On Quasifree Representations of Infinite Dimensional Symplectic Group

Taku Matsui and Yoshihito Shimada

Graduate School of Mathematics
Kyushu University
1-10-6 Hakozaki, Fukuoka 812-8581
JAPAN

We consider an infinite dimensional generalization of Metaplectic representations (Weil representations) for the (double covering of) symplectic group. Given quasifree states of an infinite dimensional CCR algebra, projective unitary representations of the infinite dimensional symplectic group are constructed via unitary implementors of Bogoliubov automorphisms. Complete classification of these representations up to quasi-equivalence is obtained.

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`e-mail: matsui@math.kyushu-u.ac.jp, shimada@math.kyushu-u.ac.jp`

1 Introduction

In this paper, we consider unitary representations of an infinite symplectic group. The group $Sp(\infty)$ we deal with here is the set of invertible operators $g = 1 + A$ where g preserves a symplectic form on an infinite dimensional vector space and A is of finite rank. This group is essentially same as the inductive limit of the classical symplectic group in the sense that the latter group is dense in operator norm topology.

First, we present a class of unitary representations of the Lie algebra $sp(\infty)$ on GNS spaces of quasifree states of the CCR (canonical commutation relations) algebra (= the infinite dimensional Heisenberg algebra). The construction of the representation is done in the same fashion as the metaplectic representation (Weil representation) of finite dimensional groups. (c.f. [7]) The representation constructed here is refered to as quasifree represetation. As in the finite dimensional case, our infinitesimal representations give rise to unitary representations of the double covering of $Sp(\infty)$. The infinite dimensional CCR algebra has infinitely many mutually non-equivalent representations, and we obtain a huge number of metaplectic representations of $Sp(\infty)$. This class of representations contains uncountably many irreducible representations as well as non type I factor representations.

The theory of unitary representation for infinite dimensional groups is a field of interplay between Ergodic Theory, the measure theory of infinite dimensional space, operators algebras, and mathematical physics, in particular, quantum field theory. So far two classes of infinite dimensional groups are considered. (1) groups whose matrix elements are functions: Examples are loop groups and the diffeomorphism group of the circle (See [13], [6] and the references therein), and their higher dimensional analogue. (2) inductive limit of classical groups, $O(\infty)$ or $U(\infty)$. See [5], [14] and [16].

So far the construction of unitary representations has been carried out in two ways. One way is to construct measures on an infinite dimensional space quasi-invariant under the group in question ([11]). Another method is to use Fock spaces of the quantum field theory and unitary implementors of Bogoliubov automorphisms. (c.f. [6] and [16]).

Turning into inductive limit $O(\infty)$ and $U(\infty)$ certain representations of these groups are closely connected with the gauge invariant part of CAR (canonical anticommutation relations) algebra. In general the inductive limit procedure of compact groups yields an inductive limit of group C^*-algebras which are approximately finite dimensional. Then there is a one to one correspondence of primitive ideals of the AF algebra and factor representations of the group. The factor representation of $U(\infty)$ constructed on GNS spaces of quasifree states of the CAR algebra corresponds to the U(1) gauge invariant part of the CAR algebra as the quotient by the primitive ideal. (c.f. [16]) In the same manner, spin representations of $O(\infty)$ corresponds to the Z_2 invariant of the CAR algebra. Quasi-equivalence of quasifree states for gauge invariant CAR algebras was investigated in [16], [8] and [9] and these results leads to classification of representations of $O(\infty)$ and $U(\infty)$ on GNS spaces of quasifree states on CAR algebras.

In the same spirit, we can introduce quasifree representation (= Metaplectic or Weil representation) for $Sp(\infty)$ with the aid of quasifree states of the CCR algebra. However, there is a crucial difference. The symplectic group $Sp(N)$ is non compact on one hand, and the CCR algebra is an unbounded operator algebra. The unitary representative of $O(\infty)$ or $U(\infty)$ is an element of the gauge invariant CAR algebra while this is not the case for $Sp(\infty)$. In this sense, it is not correct that classification Metaplectic representations reduces to the representation theory of the Z_2 gauge invariant part of the CCR algebra. Nevertheless we succeeded complete classification of generalized Metaplectic representations on GNS spaces associated with quasifree states of the CCR algebra.

The main result of this paper is Theorem 7.1 where we obtained the complete classification of quasifree representation constructed in the GNS representations associated with quasifree states of the CCR algebra. To achieve our object we found it necessary to use Modular theory of von Neumann algebra for quasifree states of CCR algebras. This machinery was established by H.Araki in [1], [2], [3] and [4].

Next we mention the organization of this paper. In Section 2 and 3 we introduce quasifree states of CCR algebras and Fock spaces in an abstract way.

The infinitesimal quasifree representation of the Lie algebra $sp(\infty)$ is defined in Section4.

If the quasifree state of the CCR algebra is pure, the associated representation of $sp(\infty)$ decomposes into two mutually non-equivalent irreducible representations. This fact is proved in Section 5.

Section 6 is devoted to an analysis of von Neumann algebras generated by $sp(\infty)$. Using results of Section 6 our main result Theorem 7.1 is proved in Section 7.

In the final section we show that our irreducible quasifree representation is extendible to a projective unitary representation of a larger symplectic group $Sp(P,\infty)$ where $Sp(P,\infty)$ is a symplectic transformation commuting with a fixed projection P modulo Hilbert Schimidt class operators. This result is closely connected with another result of D.Pickrell in [14] where he introduced the notion of spherical representations and examined the same extension property.

2 Quasifree Representations of CCR algebra

We briefly sketch GNS representations of CCR algebra associated to quasifree states.

2.1 Definition. *Let K be a complex vector space and $\gamma(f,g)$ be a non-degenerate hermitian form for $f,g \in K$. Let Γ be an antilinear involution satisfying $\Gamma^2 = 1, \gamma(\Gamma f, \Gamma g) = -\gamma(g,f)$. A self-dual CCR algebra $\mathfrak{A}(K,\gamma,\Gamma)$ is a complex *-algebra generated by identity 1 and $\{B(f) \mid f \in K\}$ where $B(f)$ is complex linear in $f \in K$ and satisfies $B(f)^* = B(\Gamma f)$, $B(f)^*B(g) - B(g)B(f)^* = \gamma(f,g)$.*

2.2 Definition. *A state φ on $\mathfrak{A}(K,\gamma,\Gamma)$ is a called quasifree state if*

$$\varphi(B(f_1)\ldots B(f_{2n-1})) = 0,$$
$$\varphi(B(f_1)\ldots B(f_{2n})) = \sum_{\sigma\in\mathfrak{S}}\prod_{j=1}^{n}\varphi(B(f_{\sigma(j)})B(f_{\sigma(n+j)})).$$

$\mathfrak{S}$ is the set of all permutations of $\{1,2,\ldots,n\}$ satisfying

$$\sigma(1) < \sigma(2) < \ldots < \sigma(n),\ \sigma(j) < \sigma(j+n), j = 1,2,\ldots,n.$$

The cardinal number of $\mathfrak{S}$ equals $(2n)!2^{-n}(n!)^{-1}$.

For any quasifree state φ, let

$$S(f,g) := \varphi(B(f)^* B(g)).$$

Then a positive semi-definite hermitian form $S(\cdot,\cdot)$ satisfies

$$S(f,g) - S(\Gamma g, \Gamma f) = \gamma(f,g). \tag{2.1}$$

Conversely, a positive semi-definite hermitian form $S : K \times K \to \mathbf{C}$ satisfying (2.1) is given. Then there exists an unique quasifree state φ_S on $\mathfrak{A}(K,\gamma,\Gamma)$ such that

$$\varphi_S(B(f)^* B(g)) = S(f,g)$$

for all $f, g \in K$. That is to say, a quasifree state is completely specified by a positive semi-definite hermitian form satisfying (2.1). (See [1].)

We define a bounded operator "S" induced by a positive semi-definite hermitian form S satisfying (2.1). Due to the non-degeneracy of γ, a hermitian form

$$(f,g)_S := S(f,g) + S(\Gamma g, \Gamma f)$$

is positive definite. In other words $(\cdot,\cdot)_S$ is an inner product. Let K_S be the completion of K with respect to $(\cdot,\cdot)_S$. Then there exists a bounded operator S on K_S such that

$$S(f,g) = (f, Sg)_S$$

for all $f, g \in K$. Let Γ_S be an antiunitary involution on K_S such that $\Gamma_S f = \Gamma f$ for all $f \in K$. The bounded operator S satisfies $S^* = S$, $\Gamma_S S \Gamma_S = 1 - S$ and $0 \leq S \leq 1$. $\gamma_S := 2S - 1$ satisfies $\gamma(f,g) = (f, \gamma_S g)_S$ for all $f, g \in K$.

If the bounded operator S on K_S induced by a positive semi-definite hermitian form satisfying (2.1) is a projection, we call S a *basis projection.*

Let $(\mathcal{H}_S, \pi_S, \Omega_S)$ be a GNS representation of CCR algebra $\mathfrak{A}(K,\gamma,\Gamma)$ associated to φ_S. The Hilbert space given by GNS construction is abstract, however in case that S is a basis projection, it can be written concretely.

Let L be a Hilbert space and consider the Boson Fock space :

$$\mathcal{F}_{\mathrm{b}}(L) := \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^n L, \quad \otimes_{\mathrm{s}}^0 L := \mathbf{C}, \quad \Psi = 1,$$

$$\langle f_1 \otimes_{\mathrm{s}} \ldots \otimes_{\mathrm{s}} f_n, g_1 \otimes_{\mathrm{s}} \ldots \otimes_{\mathrm{s}} g_m \rangle = \delta_{mn} \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \prod_{j=1}^{n} (f_{\sigma(j)}, g_{\sigma(j)}).$$

$\otimes_{\mathrm{s}}$ is the symmetric tensor product and $\mathfrak{S}_n$ is the set of all permutations of $\{1, 2, \ldots, n\}$. Now we define annihilation operators $b(f), f \in L$ on $\mathcal{F}_{\mathrm{b}}(L)$ as follows :

$$b(f) f_1 \otimes_{\mathrm{s}} \ldots \otimes_{\mathrm{s}} f_n := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (f, f_j) f_1 \otimes_{\mathrm{s}} \ldots \otimes_{\mathrm{s}} f_{j-1} \otimes_{\mathrm{s}} f_{j+1} \ldots \otimes_{\mathrm{s}} f_n,$$
$$b(f)\Psi := 0$$

and creation operators $b^\dagger(f), f \in L$ on $\mathcal{F}_{\mathrm{b}}(L)$ as follows :

$$b^\dagger(f_1) f_2 \otimes_{\mathrm{s}} \ldots \otimes_{\mathrm{s}} f_{n+1} := \sqrt{n+1} f_1 \otimes_{\mathrm{s}} \ldots \otimes_{\mathrm{s}} f_{n+1},$$
$$b^\dagger(f)\Psi := f.$$

2.3 Lemma. *Suppose that $S : K_S \to K_S$ is a basis projection.*

(1) $b^\dagger(f), b(g), f, g \in SK_S$ *are closable operators. Let $\overline{A}$ be the closure of operator A. The finite particle vector subspace of $\mathcal{F}_\mathrm{b}(SK_S)$ is a core for all $\overline{b^\dagger(f)}, \overline{b(g)}, f, g \in SK_S$.*

(2) *Due to* (1)*, we can define the addition and multiplication of creation and annihilation operators on the finite particle vector subspace of $\mathcal{F}_\mathrm{b}(SK_S)$. Let $\mathfrak{A}_{\mathbf{CCR}}(SK_S)$ be a *-algebra generated by all annihilation and creation operators. Let $\alpha(S) : \mathfrak{A}(K, \gamma, \Gamma) \to \mathfrak{A}_{\mathbf{CCR}}(SK_S)$ be a *-homomorphism satisfying the following relation :*

$$\alpha(S)(B(f)) := b^\dagger(Sf) + b(S\Gamma f), f \in K.$$

*Then $(\mathcal{F}_\mathrm{b}(SK_S), \alpha(S), \Psi)$ is a *-representation of CCR algebra $\mathfrak{A}(K, \gamma, \Gamma)$. Moreover, it is unitary equivalent to the GNS representation $(\mathcal{H}_S, \pi_S, \Omega_S)$.*

Proof. (1) See chapter X section 7 of [10]. (2) An unitary operator $u : \mathcal{H}_S \to \mathcal{F}_\mathrm{b}(SK_S)$, $\pi_S(X)\Omega_S \mapsto \alpha(S)(X)\Psi$ satisfies $u^*\alpha(S)(X)u = \pi_S(X)$ for all $X \in \mathfrak{A}(K, \gamma, \Gamma)$. ∎

By Lemma 2.3, we call π_S a *Fock representation* and φ_S a *Fock state* if S is a basis projection. In case that φ_S is a Fock state, we have the following important lemma. (See Lemma 5.4 and 5.5 of [1].)

2.4 Lemma. *Suppose that $S : K_S \to K_S$ is a basis projection.*

(1) $\pi_S(B(f)), f \in \mathrm{Re}K$ *is an essentially self-adjoint operator and set*

$$W_S(f) := \exp\left(i\overline{\pi_S(B(f))}\right).$$

Then $W_S(f)$ satisfies the following relations :

$$W_S(f_1)W_S(f_2) = \exp\left(-\frac{1}{2}\gamma(f_1, f_2)\right) W_S(f_1 + f_2).$$

(2) *If $f \in \mathrm{Re}K_S$, we define $W_S(f)$ via the following limit*

$$W_S(f) := \text{s-}\lim_{n\to\infty} W_S(f_n) \tag{2.2}$$

where $\{f_n\}$ is a sequence in $\mathrm{Re}K$ *satisfying $\|f - f_n\| \to 0$. Note that the limit* (2.2) *does not depend on the choice of $\{f_n\}$.*

(3) *Let $\mathrm{Re}K_S := \{f \in K_S \,|\, \Gamma_S f = f\}$. The restriction of $(\cdot, \cdot)_S$ to $\mathrm{Re}K_S$ is an inner product of $\mathrm{Re}K_S$. $f \mapsto W_S(f)$ is continuous with respect to the norm on $\mathrm{Re}K_S$ and the strong operator topology of bounded operators on $\mathcal{H}_S$.*

(4) *Let L be a subspace of $\mathrm{Re}K_S$. Let $L^\vee$ be the set of vectors $f \in \mathrm{Re}K_S$ such that $(f, \gamma_S g)_S = 0$ for all $g \in L$ and let $\overline{L}$ be the closure of L in $\mathrm{Re}K_S$. Let $\mathcal{R}_S(L)$ be a von Neumann algebra generated by $W_S(f), f \in L$. Then we obtain the following relations :*

(i) $\mathcal{R}_S(L) = \mathcal{R}(\overline{L})$,

(ii) $\mathcal{R}_S(L)' = \mathcal{R}_S(L^\vee)$,

(iii) $\{\mathcal{R}_S(L_1) \cup \mathcal{R}_S(L_2)\}'' = \mathcal{R}_S(L_1 + L_2)$,

(iv) $\mathcal{R}_S(L_1) \cap \mathcal{R}_S(L_2) = \mathcal{R}_S(\overline{L_1} \cap \overline{L_2})$.

