M}{M!} \int_{\Lambda^M} dx_{n+p+1} \dots dx_{n+p+M} \sum_{(\tilde{F},\tilde{R})} \int_{\mathbb{C}^M} \left(\frac{d^2 z}{-\pi}\right)^M \prod_{ij \in \tilde{F}} U'_{ij} \\
&\cdot \prod_{i \in \tilde{R}} [\epsilon g'(\epsilon t_i)] \prod_{i \notin R} g(\epsilon t_i) \prod_{ij \notin F} U_{ij} + o(1),
\end{aligned} \tag{2.5}$$

where $o(1)$ denotes a quantity which tends to zero with ϵ. Here $\tilde{R}$ is a subset of $\{n + p + 1, \dots, n + p + M\}$ and as before, each tree of $\tilde{F}$ has exactly one root from $\tilde{R}$. We have eliminated the integrals over $z_1, \dots, z_n$ because $(-\pi\epsilon)^{-1} d^2 z g'(t_i/\epsilon)$ tends to $\delta(z_i)$. The factors $g(\epsilon t_i)$ with $n + 1 \le i \le n + p$ can be replaced with 1 because the decrease of U' in essence forces the corresponding z_i's to remain bounded. Any errors from these approximations are $o(1)$.

The only barrier to writing (2.5) as a product is the presence of interactions U_{ij} linking $F^{(n)}$ and $\tilde{F}$. However, for small ϵ the trees in F are rarely close to each other or to the trees of $F^{(n)}$. Using again the decrease of U' we see that all the vertices of a tree are in a bounded cluster and that $g(\epsilon t_i)$ can be replaced with $g(\epsilon t_r)$, where r is the root of the tree. Then the sum over roots leads to a factor $N(T)$, the number of vertices in T. Observe that $(-\epsilon/\pi) N(T) g(\epsilon t_r)^{N(T)-1} g'(\epsilon t_r) d^2 z_r$ is a probability measure which becomes very wide as $\epsilon \to 0$. Thus with high probability, the interactions U_{ij} between $F^{(n)}$ and $\tilde{F}$ can be replaced with 1, with an additional contribution to the $o(1)$ error. As a result, (2.5) can be rewritten as

$$\left[\sum_{p=0}^{\infty}\frac{\mathrm{z}^{p+n}}{p!}\int_{(\Lambda\times\mathbb{C})^p}\prod_{i=n+1}^{n+p}\frac{d^{D+2}y_i}{-\pi}\sum_{F^{(n)}}\prod_{ij\in F^{(n)}}U'_{ij}\prod_{ij\notin F^{(n)}}U_{ij}\right]Z_{\mathrm{HC}}(\mathrm{z})+o(1). \tag{2.6}$$

Taking the limit as $\epsilon\to 0$, we obtain a relation for finite D-dimensional volume Λ:

$$g^{(n)}_{\mathrm{HC}}(x_1,\dots,x_n;\mathrm{z})=g^{(n)}_{\mathrm{BP}}\left(x_1,\dots,x_n;-\frac{\mathrm{z}}{2\pi}\right)(-2\pi)^n. \tag{2.7}$$

The limit $\Lambda\nearrow\mathbb{R}^D$ exists for each term in the sum over p, by monotone convergence. By dominated convergence, the sum on p may be interchanged with the infinite volume limit, and we obtain the first part of Theorem 1.1.

Proof of (1.4). A similar factorization occurs in $Z_{\mathrm{HC}}(\mathrm{z})$, so that all the terms with k trees can be written as $1/k!$ times the k^{th} power of

$$\sum_{N=1}^{\infty}\frac{\mathrm{z}^N}{N!}\sum_T\int_\Lambda dx_1\int_{(\Lambda\times\mathbb{C})^{N-1}}\prod_{i=2}^{N}\frac{dy_i}{-\pi}\prod_{ij\in T}U'_{ij}\prod_{ij\notin T}U_{ij}, \tag{2.8}$$

so that, as argued in [BI01],

$$\lim_{\Lambda\nearrow\mathbb{R}^D}\frac{1}{|\Lambda|}\log Z_{\mathrm{HC}}(\mathrm{z})=-2\pi Z_{\mathrm{BP}}\left(-\frac{\mathrm{z}}{2\pi}\right). \tag{2.9}$$

