Curved planar quantum wires with Dirichlet and Neumann boundary conditions

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We investigate the discrete spectrum of the Hamiltonian describing a quantum particle living in the two–dimensional curved strip. We impose the Dirichlet and Neumann boundary condition on opposite sides of the strip. The existence of the discrete eigenvalue below the essential spectrum threshold depends on the sign of the total bending angle for the asymptotically straight strips.

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I Introduction

The spectral properties of curved quantum wires with Dirichlet boundary condition were widely investigated (*e.g.* [1], [2], [3]). It was shown that any small curvature of the tube in dimensions 2 and 3 produces at least one positive eigenvalue below the essential spectrum threshold. The problem of the existence of such eigenvalues in the straight quantum waveguides with a combination of Dirichlet and Neumann boundary conditions were also studied (*e.g.* [4], [5]). In the present paper we consider curved planar quantum wires, where Dirichlet boundary condition is imposed on one side of the wire, while the Neumann boundary condition is imposed on the opposite side. Both boundary conditions represent an impenetrable wall in the sense that there is no current through the boundary. They can model two types of interphases in a solid, *e.g.* in a superconductor, in principle. It is worth to know whether the presence of two types of interphases leads to new nanoscopic phenomena.

We prove that the existence of the discrete eigenvalue essentially depends on the direction of the total bending of the strip. Roughly speaking at least one bound state always exists if the Neumann boundary condition is imposed on the "outer side" of the boundary, *i.e.* the one which is locally longer. On the other hand we show that there is no eigenvalue below the essential spectrum threshold provided that the curvature does not change its direction and the Neumann boundary condition is imposed on the inner part of the boundary.

We consider a Schrödinger particle whose motion is confined to a curved planar strip of the width d. For definiteness, let a curve $\Gamma : \mathbb{R} \to \mathbb{R}^2$ be a C^3-diffeomorfism of the real axis onto $\Gamma(\mathbb{R})$. Without loss of generality we can assume $\dot\Gamma_1(s)^2 + \dot\Gamma_2(s)^2 = 1$, so s is the arc length of the curve.

We define the normal vector N to the curve Γ and the signed curvature γ in the standard way $N_1(s) = -\dot\Gamma_2(s)$, $N_2(s) = \dot\Gamma_1(s)$, $\gamma(s) = -\ddot\Gamma(s)\cdot\dot N(s)$. Let there exist real positive numbers γ_-, γ_+ such that $-\gamma_- \le \gamma(s) \le \gamma_+ < 1/d$ for all $s \in \mathbb{R}$ and let $\dot\gamma$ be bounded. Let the strip which is the configuration space of the considered particle, $\Omega \subset \mathbb{R}^2$, be defined by

$$\Omega = \left\{ \langle x,y\rangle \in \mathbb{R}^2 \,\middle|\, \langle x,y\rangle = \Gamma(s) + uN(s),\ s\in\mathbb{R},\ u\in(0,d)\right\}.$$

We are going to consider only strips Ω which are not selfintersecting. We shall denote the parts of the boundary of the region Ω by $\mathcal{D} = \left\{\langle x,y\rangle \in \mathbb{R}^2 \,\middle|\, \langle x,y\rangle = \Gamma(s),\ s\in\mathbb{R}\right\}$ and $\mathcal{N} = \left\{\langle x,y\rangle \in \mathbb{R}^2 \,\middle|\, \langle x,y\rangle = \Gamma(s) + dN(s),\ s\in\mathbb{R}\right\}$. We impose the Dirichlet boundary condition on the set $\mathcal{D}$ and the Neumann one on $\mathcal{N}$.