We introduce an another hermitian form $\widehat{\gamma}_S$ on $K_S \oplus K_S$ via the following relation :

$$\widehat{\gamma}_S(f_1 \oplus g_1, f_2 \oplus g_2) := (f_1, \gamma_S f_2)_S - (g_1, \gamma_S g_2)_S$$

for all $f_i, g_i \in K_S$. Set $\widehat{\Gamma}_S := \Gamma_S \oplus \Gamma_S$. Then $\widehat{\gamma}_S$ satisfies $\hat{\gamma}_S(\hat{\Gamma}_S h_1, \hat{\Gamma}_S h_2) = -\hat{\gamma}_S(h_1, h_2)$ for all $h_i \in K_S \oplus K_S$.

$$P_S(f_1 \oplus f_2, g_1 \oplus g_2) := (f_1, Sf_2 + \sqrt{S(1-S)} g_2)_S + (g_1, \sqrt{S(1-S)} f_2 + (1-S) g_2)_S$$

is a positive semi-definite hermitian form on $K_S \oplus K_S$ satisfying

$$P_S(h_1, h_2) - P_S(\widehat{\Gamma}_S h_2, \hat{\Gamma}_S h_1) = \hat{\gamma}_S(h_1, h_2).$$

We denote the completion of $K_S \oplus K_S$ with respect to the inner product $(h_1, h_2)_{P_S} := P_S(h_1, h_2) + P_S(\hat{\Gamma}_S h_2, \hat{\Gamma}_S h_1)$ by K_{P_S}.

2.5 Lemma. *The bounded operator P_S on K_{P_S} satisfying $P_S(h_1, h_2) = (h_1, P_S h_2)_{P_S}$ for all $h_i \in K_S \oplus K_S$ is a basis projection.*

Proof. Let $D(\gamma_S^{-1})$ be a domain of γ_S^{-1}. By the non-degeneracy of γ, $D(\gamma_S^{-1})$ is a dense set of K_S. Let $\widehat{\gamma}_S(f_1 \oplus g_1, f_2 \oplus g_2) = (f_1 \oplus g_1, h \oplus k)_{P_S}$. Then $\gamma_S f_2 = h + 2\sqrt{S(1-S)}k$, $-\gamma_S g_2 = 2\sqrt{S(1-S)}h + k$. If $f_2, g_2 \in D(\gamma_S^{-1})$, we have

$$\gamma_{P_S}(f_2 \oplus g_2) = h \oplus k = \gamma_S^{-1}(f_2 + 2\sqrt{S(1-S)} g_2) \oplus -\gamma_S^{-1}(2\sqrt{S(1-S)} f_2 + g_2).$$

By $P_S = \frac{1}{2}(\gamma_{P_S} + 1)$, P_S is written explicitly on $D(\gamma_S^{-1}) \oplus D(\gamma_S^{-1})$ as follows :

$$P_S(f \oplus g) = \gamma_S^{-1}(Sf + \sqrt{S(1-S)} g) \oplus -\gamma_S^{-1}(\sqrt{S(1-S)} f + (1-S) g).$$

It is easily checked that P_S is a projection on K_{P_S}. ∎

Remark. Let L be a dense set of K_S with respect to $(\cdot, \cdot)_S$, then $L \oplus L$ is a dense set of $K_S \oplus K_S$ with respect to $(\cdot, \cdot)_{P_S}$. Indeed, for any $f \oplus g \in K_S \oplus K_S$, there exist $f_n, g_n \in L$ such that $\| f_n - f \|_S, \| g_n - g \|_S \to 0 (n \to \infty)$. By the following equation

$$\| f \oplus g \|_{P_S}^2 = \| \sqrt{S} f + \sqrt{1-S} g \|_S^2 + \| \sqrt{1-S} f + \sqrt{S} g \|_S^2, \tag{2.3}$$

we have $\| (f_n \oplus g_n) - (f \oplus g) \|_{P_S} \to 0 (n \to \infty)$.

By Lemma 2.5, φ_{P_S} is a Fock state on CCR algebra $\mathfrak{A}(K_S \oplus K_S, \widehat{\gamma}_S, \widehat{\Gamma}_S)$. We denote a GNS representation of CCR algebra $\mathfrak{A}(K_S \oplus K_S, \hat{\gamma}_S, \hat{\Gamma}_S)$ associated to φ_{P_S} by $(\mathcal{H}_{P_S}, \pi_{P_S}, \Omega_{P_S})$. The following corollary is a consequence of the direct application of lemma 2.4 to Fock representation $(\mathcal{H}_{P_S}, \pi_{P_S}, \Omega_{P_S})$.

2.6 Corollary. $\mathcal{R}_{P_S}(\mathrm{Re}K_S \oplus 0)' = \mathcal{R}_{P_S}(0 \oplus \mathrm{Re}K_S)$ *and* $\mathcal{R}_{P_S}(\mathrm{Re}K_S \oplus 0)$ *is a factor.*

2.7 Remark. Let $\alpha : \mathfrak{A}(K,\gamma,\Gamma) \to \mathfrak{A}(K_S \oplus K_S, \widehat{\gamma}_S, \widehat{\Gamma}_S)$ be a *-homomorphism defined by $\alpha(B(f)) = B(f\oplus 0)$ and $u_\alpha : \mathcal{H}_S \to \mathcal{H}_{P_S}$ be a linear operator defined by $u_\alpha(\pi_S(A)\Omega_S) = \pi_{P_S}(\alpha(A))\Omega_{P_S}$ for all $A \in \mathfrak{A}(K,\gamma,\Gamma)$. Then u_α preserves the inner product. In fact, since φ_{P_S} and φ_S are quasifree states and

$$\varphi_{P_S}(B(f\oplus 0)^*B(g\oplus 0)) = P_S(f\oplus 0, g\oplus 0) = S(f,g) = \varphi_S(B(f)^*B(g))$$

for all $f, g \in K$, we have $\varphi_{P_S}(\alpha(A)) = \varphi_S(A)$ for all $A \in \mathfrak{A}(K,\gamma,\Gamma)$. If X and Y are elements of $\mathfrak{A}(K,\gamma,\Gamma)$ and set $A := X^*Y$, then

$$\langle u_\alpha \pi_S(X)\Omega_S, u_\alpha\pi_S(Y)\Omega_S\rangle = \varphi_{P_S}(\alpha(X^*Y)) = \varphi_S(X^*Y) = \langle \pi_S(X)\Omega_S, \pi_S(Y)\Omega_S\rangle\,.$$

If we identify $u_\alpha\mathcal{H}_S$ with $\mathcal{H}_S$, $\mathcal{H}_S$ is a closed subspace of $\mathcal{H}_{P_S}$ and $\mathcal{H}_S = \mathcal{F}_{\mathrm{b}}(P_S(K_S\oplus 0))$. Moreover, since $u_\alpha\pi_S(A) = \pi_{P_S}(\alpha(A))u_\alpha$ on $D(\pi_S) := \pi_S(\mathfrak{A}(K,\gamma,\Gamma))\Omega_S$, we can identify $\pi_S(A)$ with $\pi_{P_S}(\alpha(A))|D(\pi_S)$ for all $A \in \mathfrak{A}(K,\gamma,\Gamma)$.

3 Fock Space and Exponential Vectors

In Remark 2.7, $\mathcal{H}_S$ is regarded as a closed subspace of $\mathcal{H}_{P_S}$. We explain the point in detail.

Let L_1, L_2 be Hilbert spaces and $L := L_1 \oplus L_2$ and $e(u) := \sum_{n=0}^\infty (\sqrt{n!})^{-1} \otimes_{\mathrm{s}}^n u$ for all $u \in L$. We call $e(u)$ an *exponential vector*.

3.1 Lemma. If $u_1 \in L_1$ and $u_2 \in L_2$, then there exists an unique unitary operator $U : \mathcal{F}_{\mathrm{b}}(L) \to \mathcal{F}_{\mathrm{b}}(L_1)\otimes\mathcal{F}_{\mathrm{b}}(L_2)$ such that $Ue(u_1+u_2) = e(u_1)\otimes e(u_2)$. This shows

$$\mathcal{F}_{\mathrm{b}}(L) = \mathcal{F}_{\mathrm{b}}(L_1)\otimes\mathcal{F}_{\mathrm{b}}(L_2).$$

(See chapter II section 19 of [12].)

3.2 Lemma. $P_SK_{P_S} = [P_S(K_S\oplus 0)]\oplus[0\oplus E_S(\{0\})K_S] = [P_S(0\oplus K_S)]\oplus[E_S(\{1\})K_S\oplus 0]$ *where $E_S(B)$ is the spectral projection of S for a Borel set $B \subset \mathbf{R}$.*

Proof. The restriction of $(\cdot,\cdot)_{P_S}$ to $P_SK_{P_S}$ is an inner product of $P_SK_{P_S}$. Let $[P_S(K_S\oplus 0)]^\perp$ be the orthogonal complement of $P_S(K_S\oplus 0)$ in P_SK_S. Let $u\oplus v \in [P_S(K_S\oplus 0)]^\perp$, then we have

$$0 = (u\oplus v, P_S(f\oplus 0))_{P_S} = (Su + \sqrt{S(1-S)}v, f)_S$$

for all $f \in K_S$. This implies $\sqrt{S}(\sqrt{S}u + \sqrt{1-S}v) = 0$, i.e. $Su + \sqrt{1-S}v \in E_S(\{0\})K_S$. By $P_S(u\oplus v) = u\oplus v$ and $\gamma_S^{-1} = -1, \sqrt{1-S} = 1, \sqrt{S} = 0$ on $E_S(\{0\})K_S$, we have

$$u\oplus v = P(u\oplus v) = 0\oplus(\sqrt{S}u + \sqrt{1-S}v).$$

Thus $u = 0$, $v \in E_S(\{0\})K_S$. We obtain $[P_S(K_S\oplus 0)]^\perp \subset 0\oplus E_S(\{0\})K_S$. Converse relation is seen from direct computation. ∎

From Lemma 3.1, Lemma 3.2 and Remark 2.7, we obtain the factorization of the Fock space :

$$\mathcal{H}_{P_S} = \mathcal{H}_S\otimes\mathcal{L}_S, \tag{3.1}$$

$\mathcal{L}_S := \mathcal{F}_{\mathrm{b}}(0\oplus E_S(\{0\})K_S)$. In particular, if $0 < S < 1$, then $\mathcal{H}_{P_S} = \mathcal{H}_S$.

Let L be a Hilbert space and

$$\mathcal{F}_{\mathrm{b}}^{+}(L) := \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{2n} L, \quad \mathcal{F}_{\mathrm{b}}^{-}(L) := \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{2n+1} L.$$

We call $\mathcal{F}_{\mathrm{b}}^{+}(L)$ the *even part of the Boson Fock space* $\mathcal{F}_{\mathrm{b}}(L)$ and $\mathcal{F}_{\mathrm{b}}^{-}(L)$ the *odd part of the Boson Fock space* $\mathcal{F}_{\mathrm{b}}(L)$.

3.3 Lemma. *Let*

$$e^{+}(u) := \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{(2n)!}} \otimes_{\mathrm{s}}^{2n} u, \quad e^{-}(u) := \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{(2n+1)!}} \otimes_{\mathrm{s}}^{2n+1} u$$

for all $u \in L$. Let $u_j \in L$, $j = 1, 2, \ldots, N$, $N \in \mathbf{N}$ satisfy $u_i \neq \pm u_j (i \neq j)$. Then $\{e^{\sigma}(u_j)\}_{j=1}^{N}$ is linearly independent. Moreover, $\{e^{\sigma}(u) \,|\, u \in L\}$ generates $\mathcal{F}_{\mathrm{b}}^{\sigma}(L)$ where $\sigma = +$ or $-$.

Proof. First, we prove the linear independence of $\{e^{+}(u_j)\}_{j=1}^{N}$. Let $\sum_{j=1}^{N} \alpha_j e^{+}(u_j) = 0$, $\alpha_j \in \mathbf{C}$. We have $0 = \left\langle e^{+}(x), \sum_{j=1}^{N} \alpha_j e^{+}(u_j) \right\rangle = \sum_{j=1}^{N} \alpha_j \cosh(x, u_j)$ for all $x \in L$. If there exist $i, j (i \neq j)$ such that $(x, u_i)^2 = (x, u_j)^2$ for all $x \in L$, then $u_i = u_j$ or $u_i = -u_j$. Thus there exists $x_0 \in L$ such that $(x_0, u_i)^2 \neq (x_0, u_j)^2$ for all $i, j (i \neq j)$. Let $x = \overline{z} x_0$, $z \in \mathbf{C}$, $\beta_j := (x_0, u_j)$, then we have $\sum_{j=1}^{N} \alpha_j \cosh(\beta_j z) = 0$ for all $z \in \mathbf{C}$. By the $2k$ th differential $\sum_{j=1}^{N} \alpha_j \beta_j^{2k} \cosh(\beta_j z) = 0$, $k = 1, 2, \ldots, N-1$, we have

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \beta_1^2 & \beta_2^2 & \cdots & \beta_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^{2(N-1)} & \beta_2^{(N-1)} & \cdots & \beta_N^{2(N-1)} \end{pmatrix} \begin{pmatrix} \alpha_1 \cosh(\beta_1 z) \\ \alpha_2 \cosh(\beta_2 z) \\ \vdots \\ \alpha_N \cosh(\beta_N z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.2}$$

for all $z \in \mathbf{C}$. Since the matrix of the left hand side of (3.2) is a Vandermonde matrix and $\beta_i^2 \neq \beta_j^2 (i \neq j)$, its determinant does not vanish and we obtain $\alpha_j \cosh(\beta_j z) = 0$ for all $z \in \mathbf{C}$ and j. Therefore, $\alpha_j = 0$ for all j. Linear independence of $\{e^{-}(u_j)\}_{j=1}^{N}$ is verified by putting "sinh" to the place of "cosh" in the above proof.

