THE LARGE RADIUS LIMIT FOR COHERENT STATES ON SPHERES

BRIAN C. HALL AND JEFFREY J. MITCHELL

Abstract. This paper concerns the coherent states on spheres studied by the authors in [J. Math. Phys. **43** (2002), 1211-1236]. We show that in the odd-dimensional case the coherent states on the sphere approach the classical Gaussian coherent states on Euclidean space as the radius of the sphere tends to infinity.

1. Introduction

In our earlier work [4] we constructed coherent states and an associated resolution of the identity for a quantum particle whose classical configuration space is a d-dimensional sphere. Although the main result of [4] is a special case of results of Stenzel [6] (building on results of Hall [1, 2]), we give a substantially different description based on the "complexifier" approach of Thiemann [7] and the "polar decomposition" approach of Kowalski and Rembieliński [5]. In [4] we also give self-contained elementary proofs of all the main results. See also [3] for a survey of related results.

We consider S^d, the sphere of radius r in $\mathbb{R}^{d+1}$, viewed as the configuration space for a classical particle. We consider also the corresponding phase space, the cotangent bundle $T^*(S^d)$, which we describe as

$$T^*(S^d) = \left\{(\mathbf{x}, \mathbf{p}) | x^2 = r^2, \, \mathbf{x} \cdot \mathbf{p} = 0\right\}.$$

Here $\mathbf{p}$ is the *linear* momentum which must be tangent to the sphere at $\mathbf{x}$, that is, orthogonal to $\mathbf{x}$.

We now briefly review the results of [4]. Following the complexifier approach of Thiemann we first choose a constant ω with units of frequency, whose significance will be discussed in Section 2. Then we consider the "complexifier" function on $T^*(S^d)$, defined by

$$\text{complexifier} = \frac{\text{kinetic energy}}{\omega} = \frac{p^2}{2m\omega} = \frac{j^2}{2m\omega r^2},$$

where j^2 is the total angular momentum. Then we define complex-valued functions $a_1, \cdots, a_{d+1}$ on $T^*(S^d)$ by the formula

$$\begin{aligned} a_k &= e^{i\{\cdot,\text{complexifier}\}} x_k \\ &= \sum_{n=0}^{\infty} \left(\frac{i}{2m\omega r^2}\right)^n \frac{1}{n!} \underbrace{\left\{\cdots\left\{\left\{x_k, j^2\right\}, j^2\right\}, \cdots, j^2\right\}}_{n} \end{aligned} \tag{1}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket. A calculation gives the explicit formula

$$\mathbf{a}\left(\mathbf{x},\mathbf{p}\right)=\cosh\left(\frac{j}{m\omega r^2}\right)\mathbf{x}+i\frac{r^2}{j}\sinh\left(\frac{j}{m\omega r^2}\right)\mathbf{p} \tag{2}$$

The functions a_k satisfy $\{a_k,a_l\}=0$ and $a_1^2+\cdots+a_{d+1}^2=r^2$. The map $(\mathbf{x},\mathbf{p})\rightarrow\mathbf{a}\left(\mathbf{x},\mathbf{p}\right)$ is a diffeomorphism of $T^*(S^d)$ with the *complex sphere* $S_{\mathbb{C}}^d$, where

$$S_{\mathbb{C}}^d=\left\{\mathbf{a}\in\mathbb{C}^{d+1}|a_1^2+\cdots+a_{d+1}^2=r^2\right\}.$$

We now consider a quantum particle moving on the sphere. We take the quantum Hilbert space to be the position Hilbert space $L^2(S^d)$. (See [4] for a more abstract approach.) We consider the quantum complexifier given by

$$\text{complexifier}=\frac{\text{kinetic energy}}{\omega}=\frac{J^2}{2m\omega r^2},$$

where J^2 is the total angular momentum operator given by

$$J^2=-\hbar^2\sum_{k<l}\left(x_k\frac{\partial}{\partial x_l}-x_l\frac{\partial}{\partial x_k}\right)^2.$$

By analogy to (1) (replacing the Poisson bracket with the commutator divided by $i\hbar$) we define non-self-adjoint operators A_k by the formula

$$\begin{aligned}A_k&=e^{i[\cdot,\ \text{complexifier}]/i\hbar}X_k\\&=\sum_{n=0}^{\infty}\left(\frac{1}{2m\omega r^2\hbar}\right)^n\frac{1}{n!}\underbrace{\left[\cdots\left[\left[X_k,J^2\right],J^2\right],\cdots,J^2\right]}_{n}.\end{aligned} \tag{3}$$

