Singular Continuous Floquet Operator for Systems with Increasing Gaps

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Abstract

Consider the Floquet operator of a time independent quantum system, periodically perturbed by a rank one kick, acting on a separable Hilbert space: $e^{-iH_0T}e^{-i\kappa T|\phi\rangle\langle\phi|}$ where T and κ are the period and the coupling constant respectively. Assume the spectrum of the self adjoint operator H_0 is pure point, simple, bounded from below and the gaps between the eigenvalues (λ_n) grow like: $\lambda_{n+1} - \lambda_n \sim Cn^d$ with $d \geq 2$. Under some hypotheses on the arithmetical nature of the eigenvalues and on the vector ϕ, cyclic for H_0, we prove the Floquet operator of the perturbed system has purely singular continuous spectrum.

1 Introduction

The long time behaviour of a periodic time dependent quantum system is linked to the spectral properties of its Floquet operator [1]. Up to now, most works in the field focused on time independent systems with pure point spectrum perturbed by a periodic and smooth time dependent potential or by periodic kicks.

In the smooth perturbative case, the analysis of the spectrum of the Floquet operator is essentially based on two strategies. On the one hand, assuming the multiplicity of the eigenvalues of the stationnary Hamiltonian is uniformly bounded and the spacings between them increase fast enough with the quantum numbers, KAM algorithms aim at proving that the spectrum remains pure point at least for small perturbations and non resonant frequencies [2], [4], [3]. In a complementary way (frequency independent and non-perturbative), adiabatic techniques exclude any absolutely continuous part in the spectrum of the Floquet operator. However, this approach gives no information on the singular part of the spectrum or it does at most generically [5], [6], [7].

By contrast, the form of the Floquet operator of kicked systems is explicit. The model is even exactly solvable if the system is kicked periodically by a rank one pertubation. Combescure studies this case with an adaptation of the Simon-Wolff machinery [8], [9]. She determines the nature of the spectrum of the Floquet operator with precision in a non perturbative way. However, her analysis is specific to the form of the operator. This work received several extensions [10] ([11] for a review).

We are interested in these systems from a different point of view. This article enhances the influence of increasing gaps in the spectrum of the unperturbed Hamiltonian on the spectrum of the Floquet operator for the perturbed system. Combescure conjectured that some singular continuous spectrum may appear as soon as the eigenvalues of the unperturbed Hamiltonian grow polynomially provided some coefficients are Diophantine ([9], remark c, p. 682). She proved it in the harmonic oscillator case. This article proves it is still true if the eigenvalues are simple and the gaps between them increase polynomially with an arbitrary growth rate.

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2 Context and main results

Let us consider a discrete time independent quantum system periodically perturbed by a rank one kick. Its evolution is described by the following Floquet operator:

$$V_\kappa = e^{-iH_0T}e^{-iT\kappa|\phi\rangle\langle\phi|} = e^{-iH_0T}(1+\mu|\phi\rangle\langle\phi|) \quad \text{with } \mu = e^{-i\kappa T}-1 \ , \ \ \kappa \in \mathbb{R}$$

acting on a separable Hilbert space $\mathcal{H}$, where H_0 is a self adjoint operator on $\mathcal{H}$ and ϕ a vector of $\mathcal{H}$. The real numbers κ and T are respectively the coupling constant and the period of the system.
Remark : Formally, the unitary operator V_κ can be considered as the Schrödinger evolution associated to the following time dependent Hamiltonian [9]:

$$H(t) = H_0 + \kappa|\phi\rangle\langle\phi| \sum_{n=-\infty}^{+\infty} \delta(t-nT) \ .$$

Assume that:
H1: H_0 is bounded from below, with discrete spectrum of multiplicity one. Let us denote its eigenvalues by $(\lambda_m)_{m\in\mathbb{N}^*}$ and let $(\phi_m)_{m\in\mathbb{N}^*}$ be an orthonormal basis of corresponding eigenvectors, where $\mathbb{N}^*$ denote the set of positive integers.
H2: The vector ϕ is cyclic for H_0.
Our main concern is to know if the spectrum of V_κ remains pure point when κ is non zero and the value of the period T is fixed. Notice that it does whenever $\kappa T = 0 \ mod(2\pi)$. Such values of κ will be excluded from our discussion in the sequel. In [9], Combescure established two types of spectral results. First, the spectrum remains pure point in the following case:

Theorem 2.1 *Under the hypotheses* **H1**, **H2**, *assume that* $(\langle\phi_m|\phi\rangle)_{m\in\mathbb{N}^*}$ *is* l^1. *Then, the spectrum of* V_κ *is pure point for almost every* κ *with respect to the Lebesgue measure on* $\mathbb{R}$.

Secondly, in the case of the harmonic oscillator with frequency ω_0, Combescure exhibits examples where the spectrum of V_κ may be singular continuous for some Diophantine ω_0. Such critical values of ω_0 and the equidistance of the simple eigenvalues of the harmonic oscillator make transitions towards higher energy levels of the system relatively easy, which may explain the appearance of the singular continuous subspace. Defining the set:

$$\mathcal{Z} = \{\kappa \in \mathbb{R}; \kappa T = 2k\pi, k \in \mathbb{Z}\},$$

Combescure proved:

Theorem 2.2 *Let H_0 be the Hamiltonian of the harmonic oscillator with frequency ω_0. Assume the vector ϕ is cyclic with respect to H_0 and:*

$$|\langle\phi_m|\phi\rangle| \geq cm^{-\gamma} \ , \ \frac{1}{2} < \gamma < 1$$

for some c positive. Then, if $\frac{\omega_0 T}{2\pi}$ is Diophantine, the spectrum of V_κ is purely singular continuous for any κ in $\mathbb{R} \setminus \mathcal{Z}$.

This result turns out to be still true if the gaps between the eigenvalues increase polynomially. Our main result is the following:

Theorem 2.3 *Under the hypotheses* **H1**, **H2**, *assume the eigenvalues of H_0 are given by:*

$$\forall m \in \mathbb{N}^* \ , \ \lambda_m = \sum_{k=0}^{d} p_k m^k$$

where $d \geq 3$. Suppose $\frac{T}{2\pi}p_r$ is irrational for an r in $\{1, \ldots, d\}$. If $rd > 3$ and

$$|\langle\phi_m|\phi\rangle| \geq cm^{-\frac{1+\epsilon}{2}} \text{ with } 0 < \epsilon < \rho \ , \ \rho = \frac{1}{8d^2(\ln d + 1.5 \ln\ln d + 4.2)}$$

and a positive constant c, then the spectrum of the unitary operator V_κ is purely singular continuous for any κ in $\mathbb{R} \setminus \mathcal{Z}$.

If $d = 3$, Theorem 2.3 applies as soon as one of the coefficients a_2 or a_3 is irrational. If $d \geq 4$, this condition is relaxed. It is enough to have an irrational number a_r with $r \geq 1$. As $\epsilon < \rho$, notice that the greater d is the smaller ρ and ϵ are. Because of the condition $d \geq 3$, these exponents do not exceed 3.10^{-3}.

Remark 1 : The conditions on the degree of the polynomial look artificial. From an intuitive point of view, the singular continuous spectrum should appear as soon as the degree obeys $d \geq 1$ (i.e. for gaps with lower increasing rates) and one of the coefficient is irrational. However, these assumptions are imposed by the technical machinery of analytic number theory used thereafter.

Remark 2 : The hypotheses on the Diophantine nature of some coefficients in Theorem 2.2 are relaxed in Theorem 2.3. In order to get this improvement, the scheme of proof given in [9] has been overhauled. Nonetheless, the irrationality condition is crucial to prove there is some singular continuous spectrum and cannot be removed. We recall what occurs in the resonant case:

Proposition 2.1 *Under the hypothesis* **H1**, *assume the eigenvalues of the Hamitonian H_0 are defined by the following:*

$$\forall m \in \mathbb{N}^* \ , \ \lambda_m = \sum_{k=0}^{d} p_k m^k \tag{2.3}$$

where $d > 1$ and all the coefficients $\frac{T}{2\pi}p_k$ are rational, except possibly $\frac{T}{2\pi}p_0$. Then, V_κ is pure point, whatever κ is.