We prove the second part of this lemma. The case of $\{e^{+}(u) \,|\, u \in L\}$ is the same as the proof of Proposition 19.4 of [12]. Since $\mathcal{F}_{\mathrm{b}}(L)$ is generated by exponential vectors $e(u), u \in L$ and $e(u) = e^{+}(u) \oplus e^{-}(u)$, $\{e^{-}(u) \,|\, u \in L\}$ generates $\mathcal{F}_{\mathrm{b}}^{-}(L)$. ∎

3.4 Lemma. *Let L_1, L_2 be Hilbert spaces. Then there exist unitary operators U_+, U_- such that*

$$U_+ : \mathcal{F}_{\mathrm{b}}^{+}(L_1 \oplus L_2) \to [\mathcal{F}_{\mathrm{b}}^{+}(L_1) \otimes \mathcal{F}_{\mathrm{b}}^{+}(L_2)] \oplus [\mathcal{F}_{\mathrm{b}}^{-}(L_1) \otimes \mathcal{F}_{\mathrm{b}}^{-}(L_2)], \tag{3.3}$$

$$U_+ e^{+}(x_1 \oplus x_2) := [e^{+}(x_1) \otimes e^{+}(x_2)] \oplus [e^{-}(x_1) \otimes e^{-}(x_2)] \tag{3.4}$$

and

$$U_- : \mathcal{F}_{\mathrm{b}}^{-}(L_1 \oplus L_2) \to [\mathcal{F}_{\mathrm{b}}^{+}(L_1) \otimes \mathcal{F}_{\mathrm{b}}^{-}(L_2)] \oplus [\mathcal{F}_{\mathrm{b}}^{-}(L_1) \otimes \mathcal{F}_{\mathrm{b}}^{+}(L_2)], \tag{3.5}$$

$$U_- e^{+}(x_1 \oplus x_2) := [e^{+}(x_1) \otimes e^{-}(x_2)] \oplus [e^{-}(x_1) \otimes e^{+}(x_2)]. \tag{3.6}$$

Proof. Let U be the unitary operator determined by Lemma 3.1. Let $U_\sigma := U|\mathcal{F}_{\mathrm{b}}^\sigma(L_1 \oplus L_2)$ where $\sigma = +$ or $-$. For all $x_1, y_1 \in L_1$ and $x_2, y_2 \in L_2$,

$$\begin{aligned}
\langle U_+ e^+(x_1+x_2), U_+ e^+(y_1+y_2)\rangle &= \left\langle U e^+(x_1+x_2), U e^+(y_1+y_2)\right\rangle \\
&= \left\langle e^+(x_1+x_2), e^+(y_1+y_2)\right\rangle \\
&= \cosh((x_1,y_1)+(x_2,y_2)) \\
&= \cosh(x_1,y_1)\cosh(x_2,y_2) + \sinh(x_1,y_1)\sinh(x_2,y_2) \\
&= \left\langle e^+(x_1), e^+(y_1)\right\rangle \left\langle e^+(x_2), e^+(y_2)\right\rangle \\
&\qquad + \left\langle e^-(x_1), e^-(y_1)\right\rangle \left\langle e^-(x_2), e^-(y_2)\right\rangle \\
&= \big\langle [e^+(x_1)\otimes e^+(x_2)] \oplus [e^-(x_1)\otimes e^-(x_2)], \\
&\qquad [e^+(y_1)\otimes e^+(y_2)] \oplus [e^-(y_1)\otimes e^-(y_2)]\big\rangle
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}_{\mathrm{b}}^+(L_1\oplus L_2) \oplus \mathcal{F}_{\mathrm{b}}^-(L_1\oplus L_2) &= \mathcal{F}_{\mathrm{b}}(L_1\oplus L_2) \\
&= \mathcal{F}_{\mathrm{b}}(L_1)\otimes \mathcal{F}_{\mathrm{b}}(L_2) \\
&= [\mathcal{F}_{\mathrm{b}}^+(L_1)\oplus \mathcal{F}_{\mathrm{b}}^-(L_1)] \otimes [\mathcal{F}_{\mathrm{b}}^+(L_2)\oplus \mathcal{F}_{\mathrm{b}}^-(L_2)] \\
&= [\mathcal{F}_{\mathrm{b}}^+(L_1)\oplus \mathcal{F}_{\mathrm{b}}^+(L_2)] \oplus [\mathcal{F}_{\mathrm{b}}^-(L_1)\oplus \mathcal{F}_{\mathrm{b}}^-(L_2)] \\
&\quad \oplus [\mathcal{F}_{\mathrm{b}}^+(L_1)\oplus \mathcal{F}_{\mathrm{b}}^-(L_2)] \oplus [\mathcal{F}_{\mathrm{b}}^-(L_1)\oplus \mathcal{F}_{\mathrm{b}}^+(L_2)].
\end{aligned}$$

Thus U_+ is the unitary operator satisfying (3.4). It is proved similarly that U_- is the unitary operator satisfying (3.6). ∎

Let $\mathcal{H}_S^\sigma := \mathcal{F}_{\mathrm{b}}^\sigma(P_S(K_S \oplus 0))$, $\mathcal{L}_S^\sigma := \mathcal{F}_{\mathrm{b}}^\sigma(0 \oplus E_S(\{0\})K_S)$, $\mathcal{H}_{P_S}^\sigma := \mathcal{F}_{\mathrm{b}}^\sigma(P_S K_{P_S})$ where $\sigma = +$ or $-$. From the argument to the above, we obtain the following relations.

$$\mathcal{H}_{P_S}^+ = (\mathcal{H}_S^+ \otimes \mathcal{L}_S^+) \oplus (\mathcal{H}_S^- \otimes \mathcal{L}_S^-), \quad \mathcal{H}_{P_S}^- = (\mathcal{H}_S^+ \otimes \mathcal{L}_S^-) \oplus (\mathcal{H}_S^- \otimes \mathcal{L}_S^+). \tag{3.7}$$

4 Quasifree Representations of $sp(\infty)$

Let K be a complex vector space and $\gamma(f,g)$ be a non-degenerate hermitian form for $f, g \in K$. Let Γ be an antilinear involution satisfying $\Gamma^2 = 1$, $\gamma(\Gamma f, \Gamma g) = -\gamma(g, f)$. Then we denote finite rank operators on K satisfying $\Gamma H \Gamma = -H$ and $H^\dagger = H$ by $sp(\infty)$. $H^\dagger$ is defined by $\gamma(H^\dagger f, g) = \gamma(f, Hg)$ for all $f, g \in K$. By the non-degeneracy of γ, $H^\dagger$ is well-defined. We call $H \in sp(\infty)$ a *Hamiltonian.* $sp(\infty)$ is a Lie algebra endowed with the Lie bracket $i[H, H'] := i(HH' - H'H)$.

4.1 Lemma. *Let K_1 be a finite dimensional subspace of K. Then there exists a Γ-invariant finite dimensional subspace $K_1^\#$ such that $K_1 \subset K_1^\#$ and the restriction of γ to $K_1^\#$ is non-degenerate.*

Proof. By the existence of the basis $\{e_j\}_{j=1}^{2k}$ of $K_1^\#$ satisfying

$$\begin{gathered}
\Gamma e_j = e_j, \quad j = 1, 2, \ldots, 2k, \\
\gamma(e_{2j-1}, e_{2j}) = i, \quad j = 1, 2, \ldots, k, \\
\gamma(e_l, e_{l'}) = 0, \ (l, l') \neq (2j-1, 2j), (2j, 2j-1), j = 1, 2, \ldots, k,
\end{gathered}$$

this lemma is proved.(See Lemma 4.1 of [2].) ∎

4.2 Lemma. *For any Hamiltonian $H \in sp(\infty)$, there exists $f_j, g_j \in (HK)^\#$ such that $Hf = \sum_{j=1}^N \gamma(g_j, f) f_j$ for all $f \in K$.*

Lemma 4.2 is verified immediately by using linearly independent vectors $\{e_j\}_{j=1}^{2k}$ of $(HK)^\#$ given by the proof of Lemma 4.1. In fact

$$Hf = i\sum_{j=1}^{2k}\{\gamma(He_{2j}, f)e_{2j-1} - \gamma(He_{2j-1}, f)e_{2j}\}$$

for all $f \in K$.

For any $H \in sp(\infty)$, we can define a second quantization of Hamiltonian, called a *bilinear Hamiltonian*, $q(H)$ as follows.

4.3 Definition. *$H \in sp(\infty)$ satisfies $Hf = \sum_{j=1}^N \gamma(g_j, f) f_j$ for all $f \in K$. Then $q(H) := \frac{1}{2}\sum_{j=1}^N B(f_j)B(g_j)^*$.*

Note that the choice of f_j, g_j is not unique for $H \in sp(\infty)$. However, $q(H)$ is independent of the choice of f_j, g_j, only depends on $H \in sp(\infty)$(Lemma 4.4 of [2]).

q is the map from Hamiltonians to bilinear Hamiltonians and satisfies $i[q(H), q(H')] = q(i[H, H'])$. Thus $q_S := \pi_S \circ q$ is a representation of $sp(\infty)$ on $\mathcal{H}_S$. $\widehat{q}$, the map from Hamiltonians on $K_S \oplus K_S$ to bilinear Hamiltonians in $\mathfrak{A}(K_S \oplus K_S, \widehat{\gamma}_S, \widehat{\Gamma}_S)$, is defined as well as $q : sp(\infty) \to \mathfrak{A}(K, \gamma, \Gamma)$. Let $\widehat{q}_{P_S} := \pi_{P_S} \circ \widehat{q}$, $q_{P_S}(H) := \widehat{q}_{P_S}(H \oplus 0)$, then q_{P_S} is a representation of $sp(\infty)$ on $\mathcal{H}_{P_S}$.

4.4 Definition. (1) *A *-representation $(\mathcal{H}, \pi)$ of $sp(\infty)$ is called a regular representation if the following two conditions hold :*

(i) *$i[\pi(H), \pi(H')] = \pi(i[H, H'])$ on a dense set $\mathcal{H}_0$ of Hilbert space $\mathcal{H}$ for all $H, H' \in sp(\infty)$.*

(ii) *$\pi(H)$ is an essentially self-adjoint operator on $\mathcal{H}_0$ for all $H \in sp(\infty)$.*

(2) *Let $(\mathcal{H}_j, \pi_j)$, $j = 1, 2$ be regular representations of $sp(\infty)$ and $\mathcal{M}_j$ denotes a von Neumann algebra generated by $\exp(i\overline{\pi_j(H)}), H \in sp(\infty)$. Then two representations, $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$, are quasi-equivalent if there exists a *-isomorphism of von Neumann algebras $\iota : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\iota(\exp(i\overline{\pi_1(H)})) = \exp(i\overline{\pi_2(H)})$ for all $H \in sp(\infty)$. Then we write $\pi_1 \sim_q \pi_2$. Moreover, if π_1 and π_2 are unitary equivalent, then we write $\pi_1 \sim \pi_2$ simply.*

Since all $q_S(H)$ are essentially self-adjoint operators on $D(\pi_S) := \pi_S(\mathfrak{A}(K, \gamma, \Gamma))\Omega_S$, $(\mathcal{H}_S, q_S)$ is a regular representation of $sp(\infty)$. Similarly, since all $q_{P_S}(H)$ are essentially self-adjoint operators on $D(\pi_{P_S}) := \pi_{P_S}(\mathfrak{A}(K_S \oplus K_S, \widehat{\gamma}_S, \widehat{\Gamma}_S))\Omega_{P_S}$, $(\mathcal{H}_{P_S}, q_{P_S})$ is a regular representation of of $sp(\infty)$. Let

$$Q_S(H) := \exp\left(i\overline{q_S(H)}\right), \quad \widehat{Q}_{P_S}(\widehat{H}) := \exp\left(i\overline{\widehat{q}_{P_S}(\widehat{H})}\right), \quad Q_{P_S}(H) := \widehat{Q}_{P_S}(H \oplus 0)$$

and

$$\mathcal{M}_S := \{Q_S(H) \,|\, H \in sp(\infty)\}'', \quad \mathcal{M}_{P_S} := \{Q_{P_S}(H) \,|\, H \in sp(\infty)\}''.$$

4.5 Lemma. $Q_{P_S}(H) = Q_S(H) \otimes 1_{\mathcal{L}_S}$ *where* $1_{\mathcal{L}_S}$ *is the identity operator on* $\mathcal{L}_S$.

Proof. Due to Remark 2.7, we obtain

$$u_\alpha Q_S(H) = Q_{P_S}(H) u_\alpha \tag{4.1}$$

for all $H \in sp(\infty)$. By (3.1) and (4.1), this lemma has been verified. ∎

Remark. The following relation is verified quite similar to Lemma 4.5 :

$$W_{P_S}(f \oplus 0) = W_S(f) \otimes 1_{\mathcal{L}_S} \tag{4.2}$$

where $W_S(f) := W_{P_S}(f \oplus 0)|\mathcal{H}_S$ for all $f \in \mathrm{Re}K_S$ and $1_{\mathcal{L}_S}$ is the identity operator on $\mathcal{L}_S$.

In Lemma 5.1 of [2], the following relation has been proved : If P is a basis projection on K, then

$$Q_P(H) W_P(f) Q_P(H)^* = W_P(e^{iH} f) \tag{4.3}$$

for all $f \in \mathrm{Re}K$ and $H \in sp(\infty)$. Due to Lemma 4.5 and (4.2), the relation (4.3) implies the following result : For the general S, we have

$$Q_S(H) W_S(f) Q_S(H)^* = W_S(e^{iH} f) \tag{4.4}$$

for all $f \in \mathrm{Re}K$ and $H \in sp(\infty)$.

Since $Q_S(H)\mathcal{H}_S^\sigma \subset \mathcal{H}_S^\sigma$ and $Q_{P_S}(H)\mathcal{H}_{P_S}^\sigma \subset \mathcal{H}_{P_S}^\sigma$ where $\sigma = +$ or $-$,

$$Q_S^\sigma(H) := Q_S(H)|\mathcal{H}_S^\sigma, \quad Q_{P_S}^\sigma(H) := Q_{P_S}(H)|\mathcal{H}_{P_S}^\sigma$$

are bounded operators on $\mathcal{H}_S^\sigma$ and $\mathcal{H}_{P_S}^\sigma$. We denote the restriction of q_S (resp. q_{P_S}) to $\mathcal{H}_S^\sigma$ (resp. $\mathcal{H}_{P_S}^\sigma$) by q_S^σ (resp. $q_{P_S}^\sigma$) and let

$$\mathcal{M}_S^\sigma := \{Q_S^\sigma(H) \,|\, H \in sp(\infty)\}'', \quad \mathcal{M}_{P_S}^\sigma := \{Q_{P_S}^\sigma(H) \,|\, H \in sp(\infty)\}''.$$

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. Then a bounded operator A on $\mathcal{H}_1 \oplus \mathcal{H}_2$ is written in the form of matrix like this :

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \; A_{ij} : \mathcal{H}_j \to \mathcal{H}_i.$$

We can verify the next lemma immediately.

