On the Algebra of Fluctuation in Quantum Spin Chains.

Taku Matsui

Graduate School of Mathematics
Kyushu University
1-10-6 Hakozaki, Fukuoka 812-8581
JAPAN

We present a proof of the central limit theorem for a pair of mutually non-commuting operators in mixing quantum spin chains. The operators are not necessarily strictly local but quasi-local. As a corollary we obtain a direct construction of the time evolution of the algebra of normal fluctuation for Gibbs states of finite range interactions on a one-dimensional lattice. We show that the state of the algebra of normal fluctuation satisfies the β-KMS condition if the microscopic state is a β-KMS state.

We show that any mixing finitely correlated state satisfies our assumption for the central limit theorem.

KEY WORDS: central limit theorem; KMS states; quantum spin chain
`e-mail: matsui@math.kyushu-u.ac.jp`

1 Summary of Results

In [6] and in [7] D.Goderis, A.Verbeure, and P.Vets investigated a central limit theorem for mixing quantum spin systems. We refer to the limit theorem of D.Goderis, A.Verbeure, and P.Vets as CLT below. In case of a single observable, their theorem is a kind of classical limit theorem for a family of spectral measures of operators appearing in quantum statistical mechanics. However if we consider more than two observables, the limit theorem suggests the emergence of Boson fields as the algebra of fluctuation of physical observables. Then, it is natural to consider the time evolution states of such Bose fields appearing in the central limit. In fact, as the states appearing in CLT are quasifree states of Bose fields, at a heuristic level, the dynamics of the fluctuation operators is the quasifree dynamics (via Bogoliubov automorphisms) i.e. unitary time evolution in the test function space of the Boson field. D.Goderis, A.Verbeure,and P.Vets in [8] investigated the construction of the time evolution of the algebra of fluctuation. They tried to introduce the time evolution for the algebra of fluctuation induced by the microscopic dynamics . As they could not prove their limit theorem to any time invariant dense subalgebra, their introduction of the time evolution in an indirect way. They have shown that the quasifree state of the algebra of fluctuation is a KMS state if the state of the microscopic system is KMS. However they failed to prove validity of their assumption for any non-trivial KMS states.

After publication of [6], [7] and [8], no concrete example of quantum spin systems has been analyzed. The difficulty lies in the mixing assumption in [7] where estimate of decay of correlation of two observables Q_1 and Q_2 located in two disjoint regions, Λ_1 and Λ_2. In many cases we may prove the following estimate:

$$|\varphi(Q_1Q_2) - \varphi(Q_1)\varphi(Q_2)| \leq C(Q_1, Q_2)\,\|Q_1\|\,\|Q_2\|\,r^{-d-\epsilon} \tag{1.1}$$

where r is the distance between the supports Λ_1 and Λ_2. To show the central limit theorem, we need information of dependence of the constant $C(Q_1, Q_2)$ on the size of Λ_1 and Λ_2. D.Goderis, A.Verbeure, and P.Vets never proved their assumption for any (non-product) Gibbs state and until recently it has been an open question whether the above results really hold for any Gibbs states.

In [12], we obtained a different proof of CLT of a single observable in quantum spin systems under a slightly different mixing condition . The advantages of our results in [12] are as follows.
(i) We can prove our assumptions for several mixing states of one-dimensional systems. The assumption of [12] is valid for Gibbs states for finite range interactions , ground states of the (massive) XY model, quasifree states of CAR algebras.
(ii) Our limit theorem is valid for (not strictly local) certain quasi-local observables, which we named *exponentially localized observables*.
In fact, instead of assuming (1.1), we can show CLT under the following condition:

$$|\varphi(Q_1Q_2) - \varphi(Q_1)\varphi(Q_2)| \leq C(Q_1)\,\|Q_2\|\,r^{-d-1-\epsilon}$$

Our proof is based on an idea of E.Bolthauzen in [2]. As far as strictly local observables concerned, we do not require any estimate of the constant $C(Q_1)$ and we did not assume the assumption on the 4th moment (CLT4) of [7] neither. To show CLT for quasi-local observables we assume stronger mixing , which is still valid in many one-dimensional examples.

In this article, we continue our study of CLT. The new result of this article is CLT for mutually non-commuting operators in the following sense:

$$\lim_N \varphi(e^{iTQ^1_{<N>}} e^{iTQ^2_{N>}} \ldots . e^{iTQ^m_{<N>}}) = e^{-\frac{t^2}{2} t(\sum_k Q^k, \sum_k Q^k)} e^{-i\frac{T^2}{2} \sum_{k<l} s(Q^k, Q^l)}$$

This CLT is a stronger statement confirming appearance of Boson fields as the algebra of normal fluctuation. We prove this limit theorem under the same condition of [12] . As we are handling a convergence of functionals rather that of measures, we had to add more argument and estimates which are absent both in our previous paper [12] and in commutative cases. Let us mention that Goderis, Verbeure and Vets have already considered the same limit theorem only for strictly local observables and again they could not prove their assumption for any non-trivial Gibbs states. We will see that our CLT is valid for a dense algebra invariant under the time evolution generated by any finite range translationally invariant Hamiltonian. As a corollary , we can introduce the time evolution for the algebra of normal fluctuation directly. The CLT of this paper implies that quasifree states obtained in CLT is a KMS state of the algebra of fluctuation provided that the microscopic state satisfies the KMS condition as well.

We concentrate here one-dimensional system. We may derive the same CLT for higher dimensional systems under a similar mixing assumption of states, though the condition looks rather difficult to verify , so we do not present its proof here. In our proof of CLT for Gibbs states, one dimensionality enters in use of the Ruelle transfer operator technique and the entire analyticity of the time evolution (in Heisenberg Picture of Quantum Mechanics).

Next we present our results more precisely. We start with explaining our notation. Here the language of Operator Algebra is used and the reader unfamiliar with this may consult [3] and [4].

By $\mathcal{A}$ we denote the UHF C^*-algebra d^∞ (the infinite tensor product of d by d matrix algebras) :

$$\mathcal{A} = \overline{\bigotimes_{\mathbf{Z}} M_d(\mathbf{C})}^{C^*} .$$

Each component of the tensor product above is specified with a lattice site $j \in \mathbf{Z}$. By $Q^{(j)}$ we denote the element of $\mathcal{A}_{\mathbf{Z}}$ with Q in the jth component of the tensor product and the unit in any other component. For a subset Λ of $\mathbf{Z}$, $\mathcal{A}_\Lambda$ is defined as the C^*-subalgebra of $\mathcal{A}_{\mathbf{Z}}$ generated by elements localized in Λ. When φ is a state of $\mathcal{A}_{\mathbf{Z}}$ the restriction to $\mathcal{A}_\Lambda$ will be denoted by φ_Λ :

$$\varphi_\Lambda = \varphi|_{\mathcal{A}_\Lambda}. \tag{1.2}$$

For simplicity we set

$$\mathcal{A}_{loc} = \cup_{|\Lambda|<\infty}\mathcal{A}_\Lambda \tag{1.3}$$

where $|\Lambda|$ is the cardinality of Λ. Let τ_j be the lattice translation (j shift to the right) determined by

$$\tau_j(Q^{(k)}) = Q^{(j+k)} \tag{1.4}$$

for any j and k in $\mathbf{Z}$.