This proposition results from the invariance of the essential spectrum of e^{-iH_0T} and V_κ (Weyl's Theorem, e.g. [12]) which is included in the finite set $\{e^{-ip_0T}e^{-i2\pi\frac{l}{Q}}; l \in \{0, \ldots, Q-1\}\}$ where Q is the greatest common divisor of the rationals $(\frac{T}{2\pi}p_k)_{k\in\{1,\ldots,d\}}$. The same argument has already been used in [1] and [13].

In the remainder of this article, we prove Theorem 2.3.

3 Proof of Theorem 2.3

3.1 Preliminaries

For any real a, $\{a\}$ and $[a]$ denote respectively its fractional and integral part : $a = [a]+\{a\}$. The fractional part of a real belongs to the unit interval $[0, 1[$.
Notice that the spectrum of V_κ is the range of the spectrum of $V_\kappa e^{ip_0 T}$ by a rotation of the unit circle. Both spectra are of the same nature. Therefore, without loss of generality, we may assume in the following that $p_0 = 0$. Moreover,

$$V_\kappa = e^{-iH_0T} + \mu e^{-iH_0T}|\phi\rangle\langle\phi| \text{ where } \mu e^{-iH_0T}|\phi\rangle\langle\phi| \text{ is trace class.}$$

So, by the Birman-Krein Theorem [14], the absolutely continuous part of the spectra of V_κ and e^{-iH_0T} are equivalent. As e^{-iH_0T} is pure point, $\sigma_{ac}(V_\kappa) = \emptyset$ for all values of κ, and the spectrum is purely singular. It remains to see that the spectrum of V_κ does not contain any eigenvalue in order to prove the Theorem. In [9], Combescure applies the Simon-Wolff method [8] to provide the following criterion:

Lemma 3.1 *Assume the vector ϕ is cyclic with respect to H_0, the complex number e^{ix} belongs to the point spectrum of V_κ iff:*

$$B(x)^{-1} = \sum_{m=1}^{+\infty} \frac{|\langle\phi_m|\phi\rangle|^2}{\sin^2(\frac{1}{2}(x-\theta_m))} < +\infty \quad \textit{and} \quad \sum_{m=1}^{+\infty} |\langle\phi_m|\phi\rangle|^2 \cot(\frac{x-\theta_m}{2}) = \cot(\frac{\kappa T}{2})$$

where θ_m is defined by: $\theta_m = 2\pi\{\frac{\lambda_m T}{2\pi}\}, m \in \mathbb{N}^*$.

Notice that the sequence $(\theta_m)_{m\in\mathbb{N}^*}$ lies in $[0, 2\pi[$. The convergence (or divergence) of $B(x)^{-1}$ is the result of a competition between the decay rate of the numerator and the distribution of the sequence $(\theta_m)_{m\in\mathbb{N}^*}$ which governs the denominator. We will prove that the hypotheses are sufficient to ensure the divergence of the series $B(x)^{-1}$ for any value of x in $[0, 2\pi[$. With this purpose in mind, some useful tools are introduced in the next section.

3.2 A flavour of analytic number theory

The notion of discrepancy introduced thereafter aims at comparing the distribution of a sequence of real numbers mod(1) with a uniform distribution measure.

Definition 3.1 *Consider a sequence $(x_m)_{m\in\mathbb{N}^*}$ of real numbers. The discrepancy of the sequence $(x_m)_{m\in\mathbb{N}^*}$ is defined by:*

$$D_N = \sup_{0\le a<b\le 1} \left| \frac{A([a,b[;N;(x_m))}{N} - (b-a) \right|$$
$$\textit{with } A([a,b[;N;(x_m)) = \#\{1 \le m \le N; \{x_m\} \in [a,b[\} \ .$$

If $\lim_{N\to+\infty} D_N = 0$ the sequence is said to be uniformly distributed mod(1).