Putting $\hbar^2/2m = 1$ we identify according to [1], [3], [6] the particle Hamiltonian with the unique self-adjoint operator acting in the Hilbert space $L^2\big(\mathbb{R}\times(0,d)\,,\ \big(1-u\gamma(s)\big)\ \mathrm{d}s\ \mathrm{d}u\big)$ associated with the quadratic form

$$q_0(f,g) = \int_{\mathbb{R}} \mathrm{d}s \int_0^d \mathrm{d}u \left(\frac{1}{1-u\gamma(s)} \overline{\frac{\partial f}{\partial s}}(s,u)\frac{\partial g}{\partial s}(s,u) + \big(1-u\gamma(s)\big)\overline{\frac{\partial f}{\partial u}}(s,u)\frac{\partial g}{\partial u}(s,u)\right) \tag{1.1}$$

defined on the domain

$$Q(q_0) = \left\{ f\in H^1\big(\mathbb{R}\times(0,d)\big) \,\middle|\, f(s,0)=0 \text{ for a.e. s} \in \mathbb{R}\right\},$$

where $H^1\big(\mathbb{R}\times(0,d)\big)$ is the standard Sobolev space and $f(s,0)$ denotes the trace of the function f on the part of the boundary $\mathbb{R}\times\{0\}$.

Remark 1 *Using similar "reflection" procedure like in the proof of Theorem 1 in [5] for trivial case of combination of Dirichlet and Neumann boundary conditions we can see that our operator acts like Laplace-Beltrami operator*

$$L_\Omega^{DN} = -\left(\frac{u\dot\gamma(s)}{\big(1-u\gamma(s)\big)^3}\frac{\partial}{\partial s} + \frac{1}{\big(1-u\gamma(s)\big)^2}\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial u^2} - \frac{\gamma(s)}{1-u\gamma(s)}\frac{\partial}{\partial u}\right) \tag{1.2}$$

on the domain $D(L_\Omega^{DN}) = \left\{ f\in H^2\big(\mathbb{R}\times(0,d)\big) \,\middle|\, f(s,0)=0,\ \frac{\partial f}{\partial u}(s,d)=0 \text{ for a.e. s}\in\mathbb{R}\right\}$.

II Existence of the discrete eigenvalue

Our main goal is to prove three propositions on the existence, resp. the absence, of the discrete spectrum in this, resp. the next section. We use the variational technique first introduced in [2] for proving the existence statements.

Proposition 1 *Let there exist a positive real number s_0 such that $\gamma(s)\le 0$ for every s satisfying $|s|\ge s_0$ and $\int_{-s_0}^{s_0}\gamma(s)\ \mathrm{d}s < 0$. Then $\inf\sigma\big(L_\Omega^{DN}\big) < \frac{\pi^2}{4d^2}$.*

PROOF: For every $\Phi\in Q(q_0)$ we define the functional

$$q[\Phi] = q_0(\Phi,\Phi) - \frac{\pi^2}{4d^2}\|\Phi\|^2_{L^2\left(\mathbb{R}\times(0,d),(1-u\gamma(s))\ \mathrm{d}u\ \mathrm{d}s\right)}. \tag{2.1}$$

According to Rayleigh-Ritz variational method (see *e.g.* [6]) it is enough to find a trial function $\Phi\in Q(q_0)$ such that $q[\Phi]<0$ to prove our proposition. We construct such trial function as follows. Let φ be an arbitrary function from the Schwartz space $\mathcal{S}$ such that $\varphi(s)=1$ for $|s|\le s_0$. We use the external scaling, *i.e.* we define a family of functions

$$\varphi_\sigma(s) = \begin{cases} \varphi(s) & \text{for } |s|\le s_0 \\ \varphi\big(\pm s_0 + \sigma(s\mp s_0)\big) & \text{for } |s|\ge s_0 \end{cases} \tag{2.2}$$

with upper signs for $s \geq s_0$, lower ones for $s \leq s_0$. Now we take a function $\Phi_\sigma(s,u) = \sqrt{\frac{2}{d}}\varphi_\sigma(s)\sin\frac{\pi u}{2d}$ as a trial function. After straightforward calculation we obtain