@Time Evolution of Local Observables

In this paper, the time evolution of (microscopic) local observables is determined by a translationally invariant finite range interaction. Our local Hamiltonian H_Λ on a finite volume Λ has the following standard form:

$$H_\Lambda = \sum_{X\subset\Lambda}\psi(X) \tag{1.5}$$

where X is an interval $[a,b]$ of $\mathbf{Z}$ and

$$\psi(X) = \psi(X)^* \in \mathcal{A}_X \quad , \quad \tau_j(\psi(X)) = \psi(X+j)$$

$$\psi(X) = 0 \qquad \text{if the diameter } d(X) \text{ of the set } X \text{ is greater than } r. \tag{1.6}$$

The time evolution α_t^Λ of the finite system in a finite set Λ of $\mathbf{Z}$ is determined by

$$\alpha_t^\Lambda(Q) = e^{itH_\Lambda}Qe^{-itH_\Lambda}.$$

From the local time evolution α_t^Λ, the global time evolution α_t is obtained via the thermodynamic limit which converges in the norm topology of $\mathcal{A}$.

$$\alpha_t(Q) = \lim_{\Lambda\to\mathbf{Z}}\alpha_t^\Lambda(Q).$$

Next,we introduce the notion of *exponentially localized observables* .

Definition 1.1 *Let θ be a positive constant $0<\theta<1$. Define $\|Q\|^{(n)}$ by the following equation:*

$$\|Q\|^{(n)} = \inf\left\{\|Q-Q_n\| \mid Q\in\mathcal{A}_{[-n,n]}\right\} \tag{1.7}$$

for n positive, $n>0$ and we set

$$\|Q\|^{(0)} = \|Q\|.$$

In terms of $\|Q\|^{(n)}$ *we introduce* $|||Q|||_\theta$ *:*

$$|||Q|||_\theta = \sum_{n=0,1,2,\ldots} \|Q\|^{(n)} \theta^{-n} \tag{1.8}$$

An element Q *of* $\mathcal{A}$ *is exponentially localized with rate* θ *if* $|||Q|||_\theta$ *is finite.*

The set of all exponentially localized elements with rate θ *is denoted by* F_θ*.*

We fix an element Q_n *of* $\mathcal{A}_{[-n,n]}$ *satisfying*

$$\|Q\|_n = \|Q - Q_n\|.$$

Note that due to compactness of a closed bounded subset of $\mathcal{A}_{[-n,n]}$ the infimum in (1.7) is attained by an element . In [11] we used slightly different definition of F_θ. But they are essentially equivalent.

The following invariance of F_θ under the time evolution is a corollary of results due to H.Araki in [1] or due to the *Propagation of Estimates* due to E.Lieb and D.Robinson (c.f. Theorem 6.2.9. and 6.2.11 of [4]) .

Proposition 1.2 *Suppose that the interaction is of finite range.*
(i) For any θ *and* θ' *with* $\theta < \theta'$*, there exists a constant* $C = C(\theta, \theta')$ *such that*

$$|||\alpha_t(Q)|||_{\theta'} \leq C|||Q|||_\theta$$

(ii) For any Q *in* F_θ*,* $\alpha_t(Q)$ *is entire analytic (analytic in the whole complex plane) as a function of* t*.*

Due to the above proposition the sets $\cup_\theta F_\theta$ and $\cap_\theta F_\theta$ are invariant under α_t.

Gibbs states

As usual, the local Gibbs state at the inverse temperature β is defined by

$$\varphi_\beta^{(\Lambda)}(Q) = \frac{tr(e^{-\beta H_\Lambda} Q)}{tr(e^{-\beta H_\Lambda})}. \tag{1.9}$$

For the finite range interaction, the following thermodynamics limit exists:

$$\lim_{N\to\infty} \varphi_\beta^{([-N,N])}(Q) = \varphi_\beta(Q).$$

φ_β is a unique Gibbs state of the time evolution α_t and φ_β satisfies the KMS boundary condition:

$$\varphi_\beta(Q_1 \alpha_{i\beta}(Q_2)) = \varphi_\beta(Q_2 Q_1)$$

for Q_1 and Q_2 in F_θ. By applying the Ruelle transfer operator technique for UHF algebras (c.f. [1], [9] , and [11]) we obtain the following estimates for the infinite volume Gibbs state φ_β.

Theorem 1.3 *Suppose that the interaction is of finite range.*
(i) There exist constants $C, m > 0$ *such that*

$$\left|\varphi_\beta^{[-k+a,b+k]}(Q) - \varphi_\beta(Q)\right| \leq Ce^{-mk} \|Q\| \tag{1.10}$$

for any Q *in* $A_{[a,b]}$.
(ii) We have exponential decay of correlation for observables localized in the half infinite intervals $(-\infty, -1]$ *and,* $[0, \infty)$.

$$|\varphi(Q_1\tau_j(Q_2)) - \varphi(Q_1)\varphi(Q_2)| \leq Ke^{-Mj} \|Q_1\| \|Q_2\| \tag{1.11}$$

for Q_1 *in* $\mathcal{A}_{(-\infty,-1]}$, Q_2 *in* $\mathcal{A}_{[0,\infty)}$ *and* $j > 0$.

Central Limit Theorem
Suppose that φ is a translationally invariant factor state. Consider the local fluctuation $Q_{<N>}$ of an observable Q :

$$Q_{<N>} = \frac{1}{\sqrt{2N+1}} \left(\sum_{|j| \leq N} (\tau_j(Q) - \varphi(Q)) \right).$$

If the observable is diagonal, the limit $\lim_N Q_{<N>}$ makes sense as convergence of spectral measures and this is nothing but a classical central limit theorem of mixing systems. The central limit does not converge in neither strong nor weak topology of operators in a Hilbert space. Nevertheless, the following weak limit exists in the GNS space of φ.

$$w - \lim_{N\to\infty} [Q_{<N>}, R_{<N>}] = s(Q, R)1 \tag{1.12}$$

where

$$s(Q, R) = \sum_{k \in \mathbf{Z}} \varphi([\tau_k(Q), R]). \tag{1.13}$$

Set

$$t(Q, R) = \lim_{N\to\infty} \varphi(Q_{<N>}R_{<N>}). \tag{1.14}$$

Definition 1.4 *Let* φ *be a translationally invariant state of* $\mathcal{A}$. *Suppose that* Q *in* $\mathcal{A}$ *is selfadjoint. We say that the central limit theorem holds for* Q *and* φ *if*

$$\lim_{N\to\infty} \varphi(e^{iTQ_{<N>}}) = e^{-\frac{T^2}{2}t(Q,Q)} \tag{1.15}$$

Theorem 1.5 *(i) Let φ be a translationally invariant state . Suppose there exist positive constants $M(j)$ such that*

$$\sum_{j=1}^{\infty} jM(j) < \infty$$

and

$$|\varphi(Q_1\tau_j(Q_2)) - \varphi(Q_1)\varphi(Q_2)| \leq K \|Q_1\| \|Q_2\| M(j)$$

for Q_1 in $\mathcal{A}_{(-\infty,-1]}$, Q_2 in $\mathcal{A}_{[0,\infty)}$ and $j > 0$.