We call these operators the annihilation operators. There is also a polar decomposition for the A_k's (generalizing a formula of [5] for the S^2 case) and an explicit formula similar to (2) but with certain quantum corrections [4, Eq. (38)]. We may also express A_k as

$$A_k=e^{-\tau\tilde{J}^2/2}X_ke^{\tau\tilde{J}^2/2}$$

where $\tilde{J}^2=J^2/\hbar^2$ is the dimensionless form of the angular momentum operator and where τ is the dimensionless parameter given by

$$\tau=\frac{\hbar}{m\omega r^2}.$$

The A_k's satisfy $[A_k,A_l]=0$ and $A_1^2+\cdots+A_{d+1}^2=r^2I$.

We now define the coherent states to be the simultaneous eigenvectors of the A_k's. There is one coherent state for each point $\mathbf{a}$ in the complex sphere $S_{\mathbb{C}}^d$, which means one for each point in the classical phase space, since we identify $S_{\mathbb{C}}^d$ with $T^*(S^d)$ by means of (2). The coherent states are given by the heuristic expression

$$|\psi_{\mathbf{a}}\rangle=e^{-\tau\tilde{J}^2/2}\left|\delta_{\mathbf{a}}\right\rangle,\quad\mathbf{a}\in S_{\mathbb{C}}^d$$

where $|\delta_{\mathbf{a}}\rangle$ is supposed to be a position eigenvector satisfying $X_k\left|\delta_{\mathbf{a}}\right\rangle=a_k\left|\delta_{\mathbf{a}}\right\rangle$. For $\mathbf{a}$ in the real sphere $|\delta_{\mathbf{a}}\rangle$ is a generalized function and $e^{-\tau\tilde{J}^2/2}\left|\delta_{\mathbf{a}}\right\rangle$ is a smooth function on S^d. For $\mathbf{a}$ in the complex sphere $|\psi_{\mathbf{a}}\rangle$ can be defined by analytic continuation with respect to $\mathbf{a}$. See [4, Prop. 1].

Explicitly we have

$$\psi_{\mathbf{a}}\left(\mathbf{x}\right)=\rho_\tau^d(\mathbf{a},\mathbf{x}),\quad\mathbf{x}\in S^d,\ \mathbf{a}\in S_{\mathbb{C}}^d.$$

where ρ_τ^d is the heat kernel on the d-sphere. Here $\rho_\tau^d(\mathbf{a},\mathbf{x})$ is initially defined for $\mathbf{a}$ and $\mathbf{x}$ in S^d, but we can extend to $\mathbf{a}\in S^d_{\mathbb{C}}$ by analytic continuation. For odd-dimensional spheres we have the formulas

$$\rho_\tau^1(\mathbf{a},\mathbf{x}) = (2\pi\tau)^{-1/2}\sum_{n=-\infty}^{\infty} e^{-(\theta-2\pi n)^2/2\tau}$$

$$\rho_\tau^3(\mathbf{a},\mathbf{x}) = (2\pi\tau)^{-3/2}e^{\tau/2}\frac{1}{\sin\theta}\sum_{n=-\infty}^{\infty}(\theta-2\pi n)e^{-(\theta-2\pi n)^2/2\tau}$$

$$\rho_\tau^{d+2}(\mathbf{a},\mathbf{x}) = -e^{d\tau/2}\frac{1}{2\pi\sin\theta}\frac{d}{d\theta}\rho_\tau^d(\mathbf{a},\mathbf{x}). \tag{4}$$

Here θ is a complex-valued quantity satisfying $\cos\theta = \mathbf{a}\cdot\mathbf{x}/r^2$. (There is precisely one such θ with $0\le \operatorname{Re}\theta\le\pi$.) See [4] for formulas in the even-dimensional case.

2. The large r limit

The purpose of this paper is to study the behavior of the coherent states in the limit $r\to\infty$. For simplicity we consider only the odd-dimensional case, although the same results almost certainly hold in the even-dimensional case as well. A region of fixed size R in a sphere of radius r will look Euclidean as long as $r\gg R$. So we expect the coherent states for large r on S^d to look like the usual coherent states on $\mathbb{R}^d$ provided that the coherent states are concentrated into a region of size $R\ll r$.