Despite this abstract definition, the following Theorem gives a practical way to estimate the value of the discrepancy of a sequence.

Theorem 3.1 *(Erdös - Turàn) For any finite sequence $(x_m)_{1\le m\le N}$ of real numbers and any positive integer n, we have:*

$$D_N \le \frac{6}{n+1} + \frac{4}{\pi}\left(\sum_{h=1}^{n}(\frac{1}{h} - \frac{1}{n+1})\left|\frac{1}{N}\sum_{m=1}^{N} e^{2\pi i h x_m}\right|\right) \quad .$$

This result is proved in [15] for example. Now, the problem is shifted to an asymptotic analysis of the exponential sum. Fortunately, this can be done for a sequence $(x_m)_{m\in\mathbb{N}^*}$ defined by a polynomial:

Definition 3.2 *Let d, N be positive integral numbers. Define the quantity ν by: $\nu = d^{-1}$. For each value of N, divide the points of the d-dimensional space $\mathbb{R}^d$ in two disjoint classes $\mathcal{C}_1(N)$ and $\mathcal{C}_2(N)$ by the following procedure : a point $(a_1,\dots,a_d)$ of $\mathbb{R}^d$ belongs to $\mathcal{C}_1(N)$ if there exists a d-uple of rational irreducible fractions $(\frac{s_1}{q_1},\dots,\frac{s_d}{q_d})$ with positive denominators whose least common multiple Q does not exceed N^ν such that :*

$$\forall r \in \{1,\dots,d\},\ \ |a_r - \frac{s_r}{q_r}| \le N^{-r+\nu} \quad .$$

A point of $\mathbb{R}^d$ which is not in $\mathcal{C}_1(N)$ belongs to $\mathcal{C}_2(N)$.

Remark : Fix any compact d-dimensional box in $\mathbb{R}^d$. The volume of the points which belong to $\mathcal{C}_1(N)$ in this box tends to 0 as N tends to infinity [16].

Theorem 3.2 *(Vinogradov) Let d, N be positive integral numbers and A be the polynomial: $A(x) = \sum_{k=1}^{d} a_k x^k$ where $a_d,\dots,a_1$ are real numbers, a_d non zero and $d \ge 3$. Then defining,*

$$\rho = \frac{1}{8d^2(\ln d + 1.5\ln\ln d + 4.2)} \quad ,$$

there exists a positive constant C_d depending on A such that:

$$\forall h \le N^{2\rho} \ \ , \ \ \left|\sum_{m=1}^{N} e^{2i\pi h A(m)}\right| \le C_d N^{1-\rho} \quad \textit{if} \quad (a_1,\dots,a_d) \in \mathcal{C}_2(N) \quad .$$

For a proof, see [16], chapter IV, Theorem 3. This theorem gives also some estimates for a point $(a_1,\dots,a_d)$ in $\mathcal{C}_1(N)$. But, they are useless for our purpose.

The fundamentals are now stated. The analytic machinery can be turned on to prove the last part of the Theorem.

3.3 Technicalities

In the sequel, the values of the real numbers x and κ are fixed respectively in $[0,2\pi[$ and $\mathbb{R}\setminus\mathcal{Z}$.

It is now time to give the right inputs to the machinery with the following notations:

$$\forall m \in \mathbb{N}^*, \theta_m = 2\pi\{\frac{\lambda_m T}{2\pi}\} = 2\pi\{P_{H_0,T}(m)\}$$

where $P_{H_0,T}$ is a polynomial defined by:

$$P_{H_0,T}(x) = \frac{T}{2\pi}\sum_{k=1}^{d} p_k x^k,$$

$x \in \mathbb{R}$. The polynomial $P_{H_0,T}$ will play the role of the polynomial A. As the coefficient $\frac{T}{2\pi}p_r$ is irrational, this sequence is uniformly distributed mod(1) (e.g. [15]). The two following steps aim at giving an estimate of the discrepancy of the sequence $(P_{H_0,T}(m))_{m\in\mathbb{N}^*}$.