The central limit theorem holds for any selfadjoint strictly local observable Q in $\mathcal{A}_{loc}$. We have convergence of the following correlation functions for local $Q(k)$ ($k = 1, 2, ...r$).

$$\lim_{N\to\infty} \varphi(\prod_{k=1}^{r} e^{iQ(k)_{<N>}}) = e^{-\frac{1}{2}t(\sum_k Q(k),\sum_k Q(k))} e^{-\frac{i}{2}\sum_{k<l} s(Q(k),Q(l))} \tag{1.16}$$

(ii) If the translationally invariant state satisfies the exponential mixing condition (1.11) , (1.16) is valid for for any exponentially localized selfadjoint $Q(k)$ in F_θ.

The convergence of (1.16) was not proved in [12] . Our proof is similar to the case of a single observable. We will give our proof in Section 3.

Time Evolution of the Algebra of Fluctuation

As the Gibbs state φ_β for any finite range interaction satisfies the assumption of Theorem 1.5, we obtain a direct construction of dynamics for the algebra of fluctuation. The set of all selfadjoint exponentially localized elements with rate θ is denoted by F_θ^{real}. Set

$$\mathcal{F} = \cup_{0<\theta<1} \ @F_\theta^{real}.$$

Due to our assumption on decay of correlation,

$$t(Q, Q) = \sum_{k\in\mathbf{Z}} \left\{\varphi_\beta(Q\tau_k(Q)) - \varphi_\beta(Q)^2\right\} \tag{1.17}$$

is finite and non-negative. Then, $t(Q, Q)$ of (1.14) is a (degenerate) positive semidefinite bilinear form. We denote the kernel of t by $\mathcal{N}$.

$$\mathcal{N} = \{Q \in \mathcal{F} \mid t(Q, Q) = 0\}$$

Next we introduce the equivalence relation on $\mathcal{F}$. A and B are equivalent if $A - B$ is in $\mathcal{N}$. By $\tilde{A}$ we denote the equivalence class for A in $\mathcal{F}$. Then $s(A, B)$ of (1.12) gives rise to a non degenerate symplectic form $\tilde{s}(\tilde{A}, \tilde{B})$ on $\tilde{F}$ where

$$\tilde{F} = \mathcal{F}/\mathcal{N} \tag{1.18}$$

and

$$\tilde{s}(\tilde{A}, \tilde{B}) = s(A, B). \tag{1.19}$$

As was already discussed in [7] and [8], the central limit gives rise to the Weyl algebra $W(\tilde{F}, \tilde{s})$ on the symplectic space $\left\{\tilde{F}, \tilde{s}\right\}$.

More precisely, we define $\mathcal{W}(\tilde{F}, \tilde{s})$ by the C^*-algebra generated by unitaries $W(\tilde{Q})$ satisfying

$$W(\tilde{Q_1})W(\tilde{Q_2}) = W(\tilde{Q_1} + \tilde{Q_2})e^{-i1/2s(\tilde{Q_1},\tilde{Q_2})}. \tag{1.20}$$

$\mathcal{W}(\tilde{F}, \tilde{s})$ is called the *algebra of normal (macroscopic) fluctuations*.

When the interaction is of finite range, the global dynamics of $\mathcal{A}$ leaves $\mathcal{F}$ invariant and we can introduce the dynamics $\tilde{\alpha}_t$ on the Weyl algebra $\mathcal{W}(\tilde{F}, \tilde{s})$ via the following equation:

$$\tilde{\alpha}_t(W(\tilde{Q}) = W(\alpha_t\tilde{(Q)}). \tag{1.21}$$

The time evolution $\tilde{\alpha}_t$ has weak continuity in the sense that $\alpha_t\tilde{(Q)}$ is continuous with respect to the topology induce by the bilinear form $t(Q, Q)$. In fact, if we have exponential clustering (1.11) , it is not difficult to show

$$\lim_{u\to 0} t(\alpha_u(Q) - Q, \alpha_u(Q) - Q) = 0. \tag{1.22}$$

Note that the left-hand side of (1.22) is equal to

$$\lim_{t\to 0} \sum_{k\in\mathbf{Z}} \varphi_\beta(\{\alpha_t(Q) - Q, \tau_k(Q)\}) = 0.$$

As was argued in [7], the central limit $\lim_N \varphi(e^{iQ_{<N>}})$ defines a quasifree state $\tilde{\varphi}$ of the algebra of normal fluctuations $\mathcal{W}(\tilde{F}, \tilde{s})$.

We now consider the GNS representation of the algebra of normal fluctuations $\mathcal{W}(\tilde{F}, \tilde{s})$ associated with $\tilde{\varphi}_\beta$. Let $\{\pi(\mathcal{W}(\tilde{F}, \tilde{s})), \Omega, \mathcal{H}\}$ be the GNS triple for $\tilde{\varphi}_\beta$. Let $\mathcal{M}$ be the von Neumann algebra generated by $\pi(\mathcal{W}(\tilde{F}, \tilde{s}))$. As $\tilde{\varphi}_\beta$ is invariant under the time evolution $\tilde{\alpha}_t$ of the algebra of normal fluctuations $\mathcal{W}(\tilde{F}, \tilde{s})$, we have a one parameter group of unitary U_t on $\mathcal{H}$ which implements $\tilde{\alpha}_t$.

$$U_t\pi(Q)U_t^* = \pi(\tilde{\alpha}_t(Q)) \quad , \quad U_t\Omega = \Omega.$$

As a consequence of the continuity (1.22), we see that the adjoint action of U_t gives rise to a weakly continuos one parameter group of automorphisms of $\mathcal{M}$. We denote this dynamics of $\mathcal{M}$ by the same symbol $\tilde{\alpha}_t$:

$$\tilde{\alpha}_t(M) = U_t M U_t^*$$

for M in $\mathcal{M}$. We state our main result of this paper.

Theorem 1.6 *Let φ_β be the unique β-KMS state for a finite range translationally invariant interaction of a one-dimensional quantum spin chain. The quasifree state $\tilde{\varphi}_\beta$ is a β-KMS state for the dynamics $\tilde{\alpha}_t$ of the von Neumann algebra $\mathcal{M}$ at the same inverse temperature β.*

It is easy to derive the KMS boundary condition of $\tilde{\varphi}_\beta$ from the central limit theorem (1.16 for mutually non-commuting observables. Note that the argument of [8] is based on a weaker version CLT i.e. CLT only for strict local observables, which makes arguments more complicated.

We explain the contents of the rest of this paper now. Section 2 is a mathematical preliminary. In Section 3, we present our proof of Theorem 1.5 (1.16) and that of Theorem 1.6.

In Section 4 we make a few comment on unsolved problems. Here, we present our proof of the uniform exponential mixing condition for finitely correlated states. Even though the result is as expected, we are not aware of any proof published elsewhere. Our proof is an simple application of *dual* transfer operator .