$$\begin{aligned} q[\Phi_\sigma] &= \frac{2}{d}\int_{\mathbb{R}}\int_0^d \left(\left|\frac{d\varphi_\sigma(s)}{ds}\right|^2 \frac{\sin^2\frac{\pi u}{2d}}{1-u\gamma(s)} + \frac{\pi^2}{4d^2}(1-u\gamma(s))\left|\varphi_\sigma(s)\right|^2\cos\frac{\pi u}{d}\right) \mathrm{d}u\ \mathrm{d}s \leq \\ &\leq \frac{\sigma}{1-d\gamma_+}\|\dot\varphi\|^2_{L^2(\mathbb{R})} + \frac{1}{d}\int_{-s_0}^{s_0}\gamma(s)\ \mathrm{d}s. \end{aligned}$$

The second term in this estimate is negative by assumption and it is independent of σ. Hence we can choose σ so small that the whole functional is negative, which finishes the proof. ∎

If we assume in addition *e.g.* that γ has a compact support, we can see using simple Dirichlet-Neumann bracketing argument (see *e.g.* [7]) that the essential spectrum begins at the value $\pi^2/4d^2$. So we can simply state the following

Corollary 1 *Let γ have a compact support and $\int_{\mathbb{R}}\gamma(s)\ \mathrm{d}s < 0$. Then the operator (1.2) has at least one discrete eigenvalue.*

Proposition 2 *Let there exist a positive real number s_0 such that $\gamma(s) = 0$ for $|s| \geq s_0$. Let $\int_{-s_0}^{s_0}\gamma(s)\ \mathrm{d}s = 0$ and $\|\gamma\|_{L^2(\mathbb{R})} > 0$. Then* $\inf\sigma\big(L^{DN}_\Omega\big) < \frac{\pi^2}{4d^2}$, i.e. *there exists at least one positive discrete eigenvalue of L^{DN}.*

PROOF: We use the same technique as in the proof of the Proposition 1. We only slightly modify the trial function. Instead of (2.2) we define for any $\varepsilon > 0$ a family of functions

$$\varphi_{\sigma,\varepsilon}(s) = \begin{cases} \varphi(s)\big(1-\varepsilon\gamma(s)\big) & \text{for} \quad |s| \leq s_0 \\ \varphi\big(\pm s_0 + \sigma(s \mp s_0)\big) & \text{for} \quad |s| \geq s_0 \end{cases} \tag{2.3}$$

and $\Phi_{\sigma,\varepsilon}(s,u) = \sqrt{\frac{2}{d}}\,\varphi_{\sigma,\varepsilon}(s)\sin\frac{\pi u}{2d}$. We substitute this function into (2.1) and we obtain

$$q[\Phi_{\sigma,\varepsilon}] \leq \sigma\|\dot\varphi\|^2_{L^2(\mathbb{R})} + \varepsilon^2\frac{\|\dot\gamma\|^2_{L^2(\mathbb{R})}}{1-d\gamma_+} + \varepsilon^2\frac{\|\gamma\|^3_{L^3(\mathbb{R})}}{d} - \varepsilon\frac{2\|\gamma\|^2_{L^2(\mathbb{R})}}{d}.$$

The term linear in ε is negative and choosing ε sufficiently small, we can make it dominating over the quadratic one. Finally, we fix this ε and choose a small enough σ to make the right hand side negative. ∎

Remark 2 *In fact we have used only one step of the original Goldstone and Jaffe construction of the trial function in the proof of the Proposition 1,* i.e. *the external scaling. We usually need in addition to deform somehow this trial function inside the interval, where original $\varphi(s) = 1$. In the present case this deformation is not necessary, because the first transverse mode function itself is a good trial function due to the different boundary conditions on both sides of the boundary. This deformation is required for the case with zero bending angle, which is seen in the proof of the Proposition 2.*

This situation is similar to the proof of Theorem 5.1 in [8]. An unbounded curved quantum layer in $\mathbb{R}^3$ with Dirichlet boundary condition is investigated there. It is proved that there exists at least one isolated eigenvalue provided the reference surface of the layer has a non-positive total Gauss curvature. If the total Gauss curvature is negative the proof does not require to add a deformation term to the "first transverse mode function" while for the zero case it is necessary.