Now, the spatial size of the coherent states is controlled by the dimensionless parameter $\tau=\hbar/m\omega r^2$. Specifically, if ΔX denotes the approximate spatial width of the coherent states then we expect that $\Delta X\approx\sqrt{\hbar/2m\omega}$ (as it is in the Euclidean case) at least if this quantity is small compared to r. In that case we expect

$$\frac{\Delta X}{r}\approx\frac{\sqrt{\hbar/2m\omega}}{r}=\sqrt{\frac{\tau}{2}}.$$

We will prove that if ω, m, and $\hbar$ remain fixed and r tends to infinity then indeed the coherent states have spatial width approximately equal to $\sqrt{\hbar/2m\omega}$ and that the coherent states become (in a sense to be described shortly) precisely the usual Gaussian wave packets on $\mathbb{R}^d$. (See (7) and (8) below.) The proof amounts to analyzing the behavior of the heat kernel for small τ, since $\tau=\hbar/m\omega r^2$ tends to zero as r tends to infinity with ω, m, and $\hbar$ fixed.

Kowalski and Rembieliński do not have a parameter comparable to our ω in [5]. This means that what they do corresponds to the $\tau=1$ case of our construction. Since the value of τ is fixed, the angular dependence of the coherent states in [5] is independent of r, that is, the coherent states in [5] simply scale proportionally to r. (See Equation (5.3) in [5].) It seems to us, therefore, that one cannot get the canonical coherent states on $\mathbb{R}^d$ in the large-r limit without introducing the parameter ω.

Another way of thinking about the significance of ω is that $m\omega$ has units of momentum divided by position. On the classical side this gives a way of writing dimensionally correct combinations of position and momentum (without using $\hbar$), as in (2). On the quantum side we expect that the ratio of the width in momentum space to the width in position space of the coherent states is approximately $m\omega$.

The coherent states are labeled by points $\mathbf{a}$ in the complex sphere, which are in one-to-one correspondence with points in the classical phase space by means of (2). We consider a coherent state where the position part of the label is at a fixed distance (independent of r) from the north pole, and we evaluate that coherent state at a point that is also at a fixed distance from the north pole. Specifically, we choose a point $(\mathbf{x}_0, \mathbf{p}_0)$ in $\mathbb{R}^d \times \mathbb{R}^d$ and we let $\tilde{\mathbf{x}}_0$ be the point in S^d given by

$$\tilde{\mathbf{x}}_0 = \left(\mathbf{x}_0, \sqrt{r^2 - x_0^2}\right), \tag{5}$$

which makes sense for all r with $r^2 \geq x_0^2$. We also let $\tilde{\mathbf{p}}_0$ be given by

$$\tilde{\mathbf{p}}_0 = \left(\mathbf{p}_0, -\frac{\mathbf{p}_0 \cdot \mathbf{x}_0}{\sqrt{r^2 - x_0^2}}\right) \tag{6}$$

so that $\tilde{\mathbf{x}}_0 \cdot \tilde{\mathbf{p}}_0 = 0$, that is, $(\tilde{\mathbf{x}}_0, \tilde{\mathbf{p}}_0) \in T^*(S^d)$. We now consider the coherent state $\psi_{\mathbf{a}(\tilde{\mathbf{x}}_0, \tilde{\mathbf{p}}_0)}$. We then fix another point $\mathbf{x}$ in $\mathbb{R}^d$, and we let $\tilde{\mathbf{x}} = (\mathbf{x}, (r^2 - x^2)^{1/2})$. We will show that $\psi_{\mathbf{a}(\tilde{\mathbf{x}}_0, \tilde{\mathbf{p}}_0)}(\tilde{\mathbf{x}})$, viewed as a function of $\mathbf{x}$, behaves for large r like a Gaussian wave packet centered at $\mathbf{x} = \mathbf{x}_0$ and with momentum $\mathbf{p}_0$.