2 Localization

We present some features of exponentially localized elements.

Lemma 2.1 *F_θ is complete in the norm topology induced by $|||Q|||_\theta$. Any bounded set of F_θ is a compact subset of $\mathcal{A}$ in norm topology.*

Lemma 2.2 *Let Q be exponentially localized with rate θ. There exists an element $\tilde{Q}_n$ of $\mathcal{A}_{[-n,n]}$ satisfying*

$$\left\| Q - \tilde{Q}_n \right\| \leq 2|||Q|||_\theta \theta^n \ , \ \left\| \tilde{Q}_n \right\| \leq \|Q\| \tag{2.1}$$

If Q is positive we can find a positive $\tilde{Q}_n$ such that

$$0 \leq \inf Q \leq \inf \tilde{Q}_n \leq \left\| \tilde{Q}_n \right\| \leq \|Q\| \tag{2.2}$$

where $\inf Q$ *is the infimum of the spectrum of Q.*

One choice of $\tilde{Q}_n$ is $tr_{(\infty,-n]\cup[n,\infty)}(Q)$ where $tr_{(\infty,-n]\cup[n,\infty)}$ is the partial trace (conditional expectation to $\mathcal{A}_{[-n,n]}$).

Lemma 2.3 *Let Q^1 and Q^2 be exponentially localized, $Q^1, Q^2 \in F_\theta$. Then the following sum is finite.*

$$\sum_{k\in\mathbf{Z}} \left\| \left[\tau_k(Q^1), Q^2 \right] \right\| < \infty \tag{2.3}$$

Proof
Fix Q_n^i in $\mathcal{A}_{[-n,n]}$ satisfying

$$\left\|Q^i - Q_n^i\right\| = \|Q\|^{(n)}.$$

for $i = 1, 2$ and set

$$R_n^i = Q_n^i - Q_{n-1}^i \ (n \geq 1) \quad , \quad R_0^i = Q_0^i.$$

Then

$$\left\|R_n^i\right\| \leq \left\|Q_n^i\right\| + \left\|Q_{n-1}^i\right\| \leq (1+\theta^{-1})\theta^n |||Q^i|||_\theta.$$

Now we have

$$Q^i = \sum_{n=0}^{\infty} R_n^i.$$

As a consequence,

$$\begin{aligned}\sum_{k\in\mathbf{Z}} \|[\tau_k(Q_1), Q_2]\| \quad &\leq \sum_{n,m=0}^{\infty} \sum_{k\in\mathbf{Z}} \left\|\left[\tau_k(R_n^1), R_m^2\right]\right\| \\ &\leq 2 \sum_{n,m=0}^{\infty} (2(m+n)+1) \left\|R_n^1\right\| \left\|R_m^2\right\| \\ &\leq 2 \sum_{n,m=0}^{\infty} (1+\theta^{-1})^2 \theta^n \theta^m |||Q^1|||_\theta |||Q^2|||_\theta < \infty \quad (2.4)\end{aligned}$$

End of Proof

The following result is due to H.Araki in [1].

Proposition 2.4 *Suppose that the interaction is of finite range. For any Q in F_θ, $\alpha_t(Q)$ is entire analytic (analytic in the whole complex plane) as a function of t.*

For any $\beta > 0$ and any θ, there exists a constant $M = M(\beta, \theta, r)$ independent of the interaction ψ such that the following estimate is valid:

$$\|\alpha_z(Q)\|_\theta \leq M F_0(2\,|\beta|\,\|h(\psi)\|) |||Q|||_{\theta e^{4|\beta|\,\|h(\psi)\|}} \qquad (2.5)$$

where any complex number z satisfying $|z| \leq \beta$ and

$$h(\psi) = \sum_{X:0\in X} \frac{\psi(X)}{d(X)},$$

$$F_0(x) = exp[(-r+1)x + 2\sum_{k=1}^{r} k^{-1}(e^{kr}-1)].$$

Thus this proposition tells us that $\cap_{0<\theta<1} F_\theta$ is a dense α_z invariant subalgebra of $\mathcal{A}$.

Next we show that $\cup_{0<\theta<1} F_\theta$ is a dense α_t invariant subalgebra of $\mathcal{A}$.

Proof of Proposition 1.2 (i). The proof of Proposition 6.2.9 of [4]. tells us the following estimate:

$$\left\|\alpha_t(A) - e^{itH_\Lambda} A e^{-itH_\Lambda}\right\| \leq \|A\| \, |\Lambda_0| \, (e^{2|t| \|\psi\|_\lambda} - 1) \sum_{j \in \Lambda^c} e^{-\lambda d(j,\Lambda_0)} \tag{2.6}$$

where A is localized in Λ_0 , λ is any positive constant, $d(j, \Lambda_0)$ is the distance of j from Λ_0 and $\|\psi\|_\lambda$ is a positive constant depending on the interaction ψ and λ .

Now take Q from F_θ and in (2.6),we set $e^{-\lambda} = \theta_0$ where $\theta < \theta_0 < \theta'$. $A = R_m = Q_m - Q_{m-1}$ in $\mathcal{A}_{[-m,m]}$, $\Lambda_0 = [-n, n]$. For n larger than m, we obtain

$$\begin{aligned}
\|\alpha_t(R_m)\|^{(n)} &\leq 2(e^{2|t|\|\psi\|_\lambda} - 1)(2m+1) \|R_m\| \sum_{k=n}^{\infty} \theta_0^{k-m} \\
&\leq 2(e^{2|t|\|\psi\|_\lambda} - 1)(2m+1) \theta^{n-m} \frac{\|R_m\|}{1-\theta_0}
\end{aligned}$$

On the other hand , in general,

$$\|Q\|^{(n)} \leq 2 \|Q\|$$

due to Lemma 2.2. As a consequence, for n smaller than m, $n < m$, we have

$$\|\alpha_t(R_m)\|^{(n)} \leq 2 \|R_m\| \leq 4\theta^n |||Q|||_\theta .$$

Then,

$$\begin{aligned}
&|||\alpha_t(Q)|||_{\theta'} \leq \sum_{m=1}^{\infty} |||\alpha_t(R_m)|||_{\theta'} \\
\leq \quad & \sum_{m=1}^{\infty} \left\{ \sum_{k=0}^{m} \theta'^{-k} \|\alpha_t(R_m)\|^{(k)} + 2(e^{2|t|\|\psi\|_\lambda} - 1) \frac{(2m+1)}{1-\theta_0} \|R_m\| \sum_{k=m+1}^{\infty} \theta'^{-k} \theta_0^{k-m} \right\} \\
\leq \quad & C_1 \sum_{m=1}^{\infty} 4(m+1) \left(\frac{\theta'}{\theta_0}\right)^{-m} \theta_0^{-m} \|R_m\| \\
\leq \quad & C_2 \sum_{m=1}^{\infty} \theta_0{}^{-m} \|R_m\| = C_3 |||Q|||_\theta .
\end{aligned} \tag{2.7}$$

Note that $\sup_m ((2m+1)\theta'^{-m} \theta_0{}^m)$ is finite. *End of Proof*

3 Proof of Theorem 1.5 and 1.6.

We present here our proof for the convergence of two point correlation for (1.16) . We concentrate the case of exponentially localized observables as this is what we need to define the time evolution and KMS states for the algebra of normal fluctuation.