Proof of Theorem 1.1 (1.16). The relation between $G^{(n)}_{\mathrm{HC}}$ and $G^{(n)}_{\mathrm{BP}}$ may be derived by differentiating (2.9) with respect to sources. Then, as explained in [BI01],

$$G^{(n)}_{\mathrm{HC}}(x_1\dots x_n;\mathrm{z})=(-2\pi)\int_{\mathbb{C}^{n-1}}\prod_{i=2}^{n}d^2z_i\;G^{(n)}_{\mathrm{BP}}\left(x_1,y_2,\dots,y_n;-\frac{\mathrm{z}}{2\pi}\right), \tag{2.10}$$

which is (1.16). In momentum space, then, $G^{(n),\mathrm{T}}_{\mathrm{HC}}$ may be obtained from $G^{(n)}_{\mathrm{BP}}$ by setting the components of momenta in the two extra dimensions to zero. This contrasts with the relation (2.7) between $g^{(n)}_{\mathrm{HC}}$ and $g^{(n)}_{\mathrm{BP}}$, in which the spatial components in the two extra dimensions are set to 0.

Proof of (1.17). Relations between G_{BP} and g_{BP} may be derived by combining (2.7) and (2.10). For example, consider the 2-point functions, which by rotation invariance can be expressed as functions of the squared-distance t:

$$\begin{aligned} G^{(2)}_{\mathrm{HC}}(t;\mathrm{z}) &:= G^{(2)}_{\mathrm{HC}}(x_1,x_2;\mathrm{z}),\ \text{where } |x_1-x_2|^2=t,\\ G^{(2)}_{\mathrm{BP}}(t;\mathrm{z}) &:= G^{(2)}_{\mathrm{BP}}(y_1,y_2;\mathrm{z}),\ \text{where } |y_1-y_2|^2=t, \end{aligned} \tag{2.11}$$

and similarly for $g_{\mathrm{HC}}^{(2)}$ and $g_{\mathrm{BP}}^{(2)}$. Since $g_{\mathrm{HC}}^{(2)}$ agrees with $G_{\mathrm{HC}}^{(2)}$ at non-coinciding points, (2.7) and (2.10) imply that

$$g_{\mathrm{BP}}^{(2),\mathrm{T}}(t;\mathrm{z}) = (-2\pi)^{-1}\int_t^\infty \pi dt'\; G_{\mathrm{BP}}^{(2)}(t',\mathrm{z}), \quad t \neq 0. \tag{2.12}$$

Differentiation yields

$$\frac{d}{dt} g_{\mathrm{BP}}^{(2)}(t;\mathrm{z}) = \frac{1}{2} G_{\mathrm{BP}}^{(2)}(t;\mathrm{z}). \tag{2.13}$$

This may be thought of as a Ward identity for the supersymmetry of our model of branched polymers.

3 Green's Function for the Hard-Core Gas in One Dimension

Laplace transforms can be used to give fairly explicit formulas for the Green's function for one-dimensional gases with only nearest neighbor interactions. We follow [FW69] in deriving the relevant expressions for the basic hard-core gas with no interactions other than a minimum separation of 1 between particles.

Let us write the grand canonical partition function in the following way (we omit the subscript HC in most of this section):

$$Z(L) = \sum_{N=0}^{\infty} \mathrm{z}^N \int_{x_1 \geq 1} dx_1 \int_{x_2 \geq x_1+1} dx_2 \cdots \int_{L-1 \geq x_N \geq x_{N-1}+1} dx_N. \tag{3.1}$$

The particles are restricted to the interval $\Lambda = [1, L-1]$, as if external particles had been placed at 0 and L. We assume $L > 1$ and put $Z(L) = 1$ for $1 < L \leq 2$. The Laplace transform can be evaluated explicitly:

$$\begin{aligned} \hat{Z}(s) &:= \int_1^\infty dL\, e^{-sL} Z(L) \\ &= \sum_{N=0}^{\infty} \mathrm{z}^N J(s)^{N+1}, \end{aligned} \tag{3.2}$$

where

$$J(s) = \int_1^\infty dx\, e^{-sx} = \frac{1}{s}\, e^{-s}. \tag{3.3}$$

Using analytic continuation as necessary to define $\hat{Z}(s)$, we have

$$\hat{Z}(s) = \frac{J(s)}{1 - \mathrm{z}J(s)} = \frac{1}{se^s - \mathrm{z}}. \tag{3.4}$$