4.6 Lemma. *For all* $H \in sp(\infty)$,

$$Q_{P_S}^+(H) = \begin{pmatrix} Q_S^+(H) \otimes 1_+ & 0 \\ 0 & Q_S^-(H) \otimes 1_- \end{pmatrix},$$
$$Q_{P_S}^-(H) = \begin{pmatrix} Q_S^+(H) \otimes 1_- & 0 \\ 0 & Q_S^-(H) \otimes 1_+ \end{pmatrix}$$

where 1_σ *is the identity operator on* $\mathcal{L}_S^\sigma$.

If S is a basis projection, we call q_S^σ a Fock representation of $sp(\infty)$ on $\mathcal{H}_S^\sigma$.

5 Structure of Fock Representations of $sp(\infty)$

In this section, we assume that S is a basis projection.

5.1 Lemma. *Assume that $K_S = K$. Let $\mathcal{E}_S$ be the set of all C.O.N.S. of SK.*

(1) *Fix $e = \{e_n\}_{n\in\mathbf{N}} \in \mathcal{E}_S$. Then $\{e_n, \Gamma e_n\}_{n\in\mathbf{N}}$ is a C.O.N.S. of K.*

(2) *For $g, h \in K$,*

$$H_{gh}f := \gamma(g,f)h + \gamma(h,f)g + \gamma(\Gamma g, f)\Gamma h + \gamma(\Gamma h, f)\Gamma g, \quad f \in K.$$

H_{gh} satisfies $\Gamma H_{gh}\Gamma = -H_{gh}$, $H_{gh}^\dagger = H_{gh}$. Let

$$H(e;1,k,l) := H_{e_k,e_l}, \quad H(e;2,k,l) := H_{e_k,\Gamma e_l},$$
$$H(e;3,k,l) := H_{ie_k,e_l}, \quad H(e;4,k,l) := H_{ie_k,\Gamma e_l}.$$

Then $sp(\infty)$ is real-linearly spanned by $H(e;j,k,l)$, $1 \le j \le 4$, $k,l \in \mathbf{N}$, $e \in \mathcal{E}_S$. Moreover,

$$q(H(e;1,k,l)) = B(e_k)B(e_l)^* + B(e_l)B(e_k)^* + \delta_{kl}, \tag{5.1}$$
$$q(H(e;2,k,l)) = B(e_k)^*B(e_l)^* + B(e_k)B(e_l), \tag{5.2}$$
$$q(H(e;3,k,l)) = i[B(e_k)B(e_l)^* - B(e_l)B(e_k)^*], \tag{5.3}$$
$$q(H(e;4,k,l)) = i[B(e_k)B(e_l) - B(e_k)^*B(e_l)^*], \tag{5.4}$$

Proof. (1) obvious. (2) For all $H \in sp(\infty)$, there exist $f_j, g_j \in (HK)^\#$ such that $Hf = \sum_{j=1}^N \gamma(g_j,f)f_j$. Let $\{e_n\}_{n=1}^M$ be a C.O.N.S. of $S(HK)^\#$. Then

$$x = \sum_{n=1}^M (e_n,x)_S e_n + (\Gamma e_n, x)_S \Gamma e_n = \sum_{n=1}^M \gamma(e_n,x)e_n - \gamma(\Gamma e_n, x)\Gamma e_n$$

for all $x \in (HK)^\#$. In particular, let $x = f_j, g_j$, then we have

$$\begin{aligned} Hf &= \sum_{j=1}^N \gamma(g_j,f)f_j \\ &= \sum_{k,l=1}^M \{\alpha(1,k,l)\gamma(e_k,f)e_l - \alpha(2,k,l)\gamma(\Gamma e_k,f)e_l \\ &\quad + \alpha(3,k,l)\gamma(\Gamma e_k,f)\Gamma e_l - \alpha(4,k,l)\gamma(e_k,f)\Gamma e_l\}. \end{aligned}$$

$\alpha(j,k,l)$ is defined as follows :

$$\alpha(1,k,l) := \textstyle\sum_{j=1}^N \overline{\gamma(e_k,g_j)}\gamma(e_l,f_j),$$
$$\alpha(2,k,l) := \textstyle\sum_{j=1}^N \overline{\gamma(\Gamma e_k,g_j)}\gamma(e_l,f_j),$$
$$\alpha(3,k,l) := \textstyle\sum_{j=1}^N \overline{\gamma(\Gamma e_k,g_j)}\gamma(\Gamma e_l,f_j),$$
$$\alpha(4,k,l) := \textstyle\sum_{j=1}^N \overline{\gamma(e_k,g_j)}\gamma(\Gamma e_l,f_j).$$

By $\gamma(Hf,g)=\gamma(f,Hg)$ for all $f,g\in K$, $\alpha(j,k,l)$ satisfies

$$\overline{\alpha(4,k,l)}=\alpha(2,l,k), \tag{5.5}$$
$$\overline{\alpha(1,k,l)}=\alpha(1,l,k), \tag{5.6}$$
$$\overline{\alpha(3,k,l)}=\alpha(3,l,k). \tag{5.7}$$

In fact, if $f=e_k$ and $g=\Gamma e_l$, then we have (5.5), if $f=e_k$ and $g=e_l$, we have (5.6), if $f=\Gamma e_k$ and $g=\Gamma e_l$, we have (5.7).

On the other hand, by $\Gamma H\Gamma=-H$, $\alpha(i,j,k)$ satisfies

$$\overline{\alpha(3,k,l)}=\alpha(1,k,l), \tag{5.8}$$
$$\overline{\alpha(4,k,l)}=\alpha(2,k,l). \tag{5.9}$$

Now let e denote a C.O.N.S. of SK satisfying $e\supset\{e_n\}_{n=1}^M$. By (5.5) $-$ (5.9), we have

$$H=\sum_{1\le k<l\le M}H(e;k,l)+\frac{1}{2}\sum_{1\le k\le M}H(e;k,k),$$
$$\begin{aligned}H(e;k,l):=&\operatorname{Re}[\alpha(1,k,l)]H(e;1,k,l)-\operatorname{Im}[\alpha(1,k,l)]H(e;3,k,l)\\&-\operatorname{Re}[\alpha(2,k,l)]H(e;2,k,l)-\operatorname{Im}[\alpha(2,k,l)]H(e;4,k,l).\end{aligned}$$

Thus $sp(\infty)$ is real-linearly spanned by all $H(e;j,k,l)$. ∎

5.2 Lemma. (1) *Ω_S is a cyclic vector for $\mathcal{M}_S^+$.*

(2) *For any $e_1\in SK$, $\pi_S(B(e_1))\Omega_S$ is a cyclic vector for $\mathcal{M}_S^-$.*

Proof. (1) Fix $e=\{e_n\}_{n=1}^\infty\in\mathcal{E}_S$. By (5.2) and (5.4), we have

$$q_S(H(e;2,k,l))+i^{-1}q_S(H(e;4,k,l))=2\pi_S(B(e_k)B(e_l))$$

for all k,l. Since $\mathcal{H}_S^+$ is linearly spanned by elements $\prod_{k,l}\pi_S(B(e_k)B(e_l))\Omega_S$, Ω_S is a cyclic vector for $\mathcal{M}_S^+$.

(2) Fix $e:=\{e_n\}_{n=1}^\infty\in\mathcal{E}_S$. By (5.1) and (5.3), we have

$$q_S(H(e;1,k,l))+i^{-1}q_S(H(e;3,k,l))=2\pi_S(B(e_k)B(e_l)^*),\ k\neq l.$$

Since

$$\{q_S(H(e;1,k,1))+i^{-1}q_S(H(e;3,k,1))\}\pi_S(B(e_1))\Omega_S=\pi_S(B(e_k))\Omega_S\ (k\ge 2),$$

SK is generated by $\pi_S(B(e_1))\Omega_S$ and all $q_S(H(e;1,k,1))$, $q_S(H(e;3,k,1))$. Therefore $\pi_S(B(e_1))\Omega_S$ is a cyclic vector for $\mathcal{M}_S^-$. ∎

5.3 Lemma. (1) *Let*

$$D(N_S):=\left\{\xi=\oplus_{n=0}^\infty\xi^{(n)}\in\mathcal{H}_S\,\middle|\,\xi^{(n)}\in\otimes_s^nSK_S,\ \sum_{n=1}^\infty n^2\parallel\xi^{(n)}\parallel^2<\infty\right\},$$
$$(N_S\xi)^{(n)}:=n\xi^{(n)},\quad \xi\in D(N_S).$$

Then the number operator N_S is an essentially self-adjoint operator on the finite particle vector subspace of $\mathcal{H}_S$.

(2) $\sigma(N_S) = \sigma_{\mathrm{p}}(N_S) = \mathbf{N} \cup \{0\}$.

(3) $e^{i\overline{N_S}} \in \mathcal{M}_S$.

Proof. (1) and (2) are well-known facts.

(3) Fix $e = \{e_n\} \in \mathcal{E}_S$. For $\Lambda \subset \mathbf{N}$, $\#\Lambda < \infty$, we define $N_\Lambda := \sum_{j\in\Lambda} \pi_S(B(e_j)B(e_j)^*)$. Since $N_\Lambda + \frac{1}{2}\#\Lambda = \frac{1}{2}\sum_{k\in\Lambda} q_S(H(e;1,k,k))$, $N_\Lambda + \frac{1}{2}\#\Lambda$ is an essentially self-adjoint operator on the finite particle vector subspace of $\mathcal{H}_S$ and $\exp(i\overline{N_\Lambda}) \in \mathcal{M}_S$. Since

$$N_\Lambda \pi_S(B(e_{\lambda_1})\dots B(e_{\lambda_n}))\Omega_S = \sum_{j\in\Lambda} \#\{k \,|\, e_{\lambda_k} = e_j\}\pi_S(B(e_{\lambda_1})\dots B(e_{\lambda_n}))\Omega_S,$$

we have

$$\begin{aligned}
&\exp(i\overline{N_\Lambda})\pi_S(B(e_{\lambda_1})\dots B(e_{\lambda_n}))\Omega_S \\
&\quad = \exp\left(i\sum_{j\in\Lambda} \#\{k \,|\, e_{\lambda_k} = e_j\}\right)\pi_S(B(e_{\lambda_1})\dots B(e_{\lambda_n})\Omega_S.
\end{aligned}$$

If $\Lambda \to \mathbf{N}\cup\{0\}$, then $\sum_{j\in\Lambda} \#\{k \,|\, e_{\lambda_k} = e_j\} \to n$. Thus

$$\begin{aligned}
\text{s-}\lim_{\Lambda\to\mathbf{N}\cup\{0\}} \exp(i\overline{N_\Lambda})\pi_S(B(e_{\lambda_1})\dots B(e_{\lambda_n}))\Omega_S &= \exp(in)\pi_S(B(e_{\lambda_1})\dots B(e_{\lambda_n}))\Omega_S \\
&= \exp(i\overline{N_S})\pi_S(B(e_{\lambda_1})\dots B(e_{\lambda_n}))\Omega_S.
\end{aligned}$$

This shows that $\exp(i\overline{N_S}) = \text{s-}\lim_{\Lambda\to\mathbf{N}\cup\{0\}} \exp(i\overline{N_\Lambda}) \in \mathcal{M}_S$. ∎

5.4 Lemma. *$\mathcal{M}_S^\sigma$ acts irreducibly on $\mathcal{H}_S^\sigma$.*

Proof. Assume that $\mathcal{H}_0$ is a non-zero intersection of $\mathcal{M}_S^+$-invariant closed subspace of $\mathcal{H}_S^+$ and the finite particle vector subspace of $\mathcal{H}_S^+$. The spectrum $\sigma(N_S|\mathcal{H}_0)$ is contained in $\sigma(N_S|\mathcal{H}_S^+) = 2\mathbf{N}\cup\{0\}$ due to the invariance of $\mathcal{H}_0$ under $\mathcal{M}_S^+$. Let $\lambda_0 := \min\sigma(N_S|\mathcal{H}_0)$.

If $\lambda_0 = 0$, we have $\Omega_S \in \mathcal{H}_0$. Thus $\overline{\mathcal{H}_0} \supset \mathcal{M}_S^+\overline{\mathcal{H}_0} \supset \mathcal{M}_S^+\Omega_S = \mathcal{H}_S^+$. This shows that $\mathcal{M}_S^+$ acts irreducibly on $\mathcal{H}_S^+$.

If $\lambda_0 = 2$, there exists $\xi \in \mathcal{H}_0$ such that $N_S\xi = 2\xi$, $\xi \in \otimes_{\mathrm{s}}^2 SK$. Then we can show that there exist $j, k \in \mathbf{N}$ such that $\pi_S(B(e_j)^*)\pi_S(B(e_k)^*)\xi$ is a non-zero element of $\mathcal{H}_0$. Since

$$\pi_S(B(e_j)^*B(e_k)^*)N_S = N_S\pi_S(B(e_j)^*B(e_k)^*) + 2\pi_S(B(e_j)^*B(e_k)^*),$$

we have

$$\begin{aligned}
N_S\pi_S(B(e_j)^*B(e_k)^*)\xi &= \{\pi_S(B(e_j)^*B(e_k)^*)N_S - 2\pi_S(B(e_j)^*B(e_k)^*)\}\xi \\
&= \pi_S(B(e_j)^*B(e_k)^*)\cdot 2\xi - 2\pi_S(B(e_j)^*B(e_k)^*\xi \\
&= 0.
\end{aligned}$$

This shows $\pi_S(B(e_j)^*B(e_k)^*)\xi \in \mathbf{C}\Omega_S$ and we have $\lambda_0 = 0$. This is contradiction. Thus $\lambda_0 \neq 2$. Moreover, $\lambda_0 \neq 2n$, $n = 1, 2, \dots$ is proved by induction.