III Absence of the discrete spectrum

The "nonexistence" counterpart to Proposition 1 could be formulated as follows. Let there exist a positive real number s_0 such that $\gamma(s) \geq 0$ for s satisfying $|s| \geq s_0$ and $\int_{-s_0}^{s_0}\gamma(s)\ \mathrm{d}s > 0$, then

$\inf \sigma\big(L_\Omega^{DN}\big) \geq \frac{\pi^2}{4d^2}$. But this statement does not hold as the following example shows. Let us consider the quantum wire with following properties. There exist real numbers $s_1 < s_2 < s_3 < s_4$ such that $\gamma(s) = 0$ for $s \in (-\infty, s_1) \cup (s_2, s_3) \cup (s_4, \infty)$, $\gamma(s) < 0$ on (s_1, s_2) and $\gamma(s) > 0$ on (s_3, s_4). According to our assumption $\int_{\mathbb{R}} \gamma(s)\ \mathrm{d}s > 0$. We take any function $\varphi(s) \in H^1(\mathbb{R})$ such that $\varphi(s) = 1$ on (s_1, s_2) and $\varphi(s) = 0$ on (s_3, s_4). Then we construct a function $\Phi = \sqrt{\frac{2}{d}}\, \varphi(s) \sin \frac{\pi u}{2d}$. We substitute this function into the functional (2.1) and we get

$$q[\Phi] = \|\dot{\varphi}\|^2_{L^2(\mathbb{R})} + \frac{1}{d} \int_{s_1}^{s_2} \gamma(s)\ \mathrm{d}s.$$

Hence $q[\Phi] < 0$ for d small enough, *i.e.* for small width of the strip, because $\gamma(s) < 0$ on (s_1, s_2) and the first term does not depend on d.

We state the weaker proposition in the sequel, the nonexistence statement is its direct corollary. The proof of the proposition will use the following lemma.

Lemma 1 *Let $\lambda_0(\alpha)$ be the lowest eigenvalue of the self-adjoint operator $-\frac{d^2}{du^2} + V(u)$ acting in the Hilbert space $L^2\big([0,d]\big)$ with the domain $D_\alpha = \big\{\psi \in AC^2\big([0,d]\big) \,\big|\, \psi(0) = 0,\ \psi'(d) + \alpha\psi(d) = 0\big\}$, where α is a real number and V is a real measurable bounded function on $[0,d]$. Then for any $\alpha_1 \geq \alpha_2$, $\lambda_0(\alpha_1) \geq \lambda_0(\alpha_2)$. More precisely*

$$\lambda_0(\alpha_2) \leq \lambda_0(\alpha_1) + (\alpha_2 - \alpha_1)\frac{\psi_0(d)^2}{\|\psi_0\|^2},$$

where ψ_0 is a real eigenfunction corresponding to $\lambda_0(\alpha_1)$.

PROOF: For $0 < \varepsilon < d$ and $3 + \alpha_2\varepsilon > 0$ we define a family of functions

$$\omega_\varepsilon(u) = \begin{cases} 0 & for \quad 0 \leq u \leq d - \varepsilon \\[2ex] \frac{\alpha_1 - \alpha_2}{(3+\alpha_2\varepsilon)\varepsilon^2}(u - d + \varepsilon)^3 \psi_0(d) & for \quad d - \varepsilon \leq u \leq d. \end{cases}$$