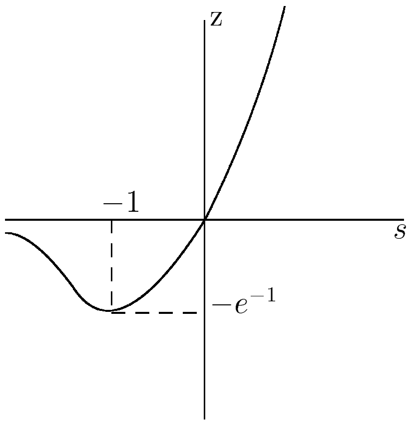

Figure 2: Graph of the function $\mathrm{z} = se^s$

We obtain $Z(L)$ by inverse transform:

$$Z(L) = \frac{1}{2\pi i} \int \frac{1}{se^s - \mathrm{z}}\, e^{sL} ds. \tag{3.5}$$

This leads to a residue formula

$$Z(L) = \sum_{n=0}^{\infty} \frac{e^{s_n(L-1)}}{(s_n + 1)}, \tag{3.6}$$

where $\{s_n\}$ are the solutions to $se^s = \mathrm{z}$, arranged in order of decreasing real part. These solutions are the branches of the Lambert W-function [CGHJK].

We will make use of some properties of the s_n. For $\mathrm{z} > 0$, there is one real solution, and for $-e^{-1} < \mathrm{z} < 0$, there are two real solutions (see Fig. 2). The complex solutions come in conjugate pairs, and all have real parts which are less than the real solutions. (This can be seen by writing $s = x + iy$ and letting $x(y)$ solve the modulus equation $(x^2 + y^2)e^{2x} = |\mathrm{z}|^2$. Then $\frac{dx}{dy^2} = -1/[2(x+x^2+y^2)] < -c < 0$ with c independent of x, y in any bounded region not intersecting $\{(x, y)|x \in (-1, 0)\}$. This shows, in fact, that the upper gap $\mathrm{Re}(s_1 - s_2) > B > 0$ with B independent of z in any interval $[-1, \mathrm{z}_0]$ with $\mathrm{z}_0 < 0$.) If we put $s_n = x_n + iy_n$, then $|y_n - n\pi| \leq$ const [CGHJK, Fig. 4]. In addition, $x_n \sim -\log|y_n/\mathrm{z}| \sim -\log|n/\mathrm{z}|$ for large n, from the modulus equation. Hence the sum in (3.6) converges for all $L > 1$.

The density, or one-point function, $G^{(1)}(x) = G^{(1)}$, is the expectation of $\rho(x) = \sum_{j=1}^{N} \delta(x - x_j)$ in the limit as $\Lambda \nearrow \infty$. If we take $\Lambda = \left[-\frac{L}{2} + 1, \frac{L}{2} - 1\right]$, then

$$G^{(1)} = \lim_{L\to\infty} \frac{Z\left(\frac{L}{2} + x\right) \mathrm{z} Z\left(\frac{L}{2} - x\right)}{Z(L)}. \tag{3.7}$$

Only the $n = 0$ term of (3.6) survives the $L \to \infty$ limit. Thus we have

$$G^{(1)} = \frac{\mathrm{z}e^{-s_0}}{s_0 + 1} = \frac{s_0}{s_0 + 1}, \tag{3.8}$$

where we have used the relation $\mathrm{z} = s_0 e^{s_0}$. If we identify $G^{(1)}$ with the density $\bar{\rho}$ and solve for the pressure $p = s_0 kT$, we obtain the equation of state for hard rods of unit length [Ton36]:

$$p = \frac{\bar{\rho} kT}{1-\bar{\rho}}. \tag{3.9}$$

The density-density correlation, or two-point function $G^{(2)}(0,x) = G^{(2)}(x)$, is the expectation of $\rho(0)\rho(x)$. For $x > 1$, this can be written as

$$\begin{aligned} G^{(2)}(x) &= \lim_{L\to\infty} \frac{Z\left(\frac{L}{2}\right) \mathrm{z} Z(x) \mathrm{z} Z\left(\frac{L}{2} - x\right)}{Z(L)} \\ &= \frac{s_0}{s_0+1} e^{-s_0 x} \mathrm{z} Z(x). \end{aligned} \tag{3.10}$$