The case of $\mathcal{M}_S^-$ is quite similar. ∎

5.5 Lemma. *Two representations q_S^+ and q_S^- are not quasi-equivalent.*

Proof. By the irreducibility of $\mathcal{M}_S^\sigma$, it suffices that q_S^+ and q_S^- are not unitary equivalent. Suppose that there exists an unitary operator $V : \mathcal{H}_S^+ \to \mathcal{H}_S^-$ such that $VQ_S^+(H) = Q_S^-(H)V$ for all $H \in sp(\infty)$. Since the left hand side of $VQ_S^+(tH)\Omega_S = Q_S^-(tH)V\Omega_S$, $t \in \mathbf{R}$ is differentiable at $t \in \mathbf{R}$, the differential of the right hand side make sense and the domain of $\overline{q_S^-(H)}$ contains $V\Omega_S$. Now let $H = H(e;1,k,l)$ and $H(e;3,k,l)$, then we have $\pi_S(B(e_k)B(e_l)^*)V\Omega_S = V\pi_S(B(e_k)B(e_l)^*)\Omega_S = 0$ for all k,l. Therefore, since $\pi_S(B(e_l)^*)V\Omega_S = 0$ for all l, we obtain $V\Omega_S = 0$. This is contradiction. ∎

6 Structure of non-Fock Representations of $sp(\infty)$

In this section, we assume that S is not a basis projection.

Lemma 6.1 is a well-known fact.

6.1 Lemma. *Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space of $\mathcal{H}$ and $E \in \mathcal{M}'$ be a projection. Let $C(\mathcal{M}) := \mathcal{M} \cap \mathcal{M}'$ and $\mathcal{M}_E = \{Q_E \,|\, Q \in \mathcal{M}\}$, $Q_E := Q|E\mathcal{H}$. Then the map $\iota : \mathcal{M} \ni Q \mapsto Q_E \in \mathcal{M}_E$ is a *-homomorphism and continuous with respect to the strong operator topologies. Moreover ι is a *-isomorphism if and only if $C(E) := \min\{F \in C(\mathcal{M}) \,|\, F \geq E, F^2 = F^* = F\} = 1$.*

6.2 Lemma. *Let E_+ be a projection of $\mathcal{H}_S$ on $\mathcal{H}_S^+$. (E_+ is a element of $\mathcal{M}'_S$.) Then $C(E_+) = 1$. Therefore, q_S, q_S^+ and q_S^- are all quasi-equivalent.*

Proof. By Lemma 6.1, we prove that $C(E_+)$ is the identity operator on $\mathcal{H}_S$. Since $C(E_+)|\mathcal{H}_S^+$ is the identity operator on $\mathcal{H}_S^+$, we have only to show that $C(E_+)|\mathcal{H}_S^-$ is the identity operator on $\mathcal{H}_S^-$. We claim that

$$C(E_+)\pi_{P_S}(B(g \oplus 0))\Omega_{P_S} = \pi_{P_S}(B(g \oplus 0))\Omega_{P_S} \tag{6.1}$$

for all $g \in K_S$.

(i) The case of $g \in E_S((0,1))K_S$.

Let $g' := \sqrt{(1-S)^{-1}S}g$ for $g \in E_S((0,1))K_S \cap D((1-S)^{-1})$. ($D(A)$ is the domain of operator A.) g' is an element of $E_S((0,1))K_S$ and satisfies $P_S(g \oplus 0) = P_S(0 \oplus g')$ and $\pi_{P_S}(B(g \oplus 0))\Omega_{P_S} = \pi_{P_S}(B(0 \oplus g'))\Omega_{P_S}$. Thus it suffices that

$$C(E_+)\pi_{P_S}(B(0 \oplus g'))\Omega_{P_S} = \pi_{P_S}(B(0 \oplus g'))\Omega_{P_S} \tag{6.2}$$

for all $g' \in E_S((0,1))K_S$. Now since $[q(H \oplus 0), B(0 \oplus g')] = 0$ for all $H \in sp(\infty)$, $C(E_+) \in \mathcal{M}_S$ "*commutes*" with $\pi_{P_S}(B(0 \oplus g'))|\mathcal{H}_S \cap D(\pi_{P_S})$. (If a bounded operator Q on a Hilbert space $\mathcal{H}$ and (unbounded) operator A on a Hilbert space $\mathcal{H}$ with a dense domain $D(A)$ satisfy $\langle QA\eta, \xi\rangle = \langle Q\eta, A^*\xi\rangle$ for all $\eta, \xi \in D(A)$, it is said that A commutes with Q.) Thus we obtain (6.2).

(ii) The case of $g \in E_S(\{1\})K_S$.

Let f be an unit vector in $E_S((0,1))K_S$. Then

$$\begin{aligned}
&\pi_{P_S}(B(g \oplus 0))\Omega_{P_S} \\
&\quad = \pi_{P_S}(B(g \oplus 0))\pi_{P_S}(B(P_S(f \oplus 0))^* B(P_S(f \oplus 0)))\Omega_{P_S} \\
&\quad = \pi_{P_S}(B(g \oplus 0))\pi_{P_S}(B(P_S(f \oplus 0))^* B(f \oplus 0))\Omega_{P_S} \\
&\quad = \pi_{P_S}(B(g \oplus 0)B(\Gamma_S\gamma_S^{-1}Sf \oplus 0)\pi_{P_S}(B(f \oplus 0))\Omega_{P_S} \\
&\qquad + \pi_{P_S}(B(g \oplus 0)B(f \oplus 0))\pi_{P_S}(B(0 \oplus \Gamma_S(-\gamma_S^{-1})2\sqrt{S(1-S)}f))\Omega_{P_S}.
\end{aligned}$$

Since $\pi_{P_S}(B(g\oplus 0)B(\Gamma_S\gamma_S^{-1}Sf\oplus 0))|\mathcal{H}_S\cap D(\pi_{P_S})$ and $\pi_S(B(g\oplus 0)B(f\oplus 0))|\mathcal{H}_S\cap D(\pi_{P_S})$ commute with $C(E_+)$ and $\Gamma_S(-\gamma_S^{-1})\sqrt{S(1-S)}f$ is an element of $E_S((0,1))K_S$, we can use the case (i) and obtain (6.1). ∎

6.3 Corollary. *q_S^σ and $q_{P_S}^\sigma$ are quasi-equivalent.*

Proof. Ω_{P_S} is a separating vector for $C(\mathcal{M}_{P_S}^+)$. Indeed, let

$$\begin{pmatrix} a_+\otimes 1_+ & 0 \\ 0 & a_-\otimes 1_- \end{pmatrix}\Omega_{P_S}=0,\ a_\sigma\in C(\mathcal{M}_S^\sigma),$$

then $a_+\Omega_S=0$. Since Ω_S is a separating vector for $C(\mathcal{M}_S^+)$, we have $a_+=0$. Moreover, by $q_S^+\sim_q q_S^-$(Lemma 6.2), we have $a_-=0$ as well.

Let $\widehat{E}_+:\mathcal{H}_{P_S}^+\to\mathcal{H}_S^+$ be a projection and $C(\widehat{E}_+):=\min\{E\in C(\mathcal{M}_{P_S}^+)\,|\,E^2=E^*=E, E\geq\widehat{E}_+\}$. Since $C(\widehat{E}_+)\Omega_{P_S}=\Omega_{P_S}$ and Ω_{P_S} is a separating vector for $C(\mathcal{M}_{P_S}^+)$, we have $C(\widehat{E}_+)=1$, that is, $q_S^+\sim_q q_{P_S}^+$. By Lemma 6.2, we obtain

$$q_S^+\sim_q q_{P_S}^+\sim_q q_S^-\sim_q q_{P_S}^-.$$

∎

If $0<S<1$, the commutant $(\mathcal{M}_{P_S}^+)'$ is written explicitly by using Tomita-Takesaki theory and we can show that $\mathcal{M}_{P_S}^+$ is a factor.

6.4 Lemma. *Let $0<S<1$. Then Ω_{P_S} is a cyclic and separating vector for $\mathcal{M}_{P_S}^+$.*

Proof. If $0<S<1$, we have already got $\mathcal{H}_{P_S}^+=\mathcal{H}_S^+$ and $Q_{P_S}(H)=Q_S(H)$. Thus Ω_{P_S} is cyclic for $\mathcal{M}_{P_S}^+$. Since $\{\widehat{Q}_{P_S}(0\oplus H)\Omega_{P_S}\,|\,H\in sp(\infty)\}$ generate $\mathcal{H}_{P_S}^+$ and $\widehat{Q}_{P_S}(0\oplus H)\in(\mathcal{M}_{P_S}^+)'$, Ω_{P_S} is cyclic for $(\mathcal{M}_{P_S}^+)'$ i.e. Ω_{P_S} is separating for $\mathcal{M}_{P_S}^+$. ∎

We define the modular conjugation $J_{\Omega_{P_S}}$ and the modular operator $\Delta_{\Omega_{P_S}}$ for $\mathcal{M}_{P_S}^+$.

For a bijective linear (resp. conjugate linear) operator U on $K_S\oplus K_S$, we define a *-automorphism (resp. conjugate *-automorphism) $\tau(U)$ of $\mathfrak{A}(K_S\oplus K_S,\widehat{\gamma}_S,\widehat{\Gamma}_S)$ satisfying $\tau(U)B(h)=B(Uh)$.

Let $\omega(f\oplus g):=\Gamma_S g\oplus\Gamma_S f$. The conjugate linear map $J_{\Omega_{P_S}}$ is defined by

$$J_{\Omega_{P_S}}\pi_{P_S}(A)\Omega_{P_S}=\pi_{P_S}(\tau(\omega)A)\Omega_{P_S},\quad A\in\mathfrak{A}(K_S\oplus K_S,\widehat{\gamma}_S,\widehat{\Gamma}_S).$$

Let $0<S<1$ and $H_S:=\log(S(1-S)^{-1})$. Let Θ_S be an infinitesimal generator defined by

$$\exp(it\Theta_S)\pi_S(A)\Omega_S=\pi_S(\tau(e^{itH_S})A)\Omega_S,\quad A\in\mathfrak{A}(K,\gamma,\Gamma).$$

α is defined in Remark 2.7. Then $\Delta_{\Omega_{P_S}}:=e^{-\Theta_S}$. Due to $0<S<1$, $\Delta_{\Omega_{P_S}}$ is defined on a dense set of $\mathcal{H}_{P_S}=\mathcal{H}_S$. (See Remark 2.7.)

6.5 Lemma. *Let $0<S<1$.*

(1) *The restriction of $J_{\Omega_{P_S}}$ on $\mathcal{H}_{P_S}^+$ is the modular conjugation associated with the pair $(\mathcal{M}_{P_S}^+,\Omega_{P_S})$ and the restriction of $\Delta_{\Omega_{P_S}}$ on $\mathcal{H}_{P_S}^+$ is the modular operator.*

(2) *$\mathcal{M}_{P_S}^+$ is a factor.*

Proof. (1) We obtain $J_{\Omega_{P_S}} Q_{P_S}(H)\Omega_{P_S} = \Delta^{1/2}_{\Omega_{P_S}} Q_{P_S}(H)^*\Omega_{P_S}$ for all $H \in sp(\infty)$. For any $A \in \mathcal{M}^+_{P_S}$, there exists a net $\{A_\nu\}$ in the linear hull of $\{Q_{P_S}(H) \,|\, H \in sp(\infty)\}$ such that $A_\nu \to A$ $(\nu \to \infty)$ with respect to the strong * operator topology :

$$\|(A_\nu - A)x\|^2 + \|(A^*_\nu - A^*)x\|^2 \to 0$$

for all $x \in \mathcal{H}^+_{P_S}$. Therefore we have

$$\langle \Delta^{1/2}_{P_S}\Psi, A^*\Omega_{P_S}\rangle = \lim_{\nu\to\infty}\langle \Delta^{1/2}_{P_S}\Psi, A^*_\nu\Omega_{P_S}\rangle = \lim_{\nu\to\infty}\langle \Psi, J_{\Omega_{P_S}} A_\nu \Omega_{P_S}\rangle = \langle \Psi, J_{\Omega_{P_S}} A\Omega_{P_S}\rangle$$

for all Ψ in the domain of $\Delta_{\Omega_{P_S}}$. This shows that all $A^*\Omega_{P_S}$ are elements of the domain of $\Delta_{\Omega_{P_S}}$. Let $T_{\Omega_{P_S}} := J_{\Omega_{P_S}}\Delta^{1/2}_{\Omega_{P_S}}$, then $T_{\Omega_{P_S}} A\Omega_{P_S} = A^*\Omega_{P_S}$ for all $A \in \mathcal{M}^+_{P_S}$. Thus $J_{\Omega_{P_S}}$ is the modular conjugation associated with the pair $(\mathcal{M}^+_{P_S}, \Omega_{P_S})$ and $\Delta_{\Omega_{P_S}}$ is the modular operator.

(2) $Q_{P_S}(H)$ satisfies $[W_{P_S}(0 \oplus f), Q_{P_S}(H)] = 0$ from the direct computation and this shows

$$\mathcal{R}_{P_S}(\mathrm{Re}K_S \oplus 0) = \mathcal{R}_{P_S}(0 \oplus \mathrm{Re}K_S)' \supset \mathcal{M}_{P_S} = \mathcal{M}^+_{P_S} \oplus \mathcal{M}^-_{P_S}. \tag{6.3}$$

By Tomita-Takesaki theory we have

$$(\mathcal{M}^+_{P_S})' = J_{\Omega_{P_S}}\mathcal{M}^+_{P_S} J_{\Omega_{P_S}} = \{\widehat{Q}_{P_S}(0 \oplus H)|\mathcal{H}^+_{P_S} \,|\, H \in sp(\infty)\}''. \tag{6.4}$$

From the quasi-equivalence of representations $q^+_{P_S}$ and $q^-_{P_S}$, (6.4) implies that $(\mathcal{M}^-_{P_S})'$ is generated by all $\widehat{Q}_{P_S}(0 \oplus H)|\mathcal{H}^-_{P_S}$, $H \in sp(\infty)$. Thus $(\mathcal{M}^+_{P_S})' \oplus (\mathcal{M}^-_{P_S})'$ is generated by all $\widehat{Q}_{P_S}(0 \oplus H)$, $H \in sp(\infty)$. Since $[W_{P_S}(f \oplus 0), \widehat{Q}_{P_S}(0 \oplus H)] = 0$, we have

$$\mathcal{R}_{P_S}(\mathrm{Re}K_S \oplus 0)' = \mathcal{R}_{P_S}(0 \oplus \mathrm{Re}K_S) \supset (\mathcal{M}^+_{P_S})' \oplus (\mathcal{M}^-_{P_S})'. \tag{6.5}$$

From Corollary 2.6 and (6.3) and (6.5), we have

$$C(\mathcal{M}^+_{P_S}) \oplus C(\mathcal{M}^-_{P_S}) \subset C(\mathcal{R}_{P_S}(\mathrm{Re}K_S \oplus 0)) = \mathbf{C}1.$$

Thus $C(\mathcal{M}^+_{P_S}) = \mathbf{C}1_+$.(And $C(\mathcal{M}^-_{P_S}) = \mathbf{C}1_-$.) ∎

7 Quasi-equivalence of Quasifree Representations of $sp(\infty)$

Let $\mathfrak{S}$ be the set of all positive semi-definite hermitian forms S on K satisfying (2.1). Now we give the main result of this paper.