Now let $\varphi_{0,\varepsilon} = \psi_0 + \omega_\varepsilon$. Then obviously $\varphi_{0,\varepsilon} \in C^2\big([0,d]\big)$, $\varphi_{0,\varepsilon}(0) = 0$ and $\varphi'_{0,\varepsilon}(d) = -\alpha_2\varphi_{0,\varepsilon}(d)$. Hence $\varphi_{0,\varepsilon} \in D_{\alpha_2}$. Using integration by parts and the definitions of $\varphi_{0,\varepsilon}$ and ω_ε we directly obtain

$$\big(\varphi_{0,\varepsilon}\,,\, -\varphi''_{0,\varepsilon} + V\varphi_{0,\varepsilon}\big)_{L^2([0,d])} = \big(\psi_0\,,\, -\psi''_0 + V\psi_0\big)_{L^2([0,d])} + (\alpha_2 - \alpha_1)\psi_0(d)^2 + \mathcal{O}(\varepsilon)$$

and in the similar way

$$\|\varphi_{0,\varepsilon}\|^2 = \|\psi_0\|^2 + \mathcal{O}(\varepsilon^{\frac{3}{2}}).$$

According to Rayleigh-Ritz variational principle

$$\lambda_0(\alpha_2) \leq \frac{\big(\varphi_{0,\varepsilon}\,,\, -\varphi''_{0,\varepsilon} + V\varphi_{0,\varepsilon}\big)_{L^2([0,d])}}{\|\varphi_{0,\varepsilon}\|^2} = \frac{\lambda_0(\alpha_1)\|\psi_0\|^2 + (\alpha_2 - \alpha_1)\psi_0(d)^2 + \mathcal{O}(\varepsilon)}{\|\psi_0\|^2 + \mathcal{O}(\varepsilon^{3/2})}.$$

Finally, for $\varepsilon \to 0^+$ we have

$$\lambda_0(\alpha_2) \leq \lambda_0(\alpha_1) + (\alpha_2 - \alpha_1)\frac{\psi_0(d)^2}{\|\psi_0\|^2}. \quad \blacksquare$$

Remark 3 *The operator defined in the previous lemma is really self-adjoint, it has the purely discrete spectrum and the eigenfunctions can be chosen real [9].*

Now we are ready to prove the following proposition.

Proposition 3 *Let $\gamma(s) \geq 0$ for every $s \in \mathbb{R}$. Then* $\inf \sigma\big(L_\Omega^{DN}\big) \geq \frac{\pi^2}{4d^2}$.

PROOF: We are going to show that for every $\Phi \in Q(q_0)$, the functional $q[\Phi]$ is nonnegative. We decompose the function Φ to the transverse orthonormal basis, *i.e.*

$$\Phi(s,u) \;=\; \sum_{k=0}^{\infty} \phi_k(s)\chi_k(s;u),$$

where $\chi_k(s;u)$ are normalized eigenfunctions of the self-adjoint operator $h(s)$ acting in the Hilbert space $L^2\big([0,d],(1-u\gamma(s))\,\mathrm{d}u\big)$ associated with the quadratic form

$$\tilde{q}_0(f,g) = \int_0^d \big(1-u\gamma(s)\big)\overline{\frac{df(s;u)}{du}}\,\frac{dg(s;u)}{du}\;\mathrm{d}u$$

defined on the domain

$$Q(\tilde{q}_0) = \big\{\psi \in AC\big([0,d]\big)\;\big|\;\psi(s;0)=0\big\}$$

for almost every values of the parameter s. Let us denote by λ_k the eigenvalues of this operator corresponding to eigenfunctions χ_k. Then we can write the functional q in the following form

$$\begin{aligned} q[\Phi] \;&=\; \int_{\mathbb{R}} \mathrm{d}s \int_0^d \mathrm{d}u\Big(\Big|\frac{\partial\Phi(s,u)}{\partial s}\Big|^2 \frac{1}{1-u\gamma(s)} + \sum_{k=0}^{\infty} |\phi_k(s)\chi_k(s;u)|^2\Big(\lambda_k - \frac{\pi^2}{4d^2}\Big)\big(1-u\gamma(s)\big)\Big) \geq \\ &\geq\; \sum_{k=0}^{\infty}\int_{\mathbb{R}} \mathrm{d}s \int_0^d \mathrm{d}u\,|\phi_k(s)\chi_k(s;u)|^2\Big(\lambda_k - \frac{\pi^2}{4d^2}\Big)\big(1-u\gamma(s)\big). \end{aligned}$$