One can insert the formula

$$Z(x) = \sum_{N=0}^{\infty} \frac{\mathrm{z}^N}{N!} (x - N - 1)^N \theta(x - N - 1) \tag{3.11}$$

to obtain the long-known expression for $G^{(2)}$ (see, for example, [SZK53, eqn. 32]) which is useful if x is not too large. Alternatively, one can insert (3.6) to obtain

$$G^{(2)}(x) = \sum_{n=0}^{\infty} \frac{s_0}{s_0+1} \, \frac{s_n}{s_n+1} \, e^{(s_n - s_0)x}, \tag{3.12}$$

and after subtracting $G^{(1)2}$, we obtain an expression for the truncated Green's function

$$G^{(2),\mathrm{T}}(x) = \sum_{n=1}^{\infty} \frac{s_0}{s_0+1} \, \frac{s_n}{s_n+1} \, e^{(s_n - s_0)x}, \tag{3.13}$$

which is convergent for $x > 1$. It is apparent from (3.11) that $\left(\frac{d}{dx}\right)^N G^{(2),\mathrm{T}}(x)$ is continuous, except for a jump at $x = N+1$. This is reflected in the divergence of the series (3.13) at $x = N+1$, when differentiated N times. For $|x| < 1$, $G^{(2),\mathrm{T}}(x) = 0$ except for a δ-function at 0 with coefficient $G^{(1)}$.

Scaling Form of the Green's Function

As $\mathrm{z} \searrow \mathrm{z}_c = -e^{-1}$, the two real solutions to $se^s = \mathrm{z}$ approach the value -1. The correlation length is given by

$$\xi := \left[\lim_{|x|\to\infty} -\frac{1}{|x|} \log G^{(2),\mathrm{T}}(x) \right]^{-1} = (s_0 - s_1)^{-1}. \tag{3.14}$$

It is clear from Fig. 2 that $s_0 - s_1 \sim (\mathrm{z} - \mathrm{z}_c)^{\frac{1}{2}}$. Hence, as $\mathrm{z} \searrow \mathrm{z}_c$, ξ diverges as $(\mathrm{z} - \mathrm{z}_c)^{-\nu}$ with a correlation exponent $\nu = \frac{1}{2}$.

If we let $\mathrm{z} \searrow \mathrm{z}_c$ and $x \to \infty$ while keeping $\hat{x} = x/\xi$ fixed, then the asymptotic form of $G^{(2),\mathrm{T}}$ is described by a scaling function

$$K(\hat{x}) = \lim_{x\to\infty, \mathrm{z}\searrow \mathrm{z}_c} x^{D-2+\eta} G^{(2),\mathrm{T}}(x). \tag{3.15}$$

Here $D = 1$, and we take the anomalous dimension $\eta = -1$ in order to get a nontrivial limit. From (3.13), and the uniform gap between s_1 and the other solutions, we have

$$K(\hat{x}) = \lim_{x\to\infty, \mathrm{z}\searrow \mathrm{z}_c} \frac{s_0}{s_0+1} \frac{s_1}{s_1+1} \frac{1}{x^2} e^{-x/\xi}. \tag{3.16}$$

A short calculation shows that $\frac{s_0}{s_0+1} \frac{s_1}{s_1+1} = -\frac{4}{(s_0-s_1)^2}(1 + O(\mathrm{z}-\mathrm{z}_c))$, and hence

$$K(\hat{x}) = -\frac{4}{\hat{x}^2} e^{-\hat{x}}. \tag{3.17}$$

We may also define a scaling function for branched polymers:

$$K_{\mathrm{BP}}(\hat{x}) = \lim_{x\to\infty, \mathrm{z}\nearrow \tilde{\mathrm{z}}_c} x^{d-2+\eta_{\mathrm{BP}}} G^{(2)}_{\mathrm{BP}}(0,x). \tag{3.18}$$