Note that for large r we have $\tilde{\mathbf{x}}_0 \approx (\mathbf{x}_0, r)$ and $\tilde{\mathbf{p}}_0 \approx (\mathbf{p}_0, 0)$. It follows from (2) that for large r we have

$$\mathbf{a}(\tilde{\mathbf{x}}_0, \tilde{\mathbf{p}}_0) \approx \left(\mathbf{x}_0 + i\frac{\mathbf{p}_0}{m\omega}, r\right).$$

That is, the first d components of $\mathbf{a}(\tilde{\mathbf{x}}_0, \tilde{\mathbf{p}}_0)$ give just $\mathbf{x}_0 + i\,\mathbf{p}_0/m\omega$ as in the Euclidean case, and the last component of $\mathbf{a}(\tilde{\mathbf{x}}_0, \tilde{\mathbf{p}}_0)$ is just a constant. Heuristically this means that on the quantum side (when applied to functions supported in a set of fixed radius R around the north pole) we should have the first d components of $\mathbf{A}$ approximately equal to $X + iP/m\omega$ and the last component of $\mathbf{A}$ approximately equal to rI. This explains heuristically why the coherent states (eigenvectors of $\mathbf{A}$) should look like their Euclidean counterparts in the large r limit.

Theorem 1. *The coherent states on odd-dimensional spheres have the following limiting property for large r. Fix $\mathbf{x}_0$ and $\mathbf{p}_0$ in $\mathbb{R}^d$, and let $\mathbf{z} = \mathbf{x}_0 + i\,\mathbf{p}_0/m\omega$. Then*

$$\lim_{r\to\infty} r^{-d}\psi_{\mathbf{a}(\tilde{\mathbf{x}}_0, \tilde{\mathbf{p}}_0)}(\tilde{\mathbf{x}}) = \left(\frac{m\omega}{2\pi\hbar}\right)^{d/2} \exp\left\{-\frac{(\mathbf{z} - \mathbf{x})^2}{2\hbar/m\omega}\right\} \tag{7}$$

where the limit is uniform for $\mathbf{x}$ in compact subsets of $\mathbb{R}^d$. Here $\tilde{\mathbf{x}}_0$, $\tilde{\mathbf{p}}_0$, and $\tilde{\mathbf{x}}$ are defined by (5) and (6) and $(\mathbf{z} - \mathbf{x})^2$ means $(z_1 - x_1)^2 + \cdots + (z_d - x_d)^2$.

The right side of (7) is the usual Gaussian coherent state, which can also be written as

$$c_z \left(\frac{m\omega}{2\pi\hbar}\right)^{d/2} \exp\left\{-\frac{(\mathbf{x} - \mathbf{x}_0)^2}{2\hbar/m\omega}\right\} e^{i\mathbf{p}_0 \cdot \mathbf{x}/\hbar}, \tag{8}$$

where $c_{\mathbf{z}} = \exp(-i\mathbf{p}_0 \cdot \mathbf{x}_0/\hbar)\exp(p_0^2/2\hbar m\omega)$.

3. Proofs

Theorem 1 would follow from standard heat kernel asymptotics if the angle θ were real. As it is, we need to verify that the expected behavior of the heat kernel holds for small τ and small *complex* θ. Recall that $\psi_{\mathbf{a}(\tilde{\mathbf{x}}_0, \tilde{\mathbf{p}}_0)}(\tilde{\mathbf{x}}) = \rho_\tau^d(\mathbf{a}(\tilde{\mathbf{x}}_0, \tilde{\mathbf{p}}_0), \tilde{\mathbf{x}})$ depends only on the complex angle between $\mathbf{a}(\tilde{\mathbf{x}}_0, \tilde{\mathbf{p}}_0)$ and $\tilde{\mathbf{x}}$. We will denote this function by $\rho_\tau^d(\theta)$. Since the heat kernel $\rho_\tau^d(\theta)$ is an even entire function of θ, and so depends only on θ^2, we need to understand the dependence of θ^2 on $\mathbf{z} = \mathbf{x}_0 + i\mathbf{p}_0/m\omega$ and $\mathbf{x}$. The following lemma provides the answer.

Lemma 2. *Fix $\mathbf{x}_0$ and $\mathbf{p}_0$ in $\mathbb{R}^d$. Let K be a compact set in $\mathbb{R}^d$ and let s be a positive number with $s < \pi$. Then for all sufficiently large r and all $\mathbf{x}$ in K there are solutions to $\cos\theta = r^{-2}(\mathbf{a}(\tilde{\mathbf{x}}_0, \tilde{\mathbf{p}}_0) \cdot \tilde{\mathbf{x}})$ with $|\theta| < s$. These solutions are unique up to a sign and satisfy*

$$\theta^2 = \frac{(\mathbf{z}-\mathbf{x})^2}{r^2} + q(r, \mathbf{z}, \mathbf{x}),$$

where $\mathbf{z} = \mathbf{x}_0 + i\frac{\mathbf{p}_0}{m\omega}$ and

$$|q(r, \mathbf{z}, \mathbf{x})| \le Cr^{-4}$$

for a fixed positive constant C and all $\mathbf{x} \in K$.