The idea of proof is as follows. Set

$$F^N_{(Q,R)}(T) = \varphi(e^{iTQ_{<N>}} e^{iTR_{<N>}}). \tag{3.1}$$

Instead of showing

$$\lim_{N\to\infty} F_N(Q,R) = e^{-1/2T^2 t(Q+R,Q+R)} e^{-i/2T^2 s(Q,R)}),$$

we prove that the following limit vanishes.

$$\lim_{N\to\infty}\left(\frac{d}{dT}F_N(Q,R)(T) + T(t(Q+R,Q+R)+is(Q,R))F_N(Q,R)(T)\right) = 0. \tag{3.2}$$

Note that the above claim is same as showing the following :

$$\begin{aligned}\lim_{N\to\infty} \quad & \{\varphi((T(it(Q+R,Q+R)-s(Q,R)) - Q_{<N>})e^{iTQ_{<N>}} e^{iTR_{<N>}}) \\ & -\varphi(e^{iTQ_{<N>}} R_{<N>} e^{iTR_{<N>}})\} = 0.\end{aligned}$$

If the second derivative of $F^N_{(Q,R)}(T)$ is bounded in a small neighborhood of the origin uniformly in N , $F^N_{(Q,R)}(T)$ and its derivative are equi-continuous. It turns out that there is at least one accumulation point $F^\infty_{(Q,R)}(T)$ in the sequence of the functions $F_N(Q,R)(T)$ $(N = 1,2,3,...)$. Due to equicontinuity of the first and the second derivative of $F^N_{(Q,R)}(T)$, $F^\infty_{(Q,R)}(T)$ is differentiable. Then (3.2) implies $F^\infty_{(Q,R)}(T)$ satisfies

$$\frac{d}{dT}F^\infty_{(Q,R)}(T) = T(t(Q+R,Q+R) - is(Q,R))F^\infty_{(Q,R)}(T). \tag{3.3}$$

The unique solution to (3.3) is $e^{-1/2T^2 t(Q+R,Q+R)} e^{-i/2T^2 s(Q,R)}$.

Now we begin the proof of the above claim. Without loss of generality we assume that

$$\varphi(Q) = \varphi(R) = 0$$

Suppose that $A = A^*$ $B = B^*$ are selfadjoint bounded operators on a Hilbert space. Then

$$\begin{aligned}&\left\| e^{i(A+B)} - e^{iB}e^{iA} \right\| \leq \| [A, e^{iB}] \| \\ &\| [A, e^{iB}] \| \leq \|[A,B]\|\end{aligned} \tag{3.4}$$

(c.f.[7] or [12].)

Lemma 3.1 *The first and the second derivative of $F^N_{(Q,R)}(T)$ are bounded uniformly in N.*

Proof : The second derivative $F^N_{(Q,R)}(T)$ has the following expression:

$$\begin{aligned}
&-\frac{d^2}{dT^2}F^N_{(Q,R)}(T) \\
&\qquad = \quad \varphi({Q_{<N>}}^2 e^{iTQ_{<N>}} e^{iTR_{<N>}}) + \varphi(e^{iTQ_{<N>}} e^{iTR_{<N>}} {R_{<N>}}^2) \\
&\qquad\quad\ \ +2\varphi(Q_{<N>} e^{iTQ_{<N>}} e^{iTR_{<N>}} R_{<N>}). \qquad (3.5)
\end{aligned}$$

Each term in (3.5) is uniformly bounded in N. We focus on $\varphi({Q_{<N>}}^2 e^{iTQ_{<N>}} e^{iTR_{<N>}})$.

Due to exponential decay of correlation, we have

$$\lim_{N\to\infty} \varphi({Q_{<N>}}^2) = \sum_{k\in\mathbf{Z}} \varphi(Q\tau_k(Q)) < \infty$$

where we used $\varphi(Q) = 0$. Thus we have a constant K_1 such that for any N

$$0 \le \varphi({Q_{<N>}}^2) \le K_1.$$

By use of (3.4) we can show uniform boundedness of the commutator of $Q_{<N>}$ and $e^{iTR_{<N>}}$:

$$\begin{aligned}
\left\| \left[Q_{<N>}, e^{iTR_{<N>}} \right] \right\| &\le T^2 \left\| [Q_{<N>}, R_{<N>}] \right\| \le T^2 \frac{1}{2N+1} \sum_{0\le|a|,|b|\le N} \left\| [Q, \tau_{b-a}(R)] \right\| \\
&\le T^2 \sum_{k\in\mathbf{Z}} \left\| [Q, \tau_k(R)] \right\| \le K_2 < \infty. \qquad (3.6)
\end{aligned}$$

Then,

$$\begin{aligned}
&\left|\varphi({Q_{<N>}}^2 e^{iTQ_{<N>}} e^{iTR_{<N>}})\right| \\
\le\ & \left|\varphi(Q_{<N>} e^{iTQ_{<N>}} e^{iTR_{<N>}} Q_{<N>})\right| + \left|\varphi(Q_{<N>} e^{iTQ_{<N>}} \left[e^{iTR_{<N>}}, Q_{<N>}\right])\right| \\
\le\ & \varphi(Q^2_{<N>})^{1/2} \left(\varphi((e^{iTR_{<N>}} Q_{<N>})^* (e^{iTR_{<N>}} Q_{<N>}))^{1/2} + \left\| \left[Q_{<N>}, e^{iTR_{<N>}} \right] \right\| \right) \\
\le\ & K_1 + (K_1)^{1/2} K_2 < \infty
\end{aligned}$$

By the same estimate,we see that other terms of (3.5) and the first derivative of $F^N_{(Q,R)}(T)$ are uniformly bounded in N. *End of Proof*

Lemma 3.2

$$\lim_{n\to\infty} \varphi((iTs(Q,R) + (e^{iTQ_{<N>}} R_{<N>} e^{-iTQ_{<N>}} - R_{<N>})) e^{iTQ_{<N>}} e^{iTR_{<N>}}) = 0 \qquad (3.7)$$

Proof: First note that

$$\begin{aligned} & \left\| e^{iTQ_{<N>}} R_{<N>} e^{-iTQ_{<N>}} - R_{<N>} - iT\left[Q_{<N>}, R_{<N>}\right] \right\| \\ \leq \quad & CT^2 \left\| \left[Q_{<N>}, \left[Q_{<N>}, R_{<N>}\right]\right] \right\| . \end{aligned} \tag{3.8}$$

The right-hand side of (3.8) is bounded from above by

$$\begin{aligned} & CT^2 \frac{1}{(2N+1)^{3/2}} \sum_{|i|,|j|,|k| \leq N} \left\| \left[\tau_{j-k}(Q), \left[\tau_{i-k}(Q), R\right]\right] \right\| \\ \leq \quad & CT^2 \frac{1}{(2N+1)^{1/2}} \sum_{|i|,|j| \leq 2N} \left\| \left[\tau_j(Q), \left[\tau_i(Q), R\right]\right] \right\| \end{aligned} \tag{3.9}$$