Here $\hat{x} = x/\xi_{\mathrm{BP}}$ with

$$\xi_{\mathrm{BP}} := \left[\lim_{|x|\to\infty} -\frac{1}{|x|} \log G^{(2)}_{\mathrm{BP}}(0,x) \right]^{-1}, \tag{3.19}$$

and η_{BP} is chosen so as to obtain a nontrivial limit for K_{BP}. As explained in [BI01], we have the relation (2.10) between $G^{(2)}_{\mathrm{BP}}$ and $G^{(2),\mathrm{T}}_{\mathrm{HC}}$, which when differentiated yields

$$G^{(2)}_{\mathrm{BP}}\left(t; -\frac{\mathrm{z}}{2\pi}\right) = \frac{1}{2\pi^2} \frac{d}{dt} G^{(2),\mathrm{T}}_{\mathrm{HC}}(t,\mathrm{z}). \tag{3.20}$$

Hence, the critical activity $\tilde{\mathrm{z}}_c$ for branched polymers is equal to $-2\pi \mathrm{z}_c = 2\pi e$, $\xi_{\mathrm{BP}}\left(-\frac{\mathrm{z}}{2\pi}\right) = \xi_{\mathrm{HC}}(\mathrm{z}), \eta_{\mathrm{BP}} = \eta_{\mathrm{HC}}$, and

$$\begin{aligned} K_{\mathrm{BP}}(\hat{x}) &= \frac{1}{4\pi^2}[\hat{x} K'_{\mathrm{HC}}(\hat{x}) - (D-2+\eta_{\mathrm{HC}}) K_{\mathrm{HC}}(\hat{x})] \\ &= \frac{1}{\pi^2 \hat{x}} e^{-\hat{x}}. \end{aligned} \tag{3.21}$$

Of course, the exponent ν_{BP} governing the divergence of ξ_{BP} as $\mathrm{z} \nearrow \tilde{\mathrm{z}}_c$ must equal ν_{HC} ($= \frac{1}{2}$ if $d = D + 2 = 3$).

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We thank Joel Lebowitz for helpful questions and comments about the multispecies case.

Appendix: Multispecies Examples

We can generate the arguments of Section 2 to multispecies examples. Define a repulsive gas partition function in D dimensions:

$$Z_{\mathrm{HC}}(\mathbf{z}) = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\alpha_1,\dots,\alpha_N} \prod_{i=1}^{N} \mathrm{z}_\alpha \int_{\Lambda^N} dx_1 \cdots dx_N \prod_{ij} U^{\alpha_i \alpha_j}(|x_i - x_j|^2), \tag{A.1}$$

where each α_i is summed over the set of species of the problem, z_α is the activity of species α, and $U^{\alpha\beta}$ is a repulsive interaction between species α and species β. The corresponding multispecies branched polymer generating function is

$$Z_{\mathrm{BP}}(\mathbf{z}) = \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{T} \sum_{\alpha_1,\dots,\alpha_N} \prod_{i=1}^{N} \mathrm{z}_\alpha \int_{(\mathbb{R}^{D+2})^{N-1}} dy_2 \cdots dy_N \prod_{ij\in T} [2U'_{ij}] \prod_{ij\notin T} U_{ij}, \tag{A.2}$$

where

$$U_{ij} = U^{\alpha_i \alpha_j}(|x_i - x_j|^2), \tag{A.3}$$

and U'_{ij} is its derivative. In particular, for the hard-core model with minimum separation $R_{\alpha\beta}$ between species α and species β, we would have $U^{\alpha\beta}(t) = \theta(t - R^2_{\alpha\beta})$ and

$$2U^{\alpha\beta\prime}(|x_i - x_j|^2) = \frac{1}{R_{\alpha\beta}} \delta(|x_i - x_j| - R_{\alpha\beta}). \tag{A.4}$$

Assume that $U^{\alpha\beta}$ satisfies the usual conditions ($U^{\alpha\beta}, U^{\alpha\beta\prime}$ positive, $U^{\alpha\beta} \to 1$ at ∞, $U^{\alpha\beta\prime}$ integrable in $\mathbb{R}^{D+2}$). Then, provided $Z_{\mathrm{BP}}(\mathbf{z})$ is absolutely convergent, we obtain a reduction formula

$$\lim_{\Lambda \nearrow \mathbb{R}^D} \frac{1}{|\Lambda|} \log Z_{\mathrm{HC}}(\mathbf{z}) = -2\pi Z_{\mathrm{BP}}\left(-\frac{\mathbf{z}}{2\pi}\right). \tag{A.5}$$

We also obtain results as in Theorem 1.1 for correlation functions.

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