7.1 Theorem. *Assume that K is separable. Let S, $S' \in \mathfrak{S}$. Two quasifree representations q^σ_S and $q^\sigma_{S'}$ of $sp(\infty)$ are quasi-equivalent if and only if the following two conditions hold :*

(1) *The topologies induced by $\| f \|_S$ and $\| f \|_{S'}$ on K are equivalent, i.e. there exists $\beta > \alpha > 0$ such that $\alpha \| f \|_S \leq \| f \|_{S'} \leq \beta \| f \|_S$ for all $f \in K$.*

(2) *$1 - \rho(S)e^{-\chi(S)}e^{\chi(S')}\rho(S')$ is a Hilbert-Schmidt class operator on K_S where $\chi(S) := \tanh^{-1} 2\sqrt{S(1-S)}$ and $\rho(S) := (2S-1)^{-1}|2S-1|$.*

By the equivalence of norms $\| \cdot \|_S$ and $\| \cdot \|_{S'}$, we can see a bounded operator S' on $K_{S'}$ as a bounded operator on K_S.

7.2 Lemma. *If* $\alpha \| \cdot \|_S \leq \| \cdot \|_{S'} \leq \beta \| \cdot \|_S$, *there exists* $0 < \alpha' < \beta'$ *such that* $\alpha' \| \cdot \|_{P_S} \leq \| \cdot \|_{P_{S'}} \leq \beta' \| \cdot \|_{P_S}$.

Proof. Immediate from (2.3). ∎

7.3 Lemma. *Let* S, $S' \in \mathfrak{S}$ *and the topologies induced by* $\| f \|_S$ *and* $\| f \|_{S'}$ *on* K *are equivalent. Then the following conditions are equivalent.*

(1) $P_S - P_{S'}$ *is a Hilbert-Schimdt class operator,*

(2) $1 - \rho(S)e^{-\chi(S)}e^{\chi(S')}\rho(S')$ *is a Hilbert-Schmidt class operator,*

(3) $1 - \rho(S')e^{-\chi(S')}e^{\chi(S)}\rho(S)$ *is a Hilbert-Schmidt class operator.*

Proof. See Lemma 6.5 of [2]. ∎

7.4 Lemma. *Let* $S, S' \in \mathfrak{S}$ *be basis projections and assume that* $K = K_S = K_{S'}$.

(1) *Let* $\theta(S, S')$ *be a non-negative hermitian operator on* K *satisfying* $\sinh^2 \theta(S, S') = -(S - S')^2$. *Let*

$$\begin{aligned} u_{12}(S/S') &:= (\sinh \theta(S, S') \cosh \theta(S, S'))^{-1} SS'(1 - S), \\ u_{21}(S/S') &:= -(\sinh \theta(S, S') \cosh \theta(S, S'))^{-1} (1 - S)S'S, \\ H(S/S') &:= -i\theta(S, S')\{u_{12}(S/S') + u_{21}(S/S')\}. \end{aligned}$$

Then $u_{ij}(S/S')^* = u_{ji}(S/S')$ *and* $H(S/S')$ *satisfies*

$$\begin{gathered} H(S/S')^\dagger = H(S/S'), \quad \Gamma H(S/S')\Gamma = -H(S/S'), \\ (iH(S/S'))^* = iH(S/S'). \end{gathered}$$

($*$ *is relative to* $(\cdot, \cdot)_S$.) *Let*

$$U(S/S') := \exp(iH(S/S')).$$

$U(S/S')$ *satisfies*

$$\begin{gathered} U(S/S')^\dagger U(S/S') = U(S/S')U(S/S')^\dagger = 1, \quad [\Gamma, U(S/S')] = 0, \\ U(S/S')^\dagger S U(S/S') = S'. \end{gathered}$$

(2) $S - S'$ *is a Hilbert-Schmidt class operator if and only if* $\theta(S, S')$ *is a Hilbert-Schmidt class operator.*

(3) *Let* $\theta(S, S')$ *be a Hilbert-Schmidt class operator. Then there exists an unique unitary operator* $T(S, S') \in \mathcal{M}_S$ *such that*

$$T(S, S')^* \overline{\pi_S(A)} T(S, S') = \pi_S[\tau(U(S/S'))A]$$

on $D(\pi_S)$ *and*

$$\langle \Omega_S, T(S, S')\Omega_S \rangle = \det{}_{SK} \left(\frac{1}{\sqrt{\cosh \theta(S, S')}} \right) \tag{7.1}$$

where $\det_{SK}$ *is the determinant of* SK.*(Since* $\theta(S, S')$ *commutes with* S*, the right hand side of* (7.1) *is well-defined.)*

Proof. (1) See Lemma 5.4 of [2]. (3) See Lemma 5.5 of [2]. ∎

7.5 Lemma. *Assume that* S, $S' \in \mathfrak{S}$ *are basis projections and the topologies induced by* $\| f \|_S$ *and* $\| f \|_{S'}$ *on* K *are equivalent. If* $S - S'$ *is a Hilbert-Schmidt class operator, then* q_S^σ *and* $q_{S'}^\sigma$ *are unitary equivalent.*

Proof. Let

$$V\pi_{S'}(A)\Omega_{S'} = \overline{\pi_S(A)}T(S,S')\Omega_S, \quad A \in \mathfrak{A}(K,\gamma,\Gamma).$$

Since $U(S/S')^\dagger SU(S/S') = S'$, we have $\varphi_{S'} = \varphi_S \circ \tau(\overline{U(S/S')})$. This shows that V is an unitary operator from $\mathcal{H}_{S'}$ to $\mathcal{H}_S$ and satisfies $V\pi_{S'}(A) = \overline{\pi_S(A)}V$ on $D(\pi_{S'})$ for all $A \in \mathfrak{A}(K,\gamma,\Gamma)$. By $V\mathcal{H}_{S'}^\sigma \subset \mathcal{H}_S^\sigma$, the restriction of V to $\mathcal{H}_{S'}^\sigma$ is an unitary operator from $\mathcal{H}_{S'}^\sigma$ to $\mathcal{H}_S^\sigma$ and we have $VQ_{S'}^\sigma(H) = Q_S^\sigma(H)V$ for all $H \in sp(\infty)$. Thus q_S^σ is unitary equivalent to $q_{S'}^\sigma$. ∎

The next corollary is directly seen from the above lemma.

7.6 Corollary. *If the topologies induced by* $\| f \|_S$ *and* $\| f \|_{S'}$ *on* K *are equivalent and* $P_S - P_{S'}$ *is a Hilbert-Schmidt class operator, then* $q_{P_S}^\sigma$ *and* $q_{P_{S'}}^\sigma$ *are unitary equivalent.*

7.7 Lemma. *Assume that* S, $S' \in \mathfrak{S}$ *are not projection. If* $1 - \rho(S)e^{-\chi(S)}e^{\chi(S')}\rho(S')$ *is a Hilbert-Schmidt class operator, then* q_S^σ *and* $q_{S'}^\sigma$ *are quasi-equivalent.*

Proof. Immediate from Corollary 6.3, Lemma 7.3 and Corollary 7.6. ∎

7.8 Lemma. *Let* $S \in \mathfrak{S}$ *be a projection and* $S' \in \mathfrak{S}$ *be not a projection. Then* $1 - \rho(S)e^{-\chi(S)}e^{\chi(S')}\rho(S')$ *is not a Hilbert-Schmidt class operator.*

Proof. Suppose $q_S^+ \sim_q q_{S'}^+$ and $q_S^- \sim_q q_{S'}^-$. Then since $q_S^+ \sim_q q_S^-$ and the irreducibility of q_S^σ, we have $q_S^+ \sim q_S^-$. However, since S is a projection, $q_S^+ \not\sim q_S^-$. This is contradiction. So $q_S^+ \not\sim_q q_{S'}^+$ or $q_S^- \not\sim_q q_{S'}^-$. If $q_S^+ \sim_q q_{S'}^+$ and $q_S^- \not\sim_q q_{S'}^-$, then $P_S - P_{S'}$ is not a Hilbert-Schmidt class operator from $q_{P_{S'}}^+ \sim_q q_{S'}^+ \sim_q q_S^+ \not\sim_q q_{P_S}^+$. Thus $1 - \rho(S)e^{-\chi(S)}e^{\chi(S')}\rho(S')$ is not a Hilbert-Schmidt class operator. The case of $q_S^- \sim_q q_{S'}^-$ and $q_S^+ \not\sim_q q_{S'}^+$ is quite similar. If $q_S^+ \sim_q q_{S'}^-$ and $q_S^+ \not\sim_q q_{S'}^+$ and $q_S^- \not\sim_q q_{S'}^-$, we have $q_{P_{S'}}^+ \sim_q q_{P_{S'}}^- \sim_q q_{S'}^- \sim_q q_S^+ \not\sim_q q_{P_S}^+$. Thus $1 - \rho(S)e^{-\chi(S)}e^{\chi(S')}\rho(S')$ is not a Hilbert-Schmidt class operator. The case of $q_S^+ \not\sim_q q_{S'}^-$ and $q_S^+ \not\sim_q q_{S'}^+$ and $q_S^- \not\sim_q q_{S'}^-$ is trivial. Therefore, $1 - \rho(S)e^{-\chi(S)}e^{\chi(S')}\rho(S')$ is not a Hilbert-Schmidt class operator. ∎

From the above lemmas, we have the necessary condition of Theorem 7.1.

7.9 Lemma (Necessity of Theorem 7.1). *Suppose that* S, $S' \in \mathfrak{S}$ *satisfy the following two conditions.*

(1) *The topologies induced by* $\| f \|_S$ *and* $\| f \|_{S'}$ *on* K *are equivalent,*

(2) $1 - \rho(S)e^{-\chi(S)}e^{\chi(S')}\rho(S')$ *is a Hilbert-Schmidt class operator.*

Then two representations q_S^σ *and* $q_{S'}^\sigma$ *are quasi-equivalent.*

Next, we prove the sufficiency of Theorem 7.1.

A state φ_{P_S} on CCR algebra $\mathfrak{A}(K_S \oplus K_S, \widehat{\gamma}_S, \widehat{\Gamma}_S)$ can be viewed as a state on $\mathcal{M}_S^+$ satisfying $\varphi_{P_S}(Q) := \langle \Omega_{P_S}, Q\Omega_{P_S} \rangle$, $Q \in \mathcal{M}_{P_S}^+$.

Now let $\dim K < \infty$. Since $P_S - P_{S'}$ is a Hilbert-Schmidt class operator, $q_{P_S}^+$ and $q_{P_{S'}}^+$ are unitary equivalent and we can identify $\mathcal{M}_{P_S}^+$ with $\mathcal{M}_{P_{S'}}^+$. Therefore, a state $\varphi_{P_{S'}}$ on CCR algebra $\mathfrak{A}(K_{S'} \oplus K_{S'}, \widehat{\gamma}_{S'}, \widehat{\Gamma}_{S'})$ is regarded as a state on $\mathcal{M}_{P_S}^+$ satisfying $\varphi_{P_{S'}}(Q) := \langle \Omega', Q\Omega' \rangle$, $\Omega' := T(P_S, P_{S'})\Omega_{P_S}$, $Q \in \mathcal{M}_{P_{S'}}^+ = \mathcal{M}_{P_S}^+$ where $T(P_S, P_{S'})$ is an unitary operator determined by (3) of Lemma 7.4.

We quote the following two results to prove the sufficiency of the main theorem.

7.10 Lemma. *Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector Ψ. Let V_Ψ be a natural positive corn associated with the pair $(\mathcal{M}, \Psi)$. If Φ is an another cyclic and separating vector for $\mathcal{M}$, then $\Phi \in V_\Psi$ if and only if the following 2 conditions hold :*

(1) $J_\Phi = J_\Psi$,

(2) $\langle \Phi, Q_+\Psi \rangle \geq 0$ *for all* $Q_+ \in \mathcal{M} \cap \mathcal{M}'$, $Q_+ \geq 0$.

Proof. See THEOREM 4 (5) of [4]. ∎

7.11 Lemma. *Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector Ψ and let V_Ψ be the natural positive corn for Ψ. Let $\Phi_i \in V_\Psi (i = 1, 2)$ and φ_{Φ_i} be a vector state for Φ_i. Then*

$$\| \varphi_{\Phi_1} - \varphi_{\Phi_2} \| \geq \| \Phi_1 - \Phi_2 \|^2 .$$

Proof. See THEOREM 4 (8) of [4]. ∎

7.12 Lemma. *Let* $\dim K < \infty$ *and* S, $S' \in \mathfrak{S}$ *be* $0 < S < 1$, $0 < S' < 1$. *Then*

$$\| (\varphi_{P_S} - \varphi_{P_{S'}})|\mathcal{M}_{P_S}^+ \| \geq 2\left\{1 - \det{}_{P_S K P_S}\left(\frac{1}{\sqrt[4]{P_S P_{S'} P_S}}\right)\right\} \tag{7.2}$$

Proof. Let $J_{\Omega'}$ and $\Delta_{\Omega'}$ be the modular conjugation and modular operator associated with the pair $(\mathcal{M}_{P_S}^+, \Omega')$ and let $V_{\Omega_{P_S}}$ be the natural positive corn associated with the pair $(\mathcal{M}_{P_S}^+, \Omega_{P_S})$. We show $J_{\Omega'} = J_{\Omega_{P_S}}$ and $\langle \Omega', Q\Omega_{P_S} \rangle \geq 0$ for all $Q \in C(\mathcal{M}_{P_S}^+)$, $Q \geq 0$ with help of Lemma 7.10 to prove $\Omega' \in V_{\Omega_{P_S}}$.