If Q and R are local, the non vanishing terms in the right-hand side of (3.9) are finite:

$$[\tau_j(Q), [\tau_i(Q), R]] = 0$$

for $|i|, |j| \geq r$. When Q and R are exponentially localized, the commutator $[\tau_i(Q), R]$ is also exponentially localized and

$$\left\| [\tau_j(Q), [\tau_i(Q), R]] \right\| \leq C\theta^{|j|+|i|}.$$

Thus

$$\sum_{i,j \in \mathbf{Z}} \left\| [\tau_j(Q), [\tau_i(Q), R]] \right\| < \infty.$$

and we obtain the following convergence:

$$\lim_{N \to \infty} \left\| e^{iTQ_{<N>}} R_{<N>} e^{-iTQ_{<N>}} - R_{<N>} - iT\left[Q_{<N>}, R_{<N>}\right] \right\| = 0. \tag{3.10}$$

Finally we show the following convergence which implies the claim of Lemma 3.2.

$$\lim_{N \to \infty} \varphi((s(Q,R) - [Q_{<N>}, R_{<N>}]) e^{iTQ_{<N>}} e^{iTR_{<N>}}) = 0 \tag{3.11}$$

Then,

$$[Q_{<N>}, R_{<N>}] = \frac{1}{2N+1} \sum_{|i| \leq N} \sum_{|j| \leq N} \tau_i([\tau_{j-i}(Q), R])$$

and

$$\lim_{N \to \infty} \sum_{|j| \leq N} [\tau_{j-i}(Q), R] = \sum_{j \in \mathbf{Z}} [\tau_j(Q), R] . \tag{3.12}$$

Note that the right-hand side of (3.12) converges in the norm topology due to exponential localization. We define A via the following equation:

$$A = \sum_{j \in \mathbf{Z}} [\tau_j(Q), R] .$$

Note that

$$s(Q,R) = \varphi(A).$$

Instead of showing (3.11) , we have only to prove

$$\lim_{N\to\infty} \varphi\left(\left(s(Q,R) - \lim_{N\to\infty} \frac{1}{2N+1}\{\sum_{|j|\le N} \tau_j(A)\} \right) e^{iTQ_{<N>}} e^{iTR_{<N>}} \right) = 0 \quad (3.13)$$

It is not difficult to see that the following identity implies (3.13).

$$\lim_{N\to\infty} \varphi\left(\left(\frac{1}{2N+1} \sum_{|j|\le N} \tau_j(A) - \varphi(A)\right)^2\right) = 0. \quad (3.14)$$

The left-hand side of (3.14) is equal to

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{(2N+1)^2} \sum_{|i|,|j|\le N} \varphi(\tau_{i-j}(A)A) - \varphi(A)^2 \\ &= \lim_{N\to\infty} \frac{1}{(2N+1)} \sum_{|i|\le N} \varphi(\tau_i(A)A) - \varphi(A)^2 = 0. \quad (3.15)\end{aligned}$$

The last identity is due to ergodicity of the state φ. (3.15) implies the claim of Lemma. *End of Proof*

We can find an increasing sequence $m(N)$ of positive integers such that

$$\lim_{N\to\infty} e^{-m(N)}\sqrt{2N+1} = 0\,,\ \lim_{N\to\infty} m(N)^{-1}(2N+1)^{1/4} = \infty. \quad (3.16)$$

We set

$$Q_{j,N} = \sum_{|k|\le N, |k-j|\le m(N)} \tau_k(Q) \quad , \quad \beta_N = \sum_{|k|\le N} 1/2\varphi(\{\tau_k(Q)\ ,\ Q_{k,N}\})$$

$$\overline{Q}_{<N>} = \beta_N^{-1/2} \left\{ \sum_{|k|\le N} \tau_k(Q) \right\} \quad , \quad \overline{Q}_{j,N} = \beta_N^{-1/2} Q_{j,N}.$$

Then $\beta_N = (2N+1)(1+o(1))$ due to summability of two point correlation functions. Thus we have only to consider

$$\varphi(e^{iT\overline{Q}_{<N>}} e^{iT\overline{R}_{<N>}})$$

and its derivative.

Furthermore, in what follows, we assume that $t(Q+R, Q+R) = 1$ for simplicity of exposition. (The case $t(Q+R, Q+R) = 0$ can be handled in the same manner.) We set

$$P = Q + R.$$

Lemma 3.3

$$\lim_{n\to\infty} \varphi((iT - \overline{P}_{<N>})e^{iT\overline{Q}_{<N>}} e^{iT\overline{R}_{<N>}}) = 0 \tag{3.17}$$

Proof: We start with the following identity:

$$\begin{aligned}
& (iT - \overline{P}_{<N>})e^{iT\overline{Q}_{<N>}} e^{iT\overline{R}_{<N>}} \\
= \quad & iT(1 - \beta_N^{-1} \sum_{|k|\le N} \tau_k(P)P_{k,N})e^{iT\overline{Q}_{<N>}} e^{iT\overline{R}_{<N>}} \\
- \quad & \beta_N^{-1/2} \sum_{|k|\le N} \tau_k(P)(1 - e^{-iT\overline{P}_{k,N}} - iT\overline{P}_{k,N})e^{iT\overline{Q}_{<N>}} e^{iT\overline{R}_{<N>}} \\
- \quad & \beta_N^{-1/2} \sum_{|k|\le N} \tau_k(P) \left(e^{-iT\overline{P}_{k,N}} - e^{-iT\overline{Q}_{k,N}} e^{-iT\overline{R}_{k,N}} \right) e^{iT\overline{Q}_{<N>}} e^{iT\overline{R}_{<N>}} \\
- \quad & \beta_N^{-1/2} \sum_{|k|\le N} \tau_k(P) e^{-iT\overline{Q}_{k,N}} e^{-iT\overline{R}_{k,N}} e^{iT\overline{Q}_{<N>}} e^{iT\overline{R}_{<N>}} \\
= \quad & (A_1 + A_2 + A_3)e^{iT\overline{Q}_{<N>}} e^{iT\overline{R}_{<N>}} + A_4.
\end{aligned} \tag{3.18}$$

Using the estimates of proof of Lemma 2.2 in [12] (c.f. (2.8) and (2.10) of [12]) we obtain

$$\lim_{N\to\infty} \varphi(A_a e^{iT\overline{Q}_{<N>}} e^{iT\overline{R}_{<N>}}) = 0$$

for $a = 1, 2$. On the other hand,

$$\begin{aligned}
\|A_3\| \quad & \le \quad \beta_N^{-1/2} \sum_{|k|\le N} \|\tau_k(P)\| \, T^2 \left\| \left[\overline{Q}_{k,N}, \overline{R}_{k,N} \right] \right\| \\
& \le \quad T^2 \beta_N^{-3/2} (2N+1) \, \|P\| \, m(N) \sum_{j\in\mathbf{Z}} \|[\tau_j(Q), R]\|
\end{aligned}$$