So it is enough to show that the lowest eigenvalue $\lambda_0(s)$ of the operator $h(s)$ satisfies the condition $\lambda_0 \geq \frac{\pi^2}{4d^2}$. It is easy to see that $h(s) = -\frac{d^2}{du^2} + \frac{\gamma(s)}{1-u\gamma(s)}\frac{d}{du}$ with the domain [9]

$$D\big(h(s)\big) = \big\{\psi \in AC^2\big([0,d]\big)\;\big|\;\psi(s;0)=0,\;\psi'(s;d)=0\big\}.$$

Therefore $\chi_k(s;\cdot)$ are solutions of the equation

$$-\chi''(s;u) + \frac{\gamma(s)}{1-u\gamma(s)}\,\chi'(s;u) = \lambda(s)\chi(s;u). \tag{3.1}$$

They are linear combinations of Bessel functions

$$\chi(s;u) = A(s)\,J_0\bigg(\frac{\sqrt{\lambda(s)}}{\gamma(s)}\,\big(1-u\gamma(s)\big)\bigg) \;+\; B(s)\,N_0\bigg(\frac{\sqrt{\lambda(s)}}{\gamma(s)}\,\big(1-u\gamma(s)\big)\bigg).$$

Imposing the prescribed boundary conditions we obtain the equation for $\lambda(s)$. So $\lambda_0(s)$ is the lowest solution of the equation

$$J_0\bigg(\frac{\sqrt{\lambda(s)}}{\gamma(s)}\bigg)\,N_1\bigg(\frac{\sqrt{\lambda(s)}}{\gamma(s)}\,\big(1-d\gamma(s)\big)\bigg) - N_0\bigg(\frac{\sqrt{\lambda(s)}}{\gamma(s)}\bigg)\,J_1\bigg(\frac{\sqrt{\lambda(s)}}{\gamma(s)}\,\big(1-d\gamma(s)\big)\bigg) = 0, \tag{3.2}$$

which will be used later.

Unitary transformation $\psi(s;u) = \sqrt{1-u\gamma(s)}\,\chi(s;u)$ from $L^2\Big([0,d],\big(1-u\gamma(s)\big)\,\mathrm{d}u\Big)$ onto $L^2\big([0,d],\,\mathrm{d}u\big)$ transforms our problem to the one of searching for the lowest eigenvalue of the operator

$$\tilde{h}(s) = -\frac{d^2}{du^2} - \frac{\gamma(s)^2}{4\big(1-u\gamma(s)\big)^2}$$

on the domain

$$D\big(\tilde{h}(s)\big) = \big\{\psi \in AC^2\big([0,d]\big)\;\big|\;\psi(s;0)=0,\;\psi'(s;d) + \frac{\gamma(s)}{2\big(1-d\gamma(s)\big)}\psi(s;d)=0\big\}. \tag{3.3}$$

Let us introduce two more operators acting in the Hilbert space $L^2\big([0,d]\big)$, $h_0 = -\frac{d^2}{du^2}$ with the domain $D\big(h(s)\big)$ and $h_1(s) = -\frac{d^2}{du^2}$ with the domain $D\big(\tilde{h}(s)\big)$. It is known that the lowest eigenvalue of h_0 is $\frac{\pi^2}{4d^2}$. Let the lowest eigenvalue of $h_1(s)$ is denoted by $\lambda_{0,1}(s)$ and the corresponding real eigenfunction by ψ_0. The normalized eigenfunction corresponding to the lowest eigenvalue $\lambda_0(s)$ of the operator $\tilde{h}(s)$ is denoted by ψ_1. Using Lemma 1 and the Rayleigh-Ritz variational principle we can write