Lemma 3.2 *Let $(a_1,\ldots,a_d)$ be a d-uple of $\mathbb{R}^d$ with rational coefficients. There exists an integer N_0 such that: $\forall N \geq N_0, (a_1,\ldots,a_d) \in \mathcal{C}_1(N)$.*
Suppose now, there exists an r in $\{1,\ldots,d\}$ such that $rd > 3$ and a_r is irrational: there exists a subsequence $(N_k)_{k\in\mathbb{N}^*}$ such that:

$$(a_1,\ldots,a_d) \in \mathcal{C}_2(N_k) \text{ with } \lim_{k\to+\infty} N_k = +\infty \ .$$

Proof : The first part of the lemma relies on the following remark: the rational coefficients $(a_i)_{i\in\{1,\ldots,d\}}$ can be written as irreducible fractions $(\frac{s_i}{q_i})_{i\in\{1,\ldots,d\}}$ with positive denominators whose least common multiple Q is fixed. Q is necessarily lower than N^ν for N large enough. Now, assume there exists $r \in \{1,\ldots,d\}$ such that $rd > 3$ and a_r is an irrational number. Let us prove the second part of the lemma *ad absurdum*. Suppose that for each value of N large enough $(a_1,\ldots,a_d)$ belongs to $\mathcal{C}_1(N)$, i.e. there exists a d-uple of irreducible fractions with positive denominators $(\frac{s_{1,N}}{q_{1,N}},\ldots,\frac{s_{d,N}}{q_{d,N}})$ with a least common multiple Q_N such that:

$$\begin{cases} q_{r,N} \leq Q_N \leq N^\nu \\ |a_r - \frac{s_{r,N}}{q_{r,N}}| \leq N^{-r+\nu} \end{cases} \tag{3.9}$$

The first condition of (3.9) implies : $q_{r,N}^{-2} \geq N^{-2\nu} \geq N^{-r+\nu}$. The condition $rd > 3$ (i.e. $r > 3\nu$) ensures that for N large enough,

$$N^{-r+\nu} < \frac{N^{-2\nu}}{2}.$$

Therefore, by the second inequality of (3.9):

$$\exists N_0 \in \mathbb{N}^*, \forall N \geq N_0, \ \ |a_r - \frac{s_{r,N}}{q_{r,N}}| < \frac{1}{2q_{r,N}^2} \ \ .$$

Therefore, the sequence $(\frac{s_{r,N}}{q_{r,N}})_{N\geq N_0}$ is a subsequence of the sequence of irreducible rational fractions obtained by the expansion of a_r in continued fractions (theorem 19, [17]). It follows from theorem 12 in [17] that necessarily :

$$Q_N \geq q_{r,N} \geq 2^{\frac{N-1}{2}}$$

for sufficiently large N, which contradicts the first inequality of (3.9). □

Lemma 3.3 *If there exists an integer r in $\{1,\ldots,d\}$ such that $rd > 3$ and the coefficient $\frac{T}{2\pi}p_r$ is irrational, we can extract a subsequence $(N_k)_{k\in\mathbb{N}^*}$ such that: $\exists C > 0, \forall k \in \mathbb{N}^*$,*

$$D_{N_k} \leq \frac{C}{N_k^\beta}$$

with $0 < \beta < \rho$.

Proof : If $\frac{T}{2\pi}p_r$ is irrational and $r > 3\nu$, then by Lemma 3.2 and Theorem 3.2 we get: $\exists C_{1,d} > 0, \forall k \in \mathbb{N}^*, \forall h \leq N_k^{2\rho}$,

$$\left| \sum_{m=1}^{N_k} \mathrm{e}^{2i\pi h P_{H_0,T}(m)} \right| \leq C_{1,d} N_k^{1-\rho} \quad .$$

Combining this result with Theorem 3.1, it follows that: $\exists C_{2,d} > 0, \forall k \in \mathbb{N}^*, \forall n \leq N_k^{2\rho}$,

$$D_{N_k} \leq C_{2,d} \left(\frac{1}{n+1} + \frac{1}{N_k^{\rho}} \sum_{h=1}^{n} (\frac{1}{h} - \frac{1}{n+1}) \right).$$