When we take N to ∞ we have $\beta_N^{-3/2}(2N+1)m(N) \to 0$. As a result

$$\lim_{N\to\infty} \varphi(A_3 e^{iT\overline{Q}_{<N>}} e^{iT\overline{R}_{<N>}}) = 0.$$

We now look at A_4. We claim that

$$\lim_{N\to\infty} \beta_N^{1/2} \left\| e^{-iT\overline{Q}_{k,N}} e^{-iT\overline{R}_{k,N}} e^{iT\overline{Q}_{<N>}} e^{iT\overline{R}_{<N>}} - e^{iT(\overline{Q}_{<N>} - \overline{Q}_{k,N})} e^{iT(\overline{R}_{<N>} - \overline{R}_{k,N})} \right\| = 0. \tag{3.19}$$

To see this, we point out the following inequalities.

$$\begin{aligned}
& \beta_N^{1/2} \left\| \left[e^{-iT\overline{R}_{k,N}}, e^{iT\overline{Q}_{<N>}} \right] \right\| \le 2T^2 \beta_N^{-1/2} m(N) \sum_{j\in\mathbf{Z}} \|[\tau_j(Q), R]\| \\
& \beta_N^{1/2} \left\| e^{-iT\overline{A}_{k,N}} e^{iT\overline{A}_{<N>}} - e^{iT(\overline{A}_{<N>} - \overline{A}_{k,N})} \right\| \le T^2 \beta_N^{-1/2} m(N) \sum_{j\in\mathbf{Z}} \|[\tau_j(A), A]\|
\end{aligned} \tag{3.20}$$

for $A = P$ or Q. Again for exponentially localized A, $\sum_{j\in\mathbf{Z}} \|[\tau_j(A), A]\|$ is finite. Thus due to choice of $m(N)$ satisfying (3.16) , the right-hand side of (3.20) vanishes in the limit of taking N to ∞.

The convergence (3.19) tell us that

$$-\lim_{N\to\infty} \varphi(A_4) = \lim_{N\to\infty} \beta_N^{-1/2} \sum_{|k|\le N} \varphi(\tau_k(P) e^{iT(\overline{Q}_{<N>} - \overline{Q}_{k,N})} e^{iT(\overline{R}_{<N>} - \overline{R}_{k,N})}).$$

Now we concentrate on the following limit:

$$\lim_{N\to\infty} \varphi(\tau_k(P) e^{iT(\overline{Q}_{<N>} - \overline{Q}_{k,N})} e^{iT(\overline{R}_{<N>} - \overline{R}_{k,N})}). \tag{3.21}$$

We can estimate decay of (3.21) as in [12]. The observable $\tau_k(P)$ is localized around the site k with exponential tail while $e^{iT(\overline{Q}_{<N>} - \overline{Q}_{k,N})} e^{iT(\overline{R}_{<N>} - \overline{R}_{k,N})}$ is localized in $(-\infty, k - m(N)] \cup [k + m(N), \infty)$ We have to approximate Q and R by strictly local elements as in our proof of CLT for non local observables of [12]. (We omit the proof as the estimates is same as that of [12].) We arrive at the following estimate.

$$\left| \varphi(\tau_k(P) e^{iT(\overline{Q}_{<N>} - \overline{Q}_{k,N})} e^{iT(\overline{R}_{<N>} - \overline{R}_{k,N})}) \right| \le C\alpha(m(N))$$

This inequality shows that the limit (3.21) vanishes and the claim of Lemma. *End of Proof*

Theorem 1.5 follows from Lemma 4.3 and Lemma 4.4.

Next we consider the KMS condition. Theorem 1.7. follows easily from Theorem 1.5. in the same manner as [8]. We repeat the argument briefly. As the state $\tilde{\varphi}_\beta$ is quasifree, it suffices to show KMS condition for $\tilde{\varphi}_\beta(W(\tilde{Q}))$. However as CLT is valid only for the selfadjoint part of F_θ we cannot consider the complex time evolution α_z in the central limit. In fact , we can use another equivalent condition to the KMS condition. (c.f. Proposition 5.3.6. of [4].)

Proposition 3.4 *(i) If the state φ is KMS, the time dependent correlation function $F(t) = \varphi(Q_1\alpha_t(Q_2))$ has a analytic extension to the strip $I_\beta = \{z | 0 < Imz < \beta\}$ $F(t)$ is bounded continuous on $\overline{I}_\beta = \{z | 0 \le Imz \le \beta\}$. and*

$$F(t + i\beta) = \varphi(\alpha_t(Q_2)Q_1). \tag{3.22}$$

(ii) Conversely if $F(t)$ admits a analytic extension to I_β and is bounded continuous on $\overline{I}_\beta$ satisfying (3.22)for any Q_1 and Q_2 φ is a KMS state.

Furthermore we have the following bound for $F(z)$ on $\overline{I}_\beta$:

$$|F(z)| \le \|Q_1\| \|Q_2\|. \tag{3.23}$$

Let φ_β be the unique Gibbs state of a finite range interaction. Set

$$F_N(t) = \varphi_\beta(e^{(i(Q_1)_{<N>}}e^{(i\alpha_t(Q_2))_{<N>}})$$

Due to (3.23), for any N we have

$$|F_N(t)| \leq 1.$$

Then we can choose a subsequence of $F_N(t)$ which converges to a bound continuos function $F_\infty(t)$ on $\overline{I}_\beta$ and analytic on I_β. $F_\infty(t)$ satisfies the KMS condition on $W(\tilde{F})$ which implies the same condition on the von Neumann algebra generated by $\pi(W(\tilde{F}))$.

4 Remarks

We include here one example of CLT which was not discussed in [12]. We consider finitely correlated states of M.Fannes,B.Nachtergaele and R.Werner . (c.f. [5])

The finitely correlated state is a non-commutative analogue of the (function of) Markov measure.

Let φ be a state of $\mathcal{A}$. Consider the linear functional φ_Q on $\mathcal{A}_{(-\infty,k]}$ defined by

$$\varphi_Q(R) = \varphi(QR) \qquad \text{for } Q \text{ in } \mathcal{A}_{[k+1,\infty)} \ . \tag{4.1}$$

Set

$$\mathcal{S}_k(\varphi) = \left\{\varphi_Q \mid Q \in \mathcal{A}_{[k+1,\infty)}\right\}. \tag{4.2}$$

Definition 4.1 *φ is a finitely correlated state if $\mathcal{S}_k$ defined in (4.2) is a finite dimensional subspace of the set of all linear functionals on $\mathcal{A}_{(-\infty,k]}$ for any k.*

Proposition 4.2 *Let φ be a translationally invariant finitely correlated state. Suppose that φ is mixing in the following sense:*

$$\lim_{k\to\infty} \varphi(Q_1\tau_k(Q_2)) = \varphi(Q_1)\varphi(Q_2).$$

Then, the uniform exponential mixing (1.10) holds and the central limit theorem is valid.