We prove the first part, $J_{\Omega'} = J_{\Omega_{P_S}}$. Since we have

$$[J_{\Omega_{P_S}}, T(P_S, P_{S'})] = 0$$

(See (6.2) of [2]), the following relation holds :

$$J_{\Omega_{P_S}}\Omega' = \Omega'. \tag{7.3}$$

We remark that we have already obtained the following relations :

$$V^* Q_{P_S}(H)V = Q_{P_{S'}}(H), \tag{7.4}$$
$$J_{\Omega'} = V J_{\Omega_{P_{S'}}} V^*, \tag{7.5}$$
$$J_{\Omega_{P_S}} Q_{P_S}(H) J_{\Omega_{P_S}} = \widehat{Q}_{P_S}(0 \oplus H)^*, \tag{7.6}$$
$$J_{\Omega_{P_{S'}}} Q_{P_{S'}}(H) J_{\Omega_{P_{S'}}} = \widehat{Q}_{P_{S'}}(0 \oplus H)^* \tag{7.7}$$

where V is the unitary operator defined in Lemma 7.5. We have

$$J_{\Omega'}Q_{P_S}(H)J_{\Omega'} = \widehat{Q}_{P_S}(0\oplus H)^* \tag{7.8}$$

from (7.4), (7.5) and (7.7). It follows

$$[J_{\Omega'}J_{\Omega_{P_S}}, Q_{P_S}(H)] = 0,$$
$$[J_{\Omega'}J_{\Omega_{P_S}}, \widehat{Q}_{P_S}(0\oplus H)] = 0$$

for all $H \in sp(\infty)$ from (7.6) and (7.8). Now the center $C(\mathcal{M}_{P_S})$ of $\mathcal{M}_{P_S}$ is trivial :

$$C(\mathcal{M}_{P_S}) = \begin{pmatrix} \mathcal{M}^+_{P_S} & 0 \\ 0 & \mathcal{M}^-_{P_S} \end{pmatrix} \cap \mathcal{M}'_{P_S} = \begin{pmatrix} C(\mathcal{M}^+_{P_S}) & 0 \\ 0 & C(\mathcal{M}^-_{P_S}) \end{pmatrix} = \mathbf{C}1.$$

Thus $J_{\Omega'}J_{\Omega_{P_S}} \in C(\mathcal{M}_{P_S}) = \mathbf{C}1$, that is,

$$J_{\Omega'} = \lambda J_{\Omega_{P_S}}, \quad \lambda \in S^1 := \{\lambda \in \mathbf{C} \mid |\lambda| = 1\}. \tag{7.9}$$

Due to (7.3) and (7.9), $\lambda = 1$.

The second part is verified by (7.1) and the factoriality of $\mathcal{M}^+_{P_S}$ and

$$\langle \Omega', \Omega_{P_S}\rangle = \langle \Omega_{P_S}, T(P_S, P_{S'})\Omega_{P_S}\rangle = \prod_{\lambda\in\sigma(\theta(P_S,P_{S'}))} \frac{1}{\sqrt{\cosh\lambda}} \geq 0.$$

Therefore $\Omega' \in V_{\Omega_{P_S}}$. Now from (7.1) and Lemma 7.11 and

$$\begin{aligned} P_S\cosh\theta(P_S,P_{S'}) &= P_S\sqrt{1-[\sinh\theta(P_S,P_{S'})]^2} \\ &= \sqrt{P_S\{1-(P_S-P_{S'})^2\}} \\ &= \sqrt{P_SP_{S'}P_S}, \end{aligned}$$

we obtain (7.2). ∎

7.13 Lemma. *Assume that K is separable. Let S, $S' \in \mathfrak{S}$ be $0 < S < 1$, $0 < S' < 1$ and the topologies induced by $\| f \|_S$ and $\| f \|_{S'}$ on K are equivalent. If $P_S - P_{S'}$ is not a Hilbert-Schmidt class operator, then there exists Γ-invariant finite dimensional subspaces K_n of K such that*

$$\lim_{n\to\infty} \| (\varphi_{S_n} - \varphi_{S'_n})|\mathcal{M}^+_{P_{S_n}} \| = 2.$$

$S_n(f,g)$ is the restriction of $S(f,g)$ to K_n.

Proof. Since we have the inequality (7.2), we can verify this lemma as the proof of Lemma 6.7 of [2]. ∎

7.14 Lemma (Sufficiency of Theorem 7.1). *Assume that K is separable. S, $S' \in \mathfrak{S}$ and the topologies induced by $\| f \|_S$ and $\| f \|_{S'}$ on K are equivalent. If $1 - \rho(S)e^{-\chi(S)}e^{\chi(S')}\rho(S')$ is not a Hilbert-Schmidt class operator, then q^σ_S and $q^\sigma_{S'}$ are not quasi-equivalent.*

Proof. First suppose that $0 < S < 1$ and $0 < S' < 1$. Then we can show $q^+_S \not\sim_q q^+_{S'}$ from Lemma 7.13 as the proof of Lemma 6.8 of [2]. Since $q^-_S \sim_q q^+_S$ and $q^-_{S'} \sim_q q^+_{S'}$ by Lemma 6.2, we have $q^-_S \not\sim_q q^-_{S'}$ as well.

For general S and S', this proof is quite similar to Lemma 6.8 of [2]. ∎

8 Metaplectic Representations of $Sp(\infty, P)$

8.1 Definition. *A bijective linear map U on K satisfies $\gamma(Uf, Ug) = \gamma(f, g)$ and $\Gamma U = U\Gamma$. Let $\tau(U)$ be a *-automorphism of CCR algebra $\mathfrak{A}(K, \gamma, \Gamma)$ satisfying $\tau(U)B(f) = B(Uf)$. Then we call U a Bogoliubov transformation for (K, γ, Γ) and $\tau(U)$ a Bogoliubov *-automorphism. The set of all Bogoliubov transformations is called the symplectic group and we denote the symplectic group for (K, γ, Γ) by $Sp(K, \gamma, \Gamma)$.*

We define some subgroups of $Sp(K, \gamma, \Gamma)$.

Let $Sp(\infty)$ be the group generated by $sp(\infty)$:

$$Sp(\infty) := \{e^{iH} \mid H \in sp(\infty)\}.$$

Suppose that P is a basis projection for (K, γ, Γ) satisfying $K_P = K$.

Let $Sp(\infty, P)$ be the set of all $U \in Sp(K, \gamma, \Gamma)$ satisfying

$$\|PU(1-P)\|_{\text{H.S.}} < \infty \tag{8.1}$$

where $\|\cdot\|_{\text{H.S.}}$ is the Hilbert-Schmidt norm of operators on K. (8.1) is the sufficient condition of the existence of unitary representation of U.

8.2 Lemma. *Let*

$$d_P(A, B) := \|A - B\| + \|P(A - B)(1 - P)\|_{\text{H.S.}}$$

for all $A, B \in Sp(\infty, P)$. Then $Sp(\infty, P)$ is a topological group with respect to d_P.

Proof. We have to show the continuity of multiplication and inverse. Suppose that $A, A_\nu, B, B_\nu \in Sp(\infty, P)$ satisfy

$$d_P(A_\nu, A) \to 0, \quad d_P(B_\nu, B) \to 0. \tag{8.2}$$

(8.2) says that for any $\varepsilon > 0$, if $\nu \to \infty$, then

$$\|PB_\nu(1-P)\|_{\text{H.S.}} \le \|PB(1-P)\|_{\text{H.S.}} + \varepsilon, \tag{8.3}$$

$$\|B_\nu\| \le \|B\| + \varepsilon. \tag{8.4}$$

First we claim

$$\|P(A_\nu - A)B(1-P)\|_{\text{H.S.}} \to 0. \tag{8.5}$$

(8.5) follows from (8.3) and

$$\begin{aligned}
&\|P(A_\nu - A)B(1-P)\|_{\text{H.S.}} \\
&\quad \le \|P(A_\nu - A)(1-P)B(1-P)\|_{\text{H.S.}} + \|P(A_\nu - A)PB(1-P)\|_{\text{H.S.}} \\
&\quad \le \|P(A_\nu - A)(1-P)\|_{\text{H.S.}}\|B\| + \|A_\nu - A\|\|PB(1-P)\|_{\text{H.S.}}.
\end{aligned}$$

Due to (8.5), we have $d_P(A_\nu B_\nu, AB) \to 0$ as $\nu \to \infty$.

On the other hand, Since (8.3), (8.4) and

$$\begin{aligned}
&\|P(A_\nu^{-1} - A^{-1}(1-P)\|_{\text{H.S.}} \\
&\quad = \|PA_\nu^{-1}(A_\nu - A)A^{-1}(1-P)\|_{\text{H.S.}} \\
&\quad \le \|PA_\nu^{-1}(1-P)(A_\nu - A)A^{-1}(1-P)\|_{\text{H.S.}} + \|PA_\nu^{-1}P(A_\nu - A)A^{-1}(1-P)\|_{\text{H.S.}} \\
&\quad \le \|PA_\nu^{-1}(1-P)\|_{\text{H.S.}}\|A_\nu - A\|\|A^{-1}\| + \|A_\nu^{-1}\|\|P(A_\nu - A)A^{-1}(1-P)\|_{\text{H.S.}},
\end{aligned}$$

we have $d_P(A_\nu^{-1}, A^{-1}) \to 0$ as $\nu \to \infty$. ∎

8.3 Lemma. *Assume that P_1 and P_2 are basis projections and there exist $\beta > \alpha > 0$ such that $\alpha \|f\|_{P_1} \leq \|f\|_{P_2} \leq \beta \|f\|_{P_1}$ for all $f \in K$ and $P_1 - P_2$ is a Hilbert-Schmidt class operator. Then $Sp(\infty, P_1) = Sp(\infty, P_2)$ as a topological group.*

Proof. We have only to show the equivalence of the distance d_{P_1} and d_{P_2}. In this proof, we denote the operator norm with respect to P by $\|\cdot\|_P$ and the Hilbert-Schmidt norm with respect to P by $\|\cdot\|_{\mathrm{H.S.},P}$. (In this proof, the Hilbert space norm of K and the operator norm of bounded operators on K is same notation, however we probably does not confuse the two meanings.)

$$\|A\|_{P_2} := \sup_{x \in K} \frac{\|Ax\|_{P_2}}{\|x\|_{P_2}} \leq \sup_{x \in K} \frac{\beta \|Ax\|_{P_1}}{\alpha \|x\|_{P_1}} = \frac{\beta}{\alpha} \|A\|_{P_1}.$$

In the same way, we have $\|A\|_{P_1} \leq \frac{\beta}{\alpha} \|A\|_{P_2}$.

On the other hand,

$$\begin{aligned}
&\|P_2 A(1-P_2)\|_{\mathrm{H.S.},P_2} \\
&\quad \leq \beta \|P_2 U(P_1/P_2)^\dagger U(P_1/P_2) A U(P_1/P_2)^\dagger U(P_1/P_2)(1-P_2)\|_{\mathrm{H.S.},P_1} \\
&\quad \leq \beta \|U(P_1/P_2)^\dagger P_1 U(P_1/P_2) A U(P_1/P_2)^\dagger (1-P_1) U(P_1/P_2)\|_{\mathrm{H.S.},P_1} \\
&\quad \leq \beta \|U(P_1/P_2)^\dagger\|_{P_1} \|U(P_1/P_2)\|_{P_1} \\
&\qquad \times \|P_1 U(P_1/P_2) A U(P_1/P_2)^\dagger (1-P_1)\|_{\mathrm{H.S.},P_1}
\end{aligned}$$

and

$$\begin{aligned}
&\|P_1 U(P_1/P_2) A U(P_1/P_2)^\dagger (1-P_1)\|_{\mathrm{H.S.},P_1} \\
&\quad \leq \|P_1 U(P_1/P_2)\{P_1 + (1-P_1)\} A U(P_1/P_2)^\dagger (1-P_1)\|_{\mathrm{H.S.},P_1} \\
&\quad \leq \|U(P_1/P_2)\|_{P_1} \|P_1 A U(P_1/P_2)^\dagger (1-P_1)\|_{\mathrm{H.S.},P_1} \\
&\qquad + \|P_1 U(P_1/P_2)(1-P_1)\|_{\mathrm{H.S.},P_1} \|U(P_1/P_2)^\dagger\|_{P_1} \|A\|_{P_1} \\
&\quad \leq \|U(P_1/P_2)\|_{P_1} \{\|A\|_{P_1} \|P_1 U(P_1/P_2)^\dagger (1-P_1)\|_{\mathrm{H.S.},P_1} \\
&\qquad + \|P_1 A(1-P_1)\|_{\mathrm{H.S.},P_1} \|U(P_1/P_2)^\dagger\|_{P_1}\} \\
&\qquad + \|P_1 U(P_1/P_2)(1-P_1)\|_{\mathrm{H.S.},P_1} \|U(P_1/P_2)^\dagger\|_{P_1} \|A\|_{P_1} \\
&\quad \leq M''(\|A\|_{P_1} + \|P_1 A(1-P_1)\|_{\mathrm{H.S.},P_1})
\end{aligned}$$

where M'' is the maximum value of

$$\begin{gathered}
\|P_1 U(P_1/P_2)(1-P_1)\|_{\mathrm{H.S.},P_1} \left\|U(P_1/P_2)^\dagger\right\|_{P_1}, \\
\left\|P_1 U(P_1/P_2)^\dagger (1-P_1)\right\|_{\mathrm{H.S.},P_1} \|U(P_1/P_2)\|_{P_1}, \\
\left\|U(P_1/P_2)^\dagger\right\|_{P_1} \|U(P_1/P_2)\|_{P_1}.
\end{gathered}$$

Since $P_1 U(P_1/P_2)(1-P_1) = \sinh\theta(P_1, P_2) u_{12}(P_1/P_2)$, $P_1 U(P_1/P_2)(1-P_1)$ is a Hilbert-Schmidt class operator and M'' is not infinity. From the above argument,

$$\|P_2 A(1-P_2)\|_{\mathrm{H.S.},P_2} \leq M'(\|A\|_{P_1} + \|P_1 A(1-P_1)\|_{\mathrm{H.S.},P_1})$$

where $M' := \beta \left\|U(P_1/P_2)^\dagger\right\|_{P_1} \|U(P_1/P_2)\|_{P_1} M''$.

Let $M := M' + \beta\alpha^{-1}$. Then $d_{P_2}(U_1, U_2) \leq M d_{P_1}(U_1, U_2)$. In the same way, we can show that there exists a positive number $m > 0$ such that $d_{P_1}(U_1, U_2) \leq \frac{1}{m} d_{P_2}(U_1, U_2)$. Therefore

$$m d_{P_1}(U_1, U_2) \leq d_{P_2}(U_1, U_2) \leq M d_{P_1}(U_1, U_2).$$

∎

8.4 Lemma. *Let $U \in Sp(\infty, P)$ and $P' := UPU^\dagger$.*

(1) *Let $R(U) := U(P/P')U$. Then $R(U)$ commutes with P and $R(U)$ is an unitary operator on K.*

(2) *$U(P/P')^\dagger$ is a positive and 1 + Hilbert-Schmidt class operator.*

Proof. (1) Since $P' = UPU^\dagger$ and $PU(P/P') = U(P/P')P'$, we have

$$PR(U) = U(P/P')P'U = U(P/P')UPU^\dagger U = R(U)P.$$

Thus $R(U)$ commutes with P. Due to $\gamma(R(U)f, R(U)g) = \gamma(f, g)$, we have

$$\gamma_P = R(U)^* \gamma_P R(U) = \gamma_P R(U)^* R(U).$$

($*$ is relative to $(\cdot,\cdot)_P$.) Since P is a projection and $\gamma_P^2 = 1$, we obtain

$$1 = \gamma_P^2 = \gamma_P \cdot \gamma_P R(U)^* R(U) = R(U)^* R(U).$$

This implies $R(U)R(U)^* = 1$. Thus $R(U)$ is an unitary operator on K.