The proof uses only that $\cos\theta \approx 1 - \theta^2/2$ for small θ, and is omitted. Note that as r tends to infinity both τ and θ tend to zero but that $\theta^2/2\tau$ tends to $(\mathbf{z}-\mathbf{x})^2/(2\hbar/m\omega)$ as a function of $\mathbf{x}$ uniformly on compact subsets of $\mathbb{R}^d$. This observation is essential to the proof of Theorem 1.

Lemma 3. *For odd positive integers d, let $P_d(\tau, \theta)$ be defined inductively by*

$$P_1(\tau, \theta) = (2\pi\tau)^{-1/2} e^{-\theta^2/2\tau}$$

$$P_{d+2}(\tau, \theta) = -\frac{e^{d\tau/2}}{2\pi}\frac{1}{\sin\theta}\frac{\partial P_d(\tau, \theta)}{\partial\theta}$$

for complex θ with $|\theta| < \pi$. Then $P_d(\tau, \theta)$ is an even holomorphic function of θ in the set $|\theta| < \pi$ (with a removable singularity at the origin) for all d, τ.

Let $R_d(\tau, \theta)$ be defined so that

$$\rho_\tau^d(\theta) = P_d(\tau, \theta) + R_d(\tau, \theta).$$

Then there exist positive constants s_d, B_d, and C_d such that

$$|R_d(\tau, \theta)| \le B_d e^{-C_d/\tau}$$

for all θ with $|\theta| < s_d$ and all τ with $0 < \tau < 1$.

Proof. The statement concerning $P_d(\tau, \theta)$ follows easily by induction. Therefore, we need only prove the bound on $R_d(\tau, \theta)$. In light of the formula for ρ_τ^1 we have

$$R_1(\tau, \theta) = \frac{1}{\sqrt{2\pi\tau}} \sum_{n \neq 0} e^{-\frac{(\theta - 2\pi n)^2}{2\tau}}.$$

If we take, say, $s_1 = \frac{\pi}{2}$, then $\mathrm{Re}(\theta - 2\pi n) > 3\pi/2$ for all non-zero integers n and all θ with $|\theta| < s_1$. From this it easily follows that, say,

$$|R_1(\tau, \theta)| \le Be^{-\pi^2/2\tau}$$

for some B and for all θ with $|\theta| < s_1$. This establishes the desired result in the $d = 1$ case.

Assume now we have the desired bound for $R_d(\tau, \theta)$. In light of the inductive formula (4) for the heat kernel and the definition of P_d we have

$$R_{d+2}(\tau, \theta) = -\frac{e^{d\tau/2}}{2\pi}\frac{1}{\sin\theta}\frac{\partial R_d}{\partial\theta}(\tau, \theta)$$

which we write as

$$R_{d+2}(\tau,\theta) = -\frac{e^{d\tau/2}}{2\pi}\frac{\theta}{\sin\theta}\frac{1}{\theta}\frac{\partial R_d}{\partial\theta}(\tau,\theta). \tag{9}$$

Set s_{d+2} equal to any positive number strictly less than s_d, and let $R_d(\tau,\theta) = \sum_{n=0}^{\infty} a_n(\tau)\theta^{2n}$ be the power series expansion for $R_d(\tau,\theta)$. By the Cauchy estimates, we have

$$|a_n(\tau)| \leq B_d e^{-C_d/\tau} s_d^{-2n}$$

for all $0 < \tau < 1$. Then for $|\theta| < s_{d+2}$ we have

$$\frac{1}{\theta}\frac{\partial R_d}{\partial\theta}(\tau,\theta) = \sum_{n=1}^{\infty} 2n\, a_n(\tau)\theta^{2n-2}.$$

Therefore, for $|\theta| < s_{d+2}$ and $0 < \tau < 1$ we have

$$\begin{aligned}\left|\frac{1}{\theta}\frac{\partial R_d}{\partial\theta}(\tau,\theta)\right| &\leq B_d e^{-C_d/\tau}\sum_{n=1}^{\infty} 2n\frac{|\theta|^{2n-2}}{s_d^{2n-2}} \\ &\leq B_d e^{-C_d/\tau}\sum_{n=1}^{\infty} 2n\frac{s_{d+2}^{2n-2}}{s_d^{2n-2}} = De^{-C_d/\tau}\end{aligned}$$

for a positive constant D. The bound for $R_{d+2}(\tau,\theta)$ now follows from (9). ☐

We use the notation of the previous lemma in what follows.