Then using some integral estimates on the last term: $\forall n \leq N_k^{2\rho}$,

$$\sum_{h=1}^{n} (\frac{1}{h} - \frac{1}{n+1}) = 1 + \sum_{h=2}^{n} \frac{1}{h} - (1 - \frac{1}{n+1}) \leq \sum_{h=2}^{n} \int_{h-1}^{h} \frac{dx}{x} + \frac{1}{n+1} = \ln n + \frac{1}{n+1} \quad .$$

Combining these two inequalities, choosing n equal to $[N_k^{2\rho}]$ and β in $]\epsilon, \rho[$ where ϵ is defined in Theorem 2.3, the following inequality holds: $\exists C > 0, \forall k \in \mathbb{N}^*$,

$$D_{N_k} \leq C_{2,d} \left(\frac{1}{N_k^{2\rho}} + \frac{2\rho \ln N_k}{N_k^{\rho}} + \frac{1}{N_k^{3\rho}} \right) \leq C \frac{1}{N_k^{\beta}} \quad .$$

□

Such an estimate allows us to compute roughly the number of real numbers θ_m which are close in a sense to x. It is the aim of the next lemma. The notion of nearness used in the following to compute these numbers θ_m takes the form of the family of intervals:

$$J_l(x) = [\frac{x}{2\pi}, \frac{x}{2\pi} + l^{-\epsilon} (\ln l)^{\frac{-1}{2}} [\ , \ \ l > 1$$

and the estimate of the number of real numbers θ_m close to x will be given by the cardinality of the following sets:

$$S_{1,N}(x) = \{1 \leq m \leq N ; \frac{\theta_m}{2\pi} \in J_m(x)\},$$
$$S_{2,N}(x) = \{1 \leq m \leq N ; \frac{\theta_m}{2\pi} \in J_N(x)\}.$$

Notice that in any case $S_{2,N}(x) \subset S_{1,N}(x)$.

Lemma 3.4 *Given $x \in [0, 2\pi[, \exists K_x > 0, \forall k \geq K_x$,*

$$\# S_{1,N_k}(x) \geq [\frac{N_k^{1-\epsilon}}{2} (\ln N_k)^{\frac{-1}{2}}].$$

Proof : Since $\frac{x}{2\pi} \in [0,1[, \exists N_x > 0, \forall N \geq N_x, J_N(x) \subset [0,1]$. By definition, $\forall N \geq N_x$,

$$\left| \frac{\# S_{2,N}(x)}{N} - N^{-\epsilon} (\ln N)^{-\frac{1}{2}} \right| \leq D_N$$

In particular, $N^{1-\epsilon}(\ln N)^{-\frac{1}{2}} - ND_N \leq \#S_{2,N}(x)$, for such integer N. Therefore, for any integer k such that: $N_k \geq N_x$,

$$N_k^{1-\epsilon}(\ln N_k)^{-\frac{1}{2}} - CN_k^{1-\beta} \leq \#S_{2,N_k}(x) \leq \#S_{1,N_k}(x).$$

As $\epsilon < \beta, \exists K_x > 0, \forall k \geq K_x$,

$$[\frac{N_k^{1-\epsilon}}{2}(\ln N_k)^{-\frac{1}{2}}] \leq \frac{N_k^{1-\epsilon}(\ln N_k)^{-\frac{1}{2}}}{2} \leq \#S_{1,N_k}(x).$$

which implies the Lemma. □

Gathering all the previous lemma, we are now able to bound $B(x)^{-1}$ from below and prove the final step of our Theorem. Due to Lemma 3.3, this estimate will be uniform in the parameter x.