(2) Due to (8.1), $\theta(P, P')$ defined in Lemma 7.4 is a Hilbert-Schmidt class operator. Indeed, we have

$$\|PU(1-P)\|_{\text{H.S.}}^2 = \operatorname{tr}(PU(1-P)U^\dagger P) \tag{8.6}$$

and

$$PU(1-P)U^\dagger P = -[\sinh\theta(P, P')]^2 P, \tag{8.7}$$

$$\Gamma \cdot PU(1-P)U^\dagger P \cdot \Gamma = -[\sinh\theta(P, P')]^2 (1-P) \tag{8.8}$$

from the direct computation. ((8.8) follows from $[\sinh\theta(P, P'), \Gamma] = 0$.) (8.6), (8.7) and (8.8) say that $\sinh\theta(P, P')$ is a Hilbert-Schmidt class operator, i.e. $\theta(P, P')$ is a Hilbert-Schmidt class operator. We obtain immediately that $H(P/P')$ is a Hilbert-Schmidt class operator.

Since $iH(P/P')$ is a hermitian operator, the positivity of $U(P/P')$ is obvious. ∎

From the above lemma, $U \in Sp(\infty, P)$ is written as $U = U(P/P')^\dagger R(U)$ and this is the polar decomposition of U.

We introduce some notations to define the metaplectic representations of $Sp(\infty, P)$.

Let P be a basis projection and U be the element of $Sp(K, \gamma, \Gamma)$ satisfying $[P, U] = 0$. (Since $[P, U] = 0$, U is an unitary operator.) Then the operator $T_P(U)$ on $\mathcal{H}_P$ is defined by

$$T_P(U)\pi_P(A)\Omega_P := \pi_P(\tau(U)A)\Omega_P, \quad A \in \mathfrak{A}(K, \gamma, \Gamma).$$

$T_P(U)$ is the second quantization of U. Since $\tau(U)$ is a *-automorphism of CCR algebra $\mathfrak{A}(K,\gamma,\Gamma)$ and φ_P is a quasifree state satisfying

$$\varphi_P(\tau(U)[B(f)^*B(g)]) = (Uf, PUg)_P = (f, Pg)_P = \varphi_P(B(f)^*B(g)),$$

$T_P(U)$ is an unitary operator on $\mathcal{H}_P$.

Let $T_P(\Gamma)$ be an antiunitary operator on $\mathcal{H}_P$ defined by

$$T_P(\Gamma)\pi_P(A)\Omega_P = \pi_P(\tau(\Gamma)A)\Omega_P, \quad A \in \mathfrak{A}(K,\gamma,\Gamma).$$

8.5 Lemma. *Let $U \in Sp(\infty, P)$. Then the unitary operator $Q_P(U)$ satisfying*

$$Q_P(U)W_P(f)Q_P(U)^* = W_P(Uf) \tag{8.9}$$

for all $f \in \mathrm{Re}K$ exists uniquely up to $S^1 := \{\lambda \in \mathbf{C} \,|\, |\lambda| = 1\}$.

Proof. Let

$$Q_P(U) := T(P, P')T_P(R(U)).$$

Then $Q_P(U)$ satisfies (8.9).

The uniqueness of $Q_P(U)$ follows from the irreducibility of the von Neumann algebra $\mathcal{R}_P(\mathrm{Re}K)$. In fact, if $Q'_P(U)$ is an another unitary operator satisfying (8.9), then we have

$$Q'_P(U)^*Q_P(U)W_P(f)Q_P(U)^*Q'_P(U) = W_P(f)$$

for all $f \in \mathrm{Re}K$ and this shows

$$Q'_P(U)^*Q_P(U) \in \mathcal{R}_P(\mathrm{Re}K)' = \mathbf{C}1.$$

Therefore $Q_P(U)$ is unique up to the phase factor. ∎

Remark. Since $Q_P(H)$ defined in the section 4 satisfies (4.3) and $Q_P(U)$ is unique up to the phase factor, we have $Q_P(e^{iH}) = \lambda Q_P(H)$, $\lambda \in S^1$.

Let

$$Q_P(\lambda, U) := \lambda Q_P(U)$$

for all $U \in Sp(\infty, P)$ and $\lambda \in S^1$.

8.6 Definition. *We denote the group generated by all $Q_P(\lambda, U)$ satisfying*

$$[T_P(\Gamma), Q_P(\lambda, U)] = 0 \tag{8.10}$$

by $Mp(\infty, P)$. We call $Mp(\infty, P)$ the metaplectic group of $Sp(\infty, P)$.

The elements λ in S^1 satisfying (8.10) are 1 and -1. In fact, by $[\Gamma, U(P/P')] = 0$ and $[\Gamma, R(U)] = 0$, we have $[T_P(\Gamma), T(P, P')] = 0$ and $[T_P(\Gamma), T_P(R(U))] = 0$. This shows $[T_P(\Gamma), Q_P(1, U)] = 0$. Thus

$$T_P(\Gamma)Q_P(\lambda, U) = \overline{\lambda}T_P(\Gamma)Q_P(1, U) = \overline{\lambda}\lambda^{-1}Q_P(\lambda, U)T_P(\Gamma).$$

Due to (8.10), $\overline{\lambda}\lambda^{-1} = 1$. Therefore $\lambda \in S^1 \cap \mathbf{R} = \{\pm 1\}$.

8.7 Proposition. (1) *The metaplectic representation $Mp(\infty, P)$ is a topological group with respect to the strong operator topology.*

(2) *The metaplectic representation is continuous projective representation with respect to the topology induced by the distance d_P and the strong operator topology, i.e. if $d_P(U_\nu, U) \to 0$ as $\nu \to \infty$, then $Q_P(\lambda, U_\nu) \to Q_P(\lambda, U)$ strongly.*

(3) *$Mp(\infty, P)$ is double covering of $Sp(\infty, P)$.*

Proof. (1) This claim is easily checked.

(2) We prove that $\|\{Q_P(1,U)-1\}\Omega_P\|^2 \to 0$ if $d_P(U,1) \to 0$. Since

$$\begin{aligned}
\det{}_P(\cosh\theta(P,P'))^2 &\le \exp\left(\left\|(\cosh\theta(P,P'))^2\right\|_{\mathrm{tr}}\right)\\
&= \exp\left(\left\|-P(P-P')^2P\right\|_{\mathrm{tr}}\right)\\
&= \exp\left(\left\|PU(1-P)U^\dagger P\right\|_{\mathrm{tr}}\right)\\
&= \exp\left(\|PU(1-P)\|_{\mathrm{H.S.}}\right),
\end{aligned}$$

we have

$$\begin{aligned}
\|\{Q_P(1,U)-1\}\Omega_P\|^2 &= 2(1-\mathrm{Re}\,\langle\Omega_P, T(P,P')\Omega_P\rangle)\\
&= 2\left\{1-\det{}_{PK}\left(\frac{1}{\sqrt{\cosh\theta(P,P')}}\right)\right\}\\
&= 2\left\{1-\frac{1}{\sqrt[4]{\det_P(\cosh\theta(P,P'))^2}}\right\}\\
&\le 2\left\{1-\exp\left(-\frac{1}{4}\|PU(1-P)\|_{\mathrm{H.S.}}\right)\right\}.
\end{aligned}$$

Thus the first claim has been proved.

Moreover, for any $f \in \mathrm{Re}K$,

$$\begin{aligned}
&\|\{Q_P(1,U)-1\}W_P(f)\Omega_P\|\\
&\quad= \|\{W_P(U^\dagger f)Q_P(1,U)-W_P(f)\}\Omega_P\|\\
&\quad\le \|\{W_P(U^\dagger f)Q_P(1,U)-W_P(U^\dagger f)\}\Omega_P\| + \|\{W_P(U^\dagger f)-W_P(f)\}\Omega_P\|\\
&\quad\le \|\{Q_P(1,U)-1\}\Omega_P\| + \|\{W_P(U^\dagger f)-W_P(f)\}\Omega_P\|
\end{aligned}$$

From Lemma 2.4(3), if $\|U-1\| \to 0$, we have $\left\|\{W_P(U^\dagger f)-W_P(f)\}\Omega_P\right\| \to 0$. We obtain the relation $\|\{Q_P(1,U)-1\}x\| \to 0$ for all $x \in \mathcal{H}_P$ if $d_P(U,1) \to 0$. Thus we have

$$\|\{Q_P(1,U_\nu)-Q_P(1,U)\}x\| \to 0$$

for all $x \in \mathcal{H}_P$ if $d_P(U_\nu, U) \to 0$.

(3) Let $f_P : Mp(\infty,P) \to Sp(\infty,P)$ be a group homomorphism defined by

$$f_P(Q_P(\lambda,U)) = U, \quad U \in Sp(\infty,P), \quad \lambda \in \{\pm 1\}.$$

Then f_P is a covering map and $\ker(f_P) = \{Q_P(-1,1), Q_P(1,1)\}$. Thus

$$Mp(\infty,P)/\ker(f_P) \simeq Sp(\infty,P),$$

that is, $Mp(\infty,P)$ is a double covering of $Sp(\infty,P)$. ∎

8.8 Lemma. *Let $Mp(\infty, P)_{\rm fin}$ be the group generated by $Q_P(\lambda, e^{iH}), H \in sp(\infty)$. Then the closure of $Mp(\infty, P)_{\rm fin}$ with respect to the strong operator topology is $Mp(\infty, P)$. That is, for any $U \in Sp(\infty, P)$, there exists a net $\{U_\mu\}$ in $Sp(\infty)$ such that*

$$\text{s-}\lim_{\mu\to\infty} Q_P(\lambda, U_\mu) = Q_P(\lambda, U).$$

Proof. $T(P, P')$ is written as

$$T(P, P') = \text{s-}\lim_{n\to\infty} Q_P(H(P/P')F_n) = \text{s-}\lim_{n\to\infty} Q_P(1, U(P/P')F_n)$$

where F_n is the spectral projection of a positive Hilbert-Schmidt class operator $\theta(P, P')$ for the open interval $(\frac{1}{n}, \infty)$. (See the proof of Lemma 5.5 of [2].)

On the other hand, since $R(U)$ is an unitary operator on K, there exists a hermitian operator H on K such that $R(U) = e^{iH}$. Since the set of all finite rank operators on K is dense set of all bounded operators with respect to the strong * operator topology, there exists a net $\{A'_\nu\}$ such that A'_ν is a finite rank operator and $A'_\nu \to H$ (strong * operator topology) as $\nu \to \infty$. Let $A_\nu := \frac{1}{2}(A'_\nu + (A'_\nu)^*)$. Then A_ν is a finite rank hermitian operator and $A_\nu \to H$ (strong operator topology) as $\nu \to \infty$. Let $H'_\nu := PA_\nu P + (1-P)A_\nu(1-P)$. Then H'_ν is a finite rank hermitian operator and commutes with P. Moreover, $H'_\nu \to H$ (strong operator topology) as $\nu \to \infty$. In fact, for any $x \in K$,

$$\begin{aligned}
&\|(H'_\nu - H)x\|_P \\
&\quad \le \|(PA_\nu P - PHP)x\|_P + \|((1-P)A_\nu(1-P) - (1-P)H(1-P))x\|_P \\
&\quad \le \|(A_\nu - H)Px\|_P + \|(A_\nu - H)(1-P)x\|_P \\
&\quad \to 0 (\nu \to \infty).
\end{aligned}$$

Let $H_\nu := \frac{1}{2}(H'_\nu - \Gamma H'_\nu \Gamma)$. Then H_ν is contained in $sp(\infty)$ and satisfies $[P, H_\nu] = 0$ and $H_\nu \to H$ (strong operator topology) as $\nu \to \infty$. This shows $e^{iH_\nu} \in Sp(\infty)$ and $\text{s-}\lim_{\nu\to\infty} e^{iH_\nu} = R(U)$. Moreover,

$$\begin{aligned}
\text{s-}\lim_{\nu\to\infty} T_P(e^{iH_\nu})W_P(f)\Omega_P &= \text{s-}\lim_{\nu\to\infty} W_P(e^{iH_\nu} f)\Omega_P \\
&= W_P(R(U)f)\Omega_P \\
&= T_P(R(U))W_P(f)\Omega_P
\end{aligned}$$

for all $f \in {\rm Re}K$. Thus

$$\text{s-}\lim_{\nu\to\infty} \mathrm{Q}_P(1, e^{iH_\nu}) = \text{s-}\lim_{\nu\to\infty} T_P(e^{iH_\nu}) = T_P(R(U)) = Q_P(1, R(U)).$$

Now let $U_\mu := U(P/P')^\dagger F_n e^{iH_\nu}$ where $\mu = (\nu, n)$. Then

$$Q_P(1, U_\mu) = Q_P(1, U(P/P')^\dagger F_n) Q_P(1, e^{iH_\nu})$$

and

$$\text{s-}\lim_{\mu\to\infty} Q_P(1, U_\mu) = Q_P(1, U).$$

∎

Let $Q_P^\sigma(\lambda, U)$ be the restriction of $Q_P(\lambda, U)$ to $\mathcal{H}_P^\sigma$ where $\sigma = +$ or $-$.
We obtain the following proposition immediately from Lemma 7.5 and Lemma 8.8.

8.9 Proposition. *Suppose that K is separable. Let P_1 and P_2 be basis projections satisfying $\alpha \|f\|_{P_1} \leq \|f\|_{P_2} \leq \beta \|f\|_{P_1}$ for all $f \in K$, $K = K_{P_1} = K_{P_2}$ and $P_1 - P_2$ is a Hilbert-Schmidt class operator. Then the metaplectic representations $Q^{\sigma}_{P_1}(\lambda, *)$ and $Q^{\sigma}_{P_2}(\lambda, *)$ (resp . $Q^{\sigma}_{P_1}(\lambda, *)$ and $Q^{\sigma'}_{P_2}(\lambda, *)(\sigma \neq \sigma')$) of $Sp(\infty, P_1) = Sp(\infty, P_2)$ are unitary equivalent.* (*resp. not unitary equivalent.*)

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