Lemma 4. *For all odd positive integers d, there exist positive numbers $s_d < \pi$ such that*

$$\lim_{\tau\to 0^+}\frac{P_d(\tau,\theta)}{(2\pi\tau)^{-d/2}e^{-\theta^2/2\tau}} = \left(\frac{\theta}{\sin\theta}\right)^{(d-1)/2}$$

uniformly on compact subsets of the set $|\theta| < s_d$.

Although we will not use this fact, it is worth noting that the function on the right in the lemma is $J^{-1/2}(\theta)$ where J is the Jacobian of the exponential mapping for S^d.

Proof. This is trivially true for $d = 1$. We proceed by induction. Define $g_d(\tau,\theta)$ so that

$$P_d(\tau,\theta) = (2\pi\tau)^{-d/2}e^{-\theta^2/2\tau}g_d(\tau,\theta)$$

and observe that $g_d(\tau,\theta)$ is even and holomorphic on $|\theta| < s_d$. Then

$$\begin{aligned}P_{d+2}(\tau,\theta) &= -\frac{e^{d\tau/2}}{2\pi}\frac{1}{\sin\theta}\frac{\partial P_d}{\partial\theta}(\tau,\theta) \\ &= (2\pi\tau)^{-(d+2)/2}e^{-\theta^2/2\tau}\left\{e^{d\tau/2}\frac{\theta}{\sin\theta}g_d(\tau,\theta) - \tau e^{d\tau/2}\frac{\theta}{\sin\theta}\frac{1}{\theta}\frac{\partial g_d}{\partial\theta}(\tau,\theta)\right\}. \qquad (10)\end{aligned}$$

Our induction hypothesis on $P_d(\tau,\theta)$ implies that $g_d(\tau,\theta)$ tends uniformly to the function $(\theta/\sin\theta)^{(d-1)/2}$ on compact subsets of $|\theta| < s_d$ as $\tau\to 0^+$. It follows by basic complex analysis that $(1/\theta)\partial g_d(\tau,\theta)/\partial\theta$ tends to $1/\theta$ times the derivative of $(\theta/\sin\theta)^{(d-1)/2}$ uniformly on compact sets. Thus, because of the factor of τ, the second term in brackets in (10) tends to zero uniformly on compact sets. The first term in the brackets tends to $(\theta/\sin\theta)^{(d+1)/2}$ uniformly on compact sets as $\tau\to 0^+$, which establishes the lemma for dimension $d+2$. ☐

We now put our results together to supply the proof of Theorem 1.

Proof. Recall that $\tau = \hbar/m\omega r^2$ and that we must evaluate the heat kernel at a value of θ where $\cos\theta = r^{-2}(\mathbf{a}(\tilde{\mathbf{x}}_0, \tilde{\mathbf{p}}_0) \cdot \tilde{\mathbf{x}})$. Lemma 2 shows that we may choose θ satisfying $|\theta| < s_d$ if r is sufficiently large. Therefore, by Lemma 3, we need only calculate the limit of $r^{-d}P_d(\hbar/m\omega r^2, \theta)$ as $r \to \infty$. As noted above,

$$\lim_{r\to\infty} \frac{\theta^2}{2\tau} = \frac{(\mathbf{z}-\mathbf{x})^2}{2\hbar/m\omega} \tag{11}$$

uniformly for $\mathbf{x} \in K$. Therefore, using (11), Lemma 4 with the fact that we may choose θ^2 to tend uniformly to zero as $r \to \infty$, and the fact that continuous functions are uniformly continuous on compact subsets, it follows that

$$\lim_{r\to\infty} r^{-d} P_d\left(\frac{\hbar}{m\omega r^2}, \theta\right) = \left(\frac{m\omega}{2\pi\hbar}\right)^{d/2} \exp\left\{-\frac{(\mathbf{z}-\mathbf{x})^2}{2\hbar/m\omega}\right\}$$

uniformly on K. ☐

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Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA
E-mail address: bhall@nd.edu

Department of Mathematics, Baylor University, Waco, TX 76798, USA
E-mail address: jeffrey_mitchell@baylor.edu