Lemma 3.5 *For all $x \in [0, 2\pi[$, $B(x)^{-1} = +\infty$.*

Proof : We recall the real number x is fixed in $[0, 2\pi[$. Using its definition, $B(x)^{-1}$ admits the following lower bound for all positive integer k:

$$B(x)^{-1} \geq 4 \sum_{m \in S_{1,N_k}(x)} \frac{|\langle \phi_m | \phi \rangle|^2}{(x - \theta_m)^2}.$$

But, $\forall m \in S_{1,N_k}(x) : |x - \theta_m| \leq 2\pi m^{-\epsilon}(\ln m)^{\frac{-1}{2}}$ and $|\langle \phi_m | \phi \rangle| \geq cm^{-\frac{1+\epsilon}{2}}$. It implies for all k:

$$B(x)^{-1} \geq \frac{4c^2}{(2\pi)^2} \sum_{m \in S_{1,N_k}(x)} \frac{\ln m}{m^{1-\epsilon}}.$$

Using Lemma 3.4 and the fact that the function $y \to y^{-1+\epsilon} \ln y$ decays monotonely for sufficiently large y, the following holds:

$$\begin{aligned}
\sum_{m \in S_{1,N_k}(x)} \frac{\ln m}{m^{1-\epsilon}} &\geq \sum_{m = N_k - [\frac{N_k^{1-\epsilon}}{2}(\ln N_k)^{-\frac{1}{2}}]+1}^{N_k} \frac{\ln m}{m^{1-\epsilon}} \\
&\geq \int_{N_k - [\frac{N_k^{1-\epsilon}}{2}(\ln N_k)^{-\frac{1}{2}}]+1}^{N_k+1} \frac{\ln x}{x^{1-\epsilon}} dx \\
&\geq (N_k - [\frac{N_k^{1-\epsilon}}{2}(\ln N_k)^{-\frac{1}{2}}] + 1)^{\epsilon} \int_{N_k - [\frac{N_k^{1-\epsilon}}{2}(\ln N_k)^{-\frac{1}{2}}]+1}^{N_k+1} \frac{\ln x}{x} dx
\end{aligned}$$

$$= (N_k - [\frac{N_k^{1-\epsilon}}{2}(\ln N_k)^{-\frac{1}{2}}]+1)^{\epsilon} \left(\frac{1}{2}(\ln(N_k + 1))^2 - \frac{1}{2}(\ln(N_k - [\frac{N_k^{1-\epsilon}}{2}(\ln N_k)^{-\frac{1}{2}}] + 1))^2 \right).$$

It remains to precise the asymptotics of each term when k tends to infinity:

$$\begin{aligned}
(N_k - [\frac{N_k^{1-\epsilon}}{2}(\ln N_k)^{-\frac{1}{2}}] + 1)^{\epsilon} &\sim {N_k}^{\epsilon}, \\
(\ln(N_k + 1))^2 &= (\ln N_k)^2 + \frac{2 \ln N_k}{N_k} + o(\frac{\ln N_k}{N_k}), \\
(\ln(N_k - [\frac{N_k^{1-\epsilon}}{2}(\ln N_k)^{-\frac{1}{2}}] + 1))^2 &= (\ln N_k)^2 - \frac{(\ln N_k)^{\frac{1}{2}}}{N_k^{\epsilon}} + o(\frac{(\ln N_k)^{\frac{1}{2}}}{N_k^{\epsilon}}) \ .
\end{aligned}$$

Gathering these informations, the asymptotics of the lower bound is given by:

$$(N_k - [\frac{N_k^{1-\epsilon}}{2}(\ln N_k)^{-\frac{1}{2}}] + 1)^{\epsilon} \int_{N_k - [\frac{N_k^{1-\epsilon}}{2}(\ln N_k)^{-\frac{1}{2}}]+1}^{N_k+1} \frac{\ln x}{x} dx \sim (\ln N_k)^{\frac{1}{2}}.$$

which implies:

$$\lim_{k \to +\infty} \sum_{m \in S_{1,N_k}(x)} \frac{\ln m}{m^{1-\epsilon}} = +\infty.$$

Necessarily, $B(x)^{-1} = +\infty$. $\square$

We deduce from Lemma 3.5 and 3.1 that $\sigma_{pp}(V_\kappa) = \emptyset$. So, the spectrum of V_κ is purely singular continuous.

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