$$\begin{aligned}
\frac{\pi^2}{4d^2} &\leq \lambda_{0,1}(s) - \frac{\gamma(s)}{2\big(1-\gamma(s)\big)}\,\frac{\psi_0(s;d)^2}{\|\psi_0\|^2} \leq \\
&\leq \lambda_0(s) - \frac{\gamma(s)}{2\big(1-\gamma(s)\big)}\,\frac{\psi_0(s;d)^2}{\|\psi_0\|^2} + \int_0^d \frac{\gamma(s)^2}{4\big(1-u\gamma(s)\big)^2}\big|\psi_1(s;u)\big|^2\,\mathrm{d}u \leq \\
&\leq \lambda_0(s) - \frac{\gamma(s)}{2\big(1-\gamma(s)\big)}\,\frac{\psi_0(s;d)^2}{\|\psi_0\|^2} + \frac{\gamma(s)^2}{4\big(1-d\gamma(s)\big)^2}.
\end{aligned} \tag{3.4}$$

The eigenfunction ψ_0 is given by the equation

$$-\psi_0''(s;u) = \lambda_{0,1}(s)\,\psi_0(s;u)$$

and it satisfies boundary conditions defined in (3.3). Hence $\psi_0(s;u) = \sin\sqrt{\lambda_{0,1}(s)}u$ and $\lambda_{0,1}(s)$ is the lowest solution of the equation

$$\sqrt{\lambda_{0,1}(s)} = -\,\alpha(s)\,\tan\sqrt{\lambda_{0,1}(s)}d$$

where $\alpha(s) = \frac{\gamma(s)}{2\left(1-d\gamma(s)\right)}$. Now necessarily $\lambda_{0,1}(s) \in \left(\frac{\pi^2}{4d^2}, \frac{\pi^2}{d^2}\right)$ in accordance with the first estimate in (3.4). We can calculate

$$\frac{\psi_0(s;d)^2}{\|\psi_0\|^2} = \frac{\sin^2\sqrt{\lambda_{0,1}(s)}d}{\frac{d}{2} - \frac{\sin 2\sqrt{\lambda_{0,1}(s)}d}{4\sqrt{\lambda_{0,1}(s)}}} = \frac{2\lambda_{0,1}(s)}{d\big(\alpha(s)^2+\lambda_{0,1}(s)\big)+\alpha(s)}.$$

To see that $\lambda_0(s) \geq \frac{\pi^2}{4d^2}$, which we want to prove, it is now enough to show

$$\alpha(s)\,\frac{2\lambda_{0,1}(s)}{d\big(\alpha(s)^2+\lambda_{0,1}(s)\big)+\alpha(s)} - \alpha(s)^2 \geq 0$$

due to (3.4). Using inequality $\lambda_{0,1}(s) \geq \frac{\pi^2}{4d^2}$ it can be seen that the inequality above holds provided

$$\frac{\pi^2}{4} \geq \frac{\gamma(s)^2 d^2}{4\big(1-d\gamma(s)\big)^2}\,\frac{1+\frac{d\gamma(s)}{2\left(1-d\gamma(s)\right)}}{2-\frac{d\gamma(s)}{2\left(1-d\gamma(s)\right)}} \tag{3.5}$$

and $d\gamma(s) < \frac{4}{5}$. The right hand side of the inequality (3.5) is obviously increasing function of $d\gamma(s)$ and it reaches value 0 for $d\gamma(s) = 0$ and value $2 < \frac{\pi^2}{4}$ for $d\gamma(s) = \frac{2}{3}$.

In the remaining interval $d\gamma(s) \in [\frac{2}{3}, 1)$ we can study directly the equation (3.2) using up to the first four terms of the known expansions of Bessel functions into series [10]. It can be seen that the equation cannot be satisfied for values $d\gamma(s) \in [\frac{2}{3}, 1)$ and $\lambda_0 < \frac{\pi^2}{4d^2}$. ∎

Corollary 2 *Let γ satisfy the same assumption as in Proposition 3 and let it have a compact support. Then the operator (1.2) has no discrete eigenvalue.*

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