Note that the exponential decay of two point correlation

$$|\varphi(Q_1\tau_k(Q_2)) - \varphi(Q_1)\varphi(Q_2)| \leq C(Q_1, Q_2)e^{-Mk}$$

is known. What matters here is the constant $C(A, B)$ on Q_1 and Q_2.

Proof: To prove the above proposition, we consider a translationally invariant state

φ of $\mathcal{A}$ and the GNS triple $\{\pi_\varphi(\mathcal{A}), \mathcal{H}_\varphi, \Omega_\varphi\}$ associated with φ. Let U be the unitary on $\mathcal{H}_\varphi$ implementing the shift τ_1.

$$U\pi_\varphi(Q)U^* = \pi_\varphi(\tau_1(Q)) \qquad U\Omega_\varphi = \Omega_\varphi$$

Let P be the projection with the following range :

$$\mathcal{H}_{(-\infty,-1]} = \overline{\pi_\varphi(\mathcal{A}_{(-\infty,-1]})\Omega_\varphi}.$$

As the unitary U^* leaves $\mathcal{H}_{(-\infty,-1]}$ invariant, we have

$$PU^*P = U^*P.$$

Set

$$\mathcal{N} = P\pi_\varphi(\mathcal{A}_{[0,\infty)})P.$$

As τ_1 is implemented by a unitary U on $\mathcal{H}_\varphi$, τ_1 is extended to $\mathcal{N}$ via the formula

$$\tau_1(PQP) = PUQU^*P.$$

We define the completely positive unital map L_{dual} from $\mathcal{N}$ to $\mathcal{N}$ via the following equation:

$$L_{dual}(R) = PURU^*P.$$

where R is an element of $\mathcal{N}$.

We call L_{dual} the dual transfer operator. Consider the vector state ψ_{dual} of $\mathcal{N}$ associated with Ω_φ. ψ_{dual} is faithful as $\mathcal{N}$ commutes with $\mathcal{A}_{(-\infty,-1]}$ and Ω_φ is a cyclic vector for $\mathcal{A}_{(-\infty,-1]}$ in $\mathcal{H}_{(-\infty,-1]}$.

Lemma 4.3 *The dimension of $\mathcal{N}$ is finite if and only if the state φ is finitely correlated.*

Proof: Set

$$\mathcal{K} = P\pi_\varphi(\mathcal{A}_{[0,\infty)})\Omega_\varphi.$$

Note that

$$\varphi_Q(R) = (P\pi_\varphi(Q^*)\Omega_\varphi, \pi_\varphi(R)\Omega_\varphi).$$

By the Schwartz inequality φ_Q is a bounded linear functional on the Hilbert space $\mathcal{H}_{(-\infty,-1]}$ and due to Riesz Lemma the set $\mathcal{S}_0$ is isomorphic to the subspace $\mathcal{K}$ of $\mathcal{H}_{(-\infty,-1]}$ To see the above Lemma, it suffices to notice that the dimension of $\mathcal{K}$ is finite if and only if φ is finitely correlated. *End of Proof*

Due to Lemma 4.3, for a finitely correlated φ, the dual transfer operator L_{dual} is a linear operator on the finite dimensional space $\mathcal{N}$. In fact we can extend the dual transfer operator L_{dual} to a unital completely positive map on the set of bounded linear operators on $\mathcal{K}$. The norm of L_{dual} is one, $\|L_{dual}\| = 1$. Furthermore, due to the mixing property of φ

$$\lim_{N\to\infty} L^n_{dual}(A) = \psi_{dual}(A)1$$

The eigenvalue 1 of the dual transfer operator L_{dual} is non-degenerate and there exists no other peripheral spectrum due to mixing of the state φ. Other eigenvalues of L_{dual} is strictly less than one. (c.f. [5]) Thus there exist constants C and M such that

$$\|L^n_{dual}(A) - \psi_{dual}(A)1\| \leq Ce^{-Mn} \|A\| .$$

Then for Q in $\mathcal{A}_{(-\infty,-1]}$, R in $\mathcal{A}_{[0,\infty)}$ and n positive,

$$\begin{aligned} |\varphi(A\tau_n(B)) - \varphi(A)\varphi(B)| &\leq \|L^n_{dual}(P\pi_\varphi(B)P) - \psi_{dual}(P\pi_\varphi(B)P)1\| \|A\| \\ &\leq Ce^{-Mn} \|A\| \|B\| \end{aligned} \tag{4.3}$$

End of Proof of Proposition 4.2.

In Section 1 we considered Gibbs states of finite range interactions for one-dimensional chains. The formalism can be extended to long range interactions. However, we are unable to prove necessary mixing condition for CLT. To clarify the difficulty in our approach we present the results we can prove by our methods.

First we replace F_θ with suitable non-commutative Hölder continuos functions. In stead of the weight θ^n we take $n^{-\eta}$ with the condition $\eta > 3$ and set

$$\mathcal{F}^{(\eta)} = \left\{ Q \in \mathcal{A} \mid \sum_{n=0}^{\infty} \|Q\|^{(n)} n^\eta < \infty \right\}$$

Assumption 4.4 *The interaction ψ is translationally invariant and we assume two conditions*
(i) The time evolution is well-defined on $\mathcal{A}$.

$$\lim_{N\to\infty} e^{itH_{[-N,N]}} Q e^{-itH_{[-N,N]}} = \alpha_t(Q)$$

(ii) The following limit exists and it gives rise to an element of $\mathcal{F}^{(\eta)}$.

$$\lim_{N\to\infty} e^{\beta/2H_{[0,N]}} e^{-\beta/2H_{[1,N]}} = a_\beta \in \mathcal{F}^{(\eta)} \tag{4.4}$$

Proposition 4.5 *If Assumption 4.4 is valid, the Gibbs state for the inverse temperature β is unique, satisfying the following mixing:*

$$|\varphi(Q_1\tau_n(Q_2)) - \varphi(Q_1)\varphi(Q_2)| \leq Kn^{-(\eta-1)} \|Q_1\| \|Q_2\| \tag{4.5}$$

for Q_1 in $\mathcal{A}_{(-\infty,-1]}$, Q_2 in $\mathcal{A}_{[0,\infty)}$ and $n > 0$.
Thus as a corollary, CLT holds if $\eta > 3$.

Decay of correlation (4.5) can be derived by analysis of Ruelle transfer operators (c.f. [11]), however we have no idea to prove the convergence of (4.4)

So far, we have discussed quantum spin models on one-dimensional lattice only. In higher dimensional lattice we can prove CLT if we have the following mixing for a state φ

$$|\varphi(Q_1 Q_2) - \varphi(Q_1)\varphi(Q_2)| \leq C_{\Lambda_0} e^{-Mk} \|Q_1\| \, \|Q_2\|$$

where the support of Q_1 is Λ_0 and the distance of the support of Q_1 and that of Q_2 is k and the constant C_{Λ_0} is independent of the size of the support of Q_2. We are not certain that this estimate (or the (in)dependence of the constant C_{Λ_0} on the size of the support of Q_2) is valid for high temperature Gibbs states at the moment. It is a non trivial question whether the above estimate can be shown by the high temperature expansion. Note that we do not require the assumption (CLT4) of [7].

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